# Cramér-Rao Bounds for Non-Linear Filtering with Measurement Origin Uncertainty

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**Abstract –** *We are concerned with the problem of tracking a single target using multiple sensors. At each stage the measurement number is uncertain and measurements can either be target generated or false alarms. The Cramér-Rao bound gives a lower bound on the performance of any unbiased estimator of the target state. In this paper we build on earlier research concerned with calculating Posterior Cramér-Rao bounds for the linear filtering problem with measurement origin uncertainty. We derive the Posterior Cramér-Rao bound for the multi-sensor, non-linear filtering problem. We show that under certain assumptions this measurement origin uncertainty again expresses itself as a constant information reduction factor. Moreover we discuss how these assumptions can be relaxed, and the complications that occur when they no longer hold. We present an example concerned with multi-sensor management. We show that by utilizing the Cramér-Rao bound we are able to determine the combination of sensors that will enable us to achieve the most accurate tracking performance. Simulation results, using a probabilistic data association filter confirm our predictions.*

**Keywords:** Cramér-Rao lower bound, Fisher Information Matrix, non-linear filtering, data association, resource management.

## 1 Introduction

The Cramér-Rao bound (CRLB) is defined to be the inverse of the Fisher Information Matrix (e.g. see [13]) and provides a lower bound on the performance of any unbiased estimator of an unknown parameter vector. This provides a powerful tool that, within the context of target tracking, has been used to assess the performance of unbiased estimators of track parameters (for deterministic target motion: see [3] and [4]). In the case of dynamic and uncertain target motion the *Posterior* Cramér-Rao Lower Bound (PCRLB) has been used to determine performance bounds for recursive Bayesian estimators of the uncertain target state (for example, see [10] and [11]).



Moreover, recently the use of CRLBs has been proposed for both system design ([6]) and dynamic, online sensor management ([2] and [8]). The general approach is to minimize the root mean square estimator error(s), determined from the Fisher Information Matrix, by controlling the acquisition process itself. This technique has been used to both determine optimal observer manoeuvres for bearings-only tracking [8] and determine optimal schedules and configurations for the deployment of passive sonobuoys in submarine tracking [2].

We are concerned with calculating PCRLBs for the general problem of tracking a single target when there is measurement origin uncertainty. The target motion is dynamic and a potentially non-linear function of the target state. Each sensor can generate measurements at discrete time epochs. At each stage the measurement number is uncertain and measurements can either be target generated or false alarms. A target generated measurement can also be a non-linear function of the target state (e.g. as in the case of azimuth and range measurements).

Previously [3] and [4] observed the surprising result that in calculating CRLBs for parameter estimation the Fisher Information Matrix with measurement uncertainty was equal to the Fisher Information Matrix *without* measurement uncertainty multiplied by a constant information reduction factor. [7] then provide a general framework giving sufficient conditions for this to be true. In particular, [7] showed that if target generated measurements have a symmetric distribution around the true measurement, and false alarms have a uniform distribution, then the measurement uncertainty can be expressed as a constant information reduction factor.

Determining PCRLBs represents an even more challenging problem. The reason is that in this case, in calculating the Fisher Information Matrix it is necessary to consider both the effect of measurement uncertainty, as well as uncertainty in the random target state. The excellent paper [12] provides a Riccati-like recursion that allows one to sequentially determine the PCRLBs for target state estimation for the non-linear filtering problem. This framework has been utilized by [11] to calculate the PCRLBs for bearings-only tracking with no measurement origin uncertainty. However, of greater relevance to the current research [14] provide a general analysis of the problem of calculating PCRLBs with measurement uncertainty, but with target generated measurements that are linear functions of the target state.

We introduce a general framework for determining PCRLBs that allows a marriage of non-linear measurements (unlike [14]), and uncertain dynamics (unlike [7]). We show that using the same assumptions as [7] and [14] the measurement uncertainty again expresses itself as a constant information reduction factor. This ensures that calculating the PCRLBs is relatively straightforward and computationally inexpensive, making a PCRLB analysis suitable for real-time, online sensor management, the like of which is considered in [2].

# 2 The Cramér-Rao Lower Bound

## 2.1 Background

Let $\hat{x}(z)$ be an unbiased estimator of a parameter vector $x$, based on the measurement vector $z$. Then the CRLB for the error covariance matrix is defined to be the inverse of the Fisher Information Matrix, $J$, i.e.

$$C_n \triangleq \mathbb{E}\left[[\hat{x}-x]\,[\hat{x}-x]^T\right] \geq J^{-1}. \tag{1}$$

The inequality in equation (1) means that the difference $C_n - J^{-1}$ is a positive semi-definite matrix. If $x$ is an unknown and *random* parameter vector, we seek the PCRLB, and $J$ is given by

$$J_{ij} = \mathbb{E}\left[-\frac{\partial^2 \log p(z,x)}{\partial x^i \partial x^j}\right], \tag{2}$$

where $p(z,x)$ is the joint probability density function of $(z,x)$, and the expectation $\mathbb{E}[\cdot]$ is with respect to both $z$ and $x$.

## 2.2 PCRLBs For Target State Estimation

Throughout this paper we will consider the following dynamic system.

$$X_{k+1} = a_k(X_k) + w_k, \tag{3}$$

where $X_k$ is the target state at time $k$, $a_k(\cdot)$ is a (potentially) non-linear function of $X_k$, and $\{w_k\}$ is a white noise sequence. Measurements are available at discrete time epochs, we denote this measurement sequence by

$$\{Z(k)\}_{k\geq 1}\,, \tag{4}$$

and will be detailed fully in the next section. We seek the PCRLB for unbiased estimators $\hat{X}_k(I(k))$ of the target state, $X_k$, given the available sensor measurements, $I(k) = \{Z(1), \ldots, Z(k)\}$.

Until recently, calculating PCRLBs for this problem has proved notoriously difficult. However Tichavsky et al. [12] provide the following Riccati like recursion giving the sequence of posterior Fisher information matrices, $J_k$, $k > 0$, for the unbiased estimation of $X_k$.

$$J_{k+1} = D_k^{22} - D_k^{21}(J_k + D_k^{11})^{-1} D_k^{12}, \tag{5}$$

where

$$D_k^{11} = \mathbb{E}\left[-\Delta_{X_k}^{X_k} \log p(X_{k+1}|X_k)\right], \tag{6}$$

$$D_k^{12} = \mathbb{E}\left[-\Delta_{X_k}^{X_{k+1}} \log p(X_{k+1}|X_k)\right], \tag{7}$$

$$= \left[D_k^{21}\right]^T, \tag{8}$$

$$D_k^{22} = D_k^{33} + J_Z(k+1), \tag{9}$$

and

$$D_k^{33} = \mathbb{E}\left[-\Delta_{X_{k+1}}^{X_{k+1}} \log p(X_{k+1}|X_k)\right], \tag{10}$$

$$J_Z(k+1) = \mathbb{E}\left[-\Delta_{X_{k+1}}^{X_{k+1}} \log p(Z(k+1)|X_{k+1})\right]. \tag{11}$$

$\Delta$ is a second-order partial derivative operator whose $(i,j)$th term is given by

$$\left[\Delta_\alpha^\beta\right]_{ij} \triangleq \frac{\partial^2}{\partial \alpha_i \partial \beta_j}. \tag{12}$$

Calculation of $D_k^{11}$, $D_k^{12}$ and $D_k^{33}$ is (generally) straightforward. Indeed if $a_k(X_k)$ is linear, i.e. $a_k(X_k) = A_k X_k$ and $w_k$ is Gaussian with zero mean and variance $Q_k$ it can easily be shown that (see also Ristic et al. [11])

$$D_k^{11} = A_k^T Q_k^{-1} A_k, \tag{13}$$

$$D_k^{12} = -A_k^T Q_k^{-1}, \tag{14}$$

$$D_k^{33} = Q_k^{-1}. \tag{15}$$

In this case if we apply the Matrix Inversion lemma to equation (5), using equations (13), (14) and (15) then

$$J_{k+1} = \left(Q_k + A_k J_k^{-1} A_k^T\right)^{-1} + J_Z(k+1). \tag{16}$$

The initial Fisher Information Matrix is given by

$$J_0 = \mathbb{E}\left[-\Delta_{X_0}^{X_0} \log p(X_0)\right]. \tag{17}$$

Section 4 is devoted to the calculation of the matrix $J_Z(k)$ for a scenario with multiple sensors, each with independent (non-linear) measurement processes. Each sensor can have multiple false alarms, and we are tracking a single target that is moving with variable dynamics.

In calculating $J_Z(k)$ the expectation in equation (11) is both over the state, $X_{k+1}$, and the measurement, $Z_{k+1}$ making it difficult to calculate. We show that under certain assumptions this expectation can be decomposed and a constant information reduction factor gives the effect of the measurement uncertainty.

## 3 Problem Specification

### 3.1 Multiple Independent Sensors

We consider $N \geq 1$ sensors, and let $Z^{(j)}(k)$ be the measurement vector at sensor $j$. We will assume that the sensors have independent measurement processes. Hence in the presence of false alarms, the total number of measurements can vary between sensors at each time epoch, $k$. Let $m_k^{(j)}$ be the total number of measurements at sensor $j$ at time $k$, i.e.

$$Z^{(j)}(k) = \{z_{ij}(k)\}_{i=1}^{m_k^{(j)}}, \tag{18}$$

where $z_{ij}(k)$ is the $i$th measurement at sensor $j$ at time $k$. The measurement process independence gives us

$$\log p(Z(k) \Big| X_k, M_k) = \sum_{j=1}^{N} \log p(Z^{(j)}(k)|X_k, m_k^{(j)}). \tag{19}$$

where $Z(k) = (Z^{(1)}(k), \ldots, Z^{(N)}(k))$ and $M_k = (m_k^{(1)}, \ldots, m_k^{(N)})$. Now, the multi-sensor generalization of Zhang and Willett [14] equation (22) is

$$J_Z(k) = \sum_{M_k} p(M_k) J_Z(M_k; k), \tag{20}$$

where

$$J_Z(M_k; k) = \mathbb{E}\left[-\Delta_{X_k}^{X_k} \log p(Z(k)|X_k, M_k) \Big| M_k\right]. \tag{21}$$

It is clear from equation (19) that

$$J_Z(M_k; k) = \sum_{j=1}^{N} J_Z^j(m_k^{(j)}; k), \tag{22}$$

where

$$J_Z^j(m_k^{(j)}; k) = \mathbb{E}\left[-\Delta_{X_k}^{X_k} \log p(Z^{(j)}(k)|X_k, m_k^{(j)}) \Big| m_k^{(j)}\right]. \tag{23}$$

Hence, using both the independence of the measurement processes at each sensor, and equation (22), we get

$$J_Z(k) = \sum_{j=1}^{N} \sum_{m_k^{(j)}=1}^{\infty} p(m_k^{(j)}) J_Z^j(m_k^{(j)}; k). \tag{24}$$

### 3.2 Measurement Model

Now, let us consider the measurements at one of the $N$ sensors (i.e. at sensor $j$). We use the false alarm model specified in Zhang and Willett [14] (see also [7]). The number of false alarms has a Poisson distribution with mean $\lambda V$. Hence at each stage there are $m_k$ measurements (for brevity, the index '$j$' is omitted), and two possibilities, either all $m_k$ measurements are false alarms, or there is one true observation of the target and $m_k - 1$ false alarms.

Now, it is easy to show that the prior probability, $p(m_k)$ that there are $m_k = 1, 2, \ldots$ observations at time $k$ is given by

$$p(m_k) = (1 - P_d)\frac{(\lambda V)^{m_k} \exp(-\lambda V)}{m_k!} + P_d \frac{(\lambda V)^{m_k - 1} \exp(-\lambda V)}{(m_k - 1)!}, \tag{25}$$

where $P_d$ is the probability of detecting the target. The probability that one measurement is target generated is then given by

$$\epsilon(m_k) = \frac{P_d}{p(m_k)} \frac{(\lambda V)^{m_k - 1} \exp(-\lambda V)}{(m_k - 1)!}. \tag{26}$$

It can then be shown that the PDF of the measurement $Z^{(j)}(k)$, given $(X_k, m_k)$ is given by

$$p(Z^{(j)}(k)|X_k, m_k) = \left[(1 - \epsilon(m_k)) + \frac{\epsilon(m_k)}{m_k} \sum_{i=1}^{m_k} \frac{p_1(z_{ij}(k))}{p_0(z_{ij}(k))}\right] \times \left[\prod_{i=1}^{m_k} p_0(z_{ij}(k))\right], \tag{27}$$

where $p_1$ is the PDF of a true detection (typically Gaussian), and to remind the reader, $Z^{(j)}(k)$ is given by equation (18). $p_0$ is the false alarm PDF. We assume (*assumption A1*) that false alarms have a uniform distribution across the region, $A$, (with hyper-volume, $V$) under observation, i.e.

$$p_0(z_{ij}(k)) = \frac{1}{V}, \qquad \text{for } z_{ij}(k) \in A. \tag{28}$$

Equation (27) then becomes

$$p(Z^{(j)}(k)|X_k, m_k) = \left[\frac{(1 - \epsilon(m_k))}{V^{m_k}} + \frac{\epsilon(m_k)}{m_k V^{m_k - 1}} \sum_{i=1}^{m_k} p_1(z_{ij}(k))\right]. \tag{29}$$

$z_{ij}(k)$ can encompass measurements of $n(\geq 1)$ different types (e.g. azimuth and range). Therefore we let

$$z_{ij}(k) = \left(z_{ij}^1(k), \ldots, z_{ij}^n(k)\right)^T, \tag{30}$$

for $i = 1, \ldots, m_k$. We will assume that for each target generated measurement, $z_{ij}(k)$,

$$z_{ij}(k) = f_j(X_k, k) + v_{ij}. \tag{31}$$

The measurement error, $v_{ij}$, is assumed (*assumption A2*) to be a zero mean Gaussian random variable with $n \times n$

covariance matrix, $R = diag(\sigma_1^2, \ldots, \sigma_n^2)$. Hence we can write

$$p_1(z_{ij}(k)) = \prod_{r=1}^{n} \phi_r\left(z_{ij}^r(k) - f_j^r(X_k)\right), \quad (32)$$

where $f_j^r(X_k)$ is the component of $f_j(X_k, k)$ relating to the $r$th measurement type (c.f. equation (31)). $\phi_r$ is the PDF of the Gaussian distribution with zero mean and standard deviation $\sigma_r$.

# 4 Determining $J_Z(k)$

## 4.1 General Expression

$p(Z^{(j)}(k)|X_k, m_k)$ is given by equations (29) and (32). Hence

$$\begin{aligned} &\frac{\partial}{\partial X_k^a}\left[\log p(Z^{(j)}(k)|X_k, m_k)\right] \\ &= \frac{-\epsilon(m_k)}{m_k p(Z^{(j)}(k)|X_k, m_k) V^{m_k - 1}} \\ &\quad \times \sum_{s=1}^{n}\left[\frac{\partial f_j^s}{\partial X_k^a}\right]\sum_{i=1}^{m_k} G_s^i(\underline{\phi}), \end{aligned} \quad (33)$$

where, for $s = 1, \ldots, n$

$$\begin{aligned} G_s^i(\underline{\phi}) &\triangleq \frac{\partial}{\partial z_{ij}^s(k)} \prod_{r=1}^{n} \phi_r\left(z_{ij}^r(k) - f_j^r(X_k)\right), \quad (34) \\ &= \left(\frac{\phi_s'\left(z_{ij}^s(k) - f_j^s(\underline{X}_k)\right)}{\phi_s\left(z_{ij}^s(k) - f_j^s(X_k)\right)}\right) \\ &\quad \times \prod_{r=1}^{n} \phi_r\left(z_{ij}^r(k) - f_j^r(X_k)\right). \quad (35) \end{aligned}$$

Now, it can easily be shown that

$$\mathbb{E}\left[-\frac{\partial^2 \log p(\cdot)}{\partial X_k^a \partial X_k^b}\right] = \mathbb{E}\left[\frac{\partial \log p(\cdot)}{\partial X_k^a}\frac{\partial \log p(\cdot)}{\partial X_k^b}\right], \quad (36)$$

for any PDF $p(\cdot)$. Hence if we use equations (36) and (33) we obtain equation (44). If we use the partition theorem for expectations then

$$\begin{aligned} &\left[J_Z^j(m_k; k)\right]_{ab} \quad (37) \\ &= \mathbb{E}\left[\mathbb{E}\left[-\frac{\partial^2 \log p(Z^{(j)}(k)|X_k, m_k)}{\partial X_k^a X_k^b}\middle| X_k, m_k\right]\right], \end{aligned}$$

where $[\cdot]_{ab}$ is the $(a, b)$th element of $J_Z^j(m_k; k)$. $J_Z(k)$ then follows from equation (24).

In general, both the inner expectation of equation (37) (given by equation (44)), and the outer expectation (with respect to $X_k$) are difficult to calculate. However in the next section we will show how making a simple assumption allows the problem to be simplified dramatically.

## 4.2 Simplification

As argued by Niu et al. [7] 'far out' measurements will be accorded a small weight when a reasonable data association algorithm is used. Hence we can restrict our measurements to a validation gate (see Bar-Shalom and Li, [1]), i.e.

$$|z_{ij}^r(k) - f_j^r(X_k)| < g\sigma_r, \quad (38)$$

for $i = 1, \ldots, m_k$ and $r = 1, \ldots, n$. Now, in general

$$dz_{ij} = D(z_{ij}^1, \ldots, z_{ij}^n) dz_{ij}^1 \ldots dz_{ij}^n, \quad (39)$$

where $D(\cdot)$ is a Jacobian term that accounts for the dependency between the $n$ different measurement dimensions.

Now, if we make a final assumption (*assumption A3*) we can simplify the expression for $J_Z(\cdot)$ to the point where we are able to decouple the effects of the measurement uncertainty and the target state uncertainty. This assumption is that the $n$ measurement dimensions, $(z_{ij}^1(k), \ldots, z_{ij}^n(k))$ are independent/orthogonal. This ensures that $D(\cdot) = 1$ and the gated observation region is then independent of the state $X_k$.

Clearly, $\phi_r(u)$, $r = 1, \ldots, n$ are even symmetric functions. Hence $G_s(\underline{\phi})G_{s'}(\underline{\phi})$ is an odd symmetric function of $Z^{(j)}(k)$ for $s \neq s'$. It then follows from assumption (A3) that $D(\cdot) = 1$. Hence integrating an odd symmetric function over an even symmetric range (given by equation (38)) we have

$$\begin{aligned} &\int_{z_{m_k j} \in A} \cdots \int_{z_{1j} \in A} \quad (40) \\ &\frac{G_s^i(\underline{\phi}) G_{s'}^{i'}(\underline{\phi})}{p(Z^{(j)}(k)|X_k, m_k)} dz_{1j} \ldots dz_{m_k j} = 0, \end{aligned}$$

if $s \neq s'$ and $i \neq i'$. Hence of the $nm_k \times nm_k$ terms in the integrand of equation (44) all but $nm_k$ of them are zero. Hence the integrand of (44) reduces to

$$\frac{\sum_{s=1}^{n}\sum_{i=1}^{m_k}\left[\frac{\partial f_j^s}{\partial X_k^a}\right]\left[\frac{\partial f_j^s}{\partial X_k^b}\right] G_s^i(\underline{\phi})^2}{p(Z^{(j)}(k)|X_k, m_k)}. \quad (41)$$

The $z_{ij}(k)$'s are independent and identically distributed. Therefore the integrand can be further reduced to

$$\frac{m_k \sum_{s=1}^{n}\left[\frac{\partial f_j^s}{\partial X_k^a}\right]\left[\frac{\partial f_j^s}{\partial X_k^b}\right] G_s^1(\underline{\phi})^2}{p(Z^{(j)}(k)|X_k, m_k)}, \quad (42)$$

giving us equation (45).

Now, let

$$\begin{aligned} K_s(X_k, m_k) &= \frac{\epsilon(m_k)^2}{m_k V^{2m_k - 2}} \int_{z_{m_k j} \in A} \cdots \int_{z_{1j} \in A} \\ &\quad \frac{G_s^1(\underline{\phi})^2}{p(Z^{(j)}(k)|X_k, m_k)} dz_{1j} \ldots dz_{m_k j}. \end{aligned} \quad (43)$$

$$\mathbb{E}\left[-\frac{\partial^2}{\partial X_k^a X_k^b}\left[\log\left(p(Z^{(j)}(k)|X_k,m_k)\right)\right]\middle|X_k,m_k\right]$$

$$= \frac{\epsilon(m_k)^2}{m_k^2 V^{2m_k-2}}\int_{z_{m_kj}\in A}\cdots\int_{z_{1j}\in A}\frac{\left[\sum_{s=1}^{n}\left[\frac{\partial f_j^s}{\partial X_k^a}\right]\sum_{i=1}^{m_k}G_s^i(\underline{\phi})\right]\left[\sum_{s=1}^{n}\left[\frac{\partial f_j^s}{\partial X_k^b}\right]\sum_{i=1}^{m_k}G_s^i(\underline{\phi})\right]}{p(Z^{(j)}(k)|X_k,m_k)}dz_{1j}\dots dz_{m_kj}, \tag{44}$$

$$= \frac{\epsilon(m_k)^2}{m_k V^{2m_k-2}}\sum_{s=1}^{n}\left[\frac{\partial f_j^s}{\partial X_k^a}\right]\left[\frac{\partial f_j^s}{\partial X_k^b}\right]\int_{z_{m_kj}\in A}\cdots\int_{z_{1j}\in A}\frac{G_s^1(\underline{\phi})^2}{p(Z^{(j)}(k)|X_k,m_k)}dz_{1j}\dots dz_{m_kj}, \tag{45}$$

$$K_s(m_k) = \frac{\epsilon(m_k)^2}{\sigma_s^4 m_k V^{m_k-2}}\int_{\tilde{z}_{m_kj}\in A'}\cdots\int_{\tilde{z}_{1j}\in A'}\frac{(\tilde{z}_{1j}^s)^2\prod_{r=1}^{n}\phi_r\left(\tilde{z}_{1j}^r\right)^2}{\left[(1-\epsilon(m_k))+\frac{\epsilon(m_k)V}{m_k}\sum_{i=1}^{m_k}\prod_{r=1}^{n}\phi_r(\tilde{z}_{ij}^r)\right]}d\tilde{z}_{1j}\dots d\tilde{z}_{m_kj}. \tag{46}$$

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Equation (45) then becomes

$$\mathbb{E}[\cdot|X_k,m_k] = \sum_{s=1}^{n} F_j^s(X_k)_{ab}K_s(X_k,m_k), \tag{47}$$

where

$$F_j^s(X_k)_{ab} = \frac{\partial f_j^s}{\partial X_k^a}\frac{\partial f_j^s}{\partial X_k^b}. \tag{48}$$

To proceed, $\phi_r'(u)$, is given by

$$\phi_r'(u) = -\frac{u}{\sigma_r^2}\phi_s(u), \qquad \text{for all } u. \tag{49}$$

Therefore

$$G_s^i(\underline{\phi}) = -\frac{\left(z_{ij}^s(k)-f_j^s(X_k)\right)}{\sigma_s^2} \times \prod_{r=1}^{n}\phi_r\left(z_{ij}^r(k)-f_j^r(X_k)\right). \tag{50}$$

The gating (c.f. equation (38)) ensures that the observation volume of each measurement $z_{ij}^r(k)$, is symmetric about $f_j^r(X_k)$. Hence if we make the change of variable

$$\tilde{z}_{ij}^r = \left[z_{ij}^r - f_j^r(X_k)\right], \tag{51}$$

for $i = 1,\dots,m_k$, $r = 1,\dots,n$ in equation (43) and substitute equation (50), we obtain equation (46), where $\tilde{z}_{ij} = (\tilde{z}_{ij}^1,\dots,\tilde{z}_{ij}^n)$. $A'$ is the mapping of the region, $A$, under the transformation(s) given by (51). It follows from assumption (A3) that $A'$ is given by the hypercube:

$$A' = [-g\sigma_1, g\sigma_1]\times\dots\times[-g\sigma_n, g\sigma_n]. \tag{52}$$

It is easy to see that $K_s(X_k,m_k) = K_s(m_k)$, independent of $X_k$, because both the integration region $A'$ and the integrand are independent of $X_k$. The (gated) volume of the $n$-dimensional hypercube is then

$$V = V_g = \prod_{r=1}^{n}\{2g\sigma_r\}. \tag{53}$$

Combining equations (47) and (37) it now follows that

$$\left[J_Z^j(m_k;k)\right]_{ab} = \sum_{s=1}^{n}K_s(m_k)\mathbb{E}\left[F_j^s(X_k)_{ab}\right]. \tag{54}$$

At this stage it is important to note that $K(m_k)$ is a function of $\lambda$, $V$, $P_d$ and $\sigma_s$, $s = 1,\dots,n$. Each of these parameters can be sensor specific, however, if we assume that all sensors have the same values of $\lambda$, $A$, $P_d$ and $\sigma_s$, $s = 1,\dots,n$, then

$$[J_Z(k)]_{ab} = \sum_{s=1}^{n}q_s\left\{\frac{1}{\sigma_s^2}\sum_{j=1}^{N}\mathbb{E}\left[F_j^s(X_k)_{ab}\right]\right\}, \tag{55}$$

where

$$q_s(P_d,\lambda,\sigma,V) = \sigma_s^2\sum_{m_k=1}^{\infty}p(m_k)K_s(m_k), \tag{56}$$

for $s = 1,\dots,n$. To remind the reader, $p(m_k)$ is given by equation (25) and $K_s(m_k)$ is given by equation (46).

$K_s(m_k)$ can be determined using the method of Monte Carlo integration (for example, see [9]). The expectation $\mathbb{E}[\cdot]$ in equation (55) is with respect to the target state, $X_k$ and can be estimated by generating a number, $N_P$, of (potential) target trajectories, $\tilde{X}_i^j$, $i = 1,2,\dots$; $j = 1,\dots,N_P$, from the system model (3) and averaging across them:

$$\mathbb{E}\left[F_j^s(X_k)_{ab}\right] \approx \frac{1}{N_P}\sum_{j=1}^{N_P}F_j^s(\tilde{X}_k^j)_{ab}. \tag{57}$$

## 4.3 Information Reduction Factor

If we define $Q = diag(q_1, \ldots, q_n)$, then in matrix notation we can write

$$J_Z(k) = \sum_{j=1}^{N} \mathbb{E}\left[H_j(X_k)^T Q R^{-1} H_j(X_k)\right], \qquad (58)$$

where the $(a,b)$th element of the $n \times d$ matrix $H_j(X_k)$ is given by

$$H_j(X_k)_{ab} \triangleq \frac{\partial f_j^a}{\partial X_k^b}, \qquad (59)$$

and to remind the reader, $R = diag(\sigma_1^2, \ldots, \sigma_n^2)$ is the covariance matrix of each target generated measurement $z_{ij}(k)$. $d$ gives the dimensionality of the state space.

If there is no measurement uncertainty, (i.e. $\lambda = 0$), and we are guaranteed a single measurement at each stage, (i.e. $P_d = 1$), it is easily shown that (see also [11])

$$J_Z(k) = \sum_{j=1}^{N} \mathbb{E}\left[H_j(X_k)^T R^{-1} H_j(X_k)\right]. \qquad (60)$$

Hence $Q$ can again be interpreted as a matrix of constant information reduction factors that scale the effect of the measurement uncertainty.

In the case of bearings-only tracking (i.e. $n = 1$), the information reduction factor, $q_1$ is given in table 1, for different combinations of $\lambda$ and $P_d$. In each case the bearing error standard deviation of a true detection, $\sigma = 0.1$ radians, and $g = 4$ (hence $V = 0.8$).

| $\lambda$ ($\lambda V$) | probability, $P_d$ | | | |
|---|---|---|---|---|
| | 0.7 | 0.8 | 0.9 | 1.0 |
| 0.00 (0.00) | 0.7000 | 0.8000 | 0.9000 | 1.0000 |
| 0.20 (0.16) | 0.5750 | 0.6867 | 0.8057 | 0.9364 |
| 0.40 (0.32) | 0.5021 | 0.6128 | 0.7371 | 0.8872 |
| 0.60 (0.48) | 0.4490 | 0.5568 | 0.6790 | 0.8350 |
| 0.80 (0.64) | 0.4058 | 0.5097 | 0.6304 | 0.7894 |
| 1.00 (0.80) | 0.3700 | 0.4696 | 0.5866 | 0.7464 |

Table 1: The value of the information reduction factor, $q_1(P_d, \lambda, \sigma, V)$ for bearings-only tracking.

It can be seen from table 1 that $q_1(\cdot)$ decreases as both the false alarm rate increases and the probability of detecting the target decreases. Zhang and Willett [14] showed that the *steady-state* PCRLB root mean square position errors increase as $q_1(\cdot)$ decreases. Hence as the false alarm rate increases and the probability of detecting the target decreases the theoretical bound on the performance of any filtering algorithm worsens, as one would expect.

Equation (58) is the generalization of Zhang and Willett [14] equation (41) to allow for non-linear measurements and multiple sensors. If we use a single sensor ($N = 1$) and linear model for target generated measurements, i.e. $f_k(X_k) = B_k X_k$, then $H_k(X_k) = B_k$ and equation (58) reduces to equation (41) from [14].

## 4.4 Assumption Relaxation

In the previous analysis we have made the following assumptions:

- A1: false alarms have a uniform distribution,
- A2: target generated measurements are independent and identically distributed and have a Gaussian distribution around the true measurement,
- A3: the $n$ measurement dimensions are orthogonal/independent.

Assumptions (A1) and (A2) are consistent with [7] whereas assumption (A3) is the clear distinction between our research and that of [7]. These conditions are sufficient (but not necessary) for the measurement uncertainty to express itself as a constant information reduction factor.

Assumptions (A2)-(A3) ensure that equation (40) holds, and this enables us to reduce equation (44) to equation (45). Crucially (A2)-(A3) also ensure that we can factor out $f_j^r(X_k), r = 1, \ldots, n$ from the integral (43) and this allows us to create the decomposition (54) from which we get the information reduction factor. However it is the fact that the distribution of target generated measurements is *even symmetric* around the true measurement that, together with assumption (A3), allows these simplifications. Hence as noted in [7] we can relax (A2) and use any symmetric, zero mean error distribution.

The effect of assumption (A1) is to reduce equation (27) to equation (29) and hence reduce the dependency of $p(Z^{(j)}(k)|X_k, m_k)$ on the measurements $z_{ij}(k)$, $i = 1, \ldots, m_k$. However, again as noted in [7], we can weaken this assumption, and assume that false alarms also have an even symmetric distribution around the true measurement. This will make things notationally even more complicated. However, in this case an information reduction factor will still exist. This task is left to the reader.

Assumption (A3) is critical in allowing us to express the observation region as a hypercube that is independent of the target state, $X_k$. It is easy to see that with range and bearing $(R, \theta)$ measurements this assumption does not hold. In this case, with a gate of $g$ standard deviations in each measurement dimension the volume of the gate (at sensor $j$) is

$$V_g = 4g^2 \sigma_\theta \sigma_R \sqrt{(x_k - x_j^s)^2 + (y_k - y_j^s)^2}, \qquad (61)$$

which is clearly dependent on the target location $(x_k, y_k)$. To take account of this general dependency we must include a Jacobian term $D(\cdot)$. In the case of range and bearing tracking

$$D(z_{ij}^{\theta}, z_{ij}^{R}) = \sqrt{(x_k - x_j^s)^2 + (y_k - y_j^s)^2}. \qquad (62)$$

Depending on the functional form of this Jacobian, we may, in some instances still be able to factor out the effect of the measurement uncertainty from the integrand of equation (44). However in this case any general expression for the information reduction factor will be even more complicated than before. This task is left for future work.

For specific nonlinear measurement models for which it is possible, one could use an unbiased conversion such as that in [5] to circumvent the problem of having a Jacobian that varies across the gate. However, the focus of this report is on the general case of nonlinear measurements. Since relaxing assumption (A3) greatly complicates the analysis, here we assume that the gate is sufficiently small with respect to the non-constant nature of the Jacobian that the Jacobian can *always* be taken to be unity.

We note that if neither assumptions (A1)-(A3), nor the relaxations discussed above hold, then it is generally not possible to decouple the measurement uncertainty and the target state uncertainty. In determining the PCRLBs we must then determine $J_Z^j(m_k; k)$, for each value of $m_k$ by performing Monte Carlo integration (again, see [9]), simultaneously generating samples from the target state space (at time $k$) and the $n \times m_k$ dimensional measurement space. This is computationally expensive, and in such cases PCRLBs may not be suitable for implementation in 'real-time' systems.

# 5 Example: Sensor Management

## 5.1 Model Specification

A nearly constant-velocity (CV) model [1], with power spectral density, $l = 1.0 \times 10^{-5}$, prescribes the target dynamics. A measurements is available at each sensor every 10 seconds. Measurements are bearings-only:

$$f_j(X_k) \quad = \quad \tan^{-1}\left(\frac{y_k - y_j^s}{x_k - x_j^s}\right), \qquad (63)$$

$(x_k, y_k)$ is the cartesian position of the target at time epoch $k$ and $(x_j^s, y_j^s)$ is the location of the $j$th sensor at that time.

## 5.2 Simulation Results

We compare the PCRLBs determined for two sensor configurations (see figure 1). For each sensor, the bearing error standard deviation, $\sigma = 0.3^o$; $g = 5$, $\lambda = 0.01$ and $P_d = 0.8$. We use $N_P = 200$ randomly generated target trajectories. The initial target state has a Gaussian distribution with mean, $(1000, -10/3, 1000, -10/3)$, and variance, $I_4$, where $I_4$ is the $4 \times 4$ identity matrix. Distances are in metres and speeds are in metres per second.

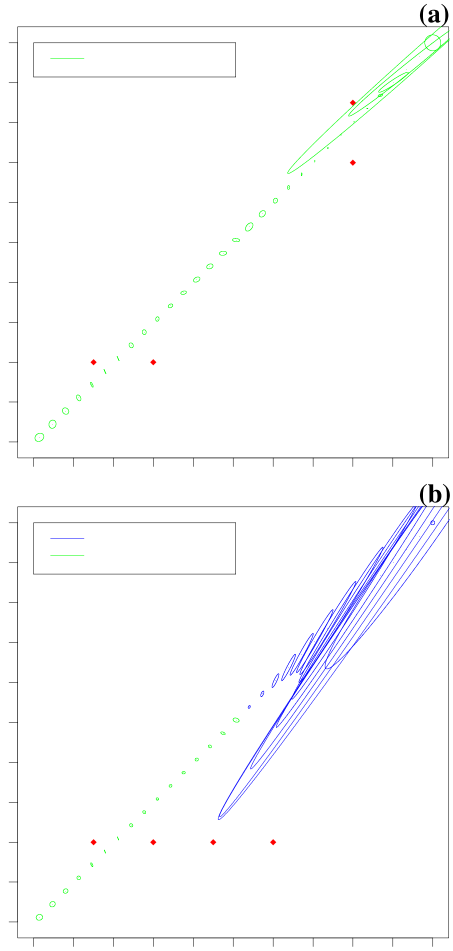


Figure 1: PCRLBs for two sensor arrays. The sensor positions are marked in red. The centre of each PCRLB ellipse gives the mean target position at each measurement epoch.

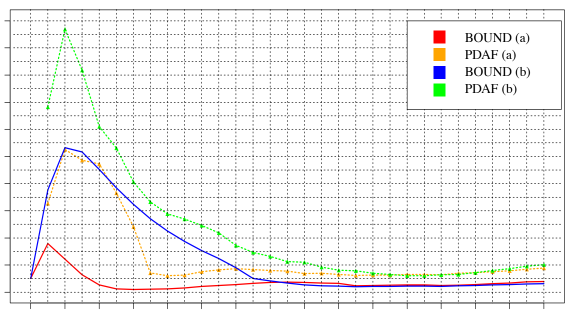


Figure 2: Probabilistic data association filter (PDAF) RMSEs compared with the RMSE bounds for the two sensor arrays given in figures 1(a) and 1(b) respectively.

In figure 2 we show the largest (of the $x, y$ coordinate) root mean square error (RMSE) bounds, plotted against time, for each of the two sensor arrays. The sudden decreases in the RMSE occur as the target approaches and passes between the sensors. It is clear that in this particular example the sensor configuration shown in figure 1(a) achieves greater control over the target uncertainty, and should be preferred (compare the red and blue lines).

A non-parametric PDAF [1] was then applied to data simulated for the $N_P = 200$ target trajectories. The track estimate RMSEs were consistent with the PCRLB predictions (again, see figure 2). This both adds further weight to the usefulness of PCRLBs for sensor array management, and validates the theory developed.

# 6 Conclusions

We have introduced a general framework for determining PCRLBs that extends previous work by [7] and [14] by allowing the marriage of non-linear measurements and uncertain dynamics. We show that under certain assumptions the measurement uncertainty expresses itself as a constant information reduction factor.

These assumptions are consistent with [7] and are sufficient conditions for the existence of an information reduction factor. Calculating PCRLBs is then relatively straightforward and computationally inexpensive. As a result, they can be determined and utilized for real-time, online problems, such as multi-sensor management (see [2]).

However, if the assumptions sufficient to produce an information reduction factor do not hold, in determining the PCRLB it is generally no longer possible to decouple the effects of measurement uncertainty and target state uncertainty. In such cases the computational expense in determining the bound is greatly increased, and Monte Carlo integration must be performed at each update.

We corroborated the PCRLB with simulation results for bearings-only tracking of a CV target. We used a probabilistic data association filter, and the performance of the filter was consistent with the bound. Moreover, the example demonstrated how PCRLBs can be used for sensor management, by enabling us to determine the sensor combination that can most effectively minimize the root mean square position errors of optimal estimators of the target state.

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