The CRC Handbook of Combinatorial Designs

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36 Nested Designs

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36.1 NBIBDs: Definition and Example

36.1 Definition If the blocks of a BIBD $(\mathcal{V}, \mathcal{D}_1)$ with v symbols in b_1 blocks of size k_1 are each partitioned into sub-blocks of size k_2, and the $b_2 = b_1k_1/k_2$ sub-blocks themselves constitute a BIBD $(V, \mathcal{D}_2)$, then the system of blocks, sub-blocks and symbols is a *nested balanced incomplete block design* (nested BIBD or NBIBD) with parameters $(v, b_1, b_2, r, k_1, k_2)$, r denoting the common replication. $(V, \mathcal{D}_1)$ and $(V, \mathcal{D}_2)$ are the *component BIBDs* of the NBIBD.

36.2 Example An NBIBD(16,24,48,15,10,5). Sub-blocks are separated by |.

$(0, 1, 2, 3, 4 \mid 5, 6, 7, 8, 9)$	$(0, 1, 2, 3, 5 \mid 4, 6, 10, 11, 12)$	$(0, 1, 2, 3, 6 \mid 4, 5, 13, 14, 15)$
$(0, 1, 10, 11, 12 \mid 2, 3, 7, 8, 9)$	$(0, 2, 13, 14, 15 \mid 1, 3, 7, 8, 9)$	$(0, 3, 13, 14, 15 \mid 1, 2, 10, 11, 12)$
$(0, 4, 5, 7, 11 \mid 1, 8, 10, 13, 14)$	$(0, 4, 5, 9, 10 \mid 1, 7, 12, 13, 15)$	$(0, 4, 5, 8, 12 \mid 1, 9, 11, 14, 15)$
$(0, 6, 7, 10, 13 \mid 2, 4, 8, 11, 14)$	$(0, 6, 9, 12, 15 \mid 2, 4, 7, 10, 13)$	$(0, 6, 8, 11, 14 \mid 2, 4, 9, 12, 15)$
$(0, 7, 8, 10, 15 \mid 3, 5, 6, 12, 14)$	$(0, 7, 9, 12, 14 \mid 3, 5, 6, 11, 13)$	$(0, 8, 9, 11, 13 \mid 3, 5, 6, 10, 15)$
$(1, 5, 7, 12, 14 \mid 2, 6, 8, 10, 15)$	$(1, 5, 9, 11, 13 \mid 2, 6, 7, 12, 14)$	$(1, 5, 8, 9, 15 \mid 2, 6, 9, 11, 13)$
$(1, 4, 6, 7, 13 \mid 3, 8, 11, 12, 15)$	$(1, 4, 6, 9, 15 \mid 3, 7, 10, 11, 14)$	$(1, 4, 6, 8, 14 \mid 3, 4, 8, 12, 13)$
$(2, 5, 7, 11, 15 \mid 3, 4, 8, 12, 13)$	$(2, 5, 9, 10, 14 \mid 3, 4, 7, 11, 15)$	$(2, 5, 8, 12, 13 \mid 3, 4, 9, 10, 14)$

36.2 NBIBDs: Existence

36.3 Remarks The necessary conditions for existence of a NBIBD are those for the two component BIBDs $(V, \mathcal{D}_1)$ and $(V, \mathcal{D}_2)$. Together they are: $b_1 \geq v$, $v | b_1k_1$, $v(v-1) | b_1k_1(k_1-1)$, and $v(v-1) | b_1k_1(k_2-1)$. The necessary conditions are sufficient for $k_1 = 4$ [4].

36.4 Remarks There are 3 non-isomorphic BIBDs with $(v, b_1, k_1) = (10, 15, 6)$ and 960 non-isomorphic BIBDs with $(v, b_2, k_2) = (10, 30, 3)$ but [5] there is no NBIBD$(10, 15, 30, 9, 6, 3)$. Thus the necessary conditions are not sufficient. This is the only case of nonexistence, where suitable component designs do exist, for $v \leq 16$ and $r \leq 30$.

36.5 Table Initial blocks for NBIBDs for $v \leq 16$ and $r \leq 16$. One solution, provided at least one exists, is listed for each set of parameters meeting the necessary conditions, except that multiples of r are not listed for fixed values of (v, k_1, k_2).

	$(v, b_1, b_2, r, k_1, k_2)$, Blocks
1.	$(5, 5, 10, 4, 4, 2)$, $(1\ 4 \mid 2\ 3)$ mod 5
2.	$(7, 7, 21, 6, 6, 2)$, $(1\ 6 \mid 2\ 5 \mid 4\ 3)$ mod 7
3.	$(7, 7, 14, 6, 6, 3)$, $(1\ 2\ 4 \mid 6\ 5\ 3)$ mod 7
4.	$(8, 14, 28, 7, 4, 2)$, $(0\ 1 \mid 4\ 2)(3\ 6 \mid 5\ \infty)$ mod 7
5.	$(9, 18, 36, 8, 4, 2)$, $(01\ 02 \mid 10\ 20)(11\ 22 \mid 12\ 21)$ mod (3,3)
6.	$(9, 12, 36, 8, 6, 2)$, $(1\ 2 \mid 3\ 6 \mid 4\ \infty)(5\ 6 \mid 7\ 2 \mid 0\ \infty)(0\ 4 \mid 1\ 7 \mid 3\ 5)$ PC(4), mod 8

7.	$(9,12,24,8,6,3)$, $(1\ 3\ 4 \mid 2\ 6\ \infty)(5\ 7\ 0 \mid 2\ 6\ \infty)(1\ 3\ 4 \mid 5\ 7\ 0)$ PC(4), mod 8
8.	$(9,9,36,8,8,2)$ $(1\ 8 \mid 2\ 7 \mid 3\ 6 \mid 4\ 5)$, mod 9
9.	$(9,9,18,8,8,4)$, $(01\ 02\ 10\ 20 \mid 11\ 22\ 12\ 21)$ mod (3,3)
10.	$(10,15,45,9,6,2)$, $(0_0\ 2_0 \mid 3_0\ 2_1 \mid 3_1\ 4_1)(2_0\ 3_0 \mid 0_0\ 3_1 \mid 4_0\ 0_1)(0_0\ 0_1 \mid 1_0\ 3_1 \mid 2_1\ 4_1)$ mod 5
11.	$(10,15,30,9,6,3)$, No NBIBD exists, see Example 36.28 for $(10,30,60,18,6,3)$
12.	$(10,10,30,9,9,3)$, $(1_0\ 2_0\ 4_1 \mid 3_0\ 4_0\ 3_1 \mid 0_1\ 1_1\ 2_1)(2_0\ 3_1\ 0_0 \mid 1_0\ 2_1\ 3_0 \mid 1_1\ 4_0\ 4_1)$ mod 5
13.	$(6,15,30,10,4,2)$, $(0\ 2 \mid 1\ 3)(\infty\ 0 \mid 3\ 4)(\infty\ 4 \mid 1\ 2)$ mod 5
14.	$(11,11,55,10,10,2)$, $(1\ 10 \mid 2\ 9 \mid 3\ 8 \mid 4\ 7 \mid 5\ 6)$ mod 11
15.	$(11,11,22,10,10,5)$, $(1\ 3\ 4\ 5\ 9 \mid 2\ 6\ 8\ 10\ 7)$ mod 11
16.	$(12,33,66,11,4,2)$, $(0\ 1 \mid 3\ 7)(10\ 2 \mid 9\ 4)(8\ 6 \mid 5\ \infty)$ mod 11
17.	$(12,22,66,11,6,2)$, $(0\ 3 \mid 1\ 5 \mid 4\ 9)(8\ 10 \mid 7\ 6 \mid 2\ \infty)$ mod 11
18.	$(12,22,44,11,6,3)$, $(0\ 1\ 3 \mid 4\ 5\ 9)(10\ 7\ \infty \mid 6\ 8\ 2)$ mod 11
19.	$(7,21,42,12,4,2)$, $(0\ 1 \mid 4\ 2)(0\ 2 \mid 1\ 4)(0\ 4 \mid 2\ 1)$ mod 7
20.	$(13,39,78,12,4,2)$, $(1\ 12 \mid 5\ 8)(2\ 11 \mid 3\ 10)(4\ 9 \mid 6\ 7)$ mod 13
21.	$(13,26,78,12,6,2)$, $(3\ 10 \mid 4\ 9 \mid 1\ 12)(5\ 8 \mid 11\ 2 \mid 6\ 7)$ mod 13
22.	$(13,26,52,12,6,3)$, $(1\ 3\ 9 \mid 4\ 12\ 10)(2\ 6\ 5 \mid 7\ 8\ 11)$ mod 13
23.	$(13,13,78,12,12,2)$, $(1\ 12 \mid 2\ 11 \mid 3\ 10 \mid 4\ 9 \mid 5\ 8 \mid 6\ 7)$ mod 13
24.	$(13,13,52,12,12,3)$, $(1\ 3\ 9 \mid 4\ 12\ 10 \mid 2\ 6\ 5 \mid 7\ 8\ 11)$ mod 13
25.	$(13,13,39,12,12,4)$, $(1\ 12\ 5\ 8 \mid 2\ 11\ 3\ 10 \mid 4\ 9\ 6\ 7)$ mod 13
26.	$(13,13,26,12,12,6)$, $(1\ 3\ 9\ 4\ 12\ 10 \mid 2\ 6\ 5\ 7\ 8\ 11)$ mod 13
27.	$(15,35,105,14,6,2)$, $(1_1\ 0_0 \mid 2_1\ 0_1 \mid 4_1\ \infty)(0_0\ 3_0 \mid 0_1\ 5_0 \mid \infty\ 6_0)(2_0\ 1_0 \mid 4_0\ 3_1 \mid 1_1\ 0_1)$ $(2_0\ 0_1 \mid 5_0\ 1_1 \mid 3_1\ 3_0)(4_0\ 1_1 \mid 5_0\ 0_0 \mid 0_1\ 3_1)$ mod 7
28.	$(15,35,70,14,6,3)$, $(1_1\ 2_1\ 4_1 \mid 0_0\ 0_1\ \infty)(0_0\ 0_1\ \infty \mid 3_0\ 5_0\ 6_0)(2_0\ 4_0\ 1_1 \mid 1_0\ 3_1\ 0_1)$ $(2_0\ 5_0\ 3_1 \mid 0_1\ 1_1\ 3_0)(4_0\ 5_0\ 0_1 \mid 1_1\ 0_0\ 3_1)$ mod 7
29.	$(15,21,105,14,10,2)$, No $\mathcal{D}_1$ exists, but $(15,42,210,28,10,2)$ does exist ([10])
30.	$(15,21,42,14,10,5)$, No $\mathcal{D}_1$ exists, but $(15,42,84,28,10,5)$ does exist ([10])
31.	$(15,15,105,14,14,2)$, $(1\ 14 \mid 2\ 13 \mid 3\ 12 \mid 4\ 11 \mid 5\ 10 \mid 6\ 9 \mid 7\ 8)$ mod 15
32.	$(15,15,30,14,14,7)$, $(0_1\ 1_1\ 2_1\ 3_1\ 4_1\ 5_1\ 6_1 \mid 0_0\ 1_0\ 2_0\ 3_0\ 4_0\ 5_0\ 6_0)$ fixed, $(\infty\ 4_0\ 1_0\ 1_1\ 2_0\ 4_1\ 2_1 \mid 0_1\ 6_1\ 5_1\ 5_0\ 3_1\ 6_0\ 3_0)$ $(0_0\ 2_0\ 4_0\ 5_1\ 1_0\ 6_1\ 3_1 \mid \infty\ 6_0\ 5_0\ 4_1\ 3_0\ 2_1\ 1_1)$ mod 7
33.	$(16,60,120,15,4,2)$, $(\infty\ 0 \mid 5\ 10)(1\ 2 \mid 4\ 8)(6\ 9 \mid 7\ 13)(11\ 3 \mid 12\ 14)$ mod 15
34.	$(16,40,120,15,6,2)$, $(0\ 1 \mid 9\ 3 \mid 5\ 12)(0\ 3 \mid 1\ 12 \mid 6\ 2)(11\ 9 \mid 1\ 3 \mid 0\ 8)$ mod 16, last block PC(8)
35.	$(16,40,80,15,6,3)$, $(0_0\ 0_1\ 0_2 \mid 1_1\ 2_1\ 3_1)(1_0\ 3_0\ 0_2 \mid 1_2\ 0_1\ 2_1)(1_0\ 3_0\ 0_1 \mid 0_0\ 3_2\ 4_1)$ $(2_0\ 3_0\ 4_1 \mid 4_0\ 0_2\ 1_2)(0_0\ 0_1\ 0_2 \mid 1_2\ 2_2\ 4_2)(\infty\ 3_0\ 4_0 \mid 0_0\ 3_1\ 4_2)(\infty\ 2_1\ 4_2 \mid 0_0\ 3_1\ 1_2)$ $(\infty\ 3_1\ 1_2 \mid 0_0\ 1_1\ 3_2)$ mod 5
36.	$(16,30,120,15,8,2)$, $(\infty\ 0 \mid 3\ 14 \mid 1\ 4 \mid 9\ 7)(2\ 8 \mid 6\ 13 \mid 5\ 10 \mid 11\ 12)$ mod 15
37.	$(16,30,60,15,8,4)$, $(0\ 1\ 3\ 7 \mid 4\ 9\ 14\ \infty)(2\ 10\ 11\ 13 \mid 5\ 6\ 8\ 12)$ mod 15
38.	$(16,24,120,15,10,2)$, $(\infty_1 0_1 \mid \infty_2 0_2 \mid 1_2 1_3 \mid 2_2 2_4 \mid 2_3 1_4)(\infty_2 0_3 \mid \infty_3 0_2 \mid 1_1 2_4 \mid 2_1 2_3 \mid 1_3 1_4)$ $(\infty_1\ 0_3 \mid \infty_3\ 0_1 \mid 1_1\ 2_2 \mid 2_1\ 1_4 \mid 1_2\ 2_4)(\infty_2\ 2_4 \mid \infty_3\ 2_3 \mid \infty_4\ 0_2 \mid 0_1\ 2_2 \mid 1_1\ 1_2)$ $(\infty_1\ 0_2 \mid \infty_3\ 2_4 \mid \infty_4\ 0_3 \mid 2_1\ 1_3 \mid 1_2\ 2_3)(\infty_1\ 2_4 \mid \infty_2\ 1_1 \mid \infty_4\ 0_1 \mid 2_1\ 0_3 \mid 2_2\ 1_3)$ $(\infty_4\ 0_4 \mid 1_1\ 2_1 \mid 1_2\ 2_2 \mid 1_3\ 2_3 \mid 1_4\ 2_4)$ mod 3, with $(\infty_1\ \infty_2 \mid \infty_3\ \infty_4 \mid 2_1\ 2_4 \mid 2_2\ 1_4 \mid 2_3\ 0_4)$ $(\infty_1\ \infty_3 \mid \infty_2\ \infty_4 \mid 0_1\ 0_4 \mid 0_2\ 2_4 \mid 0_3\ 1_4)(\infty_1\ \infty_4 \mid \infty_2\ \infty_3 \mid 1_1\ 1_4 \mid 1_2\ 0_4 \mid 1_3\ 2_4)$
39.	$(16,24,48,15,10,5)$, See Example 36.2
40.	$(16,20,120,15,12,2)$, $(1\ 2 \mid 4\ 8 \mid 6\ 13 \mid 7\ 9 \mid 0\ 5 \mid 10\ \infty)(6\ 7 \mid 9\ 13 \mid 11\ 3 \mid 12\ 14 \mid 5\ 10 \mid 0\ \infty)$ $(11\ 12 \mid 14\ 3 \mid 1\ 8 \mid 2\ 4 \mid 10\ 0 \mid 5\ \infty)(1\ 4 \mid 6\ 9 \mid 11\ 14 \mid 2\ 8 \mid 7\ 13 \mid 12\ 3)$ PC(5), mod 15
41.	$(16,20,80,15,12,3)$, $(0\ 1\ 5 \mid 2\ 8\ 10 \mid 6\ 7\ 9 \mid 13\ 4\ \infty)(5\ 6\ 10 \mid 7\ 13\ 0 \mid 11\ 12\ 14 \mid 3\ 9\ \infty)$ $(10\ 11\ 0 \mid 12\ 3\ 5 \mid 1\ 2\ 4 \mid 8\ 14\ \infty)(2\ 7\ 12 \mid 1\ 4\ 8 \mid 6\ 9\ 13 \mid 3\ 11\ 14)$ PC(5), mod 15
42.	$(16,20,60,15,12,4)$, $[(1\ 2\ 4\ 8 \mid 6\ 7\ 9\ 13 \mid 0\ 5\ 10\ \infty)(6\ 7\ 9\ 13 \mid 11\ 12\ 14\ 3 \mid 5\ 10\ 0\ \infty)$ $(11\ 12\ 14\ 3 \mid 1\ 2\ 4\ 8 \mid 10\ 0\ 5\ \infty)(1\ 2\ 4\ 8 \mid 6\ 7\ 9\ 13 \mid 11\ 12\ 14\ 3)]$ PC(5), mod 15
43.	$(16,20,40,15,12,6)$, $(0\ 5\ 1\ 4\ 7\ 13 \mid 2\ 6\ 8\ 10\ 9\ \infty)(5\ 10\ 6\ 9\ 12\ 3 \mid 7\ 11\ 13\ 0\ 14\ \infty)$ $(10\ 0\ 11\ 14\ 2\ 8 \mid 12\ 1\ 3\ 5\ 4\ \infty)(1\ 9\ 6\ 14\ 11\ 4 \mid 8\ 7\ 13\ 12\ 3\ 2)$ PC(5), mod 15
44.	$(16,16,80,15,15,3)$, $(1\ 5\ 8 \mid 2\ 10\ 12 \mid 3\ 4\ 7 \mid 6\ 11\ 13 \mid 9\ 14\ 15)$ mod 16
45.	$(16,16,48,15,15,5)$, $(3\ 14\ 10\ 2\ 1 \mid 12\ 5\ 8\ 6\ 11 \mid 9\ 15\ 4\ 7\ 13)$ mod 16

Some initial blocks taken through partial cycles, e.g. PC(5) $\Rightarrow$ subcycle of order 5

36.6 Definition An NBIBD is *resolvable* if the superblock component design $(V, \mathcal{D}_1)$ is resolvable. An NBIBD is *near-resolvable* if the superblock component design $(V, \mathcal{D}_1)$ is near-resolvable and $k_1 < v - 1$.

36.7 Remarks

1. Table 36.5 contains resolvable and near-resolvable NBIBDs whenever the necessary conditions for those designs are met.
2. In Table 36.5, the following NBIBDs are resolvable: 4, 16, 17, 18, 33, 36, 37.
3. In Table 36.5, the following NBIBDs are near-resolvable: 5, 20, 21, 22.

36.3 Relationships Between NBIBDs and Other Designs

36.8 Remark An NBIBD with $k_1 = v - 1$ is a near-resolvable BIBD.

36.9 Remarks A whist tournament design Wh$(4n)$ is a resolvable NBIBD$(4n, n(4n-1), 2n(4n-1), 4n-1, 4, 2)$. A whist tournament design Wh$(4n+1)$ is for $n > 1$ a near-resolvable NBIBD$(4n+1, n(4n+1), 2n(4n+1), 4n, 4, 2)$. Any NBIBD with $k_1 = 2k_2 = 4$ is a *balanced doubles schedule* [4].

36.10 Remarks Resolvable and near-resolvable NBIBDs have also been called *generalized whist tournaments* ([1]). A *pitch tournament design* is a resolvable or near-resolvable NBIBD$(v, v(v-1)/8, v(v-1)/4, v-1, 8, 4)$.

36.11 Remarks Table 36.5 contains these designs:

1. Near-resolvable BIBDs: 1, 2, 3, 8, 9, 12, 14, 15, 23, 24, 25, 26, 31, 32, 44, 45.
2. Whist tournaments: 1, 4, 5, 16, 20, 33.
3. Other balanced doubles schedules: 13, 19
4. Pitch tournaments: 9, 37.

36.12 Remark A partition of the rows of a perpendicular array PA$_\lambda(t, k_1, v)$ into $\frac{k_1}{k_2}$ sets of size k_2 is a NBIBD$(v, \lambda\binom{v}{t}, \lambda\binom{v}{t}k_1/k_2, \lambda\binom{v}{t}k_1/v, k_1, k_2)$.

36.4 General nesting and other nested designs

36.13 Definition Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two collections of equi-sized multisets (blocks) of elements from the same v-set $\mathcal{V}$. If there is a partition of each of the b_1 blocks of $\mathcal{D}_1$ into blocks of size k_2, so that the resulting collection of $b_2 = b_1k_1/k_2$ blocks is $\mathcal{D}_2$, then the blocks of $\mathcal{D}_2$ are *sub-blocks* of the blocks of $\mathcal{D}_1$ and the system $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2)$ is a *nested block design*.

36.14 Remarks This definition of nested block design provides a general framework for the nesting concept. Excluded, among others, are notions of nesting for which sub-blocks do not fully partition blocks [7].

36.15 Remark A resolvable BIBD (RBIBD) $(V, \mathcal{D})$ is a nested block design $(V, \mathcal{D}_1, \mathcal{D}_2)$ where the blocks of $\mathcal{D}_1$, of size $k_1 = v$, are the resolution classes of $\mathcal{D}$, and $\mathcal{D}_2 = \mathcal{D}$.

36.16 Remark Nested block designs may have more than two blocking systems and consequently more than one level of nesting. A doubly nested block design is a system $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ where both $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2)$ and $(\mathcal{V}, \mathcal{D}_2, \mathcal{D}_3)$ are nested block designs. This may be extended in the obvious fashion.

36.17 Definition A multiply nested BIBD (MNBIBD) is a nested block design $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_s)$ with parameters $(v, b_1, \ldots, b_s, r, k_1, \ldots, k_s)$ for which the systems $(\mathcal{V}, \mathcal{D}_j, \mathcal{D}_{j+1})$ are NBIBDs for $j = 1, \ldots, s-1$.

36.18 Remarks A resolvable NBIBD is a doubly nested block design. A near-resolvable NBIBD$(v, b_1, b_2, r, k_1, k_2)$ is a MNBIBD$(v, b_1k_1/(v-1), b_1, b_2, r, v-1, k_1, k_2)$.

36.19 Example $(1\ 16, 13\ 4|9\ 8, 15\ 2||3\ 14, 5\ 12|10\ 7, 11\ 6) \bmod 17$ is an initial block for a triply nested BIBD $(17, 17, 34, 68, 136, 17, 16, 8, 4, 2)$.

36.20 Remark For $v \leq 20$ and $r \leq 30$ there are 29 sets of parameters meeting the necessary conditions for existence of a doubly nested BIBD. At this writing designs are known for all of these except $(v, b_1, b_2, b_3, r, k_1, k_2, k_3) = (16, 20, 40, 80, 15, 12, 6, 3)$ [11].

36.21 Construction Let $\mathcal{M}_1$ be an MNBIBD $(\overline{v}, \overline{b}_1, \overline{b}_2, \ldots, \overline{b}_s, \overline{r}, \overline{k}_1, \overline{k}_2, \ldots, \overline{k}_s)$ with $s \geq 1$ component designs (if $s = 1$ then $\mathcal{M}_1$ is a BIBD; if $s = 2$ then an NBIBD; and if $s > 2$ then an MNBIBD). Let $\mathcal{M}_2$ be an MNBIBD $(\widehat{v}, \widehat{b}_1, \widehat{b}_2, \ldots, \widehat{b}_t, \widehat{r}, \widehat{k}_1, \widehat{k}_2, \ldots, \widehat{k}_t)$ with $t \geq 2$ component designs, and with $\widehat{k}_1/\widehat{k}_q = \overline{v}$ for some $2 \leq q \leq t$. Select one block of size $\widehat{k}_1$ from $\mathcal{M}_2$ and label its sub-blocks of size $\widehat{k}_q$ with the symbols $1, 2, \ldots, \overline{v}$, which are the treatment symbols of $\mathcal{M}_1$. Now replace each symbol in $\mathcal{M}_1$ by the correspondingly labelled sub-block of the selected block from $\mathcal{M}_2$. Each large block of the so modified $\mathcal{M}_1$ is now of size $k_1 = \overline{k}_1\widehat{k}_q$ and contains successively nested blocks of sizes $k_2, k_3, \ldots, k_{s+t-q+1}$ where $k_j = \overline{k}_j\widehat{k}_q$ for $j = 1, \ldots, s$ and $k_j = \widehat{k}_{q+j-s-1}$ for $j = s+1, \ldots, s+t-q+1$. Repeat this process $\widehat{b}_1$ times, using a new copy of $\mathcal{M}_1$ for each of the $\widehat{b}_1$ blocks of $\mathcal{M}_2$. The resulting design $\mathcal{M}$ is an MNBIBD $(v, b_1, b_2, \ldots, b_{s+t-q+1}, r, k_1, k_2, \ldots, k_{s+t-q+1})$ with $v = \widehat{v}$, $r = \overline{r}\widehat{r}$, block sizes k_j as specified above, and $b_j = \overline{b}_j\widehat{b}_1$ for $j \leq s$, and $b_j = \overline{k}_s\overline{b}_s\widehat{b}_1\widehat{k}_q/\widehat{k}_{q+j-s-1}$ for $j > s$.

36.22 Theorem Let v be a prime power of the form $v = a_0a_1a_2\cdots a_n+1$ ($a_0 \geq 1$, $a_n \geq 1$ and $a_i \geq 2$ for $1 \leq i \leq n-1$ are integers). Then there is an MNBIBD with n component designs having $k_1 = ua_1a_2\cdots a_n$, $k_2 = ua_2a_3\cdots a_n$, $\ldots$, $k_n = ua_n$, and with a_0v blocks of size k_1, for any integer u with $1 \leq u \leq a_0$ and $u > 1$ if $a_n = 1$. If integer $t \geq 2$ is chosen so that $2 \leq tu \leq a_0$, then there is an MNBIBD with $n+1$ component designs, with the same number of big blocks but of size $k_0 = tk_1$, and with its n other block sizes being $k_1, \ldots, k_n$ as given above.

36.23 Theorem With the conditions of Theorem 36.22, if a_0 is even and a_i is odd for $i \geq 1$, then MNBIBDs can be constructed with the same block sizes but with $a_0v/2$ blocks of size k_1.

36.24 Remarks NBIBD constructions arise as special cases of 36.21, 36.22, and 36.23. An example for 36.21 is $s = 1$, $t = 2$. With mild abuse of terminology, Construction 36.21 also works if either $\mathcal{M}_1$ or $\mathcal{M}_2$ is taken as a RBIBD, for instance $s = 1$, $t = 2$ and $\hat{v} = \hat{k}_1$ so that $\mathcal{M}_2$ is RBIBD and $\mathcal{M}$ is NBIBD.

36.25 Definition A *nested row-column design* is a system $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ for which (i) each of $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2)$ and $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_3)$ is a nested block design, (ii) each block of $\mathcal{D}_1$ may be displayed as a $k_2 \times k_3$ row-column array, one member of the block at each position in the array, so that the columns are the $\mathcal{D}_2$ sub-blocks in that block, and the rows are the $\mathcal{D}_3$ sub-blocks in that block.

36.26 Definition A (completely balanced) *balanced incomplete block design with nested rows and columns*, BIBRC(v, b_1, k_2, k_3), is a nested row-column design $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ for which each of $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2)$ and $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_3)$ is a NBIBD.

36.27 Example A BIBRC for five symbols in ten 2×2 nesting blocks.

$$\begin{array}{|c|c|}\hline 1&2\\\hline 3&4\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 2&3\\\hline 4&5\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 3&4\\\hline 5&1\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 4&5\\\hline 1&2\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 5&1\\\hline 2&3\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&3\\\hline 2&4\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 2&4\\\hline 3&5\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 3&5\\\hline 4&1\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 4&1\\\hline 5&2\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 5&2\\\hline 1&3\\\hline\end{array}$$

36.28 Example A BIBRC for ten symbols in thirty 2×3 nesting blocks. Initial blocks (mod 10) are

$$\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 5&6&9\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&7\\\hline 3&5&8\\\hline\end{array}\quad\begin{array}{|c|c|c|}\hline 1&2&4\\\hline 3&9&5\\\hline\end{array}$$

36.29 Remark If $k_2 = k_3$ then a nested row-column design is a BIBRC if $(\mathcal{V}, \mathcal{D}_1)$ and $(\mathcal{V}, \mathcal{D}_2 \cup \mathcal{D}_3)$ are BIBDs, loosening the complete balance requirement that $(\mathcal{V}, \mathcal{D}_2)$ and $(\mathcal{V}, \mathcal{D}_3)$ are individually BIBDs. An example is the first five blocks of Example 36.28. Further relaxations are explained in [8].

36.30 Theorem If $v = mpq+1$ is a prime power and p and q are relatively prime, then initial nesting blocks for a BIBRC(v, mv, sp, tq) are $A_l = x^{l-1}L \otimes M$ for $l = 1, \ldots, m$, where $L_{s\times t} = (x^{i+j-2})_{i,j}$, $M_{p\times q} = (x^{[(i-1)q+(j-1)p]m})_{i,j}$, s and t are integers with $st \le m$, and x is a primitive element of GF$_v$. If m is even and pq is odd, then $A_1, \ldots, A_{m/2}$ are intial nesting blocks for BIBRC$(v, mv/2, sp, tq)$

36.31 Theorem Write $x^{u_i} = 1-x^{2mi}$ where x is a primitive element of GF$_v$ and $v = 4tm+1$ is a prime power. Let A be the addition table with row margin $(x^0, x^{2m}, \ldots, x^{(4t-2)m})$ and column margin $(x^m, x^{3m}, \ldots, x^{(4t-1)m})$, and set $A_l = x^{l-1}A$. If $u_i - u_j \not\equiv m$ (mod $2m$) for $i, j = 1, \ldots, t$ then $A_1, \ldots, A_m$ are initial nesting blocks for BIBRC$(v, mv, 2t, 2t)$. Including 0 in each margin for A, if further $u_i \not\equiv m$ (mod $2m$) for $i = 1, \ldots, t$ then $A_1, \ldots, A_m$ are initial nesting blocks for BIBRC$(v, mv, 2t+1, 2t+1)$.

36.32 Definition A *bottom-stratum universally optimal nested row-column design*, BNRC(v, b_1, k_2, k_3), is a nested row-column design $(\mathcal{V}, \mathcal{D}_1, \mathcal{D}_2, \mathcal{D}_3)$ for which (i) $(\mathcal{V}, \mathcal{D}_2)$ is a BIBD or, more generally, a BBD, and (ii) the $\mathcal{D}_3$ sub-blocks within any block of $\mathcal{D}_1$ are identical as multi-sets.

36.33 Example A BNRC with 4 symbols in nesting blocks of size 2×4.

$$\begin{array}{|c|c|c|c|}\hline 1&1&2&2\\\hline 2&2&1&1\\\hline\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&4&4\\\hline 4&4&3&3\\\hline\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 3&4&2&1\\\hline\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 3&4&2&1\\\hline\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 3&4&2&1\\\hline\end{array}\quad\begin{array}{|c|c|c|c|}\hline 1&2&3&4\\\hline 3&4&2&1\\\hline\end{array}$$

36.34 Theorem The existence of BNRC(v, b_1, k_2, k_{31}) and BNRC(v, b_1, k_2, k_{32}) implies existence of BNRC$(v, b_1, k_2, k_{31} + k_{32})$. The existence of BNRC$(v, b_1, k_2, k_3)$ for which b_1 is a multiple of s implies existence of BNRC$(v, b_1/s, k_2, sk_3)$. The column-wise juxtaposition of the nesting blocks of a BNRC into a $k_2 \times b_1k_3$ array is a row-regular GYD.

36.35 Theorem If $v = mq + 1$ is a prime power and $2 \le p \le q$, initial nesting blocks for a BNRC(v, mv, p, q) are $A_l = (x^{(i+j-2)m+l-1})_{ij}$ for $l = 1, \ldots, m$ and x a primitive element of GF$_v$. If m is even and q is odd, $A_1, \ldots, A_{m/2}$ generate BIBRC$(v, mv/2, p, q)$.

36.36 Remarks

1. BIBRCs and BNRCs are statistically optimal for competing models ([8]).
2. The necessary conditions for existence of these designs are those of the component BIBDs. The necessary conditions are sufficient for $k_1 = 4$ ([2],[13]).

3. Most work on BIBRCs and BNRCs has concentrated on constructing infinite series, often employing starter blocks and the finite fields ([6],[8],[2]) as illustrated in 36.30, 36.31, and 36.35.

36.5 See Also

§II.7	General treatment of resolvable and near-resolvable BIBDs.
§VI.65.6	Details on BBDs and Generalized Youden designs.
§VI.38	Perpendicular arrays can be arranged into MNBIBDs and BIBRCs.
§VI.54	Many constructions for NBIBDs which are Whist tournaments.
§VI.51	Various tournament designs, some of which are NBIBDs.
[10]	Survey of NBIBDs; contains much of the information given here.
[8]	Survey of nested designs, including NBIBDs, BIBRCs, and BNRCs.
[9]	Exploration of nesting, crossing, and other relationships for blocking systems from an optimality perspective, with constructions.
[1]	Construction of resolvable and near-resolvable NBIBDs.
[12], [3]	Uses of NBIBDs in constructing other combinatorial designs not discussed here.

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