SHM
Structural Health Monitoring

Crack Modeling for Structural Health Monitoring

M. I. Friswell[1,*] and J. E. T. Penny[2]

[1]*Department of Mechanical Engineering, University of Wales Swansea, Singleton Park, Swansea SA2 8PP, UK*
[2]*School of Engineering and Applied Science, Aston University, Birmingham B4 7ET, UK*

There are a number of approaches to the modeling of cracks in beam structures reported in the literature, that fall into three main categories; local stiffness reduction, discrete spring models, and complex models in two or three dimensions. This paper compares the different approaches to crack modeling, and demonstrates that for structural health monitoring using low frequency vibration, simple models of crack flexibility based on beam elements are adequate. This paper also addresses the effect of the excitation for breathing cracks, where the beam stiffness is bilinear, depending on whether the crack is open or closed. Most structural health monitoring methods assume that the structure is behaving linearly, whereas in practice the response will be nonlinear to an extent that varies with the form of the excitation. This paper will demonstrate these effects for a simple beam structure.

Keywords crack · structural health monitoring · vibration · breathing crack · finite element modeling

1 Introduction

The identification of the location and depth of a crack in beam type structures is an important example of structural health monitoring, and has received considerable attention. Most of the approaches use the modal data of a structure before damage occurs as baseline data, and all subsequent tests are compared to it. Any deviation in the modal properties from this baseline data is used to estimate the crack size and location. The advantage of using this baseline data is that some allowance is made for modeling errors. Doebling et al. (1998) gave a review of the research on crack and damage detection and location in structures using vibration data. The estimation of crack size and location generally requires a mathematical model (usually a finite element model) along with experimental modal parameters of the structure. The estimation methods are predominately based on the change in natural frequencies, the change in mode shapes or measured dynamic flexibility (for example, Cawley and Adams, 1979; Friswell et al., 1994; Doebling et al., 1996). Salawu (1997) gave a review of research work on crack detection based on the change in natural frequencies. Another class of crack detection methods, also based on the change in modal parameters, uses a different identification approach based on the modification of structural model matrices (such as mass, stiffness and damping matrices)

*Author to whom correspondence should be addressed.
E-mail: m.i.friswell@swansea.ac.uk

Vol 1(2): 0139–148
[1475-9217 (200210) 1:2;139–148; 10.1106/145792102028836]

using model updating methods (Doebling et al., 1998).

There are a number of approaches to the modeling of cracks in beam structures reported in the literature, that fall into three main categories; local stiffness reduction, discrete spring models, and complex models in two or three dimensions. Dimarogonas (1996) and Ostachowicz and Krawczuk (2001) gave comprehensive surveys of crack modeling approaches. The simplest methods for finite element models reduce the stiffness locally, for example by reducing a complete element stiffness to simulate a small crack in that element. This approach suffers from problems in matching damage severity to crack depth, and is affected by the mesh density. An improved method introduces local flexibility based on physically based stiffness reductions, where the crack position may be used as a parameter for identification purposes. The second class of methods divides a beam type structure into two parts that are pinned at the crack location and the crack is simulated by the addition of a rotational spring. These approaches are a gross simplification of the crack dynamics and do not involve the crack size and location directly. The alternative, using beam theory, is to model the dynamics close to the crack more accurately, for example producing a closed form solution giving the natural frequencies and mode shapes of cracked beam directly or using differential equations with compatible boundary conditions satisfying the crack conditions. The approaches of Christides and Barr (1984) and Sinha et al. (2002), and of Lee and Chung (2001) will be considered in more detail later. Alternatively two or three dimensional finite element meshes for beam type structures with a crack may be used. Meshless approaches may also be used, but are more suited to crack propagation studies. No element connectivity is required and so the task of remeshing as the crack grows is avoided, and a growing crack is modeled by extending the free surfaces corresponding to the crack (Belytschko et al., 1995). However, the computational cost of these meshless methods generally exceeds that of conventional FEA. Rao and Rahman (2001) avoided this difficulty by coupling a meshless region near the crack with an FEA model in the remainder of the structure. The two and three dimensional approaches produce detailed and accurate models but are a complicated and computational intensive approach to model simple structures like beams. Furthermore, finite element models will contain modeling errors, the data will include measurement errors, and the use of low frequency vibration will tend to average out localized effects. The result is that these very detailed models do not substantially improve the results from crack detection and location algorithms.

The models described in this paper are for an open crack. A breathing crack, which opens and closes, can produce interesting and complicated nonlinear dynamics. Brandon (1998) and Kisa and Brandon (2000) gave an overview of some of the techniques that may be applied. Many techniques to analyze the resulting nonlinear dynamics are based on approximating the bilinear stiffness when the crack opens and closes. The approach proposed in this paper is able to approximate the stiffness matrix for the beam with an open crack. Such an approach will certainly be more efficient than those based on two or three dimensional finite element models for time integration of the equations of motion, although any realistic multi degree of freedom nonlinear analysis would have to be based on a reduced order model of the structure. However, the nonlinearity introduced by the crack is often weak, and many of the common testing techniques will tend to linearize the response (Friswell and Penny, 1992; Leonard et al., 2001; Worden and Tomlinson, 2001). Sinusoidal forcing will tend to emphasise the nonlinearity, and damage detection methods based on detecting harmonics of the forcing frequency have been proposed (Shen, 1998). In rotor dynamic applications, these approaches are useful because the forcing is inherently sinusoidal (Dimarogonas, 1996). However in structural health monitoring applications, this approach requires considerable hardware and software to implement, and also requires a lengthy experiment. The second purpose of this paper is to consider the importance of the effects of the nonlinearity in structural health monitoring of beam type structures with breathing cracks. In particular, the effects of the nonlinearity on measurements obtained using impact and random excitations will be considered.

2 Models of Open Cracks

Although the geometry of a crack can be very complicated, the contention in this paper is that for low frequency vibration only an effective reduction in stiffness is required. Thus, for comparison, a simple model of an open crack, which is essentially a saw cut, will be used. This will allow the comparison of models using beam elements, with those using plate elements. Only a selection of beam models will be used, that illustrate the fact that many beam models are able to model the effect of the crack at low frequencies.

Two standard approaches using beam elements are shown in Figure 1. In the first approach, the stiffness of a single element is reduced, which requires a fine mesh, and also the derivation of the effect of a crack on the element stiffness. In the second approach, the beam is separated into two halves at the crack location. The beam sections are then pinned together and a rotational spring used to model the increased flexibility due to the crack. Translational springs may also be used in place of the pinned constraint. The major difficulties with this approach is that a finite element node must be placed at the crack location, requiring remeshing for health monitoring applications, and the relationship between the spring stiffness and crack depth needs to be derived.

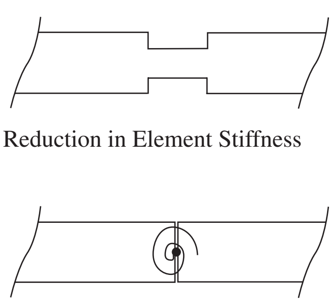

Figure 1 Simple crack models for beam elements.

For illustration, the open crack will be modeled using plate elements. The geometry is modeled by removing elements where the crack is located. Figure 2 shows this in the case of plate elements, and shows the side view of the mesh used. Clearly more complex methods may be used, and the review papers quoted earlier give further details.

2.1 The Approach of Christides and Barr

Clearly some of the material adjacent to the crack will not be stressed and thus will offer only a limited contribution to the stiffness. The actual form of this increased flexibility is quite complicated, but in this paper we approximate this phenomenon as a variation in the local flexibility. In reality, for a crack on one side of a beam, the neutral axis will change in the vicinity of the crack, but this will not be considered here. Shen and Pierre (1994) and Carneiro (2000) have extended this approach to consider single edge cracks. Christides and Barr (1984) considered the effect of a crack in a continuous beam and calculated the stiffness, EI, for a rectangular beam to involve an exponential function given by

$$EI(x) = \frac{EI_0}{1 + C\exp(-2\alpha|x - x_c|/d)} \qquad (1)$$

where $C = (I_0 - I_c)/I_c \cdot I_0 = (wd^3)/12$ and $I_c = w(d - d_c)^3/12$ are the second moment of areas of the undamaged beam and at the crack. w and d are the width and depth of the undamaged beam, and d_c is the crack depth. x is the position along the beam, and x_c the position of the crack. α is a constant that Christides and Barr estimated from experiments to be 0.667. The inclusion of the stiffness reduction of Christides and Barr (1984) into a finite element model of a structure, using

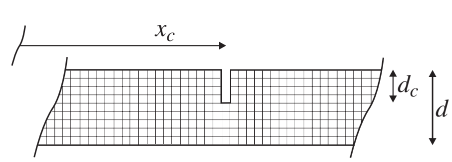

Figure 2 A simple crack model using plate elements.

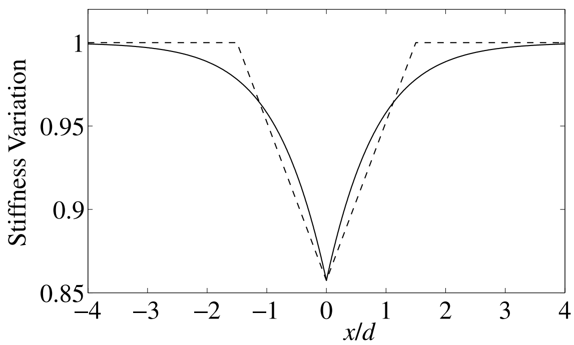

Figure 3 The variation in beam stiffness for the approaches of Christides and Barr (1984) (solid line) and Sinha et al. (2002) (dashed line).

beam elements, is complicated because the flexibility is not local to one or two elements, and thus the integration required to produce the stiffness matrix for the beam would have to be performed numerically every time the crack position changed. Furthermore, for complex structures, without uniform long beams, Equation (1) would only be approximate. Sinha et al. (2002) used a simplified approach, where the stiffness reduction of Christides and Barr was approximated by a triangular reduction in stiffness. An example of this approximation is shown in Figure 3, for a crack of depth 5%, located at $x = 0$. The advantages of this simplified model is that the stiffness reduction is now local, and the stiffness matrix may be written as an explicit function of the crack location and depth. For cracks of small depth, a good approximation to the length of the beam influenced by the crack is $2d/\alpha$.

2.2 Fracture Mechanics Approach

An alternative approach is to estimate the increased flexibility caused by the crack, using empirical expressions of stress intensity factors from fracture mechanics. Lee and Chung (2001) gave such an approach based on the relationships given by Tada et al. (1973). Only a summary of the relevant equations will be given here. The element stiffness matrix is given by

$$\mathbf{K}_c = \mathbf{T}^T \mathbf{C}^{-1} \mathbf{T} \tag{2}$$

where the transformation, $\mathbf{T}$, is

$$\mathbf{T} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}. \tag{3}$$

The flexibility matrix, $\mathbf{C}$, for an element containing the crack in the middle, is given by

$$\mathbf{C} = \frac{1}{6EI} \begin{bmatrix} 2\ell_e^3 & 3\ell_e^2 \\ 3\ell_e^2 & 6\ell_e \end{bmatrix} + \frac{18\pi(1-\nu^2)}{Ewd^2} \begin{bmatrix} \ell_e^2 & 2\ell_e \\ 2\ell_e & 4 \end{bmatrix} \int_0^{d_c/d} \beta F_I^2(\beta)\, \mathrm{d}\beta. \tag{4}$$

where ℓ_e is the element length and ν is Poisson's ratio. $F_I(\beta)$ is the correction factor for the stress intensity factor, and may be approximated as

$$F_I(\beta) = \sqrt{\frac{\tan(\pi\beta/2)}{\pi\beta/2}} \frac{0.923 + 0.199[1 - \sin(\pi\beta/2)]^4}{\cos(\pi\beta/2)}. \tag{5}$$

This formulation does give the stiffness matrix of the element containing the crack explicitly in terms of the crack depth. There are two difficulties with using this approach for structural health monitoring. The main problem is that the crack is located at the centre of the element, requiring that the finite element mesh be redefined as the crack moves. Furthermore, the stiffness matrix of the crack is a complicated function of the crack depth, and does not depend on the crack location explicitly.

2.3 A Numerical Comparison of the Models

The approaches to crack modeling will be compared using a simple example of a steel cantilever beam 1 m long, with cross section 25 × 50 mm. Bending in the more flexible plane is considered. The crack is assumed to be located at a distance 200 mm from the fixed end, and has a constant depth of 10 mm across the beam width.

The beam is modeled using 20 Euler–Bernoulli beam elements, and gives the natural frequencies shown in Table 1. For the plate elements, the length is split into 401 elements and the depth into 10 elements. Thus the elements are approximately 2.5 mm square. A large number of elements is required because an element with linear shape functions is used. Table 1 shows the estimated natural frequencies using the Quad4 element in the Structural Dynamics Toolbox (Balmes, 2000).

The damaged beam was also modeled using the approaches discussed earlier, and the results are shown in Table 2. The beam models all contain 20 elements, and the nodes are arranged such that the crack occurs in the middle of an element. Of course in the case of the discrete rotational spring a node is placed at the crack location. The reduction in the element stiffness is adjusted so that the percentage change in the first natural frequency is the same as that for the plate model. The other beam models are adjusted in a similar way. In the plate model, the crack is simulated by removing 4 elements and thus represents a saw cut 10 mm deep. The row of elements below the crack is also made thinner, so that the crack has negligible width. The differences in the lower natural frequencies are very similar for all models, and these differences are smaller than the changes that would occur due to small modeling errors, or changes due to environmental effects. Of course the accuracy at higher frequencies becomes less since the modes are influenced more by local stiffness variations.

Table 1 The natural frequencies (in Hz) for the undamaged beam.

	Beam	*Plate*
Number DoF	40	13233
Modes 1	20.709	20.707
2	129.78	129.39
3	363.40	360.62
4	712.16	701.96
5	1177.4	1150.6

Table 2 The percentage changes in the natural frequencies for the damaged beam.

	Beam				
Modes	*Element Stiffness Reduction*	*Discrete Spring*	*Sinha et al. (2002)*	*Lee and Chung (2001)*	*Plate*
1	4.18	4.18	4.18	4.18	4.18
2	0.07	0.04	0.08	0.04	0.04
3	1.24	1.23	1.24	1.20	1.22
4	2.99	3.08	2.98	2.99	3.07
5	2.37	2.45	2.37	2.34	2.69

2.4 Comparison with Experimental Results

The previous section has shown that the natural frequencies predicted from different models are very close. Of course the question is whether the differences in these predictions are smaller than the measurement errors. As a demonstration, the example of Rizos et al. (1990) will be used. Kam and Lee (1992) and Lee and Chung (2001) also used these results. The example is a steel cantilever beam of cross-section 20×20 mm and length 300 mm. Table 3 shows the measured and predicted frequencies of the uncracked beam. Rizos et al. (1990) propagated cracks at a number of different positions and depths, but here only a crack 80 mm from the cantilever root, and depths of 2 and 6 mm will be considered. Table 3 also shows the measured natural frequencies for these crack depths. The damaged cantilever beam is modeled using the beam methods described earlier. The depth of the crack is optimized so that the percentage change in the first mode matches the experimental result, to allow for possible errors in measuring the crack depth. Tables 4 and 5 show the measured and predicted frequency changes for the two crack depths. The results clearly show that the differences in the natural frequencies predicted by the models are smaller than the measurement errors.

Table 3 The natural frequencies (in Hz) for the experimental cantilever beam example.

		Experimental		
Modes	*FE Model Undamaged*	*Undamaged*	*2 mm Crack*	*6 mm Crack*
1	185.1	185.2	184.0	174.7
2	1159.9	1160.6	1160.0	1155.3
3	3247.6	3259.1	3245.0	3134.8

Table 4 The percentage changes in the natural frequencies for the damaged beam with a 2 mm crack.

Modes	*Element Stiffness Reduction*	*Discrete Spring*	*Sinha et al. (2002)*	*Lee and Chung (2001)*	*Experi-mental*
1	0.648	0.648	0.648	0.648	0.648
2	0.065	0.063	0.090	0.063	0.052
3	0.606	0.610	0.604	0.606	0.433

Table 5 The percentage changes in the natural frequencies for the damaged beam with a 6 mm crack.

Modes	*Element Stiffness Reduction*	*Discrete Spring*	*Sinha et al. (2002)*	*Lee and Chung (2001)*	*Experi-mental*
1	5.67	5.67	5.67	5.67	5.67
2	0.56	0.54	0.88	0.54	0.46
3	4.92	4.95	4.49	4.92	3.81

Thus the simple models for cracked beams may be used with confidence in health monitoring applications.

3 A Single Mode Approximation for Breathing Cracks

Suppose that the structure is forced such that its response may be approximated using a single degree of freedom approximation. Although this situation rarely occurs in practice, the following analysis will provide insight into the general response of breathing cracks that will be addressed in the next section. The equation of motion is given by

$$m\ddot{q} + c\dot{q} + k(q)q = f(t) \tag{6}$$

where m and c are the mass and damping terms, $f(t)$ is the applied force, and the stiffness is given by the bilinear function,

$$k(q) = \begin{cases} k_c & \text{if } q > q_0 \\ k_t & \text{if } q < q_0 \end{cases} \tag{7}$$

q_0 is the value of the response when the crack opens or closes. It is assumed this happens instantaneously, and for a particular mode the value of q_0 may be related to the curvature at the crack. In practice the transition from open to closed will not be immediate, leading to a weaker nonlinearity. But here the instantaneous transition is considered as the worst case.

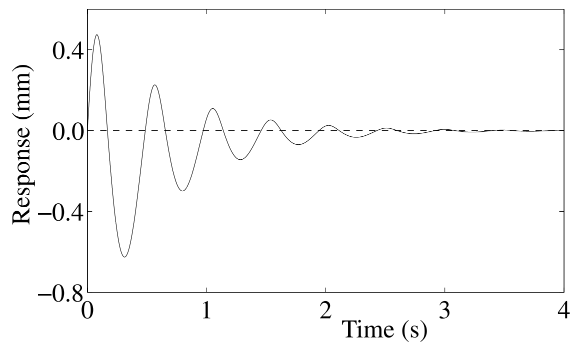

Figure 4 The impulse time response for the SDoF example.

If Equations (6) and (7) represented a single mode approximation, then the mode shape is assumed not to change. Thus the modal mass and damping coefficients are constant, and only the modal stiffness value would change. Friswell and Penny (1992) integrated these equations explicitly for an impulse response, and demonstrated the response for a sinusoidal response using numerical integration. In this paper we will use numerical integration throughout.

Figure 4 shows the impulse time response with,

$$m = 1\,\text{kg}, \quad c = 3\,\text{kg/s}, \quad k_c = 350\,\text{N/m}, \quad k_t = 100\,\text{N/m}, \quad q_0 = 0$$

This is the same example as given by Friswell and Penny (1992). The change in stiffness between the open and closed phases is larger than would occur in practice, but the example serves as an illustration. The simulation was performed in MATLAB using ode45 with the stiffness changes identified as 'events'. Figure 5 shows the Fourier Transform of this response and clearly shows a single peak at just over 2 Hz. Although the presence of the harmonics clearly indicates that the response is nonlinear, the magnitude of the response at the harmonics is quite small. Remember that the stiffness difference between

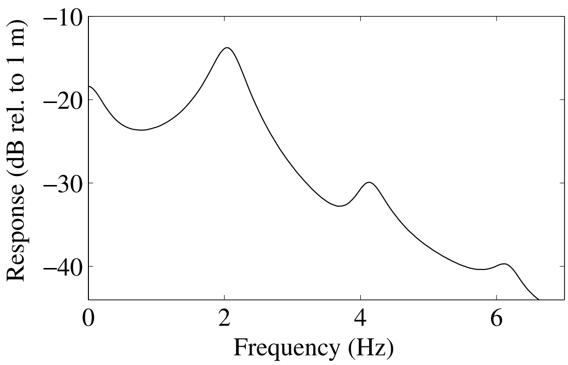

Figure 5 The Fourier transform of the impulse response for the SDoF example.

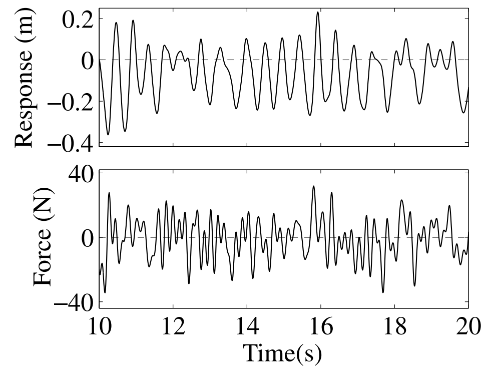

Figure 6 The applied force and time response for the SDoF example using band-limited random excitation.

the open and closed crack has been exaggerated for illustration. In practical cases, only a single natural frequency, which is based on the composite time period of the opening and closing parts of the response, would be identified.

The second example is the response of the system to a random force. Worden and Tomlinson (2001) indicated that random excitation tends to make the nonlinearity difficult to detect from the measured frequency response functions, although the measured coherence may be used. These features of random excitation will now be demonstrated for the single mode approximation to a breathing crack. Figure 6 shows the band-limited random force and the resulting response. It is clear that the response, again obtained from numerical integration, has filtered out the higher frequency content of the force. Each forced response was simulated for 50 s, and the force considered was burst random, where the force was set to zero, 8 s before the end so that any leakage effects were minimized. Twenty averages were used to calculate the frequency response function and the coherence, shown in Figures 7 and 8 respectively. The frequency response function appears to be linear, although very noisy. No noise has been added to the simulation, and this effect is purely due to the nonlinearity arising from the breathing crack. There is no increase in the response around 4 Hz, however the coherence is quite poor, with a reduction around 4 Hz. From these results it is possible that a nonlinearity would be detected, although in this example the effect of the nonlinearity is magnified. It practice, with small cracks, it is unlikely that the nonlinear effects would be distinguished from measurement noise.

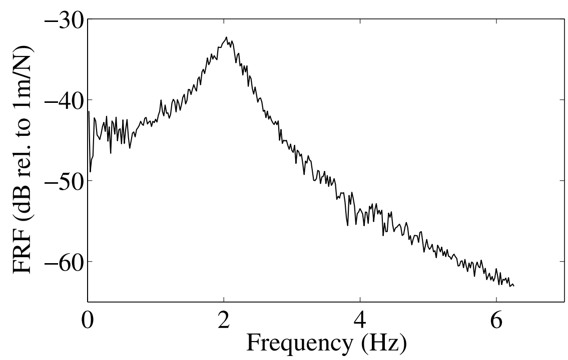

Figure 7 The estimated frequency response function for the SDoF example using band-limited random excitation.

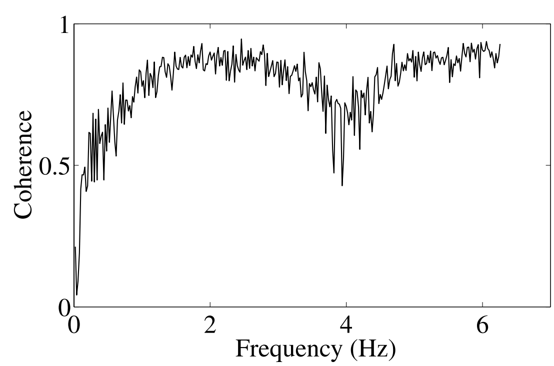

Figure 8 The coherence for the SDoF example using band-limited random excitation.

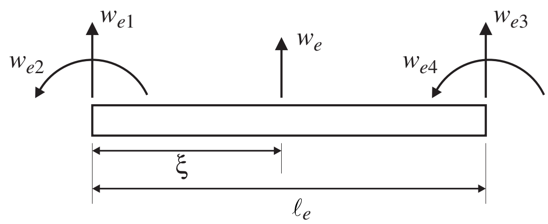

Figure 9 The definition of the local element co-ordinates.

3.1 The Response of a Beam with a Breathing Crack

Suppose that the crack opens and closes instantaneously at a known beam curvature at the crack location. Assume that the crack is located within element *j*, as shown in Figure 9. Then the displacement within the element is

$$w(\xi) = [N_{e1}(\xi)N_{e2}(\xi)N_{e3}(\xi)N_{e4}(\xi)]\begin{Bmatrix} w_{e1} \\ w_{e2} \\ w_{e3} \\ w_{e4} \end{Bmatrix} \qquad (8)$$

where $N_{ei}(\xi)$ is the *i*th cubic shape function, and w_{ei} are the displacements and rotations at the nodes.

The curvature at the crack location, γ, is then given by

$$\gamma = w''(\xi_c) = [N''_{e1}(\xi_c)N''_{e2}(\xi_c)N''_{e3}(\xi_c)N''_{e4}(\xi_c)]\begin{Bmatrix} w_{e1} \\ w_{e2} \\ w_{e3} \\ w_{e4} \end{Bmatrix} \qquad (9)$$

where ξ_c is the crack location in local co-ordinates. Thus the curvature at the crack is obtained by premultiplying the nodal coordinate vectors of the element containing the crack by a constant row vector. Thus, in terms of the global generalized coordinates of the full model, $\mathbf{q}$, the curvature is given by

$$\gamma = \mathbf{S}\,\mathbf{q} \qquad (10)$$

for some constant row vector $\mathbf{S}$, that only has 4 nonzero terms, given in Equation (9).

The estimation of curvature is more difficult for the model where the beam is pinned at the crack location and a rotational spring used to model the crack flexibility. Here the beam slope, and curvature, will be discontinuous, and it is difficult to devise a suitable estimate for curvature, although from the slope information it is easy to determine if the curvature is positive or negative. The discrete spring model has the further complication that an extra degree of freedom is added when the pinned connection is introduced into the undamaged beam model. For a fixed crack location, this may be overcome by using the pinned model with a very stiff rotary spring for the undamaged beam. However, in structural health monitoring the crack location has to be estimated, requiring that the mesh and the degrees of freedom be redefined at every iteration.

The equations of motion of the complete beam structure is then

$$\mathbf{M}\mathbf{q} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}(\gamma)\mathbf{q} = \mathbf{f}(t) \qquad (11)$$

where $\mathbf{M}$ and $\mathbf{C}$ are the mass and damping matrices, $\mathbf{f}(t)$ is the applied force, and the stiffness matrix is given by the bilinear function,

$$\mathbf{K}(\gamma) = \begin{cases} \mathbf{K}_c & \text{if } \gamma > \gamma_0 \\ \mathbf{K}_t & \text{if } \gamma < \gamma_0 \end{cases} \qquad (12)$$

where γ_0 is the value of the beam curvature at the crack, when the crack opens or closes.

For simulation purposes, it is necessary to use reduced order models of the structure with the crack open and closed. There are two possible approaches. The first is to reduce the two models (i.e., with the crack open and closed) using the lower modes of each. Although such an approach is efficient for a linear model, the mode shapes will be slightly different, and so the transformation matrix in each case will be different. Thus when the crack changes from open to closed not only will the stiffness matrix change, but the system state vector will also have to be transformed. Furthermore, the matrix, $\mathbf{S}$, required to calculate the curvature will be different in each case. The alternative is to have a

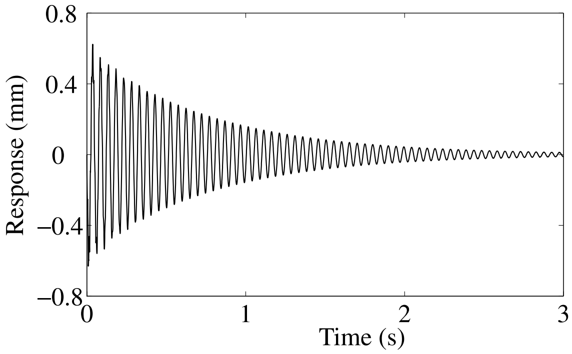

Figure 10 The impulse time response for the beam example.

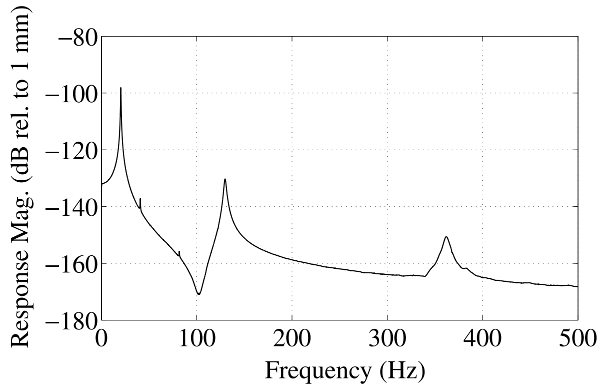

Figure 11 The Fourier transform of the impulse response for the beam example.

fixed transformation, based on the undamaged model, and use this to transform the structure when the crack is open. In this case the transformed matrices will not be diagonal, and the lower modes will only be reproduced approximately. However, since the effect of a crack on the beam dynamics is relatively small, the approximation should be reasonable. The advantage is that the system state vector does not have to be transformed, and the calculation of the curvature is the same whether the crack is open or closed.

The cracked beam discussed earlier is modeled using 20 elements, and the open crack is modeled using the approach of Sinha et al. (2002). Five modes are retained to calculate the response, and an impulse force at the tip of the beam is simulated using a half sine pulse of magnitude 100 N and duration 1 ms. One percent damping is added to all modes, based on the undamaged model. The crack is assumed to open and close when the curvature changes sign. Figure 10 shows the response of the beam at the tip, and Figure 11 shows the corresponding FFT. Although the response appears to contain only the lower mode, the higher modes contribute to the initial response, as shown in the FFT plot. The harmonics of the first mode appear in the FFT, although a small quantity of noise would mask these features. The main peaks occur at frequencies that correspond to composite natural frequencies, that is between the frequencies for the open and closed crack conditions.

4 Conclusions

This paper has considered two major features in the modeling of beam structures with breathing cracks. It was demonstrated that relatively simple beam models, with a small number of degrees of freedom, are able to model the effects of an open crack. Thus models with plate or brick elements are not necessary for structural health monitoring. Indeed two and three dimensional models are difficult to apply in health monitoring applications because not only is the number of degrees of freedom very large, the mesh must be revised as the estimated crack location changes. The second effect considered was the nonlinear dynamics of a breathing crack. It was demonstrated by using a bilinear stiffness model for the crack opening and closing that the impulse and random responses approximated a linear response with natural frequencies that were a composite of the corresponding frequencies for the beam with the crack open and closed. If these results were used for health monitoring, then the crack would be estimated at the correct location, but the crack depth estimate would be smaller than the real crack.

References

Balmes, E. (2000). Structural dynamics toolbox: for use with MATLAB, user's guide, Version 4.

Belytschko, T., Lu, Y.Y. and Gu, L. (1995). Crack propagation by element-free galerkin methods. *Engineering Fracture Mechanics*, *51*(*2*), 295–315.

Brandon, J.A. (1998). Some insights into the dynamics of defective structures. *Proceedings of the Institution of Mechanical Engineers Part C: Journal of Mechanical Engineering Science*, *212*(*6*), 441–454.

Carneiro, S.H.S. (2000). Model-based vibration diagnostic of cracked beams in the time domain. PhD Thesis, Virginia Polytechnic Institute and State University, Blacksburg, Virginia.

Cawley, P. and Adams, R.D. (1979). The locations of defects in structures from measurements of natural frequencies. *Journal of Strain Analysis*, *14*(*2*), 49–57.

Christides, S. and Barr, A.D.S. (1984). One dimensional theory of cracked bernoulli-euler beams. *International Journal of Mechanical Science*, *26*(*11*/*12*), 639–648.

Dimarogonas, A.D. (1996). Vibration of cracked structures: a state of the art review, *Engineering Fracture Mechanics*, *55*, 831–857.

Doebling, S.W., Peterson, L.D. and Alvin, K.F. (1996). Estimation of reciprocal residual flexibility from experimental modal data. *AIAA Journal*, *34*(*8*), 1678–1685.

Doebling, S.W., Farrar, C.R. and Prime, M.B. (1998). A summary review of vibration-based damage identification methods. *Shock and Vibration Digest*, *30*(*2*), 91–105.

Friswell, M.I. and Penny, J.E.T. (1992). A simple nonlinear model of a cracked beam. *10th International Modal Analysis Conference*, San Diego, California, USA. pp. 516–521.

Friswell, M.I., Penny, J.E.T. and Wilson, D.A.L. (1994). Using vibration data and statistical measures to locate damage in structures, *Modal Analysis*: *The International Journal of Analytical and Experimental Modal Analysis*, *9*(*4*), 239–254.

Kam, T.Y. and Lee, T.Y. (1992). Detection of cracks in structures using modal test data. *Engineering Fracture Mechanics*, *42*, 381–387.

Kisa, M. and Brandon, J.A. (2000). The effects of closure of cracks on the dynamics of a cracked cantilever beam. *Journal of Sound and Vibration*, *238*(*1*), 1–18.

Lee, Y.-S. and Chung, M.-J. (2001). A study on crack detection using eigenfrequency test data. *Computers and Structures*, *77*, 327–342.

Leonard, F., Lanteigne, J., Lalonde, S. and Turcotte, Y. (2001). Free vibration behaviour of a cracked cantilever beam and crack detection. *Mechanical Systems and Signal Processing*, *15*(*3*), 529–548.

Ostachowicz, W. and Krawczuk, M. (2001). On modelling of structural stiffness loss due to damage. DAMAS 2001: *4th International Conference on Damage Assessment of Structures*, Cardiff, pp. 185–199.

Rao, B.N. and Rahman, S. (2001). A coupled meshless – finite element method for fracture analysis of cracks. *International Journal of Pressure Vessels and Piping*, *78*, 647–657.

Rizos, P.F., Aspragathos, N. and Dimarogonas, A.D. (1990). Identification of crack location and magnitude in a cantilever beam from the vibration modes. *Journal of Sound and Vibration*, *138*(*3*), 381–388.

Salawu, O.S. (1997). Detection of structural damage through changes in frequencies: a review, *Engineering Structures*, *19*(*9*), 718–723.

Shen, M.-H.H. and Pierre, C. (1994). Free vibration of beams with a single-edge crack. *Journal of Sound and Vibration*, *170*(*2*), 237–259.

Shen, M.-H.H. (1998). On-line structural damage detection, in *Structronic Systems: Smart Structures, Devices and Systems, Part 1: Smart Materials and Structures*. Guran, A., Tzou, H.-S., Anderson, G.L. and Natori, M. (eds.), pp. 271–332, World Scientific, Singapore.

Sinha, J.K., Friswell, M.I. and Edwards, S. (2002). Simplified models for the location of cracks in beam structures using measured vibration data. *Journal of Sound and Vibration*, *251(1)*, 13–38.

Tada, H., Paris, P. and Irwin, G. (1973). *The stress analysis of cracks handbook*, Del Research Corporation, Hellertown, Pennsylvania.

Worden, K. and Tomlinson, G.R. (2001). *Nonlinearity in structural dynamics*: *detection*, *identification and modelling*, IoP Publishing, Bristol, UK.