Journal of Fractional Calculus and Applications,
Vol. 2. Jan. 2012, No. 2, pp. 1-9.
ISSN: 2090-5858.
http://www.fcaj.webs.com/

CRANK-NICOLSON FINITE DIFFERENCE METHOD FOR SOLVING TIME-FRACTIONAL DIFFUSION EQUATION

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Abstract. In this paper, we develop the Crank-Nicolson finite difference method (C-N-FDM) to solve the linear time-fractional diffusion equation, formulated with Caputo's fractional derivative. Special attention is given to study the stability of the proposed method which is introduced by means of a recently proposed procedure akin to the standard Von-Neumann stable analysis. Some numerical examples are presented and the behavior of the solution is examined to verify stability of the proposed method. It is found that the C-N-FDM is applicable, simple and efficient for such problems.

1. Introduction

The fractional-differential equations play a pivotal role in the modeling of number of physical phenomenon ([1]-[6], [18]). The applications of such equations include, damping laws, fluid mechanics, viscoelasticity, biology, physics, engineering and modeling of earth quakes, see ([8]-[11] and the references therein). Time fractional diffusion equations are used when attempting to describe transport processes with long memory where the rate of diffusion is inconsistent with the classical Brownian motion model. Several techniques ([3], [8], [19]-[21]) have been employed to find appropriate solutions of these equations as per their physical nature. Most of the used schemes so far encounter some inbuilt deficiencies and moreover are not compatible with the true physical nature of these problems. Numerical results reveal the complete reliability of the proposed algorithms [4]. Consequently, considerable attention has been given to the solutions of fractional ordinary/partial differential equations of physical interest. Most fractional differential equations do not have exact solutions, so approximation and numerical techniques ([7], [13]-[17]) must be used.

In the following we present some basic definitions for fractional derivatives which are used in this paper.

Definition 1 The Caputo fractional derivative $D_t^{\alpha} u(x,t)$, of order α with respect

2000 *Mathematics Subject Classification.* 65N06, 65N12, 65N15.
Key words and phrases. Crank-Nicolson finite difference method, time-fractional diffusion equation, Von-Neumann stability analysis.
Submitted April 11, 2011. Published Jan. 1, 2012.

to time is defined as [12]:

$$D_t^{\alpha}u(x,t) := \frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \begin{cases} \frac{1}{\Gamma(m-\alpha)}\int_0^t (t-s)^{m-\alpha+1}\frac{\partial^m u(x,s)}{\partial s^m}ds, & m-1<\alpha<m, \\ \frac{\partial^m u(x,t)}{\partial t^m}, & \alpha = m \in N, \end{cases} \tag{1}$$

where $\Gamma(.)$ is the Gamma function.

For more details on the fractional derivatives and its properties see ([11], [12]).

Our aim in this paper is to study the C-N-FDM for solving time-fractional diffusion equation of the form:

$$\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}} = \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0<\alpha\leq 1, \tag{2}$$

on a finite domain $0<x<1,\ 0\leq t\leq T$ and the parameter α refers to the fractional order of time derivative.

We also assume an initial condition:

$$u(x,0) = f(x), \qquad 0<x<1, \tag{3}$$

and the following Dirichlet boundary conditions:

$$u(0,t) = u(1,t) = 0. \tag{4}$$

Note that when $\alpha = 1$, Eq.(2) is the classical heat equation of the following form:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}.$$

The plan of the paper is as follows: In section 2, an approximate formula of the fractional derivative and the numerical procedure for solving time fractional diffusion equation (2) by means of the Crank-Nicolson finite difference method are given. In section 3, the stability analysis and the truncation error of the C-N-FDM scheme are introduced. In section 4, we compare exact analytical solutions with numerical ones and check the reliability of the analytical stability. Some conclusion is given in section 5.

2. Discretization for fractional diffusion equation

In this section, we use the first-order approximation method for the computation of Caputo's fractional derivative which is given by the expression [3]:

$$D_t^{\alpha}u_i^n \cong \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j}), \tag{5}$$

where

$$\sigma_{\alpha,k} = \frac{1}{\Gamma(1-\alpha)(1-\alpha)k^{\alpha}} \quad \text{and} \quad \omega_j^{(\alpha)} = j^{1-\alpha}-(j-1)^{1-\alpha}. \tag{6}$$

The formula (5) is derived in [3] for some positive integers N and M, the grid sizes in space and time for the finite difference algorithm are defined by $h=\frac{1}{N}$ and $k=\frac{T}{M}$, respectively. The grid points in the space interval [0,1] are the numbers $x_i = ih,\ i=0,1,2,...,N$ and the grid points in the time interval [0,T] are labeled $t_n = nk,\ n=0,1,2,....$

Remark 1 The quadrature formula (5) does not provide the values of the time fractional derivative at $t = 0$ which are not required by the implicit finite difference and the Crank-Nicolson method schemes that follows.

Now, Crank-Nicolson method with the discrete formula (5) is used to estimate the time α-order fractional derivative to solve numerically, the fractional diffusion equation (2). Using (5) the restriction of the exact solution to the grid points centered at $(x_i, t_n) = (ih, nk)$, in Eq.(2), satisfies for $i = 1, 2, ..., N-1$:

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j})+O(k)=$$
$$\frac{1}{2h^2}\{u_{i-1}^n-2u_i^n+u_{i+1}^n+u_{i-1}^{n-1}-2u_i^{n-1}+u_{i+1}^{n-1}\}+O(h^2),$$

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j})=\frac{1}{2h^2}\{u_{i-1}^n-2u_i^n+u_{i+1}^n+u_{i-1}^{n-1}-2u_i^{n-1}+u_{i+1}^{n-1}\}+T(x,t), \tag{7}$$

where $T(x,t)$ is the truncation term. Thus, according to Eq.(7) the numerical method is consistent, first order correct in time and second order correct in space. The resulting finite difference equations are defined by:

$$\sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j})=\frac{1}{2h^2}\{u_{i-1}^n-2u_i^n+u_{i+1}^n+u_{i-1}^{n-1}-2u_i^{n-1}+u_{i+1}^{n-1}\}$$

or

$$\sigma_{\alpha,k}\omega_1^{(\alpha)}(u_i^n-u_i^{n-1})=-\sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j})+\frac{1}{2h^2}\{u_{i-1}^n-2u_i^n+u_{i+1}^n+u_{i-1}^{n-1}$$
$$-2u_i^{n-1}+u_{i+1}^{n-1}\}.$$

Setting $\gamma=\frac{1}{2h^2}$, recalling from (6) that $\omega_1^{(\alpha)}=1$, and reordering, we finally get for $n = 1$,

$$-\gamma u_{i-1}^1+(\sigma_{\alpha,k}+2\gamma)u_i^1-\gamma u_{i+1}^1=(\sigma_{\alpha,k}-2\gamma)u_i^0+\gamma(u_{i+1}^0+u_{i-1}^0),\quad i=1,2,...,N-1,$$

for $n \geq 2$, $\quad i = 1, 2, ..., N-1$ we have:

$$-\gamma u_{i-1}^n+(\sigma_{\alpha,k}+2\gamma)u_i^n-\gamma u_{i+1}^n=$$
$$(\sigma_{\alpha,k}-2\gamma)u_i^{n-1}+\gamma(u_{i+1}^{n-1}+u_{i-1}^{n-1})-\sigma_{\alpha,k}\sum_{j=2}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j}), \tag{8}$$

with boundary conditions: $u_0^n = u_N^n = 0$, $n = 1, 2, ...,$ and initial temperature distribution: $u_i^0 = f_i = f(x_i)$, $i = 1, 2, ..., N-1$.
Eq.(8) requires, at each time step, to solve a tridiagonal system of linear equations where the right-hand side utilizes all the history of the computed solution up to that time.

Here we will show that the stability of the fractional numerical schemes can be analyzed very easily and efficiently with a method close to the well-known Von Neumann (or Fourier) method of non-fractional partial differential equations ([15], [20]).

3. Stability analysis and the truncation error

Theorem 1 The fractional Crank-Nicolson discretization, applied to the time-fractional diffusion equation (2) and defined by (8) is unconditionally stable for $0 < \alpha < 1$.

Proof. To study the stability of the method, we look for a solution of the form $u_j^n = \zeta_n e^{i\omega jh}$, $i = \sqrt{-1}$, ω real. Hence (8) becomes:

$$-\gamma\zeta_n e^{i\omega(j-1)h} + (\sigma_{\alpha,k} + 2\gamma)\zeta_n e^{i\omega jh} - \gamma\zeta_n e^{i\omega(j+1)h} = (\sigma_{\alpha,k} - 2\gamma)\zeta_{n-1} e^{i\omega jh}$$
$$+ \gamma(\zeta_{n-1} e^{i\omega(j+1)h} + \zeta_{n-1} e^{i\omega(j-1)h}) - \sigma_{\alpha,k} \sum_{j=2}^{n} \omega_j^{(\alpha)} (\zeta_{n-j+1} e^{i\omega jh} - \zeta_{n-j} e^{i\omega jh}).$$

Simplifying and grouping like terms:

$$(1+\frac{2\gamma}{\sigma_{\alpha,k}}(1-\cos(\omega h)))\zeta_n = (1-\frac{2\gamma}{\sigma_{\alpha,k}})\zeta_{n-1} + \frac{2\gamma\zeta_{n-1}}{\sigma_{\alpha,k}}\cos(\omega h) - \sum_{j=2}^{n} \omega_j^{(\alpha)} (\zeta_{n-j+1} - \zeta_{n-j}),$$

this can be reduced to:

$$\zeta_n = \frac{(1 - \frac{2\gamma}{\sigma_{\alpha,k}})\zeta_{n-1} + \frac{2\gamma\zeta_{n-1}}{\sigma_{\alpha,k}}\cos(\omega h) - \sum_{j=2}^{n} \omega_j^{(\alpha)} (\zeta_{n-j+1} - \zeta_{n-j})}{(1 + \frac{2\gamma}{\sigma_{\alpha,k}}(1 - \cos(\omega h)))}. \tag{9}$$

We observe that from Eq.(9), since $(1 + \frac{2\gamma}{\sigma_{\alpha,k}}(1 - \cos(\omega h))) \geq 1$ for all α, n, ω, h and k, it follows that:

$$\zeta_1 \leq \zeta_0(1 - \frac{2\gamma}{\sigma_{\alpha,k}}(1 - \cos(\omega h))), \tag{10}$$

and

$$\zeta_n \leq \zeta_{n-1}(1 - \frac{2\gamma}{\sigma_{\alpha,k}}(1 - \cos(\omega h))) - \sum_{j=2}^{n} \omega_j^{(\alpha)} (\zeta_{n-j+1} - \zeta_{n-j}). \tag{11}$$

Thus, for $n = 2$, the last inequality implies:

$$\zeta_2 \leq \zeta_1(1 - \frac{2\gamma}{\sigma_{\alpha,k}}(1 - \cos(\omega h))) + \omega_2^{(\alpha)} (\zeta_0 - \zeta_1).$$

Repeating the process until $\zeta_j \leq \zeta_{j-1}$, $j = 1, 2, ..., n-1$, we finally have:

$$\zeta_n \leq \zeta_{n-1}(1 - \frac{2\gamma}{\sigma_{\alpha,k}}(1 - \cos(\omega h))) - \sum_{j=2}^{n} \omega_j^{(\alpha)} (\zeta_{n-j+1} - \zeta_{n-j}) \leq \zeta_{n-j},$$

since each term in the summation is negative. This shows that the inequalities (10) and (11) imply $\zeta_n \leq \zeta_{n-1} \leq \zeta_{n-2} \leq ... \leq \zeta_1 \leq \zeta_0$.
Thus, $\zeta_n = |u_j^n| \leq \zeta_0 = |u_j^0| = |f_j|$, which entails $\|u_j^n\|_{l_2} \leq \|f_j\|_{l_2}$ and we have stability.

Remark 2 For $\alpha = 1$, the numerical scheme is reduced to the well-known convergent fully C-N algorithm for the heat equation [13]. Also, the proof of stability (and hence convergence) can be extended to other types of boundary conditions and more general time fractional diffusion equations in one and higher space dimensions.

The truncation error $T(x,t)$ of the fractional (C-N-FDM) difference scheme is:

$$
\begin{aligned}
T(x,t) &= \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}(u_i^{n-j+1}-u_i^{n-j}) - \frac{1}{2h^2}(u_{i-1}^{n}-2u_i^{n}+u_{i+1}^{n}+u_{i-1}^{n-1}-2u_i^{n-1}+u_{i+1}^{n-1})\\
&= \sigma_{\alpha,k}\sum_{j=1}^{n}\omega_j^{(\alpha)}\left[(u_i^n+(k-1)\frac{\partial u}{\partial t}+\frac{(k-1)^2}{2}\frac{\partial^2 u}{\partial t^2}+...)-(u_i^n-k\frac{\partial u}{\partial t}+\frac{k^2}{2}\frac{\partial^2 u}{\partial t^2}+...)\right]\\
&+O(k)-\frac{1}{2h^2}\left[(u_i^n+h\frac{\partial u}{\partial x}+\frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}+...)-2u_i^n+(u_i^n+h\frac{\partial u}{\partial x}+\frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}+...)\right]\\
&-\frac{1}{2h^2}\left[(u_i^{n-1}+h\frac{\partial u}{\partial x}+\frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}+...)-2u_i^{n-1}+(u_i^{n-1}-h\frac{\partial u}{\partial x}+\frac{h^2}{2}\frac{\partial^2 u}{\partial x^2}+...)\right]\\
&= O(k)+O(h^2).
\end{aligned}
$$

4. Comparison with numerical results

In this section, we implement the introduced formula (8), to solve numerically two models of time fractional differential equation (2), with different values of α.

Example 1: In this example, we solve numerically the time-fractional diffusion equation (2), with the following initial condition:

$$u(x,0)=\sin(\pi x),\quad 0<x<1, \tag{12}$$

and with boundary conditions:

$$u(0,t)=u(1,t)=0,\quad t\geq 0. \tag{13}$$

The exact solution of Eq.(2) is easily found by the method of separation of variables at $\alpha=1$ as follows:

$$u(x,t)=e^{-\pi^2 t}\sin(\pi x). \tag{14}$$

The obtained numerical results by using our proposed method are shown in Table 1. This table shows the magnitude of the maximum error at time $t=1$ between the exact solution and the numerical solution at different values of $\Delta t=k$ and $\Delta x=h$.

Table 1: Maximum error for the numerical solution using (C-N-FDM) at $t=1$.

Δx	Δt	Maximum error
0.001	2^{-3}	0.7816 e-05
0.001	2^{-4}	0.2454 e-05
0.002	2^{-5}	0.1969 e-06
0.002	2^{-6}	0.1645 e-06
0.002	2^{-7}	0.1566 e-07

Also, Figures 1 and 2 show the obtained numerical solutions using C-N-FDM with $\alpha=1$, at $t=0.5$ and $t=2$ respectively. From the obtained numerical results, we can conclude that the numerical solutions are in excellent agreement with the exact solution.

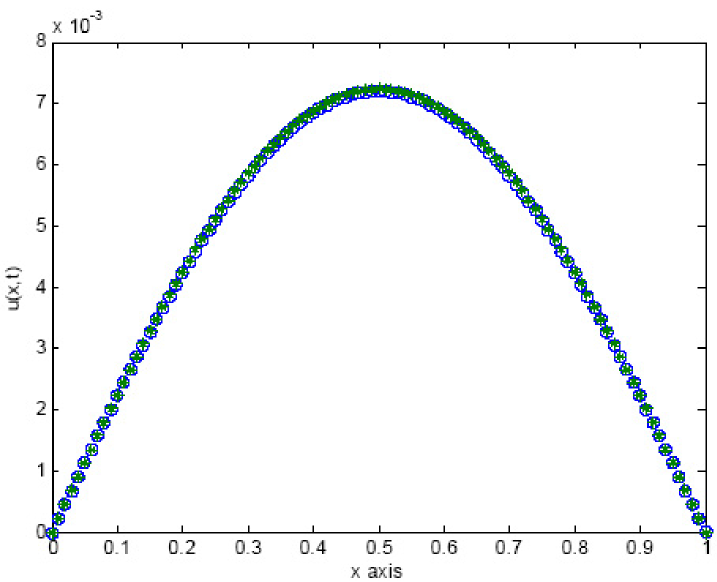

Figure 1: Comparison between the numerical solution and exact solution with $\alpha = 1$, at $t = 0.5$, $\Delta x = 0.01$ and $\Delta t = 1/2000$.

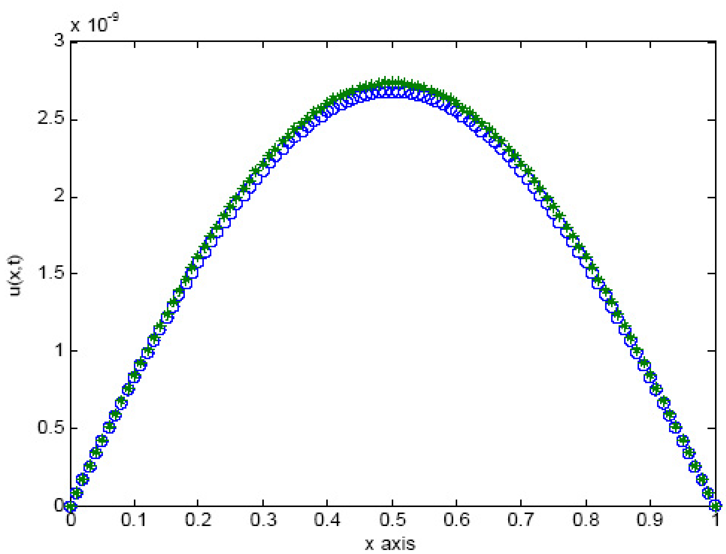

Figure 2: Comparison between the numerical solution and exact solution with $\alpha = 1$, at $t = 2$, $\Delta x = 0.01$ and $\Delta t = 1/500$.

Example 2: In this example, we solve numerically the time-fractional diffusion equation (2), with the following initial condition: $u(x, 0) = x(1 - x)$ and with boundary conditions: $u(0, t) = u(1, t) = 0$.
The obtained numerical results are shown in figures 3 and 4. Figure 3 illustrates the behavior of numerical solutions at $t = 0.5$ with $\alpha = 0.5, 0.75$ and 1 respectively. Also, figure 4 illustrates the behavior of numerical solution at $t = 2$ with $\alpha = 0.25$, 0.5, 0.75 and 1.0 respectively. We observe that the different profile behaviors as functions of α for short times, lower fractional-order solutions diffuse

"faster", figure 3, but as time increases the subdiffusion phenomena (slow asymptotic diffusion) become apparent.

5. Conclusion

In this paper, we have discussed a numerical method for the time-fractional diffusion equation on a finite slab when the partial time fractional derivative is interpreted in the sense of Caputo. The stability and the consistent of the method are proved that the method is unconditionally stable. Some test examples are given and the results obtained by the method are compared with the exact solutions. The comparison certifies that C-N-FDM gives good results. Summarizing these results, we can say that the finite difference method in its general form gives a reasonable calculations, easy to use and can be applied for the fractional differential equations in general form. All results obtained by using Matlab. Furthermore, the method can be trivially extended to dimensional problems, which is not such an easy task when C-N methods are considered.

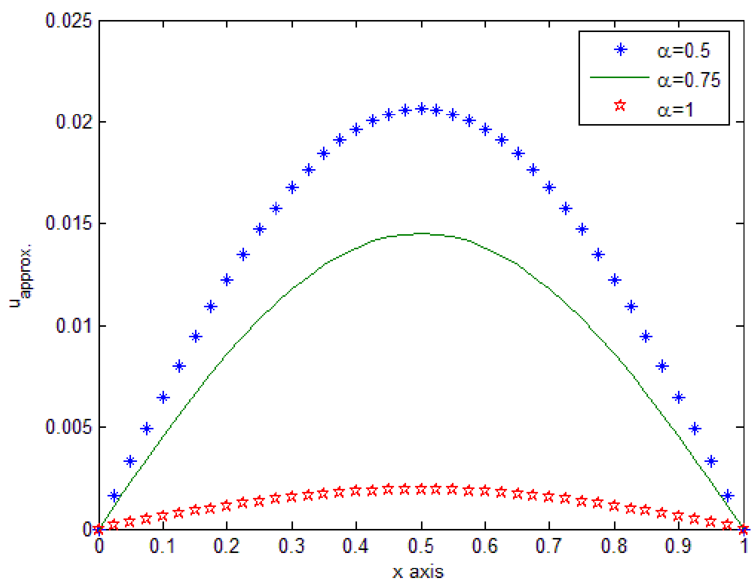

Figure 3: The behavior of numerical solution at $\alpha = 0.5, 0.75, 1$ at $t = 0.5$, $\Delta x = 0.025$ and $\Delta t = 1/256$.

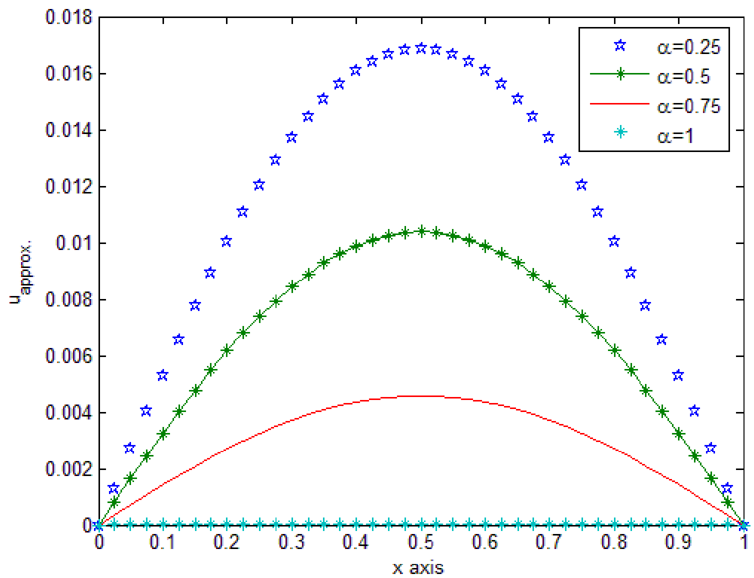

Figure 4: The behavior of numerical solution at $\alpha = 0.25$, 0.5, 0.75, 1 at $t = 2$, $\Delta x = 0.025$ and $\Delta t = 1/256$.

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