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Crawling on Simple Models of Web Graphs

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Abstract. We consider the problem of searching a randomly growing graph by a random walk. In particular we consider two simple models of "web-graphs." Thus at each time step a new vertex is added and it is connected to the current graph by randomly chosen edges. At the same time a "spider" S makes a number of steps of a random walk on the current graph. The parameter we consider is the expected proportion of vertices that have been visited by S up to time t.

1. Introduction

At the present moment, there is considerable ongoing research into the structure of large-scale real networks, and in modeling these networks as the outcomes of discrete random processes. A general introduction to this topic can be found in Hayes [Hoeffding 63] or Watts [Watts 99]. In particular, there is a strong interest in the structure of the Internet and World Wide Web (WWW). Experimental studies by, Albert, Barabasi, and Jeong [Albert et al. 99]; Broder et al. [Broder et al. 00]; and Faloutsos, Faloutsos, and Faloutsos [Faloutsos et al. 99] of the structure of the WWW have demonstrated an inverse power law for the proportion of vertices with a given degree.

To model such structures, we require a graph process which (a) evolves randomly by the addition of new vertices and/or edges at each time step t, and (b)

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whose proportional degree sequence follows a power law. Thus the proportion d_k of vertices of degree satisfies $d_k \sim Ck^{-x}$, as $k \to \infty$, where x is a constant. Such random graph processes are often referred to as web graphs, or scale-free graphs.

We can also define graph processes, which we refer to as random graph processes, in which the proportional degree sequence is similar to the degree sequence of the traditional models of random graphs introduced by Erdős and Rényi [Erdős and Rényi 59], [Erdős and Rényi 60]. Thus, in a random graph process, the proportion of vertices of a given degree is Poisson distributed, and the degree sequence drops off exponentially fast in the upper tail.

There are many models of graph processes designed to capture various aspects of the structure of the WWW found in the studies given above. See for example [Achlioptas et al. 01], [Adler and Mitzenmacher 01], [Aiello et al. 01a], [Aiello et al. 01b], [Albert et al. 99], [Barabassi et al. 99], [Barabassi and Albert 99], [Bollobás and Riordan 01], [Bollobás and Riordan to appear], [Bollobas and Riordan 02], [Bollobás and Riordan 03], [Buckley and Osthus 01], [Cooper and Frieze 01], [Chung and Lu to appear], [Chung and Lu 02], [Drinea et al. 01], [Henzinger et al. 99], [Kumar et al. 00a], [Kumar et al. 00b], and [Lu 01]. A recent survey by Bollobás and Riordan [Bollobás and Riordan 02] gives an excellent overview of the topic and many detailed structural results.

We consider a simple model of search, in which a particle (which we call a spider) makes a random walk on the nodes of an undirected graph process. It is presumed that the spider examines the data content of the nodes for some specific topic. As the spider is walking, the graph is growing, and the spider makes a random transition to whatever neighbours are available at the time. For simplicity, we assume that the growth rate of the process and the transition rate of the random walk are similar, so that the spider has at least a chance of crawling a constant proportion of the process.

We study the success of the spider's search on comparable graph processes of two distinct types: a random graph process and a web graph process. In the simple process we consider, each new vertex directs m edges toward existing vertices, either choosing vertices randomly (giving a random graph process) or copying according to vertex degree (giving a web graph process). Once a vertex has been added, the direction of the edges is ignored.

Our results are given in Theorems 1.1 and 1.2 below. Our main result is the following: For the random graph process, the expected proportion of unvisited vertices tends to 0.57. For the comparable web graph process, the expected proportion of unvisited vertices tends to 0.59.

There are several types of search which are appropriate to the WWW. Complete searches of the web, usually in a breadth first manner, are carried out by search engines such as Google [Brin and Page 98]. Link and page data for visited pages is stored, and from the link data an undirected model of the WWW can be constructed. This model may be entirely replaced when a new search is made at a future time period or may be continously updated by a continuously ongoing search. Such processes require considerable online and offline memory.

Another possibility, using a single processor, is a search by an agent which examines the semantic content of nodes for some specific topic. This type of search can be made directly on the WWW or on the model of the WWW stored by a search engine. Typical search strategies might include: moving to a random neighbour (sampling pages for content), selecting a random neighbour of large degree (locating the hub/authority vertices of the search topic), or selecting a random neighbour of low degree (favouring the discovery of newer vertices during the search).

Although the edges of a typical WWW graph are directed, the idea of evaluating models of search on an undirected process has many attractions, not least its simplicity. Examples which support the use of an undirected model are as follows.

The Google search engine [Brin and Page 98] holds a partial model of the WWW which it is continuously updating. Once a node is added to the search engine database, a list is maintained of pages in the database which point to this node. For a given node with url *PageUrl*, these links can be found by entering the **link:***PageUrl* query to Google. Thus the model of the WWW held by this search engine is equivalent to an undirected web graph. This information is used by Google to compute the page rank of a vertex. The page ranking is based on a weighted union of all directed paths leading to the vertex.

Finally, we remark that unless the web graph is strongly connected, a random walk on the directed edges soon becomes stuck at a vertex of out-degree zero. Modifications of the random walk approach which do not suffer from this defect have been studied by [Fagin et al. 00].

We then give a precise definition of the abstract scenario we consider in this paper. We have a sequence $(G(t), t = 1, 2, ...)$ of connected random graphs. The graph $G(t)$ is constructed from $G(t-1)$ by adding the vertex t, and m random edges from vertex t to $G(t-1)$. We refer to such graphs as web graphs.

There is also a spider S walking randomly from vertex to vertex on the evolving graph $G(t)$. The parameter ν_t we estimate is the expected number of vertices which have not been visited by the spider at step t, when t is large. This process is intended to model the success of a search-engine spider which is randomly crawling the World Wide Web looking for new web pages.

We consider the following models for the graph process $G(t)$. Let $m \geq 1$ be a fixed integer. Let $[t] = \{1, ..., t\}$ and let $G(1) \subset G(2) \subset \cdots \subset G(t)$.

Initially, $G(1)$ consists of a single vertex 1 plus m loops. For $t \geq 2$, $G(t)$ is obtained from $G(t-1)$ by adding the vertex t and m randomly chosen edges $\{t, v_i\}, i = 1, 2, \ldots, m$, where

Model 1. The vertices $v_1, v_2, \ldots, v_m$ are chosen independently and uniformly with replacement from $[t-1]$.

Model 2. The vertices $v_1, v_2, \ldots, v_m$ are chosen proportional to their degree after step $t-1$. Thus, if $d(v, \tau)$ denotes the degree of vertex v in $G(\tau)$, then for $v \in [t-1]$ and $i = 1, 2, \ldots, m$,

$$\mathbf{Pr}(v_i = v) = \frac{d(v, t-1)}{2m(t-1)}.$$

While vertex t is being added, the spider S is sitting at some vertex X_{t-1} of $G(t-1)$. After the addition of vertex t, and before the beginning of step $t+1$, the spider now makes a random walk of length ℓ, where ℓ is a fixed positive integer independent of t.

It seems unlikely that at time t, S will have visited every vertex. Let $\nu_{\ell,m}(t)$ denote the expected number of vertices not visited by S at the end of step t.

We will prove the following theorem:

Theorem 1.1. *In either model, if m is sufficiently large, then as $t \to \infty$,*

$$\nu_{\ell,m}(t) \sim \mathbf{E} \sum_{s=1}^{t} \prod_{\tau=s}^{t} \left(1 - \frac{d(s,\tau)}{2m\tau}\left(1 + O\left(\tfrac{1}{m}\right)\right)\right)^{\ell}. \tag{1.1}$$

We have said that m is fixed. We however have to accept errors of order $1/m$ and so in our asympotics, we let $t \to \infty$ first and then take m large.

Let

$$\eta_{\ell,m} = \lim_{t\to\infty} \frac{\nu_{\ell,m}(t)}{t}.$$

We will show that this gives the following limiting results for the models we consider.

Theorem 1.2. *Let $\eta_\ell = \lim_{m\to\infty} \eta_{\ell,m}$, then*

(a) For Model 1,

$$\eta_\ell = \sqrt{\frac{2}{\ell}} e^{(\ell+2)^2/(4\ell)} \int_{(\ell+2)/\sqrt{2\ell}}^{\infty} e^{-y^2/2}\, dy.$$

In particular, $\eta_1 = 0.57\cdots$, *and* $\eta_\ell \sim 2/\ell$ *as* $\ell \to \infty$.

(b) For Model 2,

$$\eta_\ell = e^\ell 2\ell^2 \int_\ell^\infty y^{-3} e^{-y}\, dy.$$

In particular, $\eta_1 = 0.59\cdots$, *and* $\eta_\ell \sim 2/\ell$ *as* $\ell \to \infty$.

Thus, for large m, t and $\ell = 1$, it is slightly more difficult for the spider to crawl on a web graph whose edges are generated by a copying process (Model 2) than on a uniform choice random graph (Model 1).

2. Proof of Theorem 1.1: The Main Ideas

We first consider the case where $\ell = 1$ and then generalise this case. When $\ell = 1$, the spider makes a random move to an adjacent vertex after vertex t has been added. The construction of $G(t)$ is really the construction of a digraph $D(t)$ where the direction of the arcs (x, y) satisfies $x > y$. The space $\mathcal{G}(t)$ of graphs $G(t)$ induces its measure from this space of digraphs.

Let $\Omega(t)$ denote the set of pairs $(G(t), W(t))$ where $G(t) \in \mathcal{G}(t)$ and $W(t)$ belongs to the set $\mathcal{W}_G(t)$ of t-step walks taken by the spider S which are compatible with the construction of G. Among other things, this means that the τ-th vertex of $G(t)$ visited by the walk must be in $[\tau]$.

The main idea of the proof is as follows. We fix a vertex s and estimate the probability that it is not visited by the end of step t. Thus, for $s \le \tau \le t$, we define the events

$$\mathcal{A}_s(\tau) = \{\omega \in \Omega(t) \colon \text{ Vertex } s \text{ is not visited by } \mathsf{S} \text{ during the time interval } [s, \tau]\}.$$

Let

$$t_0 = t - 100(\ln t)^3.$$

It is convenient to condition on the sequence $d(s, \tau)$ for $\tau = s, s+1, \ldots, t_0$. Let $\boldsymbol{\theta} = (\theta_\tau \colon 1 \le \tau \le t_0)$ be integers satisfying

$$\theta_1 = \cdots = \theta_s = m \le \theta_\tau \le \theta_t \le \Delta_t^* = 10(\ln t)^5 \text{ and } \theta_{\tau+1} \le \theta_\tau + 5 \text{ for } \tau \le t_0 \tag{2.1}$$

and let $\boldsymbol{\Theta} = \{\boldsymbol{\theta} \colon \text{ (2.1) holds}\}$.

Let

$$\mathcal{D}(\boldsymbol{\theta}) = \{(G(t), W(t)) \in \Omega(t) : d(s, \tau) = \theta_\tau,\ s \le \tau \le t\},$$

and for some event $\boldsymbol{C}$, let $\mathbf{Pr}_{\boldsymbol{\theta}}(\boldsymbol{C}) = \mathbf{Pr}(\boldsymbol{C} \mid \mathcal{D}(\boldsymbol{\theta}))$ be the probability of the corresponding conditional event.

We will show

Lemma 2.1. *In both Model 1 and Model 2,*

$$\mathbf{Pr}(\bigcup_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \mathcal{D}(\boldsymbol{\theta})) = 1 - \widetilde{O}(t^{-3})^1.$$

Let

$$\sigma_0 = \frac{\ln t}{100 \ln\ln t}$$

and let

$$B_t = \{s \in [t/\ln t, t]\colon\ s \text{ is within distance } \sigma_0 \text{ of a cycle of length at most } \sigma_0\}.$$

Let

$$\mathcal{G}_1(t) = \{G(t):\ |B_t| \le t^{7/8}\}.$$

We will prove that

Lemma 2.2. *If $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, then, in both Model 1 and Model 2,*

$$\mathbf{Pr}_{\boldsymbol{\theta}}(G(t) \notin \mathcal{G}_1(t)) = o(t^{-3}).$$

We then prove

Lemma 2.3. *If $s \in [t/\ln t, t)$, $s \notin B_t$ and $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, then*

(a) $$\mathbf{Pr}_{\boldsymbol{\theta}}(\overline{\mathcal{A}}_s(t) \mid \mathcal{A}_s(t-1)) = \frac{\theta_{t_0}}{2mt_0}\left(1 + O\left(\tfrac{1}{m}\right)\right) + O(t^{-3})\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1))^{-1}.$$

(b) $$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(s)) = 1 - O(s^{-1}).$$

(We condition on $\boldsymbol{\theta}$ in order to avoid some conditioning of the degree $d(s, t_0)$ due to assuming $\mathcal{A}_s(t_0)$.)

From this, we prove Theorem 1.1 as follows: If $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $s \in [t/\ln t, t]$, $s \notin B_t$, then

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t)) = \left(1 - \frac{\theta_{t_0}}{2mt}\left(1 + O\left(\tfrac{1}{m}\right)\right)\right)\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1)) + \widetilde{O}(t^{-3}).$$

[1] The $\widetilde{O}$ notation ignores polylog factors.

We see then that if $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and $s \in [t/\ln t, t)$, $s \notin B_t$ and if $\tau_0 = \tau - 100(\ln t)^3$,

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t)) = \prod_{\tau=s+1}^{t} \left(1 - \frac{\theta_{\tau_0}}{2m\tau}\left(1 + O\left(\tfrac{1}{m}\right)\right)\right) \tag{2.2}$$

$$= \prod_{\tau=s+1}^{t} \left(1 - \frac{\theta_{\tau}}{2m\tau}\left(1 + O\left(\tfrac{1}{m}\right)\right)\right). \tag{2.3}$$

Note that we can go from (2.2) to (2.3) because $\theta_\tau = \theta_{\tau_0}$ except for at most $100(\ln t)^3\Delta_t^*$ instances.

Thus, absorbing the cases where $\boldsymbol{\theta} \notin \boldsymbol{\Theta}$ into the error term, (see Lemma 2.1), summing out the conditional probabilities over degree sequences, we get that for $s \in [t/\ln t, t)$, $s \notin B_t$

$$\mathbf{Pr}(\mathcal{A}_s(t)) = \sum_{\boldsymbol{\theta}} \mathbf{Pr}(\mathcal{D}(\boldsymbol{\theta})) \prod_{\tau=s+1}^{t} \left(1 - \frac{\theta_{\tau}}{2m\tau}\left(1 + O\left(\tfrac{1}{m}\right)\right)\right)$$

$$= \mathbf{E} \prod_{\tau=s+1}^{t} \left(1 - \frac{d(s,\tau)}{2m\tau}\left(1 + O\left(\tfrac{1}{m}\right)\right)\right).$$

Note that the contribution of $s \in [1, t/\ln t] \cup B_t$ to the expectation $\nu_{\ell,m}(t)$ can only be $o(t)$ and Theorem (1.1) follows. □

3. Proof of Theorem 1.1: The Details

We emphasise that $s \geq t/\log t$ throughout and that m is a sufficiently large constant.

3.1. Proof of Lemma 2.1: Model 1

The degree $d(s,t)$ of vertex s in $G(t)$ is distributed as

$$m + B(m, (s+1)^{-1}) + \cdots + B(m, t^{-1}) \tag{3.1}$$

where the binomials $B(m, \cdot)$ are independent.

Lemma 3.1.
(a) $\mathbf{Pr}(\Delta(G(t)) \geq 2m \ln t) = \widetilde{O}(t^{-3})$ *where* $\Delta(G(t))$ *is the maximum degree in* $G(t)$.

(b) $\mathbf{Pr}(\exists \tau:\ d(s,\tau+1) - d(s,\tau) > 5) = \widetilde{O}(t^{-4})$.

Proof. $\mathbf{E}\,(d(s,t)) = m(1 + H_t - H_s) \le m(2 + \ln t/s)$ where $H_k = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$. (a) now follows from Theorem 1 of Hoeffding [Hoeffding 63]. (b) is easy, since $d(s,\tau+1) - d(s,\tau) = B(m,\tau^{-1})$. □

3.2. Proof of Lemma 2.1: Model 2

Lemma 3.2.
(a) $\mathbf{Pr}(d(s,t) \ge 10(\ln t)^5) = \widetilde{O}(t^{-3})$.

(b) $\mathbf{Pr}(\exists \tau :\ d(s,\tau+1) - d(s,\tau) > 5) = \widetilde{O}(t^{-3})$.

Proof. (a) In order to get a crude upper bound on $d(s,t)$, we divide the interval $[s,t]$ into subintervals using the points (nearest to) $s, se^{1/8}, ...se^{r/8}, ..., se^{k/8}$. Here, $se^{(k-1)/8} < t \le \lceil se^{k/8} \rceil$, so that $k \le 8 \ln\ln t$, as $s \ge t/\ln t$.

Suppose that, at the start of $I_r = (\lceil se^{r/8} \rceil, \lceil se^{(r+1)/8} \rceil]$, we have an upper bound $d(r)$ on the degree of vertex s. We prove that if $d(r) \ge 10 \ln t$, then $d(r+1) \le 2d(r)$ with probability $1 - o(t^{-3})$.

Now as long as the degree of s is $\le 2d(r)$, the number X_τ of edges acquired at step $\tau \in I_r$ is dominated by $B(m, d(r)/(m(\tau-1)))$, so that the number of edges gained during this time has expected value

$$\le 2d(r) \ln e^{1/8} = \frac{d(r)}{4}.$$

Thus, by Chernoff bounds, provided $d(r) \ge 10 \ln t$,

$$\begin{aligned} \mathbf{Pr}(d(r+1) \ge 2d(r)) &\le \mathbf{Pr}\left(\sum_{\tau \in I_r} B(m, d(r)/(m(\tau-1))) \ge d(r) \right) \\ &\le \left(\frac{e}{4}\right)^{d(r)} = o(t^{-3}) \end{aligned}$$

and thus, $d(r+1) < 2d(r)$ with probability $1 - o(t^{-3})$. Choosing $d(0) = 10 \ln t$, we see that

$$d(s,t) < d(0) 2^k \le d(0)(\ln t)^4 = 10(\ln t)^5.$$

This proves (a). For (b), we use (a) and the fact that $d(s,\tau+1) - d(s,\tau)$ can then be dominated by $B(10(\ln t)^5, (2m\tau)^{-1})$. □

3.3. Proof of Lemma 2.2: Model 1

Proof. Fix $t/\ln t \le i_1 < \cdots < i_5 \le t$ and let $I = \{i_1, \ldots, i_5\}$. We estimate $\mathbf{Pr}(I \subseteq B_t)$. For each partition $\mathcal{P}$ of I into parts $A_1, \ldots, A_k$, we consider the event

$\mathcal{E}_{\mathcal{P}} = \{\exists$ small cycles $C_1, \ldots, C_k$ and paths $P_v, v \in I$ such that

(i) $|C_i|, |P_v| \le \sigma_0$ for all i, v.

(ii) If $v \in A_i$, then P_v joins v to $C_i \cup \bigcup_{w \in A_i, w<v} P_w$;

(iii) P_v is edge disjoint from and shares one (endpoint) vertex with $C_i \cup \bigcup_{w \in A_i, w<v} P_w$;

(iv) The k collections $C_i, P_v, v \in A_i$ are pair-wise vertex disjoint. $\}$

Thus, $\{I \subseteq B_t\} \subseteq \bigcup_{\mathcal{P}} \mathcal{E}_{\mathcal{P}}$ and

$$\mathbf{Pr}(\mathcal{E}_{\mathcal{P}}) \le \sum_{\substack{C_1,\ldots,C_k \\ P_v, v \in I}} \prod_{(x,y)\in F^*} \frac{m}{\max\{x,y\}} \tag{3.2}$$

where F^* denotes the edge set of $\bigcup C_i \cup \bigcup P_v$. The term $\frac{m}{\max\{x,y\}}$ is a bound on the probability of the existence of edge (x, y) given the appearance or absence of other edges, not incident with $\max\{x, y\}$.

Thus ,

$$\mathbf{Pr}(\mathcal{E}_{\mathcal{P}}) \le \prod_{r=1}^{5} \frac{m}{i_r} \sum_{\substack{3 \le |C_i| \le \sigma_0 \\ i=1,\ldots,k}} \sum_{\substack{0 \le |P_v| \le \sigma_0 \\ v \in I}} \prod_{v \in V^*} \frac{m}{v}$$

where V^* denotes the vertex set of $\bigcup C_i \cup \bigcup P_v$, less I,

$$\le \left(\frac{m \ln t}{t}\right)^5 \sum_{\ell=1}^{10\sigma_0} \left(\sum_{v=1}^{t} \frac{m}{v}\right)^{\ell}$$

$$= o(t^{-4}).$$

Thus,

$$\mathbf{E}\left(\binom{|B_t|}{5}\right) = o(t)$$

and

$$\mathbf{Pr}(|B_t| \ge t^{7/8}) \le \frac{\mathbf{E}\left(\binom{|B_t|}{5}\right)}{\binom{t^{7/8}}{5}} = o(t^{-3}).$$

□

3.4. Proof of Lemma 2.2: Model 2

Proof. For this model, we replace 5 by 10 and let $I = \{i_1, i_2, \ldots, i_{10}\}$. Let

$$\mathcal{G}_2(t) = \{G(t) : \ d(s,t) \le 2m\sqrt{t/s}(\ln t)^2 \text{ for all } 1 \le s \le t\}.$$

It is shown below, see Lemma 4.3(a), that

$$\mathbf{Pr}(G(t) \in \mathcal{G}_2(t)) = 1 - \widetilde{O}(t^{-10}).$$

Therefore, we replace (3.2) by

$$\begin{aligned}
\mathbf{Pr}(\mathcal{E}_{\mathcal{P}} \mid G(t) \in \mathcal{G}_2(t)) &\leq \sum_{\substack{C_1,\ldots,C_k \\ P_v, v\in I}} \prod_{(x,y)\in F^*} \frac{2m(\ln t)^2}{\max\{x,y\}} \sqrt{\frac{\max\{x,y\}}{\min\{x,y\}}} \frac{1}{\mathbf{Pr}(G(t) \in \mathcal{G}_2(t))} \\
&\leq 2 \sum_{\substack{C_1,\ldots,C_k \\ P_v, v\in I}} \prod_{(x,y)\in F^*} \frac{2m(\ln t)^2}{x^{1/2}y^{1/2}}. \\
&\leq \left(\frac{2m(\ln t)^2}{t^{1/2}}\right)^{10} \sum_{\ell=1}^{20\sigma_0} \left(\sum_{v=1}^{t} \frac{2m(\ln t)^2}{v}\right)^{\ell} \\
&= \widetilde{O}(t^{-23/5}).
\end{aligned}$$

Thus, we have

$$\mathbf{E}\left(\binom{|B_t|}{10}\right) = \widetilde{O}(t^{-10}) + \widetilde{O}(t^{10-23/5}) = \widetilde{O}(t^{27/5})$$

and

$$\mathbf{Pr}(|B_t| \geq t^{7/8}) \leq \frac{\mathbf{E}\left(\binom{|B_t|}{10}\right)}{\binom{t^{7/8}}{10}} = o(t^{-3}).$$

□

3.5. Proof of Lemma 2.3

3.5.1. Rapidly mixing walks. We now consider the random walk made by the spider S. A *random walk* on an fixed undirected graph G is a Markov chain (X_t), $X_t \in V$ associated to a particle that moves from vertex to vertex according to the following rule: The probability of a transition from vertex v, of degree d, to vertex w is $1/d$ if v is adjacent to w, and 0 otherwise. Let π denote the steady state distribution of the random walk. The steady state probability $\pi_G(v)$ of the walk being at a vertex v is,

$$\pi_G(v) = \frac{d(v)}{d(G)}, \tag{3.3}$$

where $d(v)$ is the degree of v and $d(G)$ is the total degree (i.e., sum of the degrees) of the graph G.

We will need a finite time approximation of the probability distribution π_H pertaining to a random walk on a subgraph $H = G(t) - s$ of $G(t)$. We obtain this by considering the *mixing time* of the walk based on a conductance bound (3.7) of Jerrum and Sinclair [Sinclair and Jerrum 89].

Let s, t be fixed with $s \in \left[\frac{t}{\ln t}, t\right) \setminus B_t$. Let P denote the transition matrix of the random walk on H. Let $P^{i,\tau}$ denote the distribution of the τth step of a random walk on H which starts at vertex i. For $K \subset V(H) = [t] \setminus \{s\}$, let $\overline{K} = V(H) \setminus K$ and

$$\Phi_K = \frac{\sum_{i \in K, j \in \overline{K}} \pi_H(i) P(i,j)}{\pi_H(K)}.$$

It follows from (3.3) that

$$\Phi_K = \frac{e(K : \overline{K})}{d(K)}$$

where $e(K : \overline{K})$ is the number of edges from K to $\overline{K}$, and $d(K)$ is the total degree of vertices in set K.

The conductance of the walk is defined by

$$\Phi(s,t) = \min_{\pi_H(K) \leq 1/2} \Phi_K.$$

Let

$$\mathcal{G}_3(t) = \left\{ G(t) : \ \Phi(s,t) > \frac{1}{\ln t} \forall s \in [t/\ln t, t] \right\}.$$

Lemma 3.3. *If $\boldsymbol{\theta} \in \boldsymbol{\Theta}$, then, in both Model 1 and Model 2,*

$$\mathbf{Pr}_{\boldsymbol{\theta}}(G(t) \notin \mathcal{G}_3(t)) = o(t^{-3}).$$

□

3.5.2. Proof of Lemma 3.3: Model 1 Since $d(K) \leq 2m|K| + e(K : \overline{K})$ it suffices to prove a high probability lower bound on $e(K : \overline{K})$, in both models.

Lemma 3.4.

$$\mathbf{Pr}_{\boldsymbol{\theta}} \left(\Phi(s,t) \leq \frac{1}{200} \right) = o(t^{-3}).$$

Proof. For $K \subseteq [t]$, let $d(K,t) = \sum_{s \in K} d(s,t)$. Then

$$\mathbf{Pr}(\exists K \subseteq [t] : \ |K| \geq 3t/4 \text{ and } d(K,t) \leq (1.1)mt) = o(e^{-cmt}) \qquad (3.4)$$

for some absolute constant $c > 0$.

To see this, let $K \subseteq [t]$ with $|K| = k = 3t/4$. Then

$$\mathbf{E}\left(d(K,t)\right) \geq \mathbf{E}\left(d([t-k+1,t],t)\right) = mk + m \sum_{s=t-k+1}^{t} \frac{s-(t-k)}{s}$$
$$\geq 2mk - m(t-k)\ln(t/(t-k)) = \left(\frac{3}{2} - \frac{1}{4}\ln 4\right) mt \geq (1.15)mt.$$

Applying Theorem 1 of Hoeffding, we see that

$$\mathbf{Pr}(\exists K \subseteq [t] : \ |K| \geq 3t/4 \text{ and } d(K,t) \leq (1.1)mt) \leq \binom{t}{3t/4} e^{-c'mt}$$

for some absolute constant $c' > 0$. This completes the proof of Lemma 3.4. □

Now for $K, L \subset [t] \setminus \{s\}$, let $e(K : L)$ denote the number of edges of $G(t)$ which have one end in K and the other end in L (we only use this definition for $L = \overline{K} = [t] \setminus (K \cup \{s\})$ and $L = K$).

It follows from (3.4) that with probability $1 - o(t^{-3})$

$$\Phi(s,t) \geq \min_{\pi(K) \leq 1/2} \frac{e(K:\overline{K}) - 5|K|}{m|K| + e(K:\overline{K})} \geq \min_{|K| \leq 3t/4} \frac{e(K:\overline{K}) - 5|K|}{m|K| + e(K:\overline{K})}. \tag{3.5}$$

($e(K:\overline{K}) - 5|K|$ bounds the number of $K:\overline{K}$ edges in $H_s(t)$ and then observe that the degree sum of K is at most $m|K| + e(K:\overline{K})$.)

We prove the following high probability lower bound on $e(K:\overline{K})$. Together with (3.5) this proves Lemma 3.3.

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K : \ e(K:\overline{K}) \leq m|K|/150) = o(t^{-3}). \tag{3.6}$$

Suppose $K \subset [t]$, $k = |K|$ and $Y_K = e(K:\overline{K})$. Let $\kappa = \frac{1}{2}\sqrt{kt}$ and $K_- = K \cap [\kappa]$ and $K_+ = K \setminus K_-$.

Case 1: $|K_-| \geq 3k/7$.

$$\mathbf{E}_{\boldsymbol{\theta}}\left(Y_K\right) \geq \sum_{\tau=\kappa}^{t-4k/7-1} \frac{3(m-5)k/7}{\tau + 4k/7} \geq \frac{3(m-5)k}{7} \ln\left(\frac{t-1}{\kappa + 4k/7}\right).$$

Explanation: Consider the $\geq t - \kappa - 4k/7 - 1$ vertices of $[t] - [\kappa] - \{s\} - K$. Each chooses at least $m-5$ random neighbours from lower numbered neighbours (plus themselves) and the sum minimises the expected number of these choices in K_-. The 5 comes from $\theta_{\tau+1} - \theta_t \leq 5$ for $\boldsymbol{\theta} \in \boldsymbol{\Theta}$.

Applying Theorem 1 of [Hoeffding 63], we obtain

$$\mathbf{Pr}_{\boldsymbol{\theta}}(Y_K \leq \mathbf{E}_{\boldsymbol{\theta}}\left(Y_K\right)/2) \leq \exp\left\{-\frac{1}{8}\frac{3mk}{7}\ln\left(\frac{t-1}{\kappa+4k/7}\right)\right\} = \left(\frac{\kappa + 4k/7}{t-1}\right)^{3mk/56}.$$

So,

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ |K_-|\geq 3k/7,\ |K|\leq 3t/4 \text{ and } Y_K \leq \mathbf{E}(Y_K)/2) \leq$$
$$\sum_{k=1}^{3t/4}\binom{t}{k}\left(\frac{\kappa+4k/7}{t-1}\right)^{3mk/56} \leq \sum_{k=1}^{3t/4}\left(\frac{te}{k}\left(\frac{\kappa+4k/7}{t-1}\right)^{3m/56}\right)^{k}$$
$$\leq \sum_{k=1}^{3t/4}\left(\frac{3t}{k}\left(\sqrt{\frac{k}{t}}\left(\frac{1}{2}+\frac{4}{7}\sqrt{\frac{3}{4}}\right)\right)^{3m/56}\right)^{k} = o(t^{-3}).$$

This yields (3.6) for this case.

Case 2: $|K_-| \leq 3k/7$. Assume first that $k \geq 1000$. Now let Z_K denote the number of edges from the set W of $\lceil k/15\rceil$ lowest numbered vertices of K_+ which have their lower numbered endpoints also in K. Z_K is dominated by $B(m\lceil k/15\rceil, \sqrt{k/t})$ since there are at most $3k/7+\lceil k/15\rceil \leq k/2$ vertices of K below any vertex w of W and there are at least κ vertices in all below such a w. We use $Y_K = e(K:\overline{K}) \geq M - Z_K$ where $M=(m-5)\lceil k/15\rceil$. For $|K|\leq t/1000$, we write

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ 1000\leq |K|\leq t/1000,\ Z_K\geq M/2) \leq \sum_{k=1000}^{t/1000}\binom{t}{k}2^M\left(\frac{k}{t}\right)^{M/2}$$
$$\leq \sum_{k=1000}^{t/1000}\left(\frac{te}{k}\left(\frac{4k}{t}\right)^{(m-5)/30}\right)^{k} = o(t^{-3}).$$

For $|K| > t/1000$, we use Chernoff bounds and write, for some absolute positive constant $c>0$

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ t/1000\leq |K|\leq 3t/4,\ Z_K\geq 9M/10) \leq \sum_{k=t/1000}^{3t/4}\binom{t}{k}e^{-cM} = o(t^{-3}).$$

For $|K|\leq 1000$, we can write

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K:\ e(K,K)\geq 3mk/4) \leq \sum_{k=1}^{1000}\binom{t}{k}\binom{mk}{3mk/28}\left(\frac{1000}{t^{1/2}}\right)^{3mk/28} = o(t^{-3}).$$

Note that if $e(K,K)\geq 3mk/4$ then at least $3mk/4-3mk/7$ of these edges must have one end in K_+.

This completes the proof of (3.6). □

3.5.3. Proof of Lemma 3.3: Model 2.

Lemma 3.5. *There is an absolute constant $\xi > 0$ such that*

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\exists K \subseteq [t] - \{s\}, |K| \geq (1-\xi)t : \ d(K,t) \leq (1+\xi)mt) = o(t^{-3}).$$

Proof. Let ζ be a small positive constant and divide $[t]$ into approximately $1/\zeta$ consecutive intervals $I_1, I_2, \ldots$ of size $\lceil \zeta t \rceil$ plus an interval of $t - \lfloor 1/\zeta \rfloor \lceil \zeta t \rceil$. We put a high probability bound on the total degree $d(I_1, t)$. Now consider the random variables $\beta_k, k = 1, 2, \ldots$ where $\beta_k = d(I_1, k\lceil \zeta t \rceil)/(m\lceil \zeta t \rceil)$. Now $\beta_1 = 2$ and conditional on the value of β_k,

$$(\beta_{k+1} - \beta_k)m\lceil \zeta t \rceil \text{ is dominated by } B\left(m\lceil \zeta t \rceil, \frac{\beta_k + 1}{2k}\right).$$

It follows that we can find an absolute constant $c > 0$ such that

$$\mathbf{Pr}_{\boldsymbol{\theta}}\left(\beta_{k+1} \leq \beta_k\left(1 + \frac{3}{4k}\right)\right) \leq e^{-cm\zeta t}.$$

So, with probability $1 - O(e^{-cm\zeta t})$, we find that

$$d(I_1, t) \leq 2m\lceil \zeta t \rceil \prod_{k=1}^{\lceil 1/\zeta \rceil}\left(1 + \frac{3}{4k}\right) \leq 2m\lceil \zeta t \rceil \times e^{3/4}\lceil 1/\zeta \rceil^{3/4} \leq 6m\zeta^{1/4}t,$$

for small enough ζ.

Now $d([\lceil \zeta t \rceil], t)$ dominates $d(L, t)$ for any set L of size $\lceil \zeta t \rceil$. So, if $m > 1/(c\zeta)$, then the probability there is a set of size $\lceil \zeta t \rceil$ which has degree exceeding $6m\zeta^{1/4}$ is exponentially small ($\leq \binom{t}{\lceil \zeta t \rceil}e^{-t}$)). In this case, every set K of size at least $t - \lceil \zeta t \rceil$ has total degree $d(K, t) \geq 2mt - 6m\zeta^{1/4}t$ and the lemma follows by taking ζ sufficiently small. □

Lemma 3.6. *If m is sufficiently large, then*

$$\mathbf{Pr}_{\boldsymbol{\theta}}\left(\Phi(t) < \frac{1}{\ln t}\right) = O(t^{-3}).$$

Proof. For $K \subseteq [t]$, $|K| = k$, we say K is *small* if $\ln t \leq k \leq ct$ and K is large otherwise, where $c = e^{-8}$.

3.5.4. Case of K *small.* Let $K_- = K \cap [\sqrt{kt}]$ and let $K_+ = K \setminus K_-$.

Case of $|K_-| \geq k/2$*:* Let $X_t = X_t(K_-)$ be the number of those edges directed into K_- from vertices created after time $\sqrt{kt}$. The number of such edges generated at step $\tau \geq \sqrt{kt}$ dominates $B(m-5, mq/(2m\tau))$, independently of any previous step. (Here $q = |K_-|$). Thus,

$$\mathbf{E}\left(X_t\right) \geq \sum_{\tau > \sqrt{kt}}^{t} \frac{(m-5)q}{2\tau} = \frac{(m-5)q}{4} \ln \frac{t}{k}(1+o(1)).$$

Hence,

$$\mathbf{Pr}\left(X_t \leq \frac{mq}{6} \ln \frac{t}{k}\right) \leq \exp\left(-\frac{mq}{73} \ln \frac{t}{k}\right).$$

Thus,

$$\begin{aligned}
\mathbf{Pr}\left(\exists K_- : \ X_t(K_-) \leq \frac{mq}{6} \ln \frac{t}{k}\right) &\leq \binom{\sqrt{kt}}{q} \exp\left(-\frac{mq}{73} \ln \frac{t}{k}\right) \\
&\leq \exp\left(-q\left(\frac{m}{73} \ln \frac{t}{k} - \ln\left(2e\sqrt{\frac{t}{k}}\right)\right)\right) \\
&\leq t^{-4}
\end{aligned}$$

provided m is sufficiently large. Thus, **whp** the set K_- has at least $\frac{mk}{12} \ln t/k$ edges directed into it, of which at most $mk/2$ are incident with K_+. This completes the analysis of this case.

Case of $|K_+| \geq k/2$*:* We consider the evolution of the set $K_+ = \{u_1, u_2, \ldots, u_r\}$ from step $T = \sqrt{kt}$ onward. Assume that at the final step t, there are δk edges directed into K from $\overline{K}$. We can assume w.l.o.g. that $\delta \leq m/10$, for otherwise there is nothing to prove.

The number Y_{j+1} of $K : K$ edges generated by vertex u_{j+1} is a binomial random variable with expectation at most

$$\mu_{j+1} = m\frac{2mk + \delta k}{2mt_{j+1}}.$$

The numerator in the above fraction is a bound on the total degree of K.

If $Z = Z(K_+) = \sum_{j=1}^{r} Y_j$, then

$$\begin{aligned}
\mathbf{E}\left(Z\right) &\leq \frac{2mk + \delta k}{2}\left(\frac{1}{t_1} + \cdots + \frac{1}{t_r}\right) \\
&\leq \frac{2mk + \delta k}{2} \frac{r}{\sqrt{kt}} \\
&\leq 1.05\, \frac{mkr}{\sqrt{kt}}.
\end{aligned}$$

Thus, for $\alpha > 0$,

$$\begin{aligned}\mathbf{Pr}\left(\exists K_+ : \ Z(K_+) \geq \alpha k\right) &\leq \sum_{r=k/2}^{k} \binom{t}{r} \left(\frac{e \times 1.05 \times kmr}{\sqrt{kt} \times \alpha k}\right)^{\alpha k} \\ &\leq k\left(\left(\frac{3mk^{1/2}}{\alpha t^{1/2}}\right)^{\alpha} \frac{te}{k}\right)^{k} \\ &\leq t^{-4}\end{aligned}$$

if $\alpha = m/4$, $k \leq ct$ and m is sufficiently large. We have therefore proved that for small values of k, there are at least $mk/2 - mk/4$ out-edges generated by K_+ not incident with K on the condition that $\delta \leq m/10$, completing the analysis of this case.

3.5.5. Case of K large. Let $T = t/2$ and let $ct \leq |K|, |\overline{K}| \leq (1-\xi)t$ where ξ is as in Lemma 3.5. Let $M = [T]$ and $N = [T+1, t]$. Let $K_- = K \cap M$, $K_+ = K \cap N$, $q = |K_-|$, and $r = |K_+|$. We calculate the expected number of edges $\mu(K_-, K_+)$ of $L = (K_+ \times (M \setminus K_-)) \cup ((N \setminus K_+) \times K_-)$ generated at steps τ, $T \leq \tau \leq t$ which are directed into K. At step τ the number of such edges falling in L is an independent random variable with distribution dominating

$$1_{\tau \in N \setminus K_+} B\left(m-5, \frac{mq}{2m\tau}\right) + 1_{\tau \in K_+} B\left(m-5, \frac{(T-q)m}{2m\tau}\right).$$

Thus,

$$\begin{aligned}\mu(K_-, K_+) &\geq \frac{(m-5)q}{2} \sum_{\tau \in N \setminus K_+} \frac{1}{\tau} + \frac{(m-5)(T-q)}{2} \sum_{\tau \in K_+} \frac{1}{\tau} \\ &= \frac{m-5}{2}\left((k-r) \sum_{\tau \in N \setminus K_+} \frac{1}{\tau} + (T-(k-r)) \sum_{\tau \in K_+} \frac{1}{\tau}\right).\end{aligned}$$

Let $\mu(k) = \min_{K_-, K_+} \mu(K_-, K_+)$. Then "somewhat crudely,"

$$\begin{aligned}\sum_{\tau \in N \setminus K_+} \frac{1}{\tau} &\geq \ln \frac{t}{T+r} \\ \sum_{\tau \in K_+} \frac{1}{\tau} &\geq \ln \frac{t}{t-r}.\end{aligned}$$

Thus,

$$\mu(k) \geq \frac{m-5}{2}\left((k-r)\ln\frac{2t}{t+2r} + \left(\frac{t}{2} - (k-r)\right)\ln\frac{t}{t-r}\right).$$

Putting $k = \kappa t$ and $r = \rho t$, we see that

$$\mu(k) \geq \frac{(m-5)t}{2} g(\kappa, \rho)$$

where

$$g(\kappa, \rho) = (\kappa - \rho) \ln \frac{2}{1+2\rho} + \left(\tfrac{1}{2} - \kappa + \rho\right) \ln \frac{1}{1-\rho}.$$

We put a lower bound on g:

$$\rho \leq \frac{\xi}{2} \text{ implies } \kappa - \rho \geq \frac{\xi}{2} \text{ and so } g(\kappa, \rho) \geq \frac{\xi}{2} \ln \frac{2}{1+\xi}.$$

So we can assume that $\rho \geq \xi/2$. Then

$$\begin{array}{rcl} \kappa - \rho \leq \dfrac{1-\xi}{2} & \text{implies} & g(\kappa,\rho) \geq \dfrac{\xi}{2} \ln \dfrac{2}{2-\xi}. \\ \kappa - \rho > \dfrac{1-\xi}{2} \text{ and } \rho \leq \dfrac{1-\xi}{2} & \text{implies} & g(\kappa,\rho) \geq \dfrac{1-\xi}{2} \ln \dfrac{2}{2-\xi}. \\ \kappa - \rho > \dfrac{1-\xi}{2} \text{ and } \rho > \dfrac{1-\xi}{2} & \text{implies} & \kappa > 1 - \xi. \end{array}$$

We deduce that within our range of interest,

$$\mu(k) \geq \eta m t$$

for some absolute constant η.

Let Z be the number of edges generated within L, so that Z counts a subset of the edges between K and $\overline{K}$. Then

$$\begin{aligned} \mathbf{Pr}\left(\exists K_-, K_+ \subseteq N : \ Z \leq \frac{1}{2}\eta m t\right) &\leq 2^t e^{-\eta m t/8} \\ &\leq e^{-\eta m t/10}. \end{aligned}$$

Recall that m is sufficiently large. This completes the proof of the lemma, except for very small sets K.

For sets K of size $s \leq \ln t$, we note that, as $G(t)$ is connected, the conductance Φ_K is always $\Omega(1/|K|)$. □

3.6. Proof of Lemma 2.3: Continued

Define

$$\mathcal{G}(t) = \begin{cases} \mathcal{G}_1(t) \cap \mathcal{G}_3(t) & \text{Model 1} \\ \mathcal{G}_1(t) \cap \mathcal{G}_2(t) \cap \mathcal{G}_3(t) & \text{Model 2.} \end{cases}$$

We apply the main result of [Sinclair and Jerrum 89].

$$|P^{i,\tau}(v) - \pi_H(v)| \le \left(1 - \frac{\Phi^2}{2}\right)^{\tau} \frac{\pi_H(v)}{\pi_{\min}} \tag{3.7}$$

where $\pi_{\min} = \min_v \pi_H(v)$.

Using (3.7) and Lemma 3.3, we see that with $\mu_0 = 10(\ln t)^3$, **whp**

$$|P^{i,\mu_0}(v) - \pi_H(v)| = O(t^{-4}) \qquad \forall v \in [t] \setminus \{s\}. \tag{3.8}$$

We are glossing over one technical point here. Strictly speaking, (3.7) only holds for Markov chains in which $P(x,x) \ge 1/2$ for all states x. To get around this, one usually makes the walk flip a fair coin and stay put if the coin comes up heads. In our case, we also omit to add a new vertex if the coin is heads. So what we have been describing is the outcome, ignoring those times when the coin flip is heads.

For the moment, we fix some $\boldsymbol{\theta} \in \boldsymbol{\Theta}$ and assume that $t/\ln t \le s < t$.

Now, by definition, $t_0 = t - 10\mu_0$ and we define

$$\begin{aligned}
I &= [t_0+1, t-1]\\
J &= \{\sigma \in I :\ \exists \tau \in I \text{ such that } X_\tau = \sigma\}\\
\mathcal{E}_0 &= \{X_\tau \ne s,\ \tau \in I\}\\
\mathcal{E}_1 &= \{\exists i \in I, j \in J \colon X_i = j \text{ and } j \text{ has a neighbour in } \{X_\sigma \colon \sigma \in [t_0, i-2]\}\}\\
\mathcal{F}_k &= \{|J| = k\} \qquad k \ge 0\\
\mathcal{F}_{\ge k} &= \{|J| \ge k\} \qquad k \ge 0
\end{aligned}$$

and write

$$\begin{aligned}
&\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid \mathcal{A}_s(t-1)) =\\
&\quad \sum_{\substack{G \in \mathcal{G}(t_0)\\ w \in [t_0]\setminus\{s\}}} \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{A}_s(t_0))\\
&\qquad\qquad \cdot \mathbf{Pr}_{\boldsymbol{\theta}}(X_{t_0} = w, G(t_0) = G \mid \mathcal{A}_s(t-1))\\
&\qquad\qquad\qquad + \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s, G(t_0) \notin \mathcal{G}(t_0) \mid \mathcal{A}_s(t-1)).
\end{aligned} \tag{3.9}$$

It follows from Lemma 3.3 that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s, G(t_0) \notin \mathcal{G}(t_0) \mid \mathcal{A}_s(t-1)) = o(t^{-3}\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1))^{-1}). \tag{3.10}$$

To deal with the rest of (3.9), we write

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{A}_s(t_0)) &\\ = \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0) &\\ = \sum_{k=0}^{1} \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_k) &\\ \cdot \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_k \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0) &\\ + \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_{\geq 2}) &\\ \cdot \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_{\geq 2} \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0). & \qquad (3.11)\end{aligned}$$

Given $w, G(t_0)$, conditioning on $\mathcal{B} \stackrel{def}{=} \mathcal{E}_0 \cap \mathcal{F}_0$ is "almost" equivalent to S doing a random walk on $G(t_0) - \{s\}$ starting at w. In fact, we get Lemma 3.7.

Lemma 3.7.

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{B}) =$$
$$\frac{\theta_{t_0}}{2mt_0}\left(1 + \frac{1}{\theta_{t_0}} \sum_{y \in N(s,t_0)} \mathbf{E}_{\boldsymbol{\theta}}\left(\frac{d(y,t_0) - d(y,t)}{d(y,t)}\right)\right)\left(1 + O\left(\frac{1}{m}\right)\right). \qquad (3.12)$$

where $N(s, t_0)$ denotes the set of neighbours of s in $G(t_0)$.

Proof. We emphasize that throughout the proof of this lemma, a graph $G \in \mathcal{G}(t_0)$ is fixed as well as $X_{t_0} = w$. All probabilities are conditional on this, even if not stated explicitly. The only randomness in the graph $G(t)$ itself is due to new vertices.

Let

$$\mathcal{M} = \{\nexists v \in [t_0], v \neq s: \quad v \text{ has more than five neighbours in } I\}.$$

Then in both models

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\overline{\mathcal{M}} \mid G(t_0) = G) \leq |I|^6 \left(\frac{2mt_0^{1/2}(\ln t_0)^2}{mt_0}\right)^6 = \widetilde{O}(t^{-3}). \qquad (3.13)$$

Fix $y \in N(s, t_0)$ and let $\mathcal{W}_k(y)$ denote the set of walks in $H = G(t_0) - s$ which start at w, finish at y, are of length $\lambda_0 = t - t_0 = 100(\ln t)^3$, and which *leave* N^* exactly k times where N^* is the (random) set of neighbours of $I \cup \{s\}$ in $G(t_0)$. Let $\mathcal{W}_k = \bigcup_y \mathcal{W}_k(y)$ and let $W = (w_0, w_1, \ldots, w_{\lambda_0}) \in \mathcal{W}_k(y)$. Let

$$\rho_W = \frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(i) = w_i, i = 0, 1, \ldots, \lambda_0 \mid \mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(i) = w_i, i = 0, 1, \ldots, \lambda_0)}. \qquad (3.14)$$

Here, $X_G(i), i = 0, 1, \ldots, \lambda_0$ is the sequence of vertices visited by S at times $t_0, t_0+1, \ldots, t$ and we will use $W_G = W_{w,G}$ to denote this walk. We let $X_H(i), i = 0, 1, \ldots, \lambda_0$ is the set of vertices of H visited by a random walk $W_H = W_{w,H}$ on H with start vertex w.

Then

$$1 \geq \rho_W \geq \left(\frac{m-5}{m}\right)^k.$$

This is because a vertex can have at most five edges joining it to s and then

$$\frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(i) = w_i \mid X_G(i-1) = w_{i-1}, \mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(i) = w_i \mid X_H(i-1) = w_{i-1})} \begin{cases} \geq \frac{d_G(w_{i-1})-5}{d_G(w_{i-1})} & w_{i-1} \in N^*. \\ = 1 & w_{i-1} \notin N^*. \end{cases}$$

Furthermore,

$$\begin{aligned} \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M}) &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k} \mathbf{Pr}_{\boldsymbol{\theta}}(W_{w,G}(\lambda_0) = W \mid \mathcal{M}) \\ &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k} \rho_W \mathbf{Pr}_{\boldsymbol{\theta}}(W_{w,H}(\lambda_0) = W) \\ &\geq \sum_{k \geq 0} p_k \left(\frac{m-5}{m}\right)^k \end{aligned}$$

where

$$p_k = \sum_{W \in \mathcal{W}_k} \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) = W) = \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) \in \mathcal{W}_k).$$

We will show later that

$$p_0 + p_1 + p_2 \geq 1 - O(m^{-1}) \tag{3.15}$$

which immediately implies that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M}) \geq p_0 + p_1 \left(1 - \frac{5}{m}\right) + p_2 \left(1 - \frac{5}{m}\right)^2 \geq 1 - O(m^{-1}). \tag{3.16}$$

Now write

$$\begin{aligned} \mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y \mid \mathcal{B}, \mathcal{M}) &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k(y)} \mathbf{Pr}_{\boldsymbol{\theta}}(W_G(\lambda_0) = W \mid \mathcal{M}) \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M})^{-1} \\ &= \sum_{k \geq 0} \sum_{W \in \mathcal{W}_k(y)} \rho_W \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) = W) \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M})^{-1}. \end{aligned}$$

Now if

$$\begin{aligned} p_{k,y} &= \frac{\mathbf{Pr}_{\boldsymbol{\theta}}(W_H \in \mathcal{W}_k(y))}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y)} \\ &= \mathbf{Pr}_{\boldsymbol{\theta}}(W_H(\lambda_0) \text{ leaves } N^* \text{ exactly } k \text{ times} \mid X_H(\lambda_0) = y), \end{aligned}$$

then we have

$$\sum_{k \geq 0} p_{k,y} \left(\frac{m-5}{m}\right)^k \leq \frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y \mid \mathcal{B}, \mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y)} \leq \mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B} \mid \mathcal{M})^{-1}. \tag{3.17}$$

We need to be careful about probability spaces here. Let $N^{\#}$ denote the the set of neighbours of I in $G(t_0)$. In our conditional probability space, the $p_{k,y}$ are now to be thought of as random variables dependent on $N^{\#}$.

We will show later that

$$\mathbf{Pr}_{\boldsymbol{\theta},\#}(p_{0,y} + p_{1,y} + p_{2,y} \geq 1 - O(m^{-1})) = 1 - \widetilde{O}(t^{-1/2}) \tag{3.18}$$

where $\mathbf{Pr}_{\boldsymbol{\theta},\#}$ stresses that $N^{\#}$ is randomly chosen.

So, from (3.17), we obtain

$$\begin{aligned} \mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y) \left(1 - \frac{5}{m}\right)^2 & \mathbf{Pr}_{\boldsymbol{\theta},\#}(p_{0,y} + p_{1,y} + p_{2,y} \geq 1 - O(m^{-1})) \\ & \leq \mathbf{Pr}_{\boldsymbol{\theta},\#}(X_G(\lambda_0) = y \mid \mathcal{B}, \mathcal{M}) \\ & \leq \mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y)(1 + O(m^{-1})) \end{aligned}$$

or using (3.13)

$$\left| \frac{\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y \mid \mathcal{B}, \mathcal{M})}{\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y)} - 1 \right| = O\left(\frac{1}{m}\right).$$

Therefore,

$$\begin{aligned} \mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y \mid \mathcal{B}, \mathcal{M}) &= (1 + O(m^{-1}))\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y) \\ \mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y, \mathcal{B}, \mathcal{M}) &= (1 + O(m^{-1}))\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y)\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B}, \mathcal{M}) \end{aligned}$$

or

$$\begin{aligned} \mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y, \mathcal{B}) + \tilde{O}(t^{-3}) &= (1 + O(m^{-1}))\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y) \\ & \quad \cdot(\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B}) + \tilde{O}(t^{-3})) \end{aligned} \tag{3.19}$$

Now we will show later that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{B}) \geq \frac{1}{2} \tag{3.20}$$

and (3.7) implies

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_H(\lambda_0) = y) = \frac{d(y,t_0) - \delta_y}{2mt_0} + O(t^{-3}) = \Omega(t^{-1})$$

where $1 \leq \delta_y \leq 5$ is the number of (s, y) edges in $G(t_0)$. So from (3.19), we have

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_G(\lambda_0) = y \mid X_{t_0} = w, G(t_0) = G, \mathcal{B}) = (1 + O(m^{-1}))\frac{d(y,t_0)}{2mt_0}.$$

Thus,

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{B}) = \\ (1 + O(m^{-1}))\mathbf{E}_{\boldsymbol{\theta}}\left(\sum_{y \in N(s,t_0)} \frac{d(y,t_0)}{2mt_0}\frac{1}{d(y,t)}\right) \\ + O(m^{-1})\mathbf{Pr}_{\boldsymbol{\theta}}(X_{t-1} \in N(s, t-1) \setminus N(s, t_0)).\end{aligned}$$

Then Lemma 3.7 follows from

$$\mathbf{Pr}_{\boldsymbol{\theta}}(X_{t-1} \in N(s, t-1) \setminus N(s, t_0)) = \widetilde{O}(t^{-2}). \tag{3.21}$$

To verify this, we observe that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(|N(s, t-1) \setminus N(s, t_0)| \geq 2) = \widetilde{O}(t^{-2})$$

and so we only need to consider the case $N(s, t-1) \setminus N(s, t_0) = \{y_1\}$ where $y_1 \in I$.

If $y_1 - t_0 \leq 5\mu_0$, then we can prove $\mathbf{Pr}_{\boldsymbol{\theta}}(X_{t-1} = y_1 \mid X_{y_1} = w', \ldots) = \widetilde{O}(t^{-1})$ essentially by replacing t_0 by y_1. If $y_1 - t_0 > 5\mu_0$ then either (i) there exists $u \in I$ such that $X_u = y_1$, or (ii) we can condition on $X_u \neq y_1, u \in I$ and proceed as for (3.12). Finally, note that the probability of (i) is $\widetilde{O}(t^{-1})$ by the argument for (3.12). This completes the proof of (3.21) and hence Lemma 3.7 (modulo the proofs below).

3.6.1. Proof of (3.15) and (3.18). Clearly (3.18) implies (3.15). We therefore start with (3.18). Observe first that $N^{\#}$ the set of neighbours of I in $G(t_0)$ satisfies

$$\mathbf{Pr}(W_H(\lambda_0) \cap N^{\#} \neq \emptyset) = \widetilde{O}(t^{-1/2}). \tag{3.22}$$

To see this, we use the fact that the walk is defined on H and $N^{\#}$ is independent of H. We also need an upper bound of $\widetilde{O}(t^{1/2})$ for maximum degree in $G(t_0)$ and this follows from Lemmas 3.1(a) and 4.3(a).

Since $N^* = N^\# \cup N(s,t_0)$, we only have to show now that the probability that W_H leaves a vertex of $N(s,t_0)$ three times or more is $O(m^{-1})$. Furthermore, this event depends on $G(t_0)$, which is fixed and not on $G(t)$.

Given $G \in \mathcal{G}(t_0)$ and $W = W_H(\lambda_0)$, the total degree of the vertices of W is $\widetilde{O}(t^{1/2})$ in Model 2. In Model 1, we would have $\widetilde{O}(t^{-1})$ for the RHS of (3.22).

Now let $\mathcal{W}(a,b,\gamma)$ denote the set of walks in H from a to b of length γ and for $W \in \mathcal{W}(a,b,\gamma)$, let $\mathbf{Pr}(W) = \mathbf{Pr}(W_{a,H}(\gamma) = W)$. Then for $x \in V(H)$, we have

$$\begin{aligned}
\mathbf{Pr}&(X_{w,H}(\lambda_0/2) = x \mid X_{w,H}(\lambda_0) = y)\\
&= \sum_{\substack{W_1 \in \mathcal{W}(w,x,\lambda_0/2)\\ W_2 \in \mathcal{W}(x,y,\lambda_0/2)}} \frac{\mathbf{Pr}(W_1)\mathbf{Pr}(W_2)}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}\\
&= \pi_{x,H}^{-1} \sum_{\substack{W_1 \in \mathcal{W}(w,x,\lambda_0/2)\\ W_2 \in \mathcal{W}(x,y,\lambda_0/2)}} \frac{\mathbf{Pr}(W_1)\pi_{x,H}\mathbf{Pr}(W_2)}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}
\end{aligned}$$

and with W_3 equal to the reversal of W_2,

$$\begin{aligned}
&= \pi_{x,H}^{-1}\pi_{y,H} \sum_{\substack{W_1 \in \mathcal{W}(w,x,\lambda_0/2)\\ W_3 \in \mathcal{W}(y,x,\lambda_0/2)}} \frac{\mathbf{Pr}(W_1)\mathbf{Pr}(W_3)}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}\\
&= \frac{\pi_{x,H}^{-1}\pi_{y,H}}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}\mathbf{Pr}(\mathcal{W}(w,x,\lambda_0/2))\mathbf{Pr}(\mathcal{W}(y,x,\lambda_0/2))\\
&= \frac{\pi_{x,H}^{-1}\pi_{y,H}}{\mathbf{Pr}(\mathcal{W}(w,y,\lambda_0))}(\pi_{x,H} - O(t^{-24}))^2\\
&= \pi_{x,H} - O(t^{-23}).
\end{aligned}$$

It follows that the variation distance between the distribution of a random walk of length λ_0 from w to y and that of $W_1, W_3^{reversed}$ is $O(t^{-22})$ where W_1, W_3 are obtained by (i) choosing x from the steady state distribution, and then (ii) choosing a random walk W_1 from w to x and a random walk W_3 from y to x. Furthermore, the variation distance between distribution of W_1 and a random walk of length $\lambda_0/2$ from w is $O(t^{-24})$. Similarly, the variation distance between distribution of W_3 and a random walk of length λ_0 from y is $O(t^{-24})$.

Now consider W_1 and let Z_i be the distance of $X_H(i)$ from s. We observe that since $s \notin B_{t_0} \subseteq B_t$, while the walk is within σ_0 of s, the distance to s must go up or down in one step and that

$$\mathbf{Pr}\left(Z_{i+1} = Z_i + 1 \mid Z_i < \sigma_0\right) \geq 1 - \frac{1}{m-5}.$$

We will deduce from this that, where $N_H(s)$ is the set of $G(t_0)$ neighbours of s,

$$\mathbf{Pr}(W_1 \text{ or } W_3 \text{ make a } \textit{return} \text{ to } N_H(s)) = O(1/m) \tag{3.23}$$

and this together with (3.22) implies (3.18).

To verify (3.23), we first see that

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(\exists 1 \le i \le \lambda_0/2 : Z_i = 1 \mid Z_0 = \sigma_0) &\le \lambda_0 \sum_{k=0}^{\lambda_0/2} \binom{\sigma_0+2k}{\sigma_0+k}\left(\frac{1}{m-5}\right)^{\sigma_0+k} \\ &\le \lambda_0 \sum_{k=0}^{\lambda_0/2}\left(\frac{(\sigma_0+2k)e}{(m-5)(\sigma_0+k)}\right)^{\sigma_0+k} \\ &\le \lambda_0^2(2e/(m-5))^{\sigma_0}. \qquad (3.24)\end{aligned}$$

Then we have

$$\begin{aligned}&\mathbf{Pr}_{\boldsymbol{\theta}}(\exists i > 0 :\ Z_{2i} = 1, 1 < Z_1, Z_2, \ldots, Z_{2i-1} \le \sigma_0 \mid Z_0 = 1) \le \\ &\qquad \sum_{i>0}\binom{2i}{i}\left(\frac{1}{m-5}\right)^i < \sum_{i>0}\left(\frac{2}{m-5}\right)^i = \frac{2}{m-7}. \qquad (3.25)\end{aligned}$$

Equation (3.23) follows from (3.24) and (3.25).

3.6.2. Proof of (3.20). This follows from $\mathbf{Pr}(\mathcal{E}_0 \mid X_{t_0} = w, G(t_0) = G, \mathcal{F}_0) = 1 - O(m^{-1})$, much as in the proof of (3.18). In particular, we see that if a walk in G starts at $w \ne s \notin B_t$, then the probability it visits s in λ_0 steps is $O(m^{-1})$. Then we will see that $\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_0 \mid X_{t_0} = w, G(t_0) = G) = 1 - o(1)$ (see (3.26) below). In the proof below, we condition on $\mathcal{E}_0$ but the proof is valid without this conditioning.

This completes the proof of Lemma 3.7. □

We will next argue that

$$\begin{aligned}\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_{\ge k} \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0) &= \widetilde{O}(t^{-k}) \quad k = 1, 2. &(3.26)\\ \mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) &= \widetilde{O}(t^{-1/2}). &(3.27)\\ \mathbf{E}_{\boldsymbol{\theta}}\left(d(y,t) - d(y,t_0)\right) &= \widetilde{O}(t^{-1/2}). &(3.28)\end{aligned}$$

It follows from (3.9)–(3.12) and (3.26)–(3.28) that

$$\begin{aligned}&\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0) = \\ &\qquad \frac{\theta_{t_0}}{2mt}\left(1 + O\left(\frac{1}{m}\right)\right) + O(t^{-3}\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{A}_s(t-1))^{-1}). \qquad (3.29)\end{aligned}$$

and removing the conditioning on $X_{t_0} = w, G(t_0) = G$ yields Lemma 2.3(a).

For part (b), we see that $X_s = s$ if and only if [i] s chooses X_{s-1} as one of its m neighbours and then [ii] S moves to s. If we condition on $X_{s-1} = x$ and $d(x, s-1) = d$, then $\mathbf{Pr}_{\boldsymbol{\theta}}([i]) \le \frac{md}{2m(s-1)}$ and $\mathbf{Pr}_{\boldsymbol{\theta}}([ii] \mid [i]) \le \frac{2}{d}$ (we write $\le \frac{2}{d}$ in place of the more natural $\frac{1}{d+1}$ to account for x being chosen more than once). This proves part (b).

3.6.3. Proof of (3.26). Let us generate $X_i, i \in I$ using as little information about the edges incident with I as possible. Thus, at step i we first establish whether any of $t_0 + 1, \ldots, i$ are neighbours of X_{i-1}. If the answer is no, we do not determine these neighbours. Thus, up to the first time we get the answer yes, the conditional distribution of the neighbours of $t_0, t_0{+}1, \ldots, i$ is that they are chosen from a set of size $t - o(t)$ either randomly (Model 1) or from the same set with probabilities proportional to degree (Model 2). Let $\mathcal{Y}_i = \{\text{YES at } i \text{ and } X_i \in \{t_0 + 1, \ldots, i\}\}$. If $d(X_{i-1}, i) = d$, then

$$\mathbf{Pr}(\mathcal{Y}_i \mid d) = O\left(|I| \cdot \frac{d}{t} \cdot \frac{1}{d}\right) = O\left(\frac{|I|}{t}\right). \tag{3.30}$$

Since $\mathcal{F}_{\ge 1} \subseteq \bigcup_{i \in I} \mathcal{Y}_i$, we have (3.26) for $k = 1$.

Now assume that i_1 is the first i for which $\mathcal{Y}_i$ occurs and that $X_{i_1-1} = j_1$. Arguing as in the first paragraph of this subsection, we see that the conditional probability that $\mathcal{Y}_i$ occurs for $i_2 > i_1$, with $X_{i_2-1} = j_2 \ne j_1$ is also $\widetilde{O}(t^{-1}|I|)$ and this completes the proof of (3.26). $\square$

3.6.4. Proof of (3.27). Let $J = \{j_1\}$ and let j_1 be visited for the first at time t_1 and let $i_1 = X_{t_1-1}$. We condition now on i_1, j_1, t_1 a well. We write

$$\begin{aligned}
&\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) = \\
&\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \overline{\mathcal{E}}_1, \mathcal{F}_1)\mathbf{Pr}(\overline{\mathcal{E}}_1 \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) \\
&+\mathbf{Pr}_{\boldsymbol{\theta}}(X_t = s \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{E}_1, \mathcal{F}_1)\mathbf{Pr}(\mathcal{E}_1 \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1).
\end{aligned}$$

Observe that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{E}_1, X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) = \widetilde{O}(t^{-3/2}).$$

Use (3.30) plus an extra $\widetilde{O}(t^{-1/2})$ factor for the extra neighbour(s). Furthermore, it is easily seen that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{F}_{\ge 1} \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0,) = \tilde{\Omega}(t^{-1}).$$

Combining these two statements with (3.26) for $k = 2$, we see that

$$\mathbf{Pr}_{\boldsymbol{\theta}}(\mathcal{E}_1 \mid X_{t_0} = w, G(t_0) = G, \mathcal{E}_0, \mathcal{F}_1) = \widetilde{O}(t^{-1/2}).$$

So we can assume that $\mathcal{E}_1$ does not hold. We consider two possibilities:

(a) There is an $i < t-1$ such that $X_i = j_1$ and $X_{i+1} \neq i_1$:

Model 1. From time $i+1$ on, our walk is conditioned to walk on the graph H' induced by $([t_0] \cup \{j_1\}) \setminus \{s\}$. If $v \in H'$, then its steady state probability $\pi(v)$ for a random walk on H' is at least $1/(2t_0)$ and the probability that $X_{i+1} = v$ is at most twice this. It follows that in any subsequent step of a simple random walk, the probability S is at v is at most $2\pi(v)$. Although the walk is conditioned rather than constrained to walk on H', arguments similar to those for (3.16) show that this conditioning only changes such probabilities by a factor $1 + O(m^{-1})$. Thus, in this case the probability we arrive at a neighbour of s, at time $t-1$ is also $\widetilde{O}(t^{-1})$ and (3.27) follows for this case.

Model 2. Now if $v \in [t_0] \setminus \{s\}$, its steady state random walk probability $\pi(v)$ is asymptotically equal to the probability it is chosen as X_{i+1} and we can use the analysis for Model 1.

(b) For all $i < t-1$, $X_i = j_1$ implies $X_{i+1} = i_1$:
Now after replacing some sequences i_1, j_1, i_1 by just i_1, our walk is conditioned on $\mathcal{F}_0$ and then (3.27) is implied by Lemma 3.7.

3.6.5. Proof of (3.28). This follows from the fact that in Model 2, the maximum degree in $G(t)$ is $O(t^{1/2})$ **whp**, see e.g., [Cooper and Frieze 01]. For Model 1, the maximum degree is $O(\ln t)$ with sufficiently high probability.

3.7. $\ell \geq 1$.

We follow the above analysis and note that the degrees do not change during the spider's walk and that error estimates do not increase (no new vertices are added).

4. Proof of Theorem 1.2

4.1. Model 1

Theorem 4.1.

$$\eta_{\ell,m}(t) = (1 + O(m^{-1})) \int_0^1 \exp\left((m + \frac{\ell}{2}) \ln x + \frac{2m^2}{\ell} \left(1 - x^{\frac{\ell}{2m}}\right)\right) \, dx.$$

$$\eta_\ell = \sqrt{\frac{2}{\ell}} e^{(\ell+2)^2/(4\ell)} \int_{(\ell+2)/\sqrt{2\ell}}^{\infty} e^{-z^2/2} \, dz.$$

Thus, when $\ell = 1$, $\eta_1 = 2\sqrt{\pi} e^{9/4}(1 - \Phi(3/\sqrt{2}))$ where $\Phi(x)$ is the standard normal cumulate. Thus $\eta_1 \sim 0.5699953\ldots$. Furthermore, as $\ell \to \infty$, $\eta_\ell \sim 2/\ell$.

Proof of Theorem 4.1. We write $d(s,t)$ as

$$d(s,t) = X_s + X_{s+1} + \cdots + X_\tau + \cdots + X_t,$$

where $X_s = m$ and for $\tau > s$, the $X_\tau = B(m, \frac{1}{\tau-1})$ are independent.

Now

$$\begin{aligned}\sum_{\tau=s}^{t} \frac{d(s,\tau)}{\tau} &= \sum_{\tau=s}^{t} \frac{1}{\tau} \sum_{r=s}^{\tau} X_r \\ &= \sum_{r=s}^{t} X_r \left(\sum_{\tau=r}^{t} \frac{1}{\tau} \right).\end{aligned}$$

$$\sum_{\tau=r}^{t} \frac{1}{\tau} = \ln \frac{t}{r} + O\left(\frac{1}{r}\right).$$

Thus,

$$\prod_{\tau=s}^{t} \left(1 - \frac{d(s,\tau)}{2m\tau}\left(1 + O\left(\frac{1}{m}\right)\right)\right)^{\ell} = \exp\left(-\left(1+O\left(\frac{1}{m}\right)\right)\ell \sum_{\tau=s}^{t} \frac{d(s,\tau)}{2m\tau}\right) \tag{4.1}$$

$$= \exp\left(-\left(1+O\left(\frac{1}{m}\right)\right)\frac{\ell}{2m}\sum_{r=s}^{t} X_r \ln\frac{t}{r}\right)$$

$$= \prod_{r=s}^{t} \left(\frac{r}{t}\right)^{\frac{\ell X_r}{2m}\left(1+O\left(\frac{1}{m}\right)\right)}.$$

Then we can write

$$\prod_{r=s}^{t} \left(\frac{r}{t}\right)^{\frac{\lambda_2 X_r}{2m}} \leq \prod_{\tau=s}^{t} \left(1 - \frac{d(s,\tau)}{2m\tau}\left(1+O\left(\frac{1}{m}\right)\right)\right)^{\ell} \leq \prod_{r=s}^{t} \left(\frac{r}{t}\right)^{\frac{\lambda_1 X_r}{2m}}$$

where $\ell - \frac{A}{m} \leq \lambda_1 \leq \ell \leq \lambda_2 \leq \ell + \frac{A}{m}$ for some constant $A > 0$.

Now

$$\begin{aligned}\mathbf{E} \prod_{r=s}^{t} \left(\frac{r}{t}\right)^{\frac{\lambda_1 X_r}{2m}} &= \prod_{r=s}^{t} \mathbf{E} \left(\frac{r}{t}\right)^{\frac{\lambda_1 X_r}{2m}} \\ &= \left(\frac{s}{t}\right)^{\frac{\lambda_1}{2}} \prod_{r=s+1}^{t} \left(1 - \frac{1}{r-1} + \frac{1}{r-1}\left(\frac{r}{t}\right)^{\frac{\lambda_1}{2m}}\right)^{m} \\ &= (1+o(1))\left(\frac{s}{t}\right)^{\frac{\lambda_1}{2}} \exp\left\{m \ln\frac{s}{t} + \frac{2m^2}{\lambda_1}\left(1 - \left(\frac{s}{t}\right)^{\frac{\lambda_1}{2m}}\right)\right\}.\end{aligned}$$

Thus,

$$\begin{aligned}\nu_{\ell,m}(t) &\geq (1+o(1))t\int_0^1 \exp\left(\left(m+\frac{\lambda_1}{2}\right)\ln x + \frac{2m^2}{\lambda_1}\left(1-x^{\frac{\lambda_1}{2m}}\right)\right)\,dx\\ &= \left(1+O\left(\frac{1}{m}\right)\right)t\int_0^1 \exp\left(\left(m+\frac{\ell}{2}\right)\ln x + \frac{2m^2}{\ell}\left(1-x^{\frac{\ell}{2m}}\right)\right)\,dx.\end{aligned}$$

Replacing λ_1 by λ_2 to get an upper bound, we deduce that

$$\nu_{\ell,m}(t) = \left(1+O\left(\frac{1}{m}\right)\right)t\int_0^1 \exp\left(\left(m+\frac{\ell}{2}\right)\ln x + \frac{2m^2}{\ell}\left(1-x^{\frac{\ell}{2m}}\right)\right)\,dx.$$

The values of this integral are easily tabulated. For $\ell = 1$, they quickly reach a value of about 0.57 as m grows. The approximation is accurate to the second decimal place for $m \geq 4$.

As $m \to \infty$, by using the transformations $x = e^{-y}$ and $z = \sqrt{\ell/2}y + (l+2)/(\sqrt{2\ell})$, we obtain

$$\begin{aligned}\eta_\ell &= \int_0^\infty \exp\left(-\frac{\ell+2}{2}y - \frac{\ell}{4}y^2\right)\,dy\\ &= \sqrt{\frac{2}{\ell}}e^{(\ell+2)^2/(4\ell^2)}\int_{(\ell+2)/\sqrt{2\ell}}^\infty e^{-z^2/2}\,dz\\ &\sim \frac{2}{\ell}\end{aligned}$$

since $e^{x^2/2}\int_x^\infty e^{-y^2/2}dy \sim 1/x$ as $x \to \infty$. □

4.2. Model 2

Theorem 4.2.

$$\eta_\ell = e^\ell 2\ell^2\int_\ell^\infty y^{-3}e^{-y}\,dy.$$

When $\ell = 1$, $\eta_1 = 0.59634\ldots$. Furthermore, as $\ell \to \infty$, $\eta_\ell \sim 2/\ell$.

Lemma 4.3.

(a) $\mathbf{Pr}(\exists s,t:\ d(s,t) \geq 2m\sqrt{t/s}(\ln t)^2) = \widetilde{O}(t^{-10})$.

(b) If $t/\ln t \leq s \leq t$ and $r \leq 2m\sqrt{t/s}(\ln t)^2$, then

$$\mathbf{Pr}(d(s,t) = m+r) = \binom{m+r-1}{r}\left(\frac{s}{t}\right)^{m/2}\left(1-\left(\frac{s}{t}\right)^{\frac{1}{2}}\right)^r \cdot\left(1+O\left(\frac{(m+r)^3}{s}\right)+O\left(\frac{r}{\sqrt{s}}\right)\right).$$

Proof of Lemma 4.3. (a) Fix $s \le t$ and let $X_\tau = d(s,\tau)$ for $\tau = s, s+1, \dots, t$. Now conditional on $X_\tau = x$, we have

$$X_{\tau+1} = X_\tau + B\left(m, \frac{x}{2m\tau}\right)$$

and for $\lambda \le 1$

$$\begin{aligned} \mathbf{E}\left(e^{\lambda X_{\tau+1}} \mid X_\tau = x\right) &= e^{\lambda x}\left(1 - \frac{x}{2m\tau} + \frac{x}{2m\tau}e^{\lambda}\right)^m \\ &\le \exp\left\{\lambda x - \frac{x}{2\tau} + \frac{x}{2\tau}(1+\lambda+\lambda^2)\right\} \\ &= \exp\left\{\lambda x\left(1 + \frac{1+\lambda}{2\tau}\right)\right\}. \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{E}\left(e^{\lambda X_{\tau+1}}\right) &\le \mathbf{E}\left(\exp\left\{\lambda\left(1 + \frac{1+\lambda}{2\tau}X_\tau\right)\right\}\right) \\ &\le \exp\left\{m\lambda \prod_{\tau=s+1}^{\tau}\left(1 + \frac{1+\lambda}{2\tau}\right)\right\} \\ &\le \exp\{2m\lambda(t/s)^{(1+\lambda)/2}\} \\ &\le \exp\{2m\lambda(t/s)^{1/2}\ln t\}. \end{aligned}$$

Putting $u = 2m(t/s)^{1/2}(\ln t)^2$, $\lambda = \frac{2\ln\ln t}{\ln t}$, we get

$$\begin{aligned} \mathbf{Pr}(X_t \ge u) &\le \exp\{\lambda(2m(t/s)^{1/2}\ln t - u)\} \\ &\le \exp\{-m(t/s)^{1/2}\ln t)^2\} \end{aligned}$$

and part (a) follows.

(b) Let $\boldsymbol{\tau} = (\tau_1, ..., \tau_r)$ where τ_j is the step at which the transition from degree $m+j$ to degree $m+j+1$ occurs. Let $\tau_0 = s$ and let $\tau_{r+1} = t$. Let $p(s,t,r : \boldsymbol{\tau}) = \mathbf{Pr}(d(s,t) = m+r \text{ and } \boldsymbol{\tau})$. Then

$$p(s,t,r:\boldsymbol{\tau}) = \prod_{j=0}^{r}\left(\Phi_j(\tau_j)\prod_{\tau_j < T < \tau_{j+1}}\left(1 - \frac{m+j}{2mT}\right)^m\right),$$

where $\Phi_0 = 1$ and

$$\Phi_j = \left(1 + O\left(\tfrac{m+j}{\tau_j}\right)\right)\frac{m(m+j-1)}{2m\tau_j}\left(1 - \frac{m+j-1}{2m\tau_j}\right)^{m-1}.$$

In the above and in the following we use the fact that $\tau_j \geq s \gg (m+r)^3$ and $r = o(s^{1/2})$. The events that an edge sends two edges to s only contribute to the error terms here.

Now

$$\begin{aligned}
\prod_{\tau_j < T < \tau_{j+1}} \left(1 - \frac{m+j}{2mT}\right)^m &= \exp\left(-\frac{m+j}{2} \sum_{\tau_j < T < \tau_{j+1}} \left(\frac{1}{T} + O\left(\frac{m+j}{T^2}\right)\right)\right) \\
&= \exp\left(-\frac{m+j}{2}\left(\log \frac{\tau_{j+1}}{\tau_j} + O\left(\frac{m+j}{\tau_j}\right)\right)\right) \\
&= \left(\frac{\tau_j}{\tau_{j+1}}\right)^{\frac{m+j}{2}} \left(1 + O\left(\tfrac{(m+j)^2}{\tau_j}\right)\right).
\end{aligned}$$

Thus,

$$\mathbf{Pr}(d(s,t) = m+r) = \sum_{\boldsymbol{\tau}} p(s,t,r:\boldsymbol{\tau})$$

where

$$\begin{aligned}
p(s,t,r:\boldsymbol{\tau}) = \left(1 + O\left(\tfrac{(m+r)^3}{s}\right)\right) \frac{m(m+1)\cdots(m+r-1)}{2^r} \\
\cdot \left(\frac{s}{t}\right)^{m/2} \frac{1}{t^{r/2}} \frac{1}{\sqrt{\tau_1}} \cdots \frac{1}{\sqrt{\tau_r}}.
\end{aligned} \tag{4.2}$$

Now

$$\begin{aligned}
\sum_{\boldsymbol{\tau}} \frac{1}{\sqrt{\tau_1}} \cdots \frac{1}{\sqrt{\tau_r}} &= \frac{1}{r!}\left(\int_s^t \frac{1}{\sqrt{\tau}}\, d\tau + O\left(\frac{1}{\sqrt{s}}\right)\right)^r \\
&= \left(1 + O\left(\tfrac{r}{\sqrt{s}}\right)\right) \frac{2^r}{r!}\left(\sqrt{t} - \sqrt{s}\right)^r.
\end{aligned}$$

The result follows. □

Assuming the same conditions on r, s as in Lemma 4.3(b), define

$$\rho(s,t) = \prod_{\tau=s}^{t} \exp\left(-\ell \frac{d(s,\tau)}{2m\tau}\right).$$

As in the proof of Lemma 4.3, let $d(s,t) = m+r$ and let $\boldsymbol{\tau} = (\tau_1, ..., \tau_r)$ denote the transition steps of $d(s,t)$ from m to $m+r$. As before, let $\tau_0 = s$ and $\tau_{r+1} = t$. Let $\rho(s,t:\boldsymbol{\tau})$ be the value of ρ given $\boldsymbol{\tau}$.

Then

$$
\begin{aligned}
\rho(s,t:\boldsymbol{\tau}) &= \exp\left(-\frac{l}{2m}\sum_{j=0}^{r}\sum_{\tau_j\le T<\tau_{j+1}}\frac{m+j}{T}\right)\\
&= \exp\left(-\frac{\ell}{2m}\sum_{j=0}^{r}(m+j)\left(\log\frac{\tau_{j+1}}{\tau_j}+O\left(\frac{1}{\tau_j}\right)\right)\right)\\
&= \left(1+O\left(\tfrac{(m+r)^2}{s}\right)\right)\left(\frac{s}{t}\right)^{\frac{\ell}{2}} t^{-r\ell/2m}{\tau_1}^{\ell/2m}\ldots{\tau_r}^{\ell/2m}.
\end{aligned}
$$

Thus, combining $\rho(s,t:\boldsymbol{\tau})$ with $p(s,t,r:\boldsymbol{\tau})$ (as given in (4.2)) for $\boldsymbol{\tau}$ of length r and summing over $\boldsymbol{\tau}$, we have

$$
\begin{aligned}
\mathbf{E}\,\rho(s,t) &= \sum_r\sum_{\boldsymbol{\tau}}\rho(s,t:\boldsymbol{\tau})p(s,t,r:\boldsymbol{\tau})\\
&= \sum_r\left(1+O\left(\tfrac{(m+r)^3}{s}\right)\right)\left(\frac{s}{t}\right)^{(m+\ell)/2}\binom{m+r-1}{r}\frac{r!}{2^r}\\
&\qquad\cdot\ \frac{1}{t^{r(1+\ell/m)/2}}\sum_{\boldsymbol{\tau}}\prod_{j=1}^{r}\frac{1}{\tau_j^{(1-\ell/m)/2}}\\
&= \sum_r\left(1+O\left(\tfrac{(m+r)^3}{s}\right)+O\left(\tfrac{r}{s^{(1-\ell/m)/2}}\right)\right)\\
&\qquad\cdot\ \binom{m+r-1}{r}\left(\frac{1-\left(\frac{s}{t}\right)^{(1+\ell/m)/2}}{1+\ell/m}\right)^r\\
&= (1+o(1))\left(\frac{1+\frac{\ell}{m}}{1+\frac{\ell}{m}\left(\frac{t}{s}\right)^{(1+\ell/m)/2}}\right)^m.
\end{aligned}
$$

Thus, using the transformations, $x=s/t$ and $y=\ell/\sqrt{x}$, we find

$$
\begin{aligned}
\lim_{m,t\to\infty}\frac{\mathbf{E}\,\nu_{\ell,m}(t)}{t} &= \lim_{m,t\to\infty}\frac{1}{t}\sum_{s=1}^{t}\left(\frac{1+\frac{\ell}{m}}{1+\frac{\ell}{m}\left(\frac{t}{s}\right)^{(1+\ell/m)/2}}\right)^m\\
&= e^{\ell}\int_0^1 e^{-\ell/\sqrt{x}}\,dx\\
&= e^{\ell}2\ell^2\int_{\ell}^{\infty}y^{-3}e^{-y}\,dy
\end{aligned}
$$

as required.

5. Extensions and Further Research

There are some natural questions to be explored in the context of the above models.

- It should be possible to extend the analysis to other models of web graphs e.g., [Broder et al. 00] and [Cooper and Frieze 01]. In principal, one should only have to establish that random walks on these graphs are rapidly mixing.

- One can consider nonuniform random walks. Suppose, for example, that each $v \in [t]$ is given a weight $\lambda(v)$ and when at a vertex v the spider chooses its next vertex with probability proportional to $\lambda(v)$. If $\Lambda(v) = \sum_{N(v)} \lambda(v)$ ($N(v)$ denotes the neighbours of v), then the steady state probability $\pi(v)$ of being at v in such a walk is proportional to $\Theta(v) = \lambda(v)\Lambda(v)$. Again, once one shows rapid mixing, it should be possible to obtain an expression like (1.1) for the number of unvisited vertices.

- We have only estimated the expectation of the number of unvisited vertices. It would be interesting to establish a concentration result.

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