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Creating a Context for Learning: Activating Children's Whole Number Knowledge Prepares Them to Understand Fraction Division

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Abstract

When children learn about fractions, their prior knowledge of whole numbers often interferes, resulting in a whole number bias. However, many fraction concepts are generalizations of analogous whole number concepts; for example, fraction division and whole number division share a similar conceptual structure. Drawing on past studies of analogical transfer, we hypothesize that children's whole number division knowledge will support their understanding of fraction division when their relevant prior knowledge is activated immediately before engaging with fraction division. Children in 5th and 6th grade modeled fraction division with physical objects after modeling a series of addition, subtraction, multiplication, and division problems with whole number operands and fraction operands. In one condition, problems were blocked by operation, such that children modeled fraction problems immediately after analogous whole number problems (e.g., fraction division problems followed whole number division problems). In another condition, problems were blocked by number type, such that children modeled all four arithmetic operations with whole numbers in the first block, and then operations with fractions in the second block. Children who solved whole number division problems immediately before fraction division problems were significantly better at modeling the conceptual structure of fraction division than those who solved all of the fraction problems together. Thus, implicit analogies across shared concepts can affect children's mathematical thinking. Moreover, specific analogies between whole number and fraction concepts can yield a positive, rather than a negative, whole number bias.

Keywords: analogical transfer, analogical priming, mathematics learning, fraction division, whole number bias

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In learning mathematics, children's new knowledge often builds on what they already know. Many studies demonstrate positive effects of prior knowledge on later mathematics learning (e.g., Hecht & Vagi, 2012; Siegler et al., 2012). However, some aspects of children's prior mathematical experiences may bias them in ways that are counterproductive for new learning (e.g., McNeil & Alibali, 2005; Ni & Zhou, 2005). Given that some aspects of prior knowledge are beneficial for new learning and other aspects of prior knowledge negatively bias new learning, there is a need for a theory-based framework for predicting what kinds of prior knowledge teachers should actively draw on during new lessons, and how they should do so.

Fractions and the Whole Number Bias

These questions are particularly relevant in the domain of fraction learning. A large body of work demonstrates that children's prior knowledge of whole numbers negatively biases their understanding of fraction symbols (e.g., Hartnett & Gelman, 1998; Mack, 1995; Meert, Grégoire, & Noël, 2010; Smith, Solomon, & Carey, 2005; Stafylidou & Vosniadou, 2004; Van Hoof, Verschaffel, & Van Dooren, 2015) and operations (Obersteiner, Van Hoof, Verschaffel, & Van Dooren, 2016; Siegler & Pyke, 2013; Van Hoof, Vandewalle, Verschaffel, & Van Dooren, 2015). This well-documented phenomenon is known as the *whole number bias* (e.g., Ni & Zhou, 2005). Based on this evidence, some have argued that children's early knowledge of whole numbers is inconsistent with what they later learn about fractions and operations on fractions (e.g., Vamvakoussi & Vosniadou, 2010). However, despite the problems that children's whole number knowledge seems to pose for fraction learning, some researchers have advanced an alternative view, namely, that children's whole number concepts uniquely support later-acquired fraction concepts. For example, Siegler, Thompson, and Schneider (2011) have proposed an integrated framework for the development of whole number and fraction concepts. They argue that children's whole number and fraction concepts are supported by similar mental representations (i.e., a mental number line on which numerical representations become increasingly linear with development). In support of this view, recent empirical studies have provided evidence that individual differences in children's early whole number representations predict individual differences in later fraction understanding (Vukovic et al., 2014).

Taken together, these lines of work on fraction understanding paint a mixed picture concerning the effects of children's prior whole number knowledge. One possible explanation for these mixed findings is that some aspects of children's prior knowledge of whole numbers support fraction understanding, whereas other aspects do not. Thus, children's whole number knowledge may not generally support fraction learning, though specific, prior whole number concepts may be leveraged to support specific, novel fraction concepts. However, the specific pathways through which children's whole number knowledge may support (or bias) fraction understanding remain poorly understood.

Analogies in Mathematics

In this work, we apply an analogical transfer framework (e.g., Day & Goldstone, 2011; Gentner, 1983; Gick & Holyoak, 1980; Kaminski, Sloutsky, & Heckler, 2013) to mathematics learning to predict which aspects of prior knowledge should best support the learning of new concepts. Analogical transfer is the transfer of strategies or mental models from one problem to another, and it is guided by learners' perception of similarity between the two problems (e.g., Gick & Holyoak, 1980). Certain types of similarity lead to positive, or beneficial transfer whereas other types of similarity lead to negative, or harmful transfer (e.g., Gentner, 1983).

For example, in one classic study of analogical problem solving among adults, Novick (1988) demonstrated that both mathematics experts and novices often transferred earlier learned solution procedures to later target problems. However, the beneficial or harmful effects of that transfer depended on the nature of the similarity between problems. One of the given solution procedures was couched in a story problem that included similar types of objects as the target story problem; this procedure resulted in an incorrect answer for the target problem. Another given solution procedure was couched in a story with different types of objects, but that had the same *conceptual structure* as the target problem; this procedure resulted in a correct answer for the target problem. Across a series of experiments, Novick found that when experts and novices learned a procedure with a story problem that shared surface similarities to the target problem, they were very likely to attempt to apply that procedure to the target problem, even if it yielded an incorrect solution, demonstrating *negative* transfer. In contrast, after learning a solution procedure with a problem that shared conceptual structure but not surface similarities, only experts were likely to show *positive* transfer. Furthermore, when participants learned procedures for both the structurally similar problem and the surface similar problem, experts showed positive transfer from the structurally similar problem regardless of the order in which the problems were encountered. In contrast, novices demonstrated considerably more positive transfer from the structurally similar story problem if they had encountered it more recently.

This framework can be used to make predictions about when prior knowledge should support or hinder children's mathematics learning. Specifically, children's prior knowledge of similar mathematics problems should support new learning to the extent that the problems share the same underlying relationships among elements, or *conceptual structure*. On the other hand, like novice adults, children's prior knowledge of similar problems should negatively bias new learning to the extent that the problems share similar elements, but not conceptual structure, with new problems. Note that we intend the term "conceptual structure" in a narrow sense, to refer to the relational structure of the specific mathematical operation involved in a problem, and not in the sense of "central conceptual structures" such as those described by Case and Okamoto (1996). Like adults with less sophisticated mathematics knowledge (e.g., Novick, 1988), children often have difficulty discerning which previously-learned problems are most relevant to new ones, and they instead tend to rely on what was encountered most recently (e.g., Chen & Daehler, 1989). To bypass children's difficulties in identifying the best sources of prior knowledge, we hypothesize that structuring instructional activities such that children's most relevant, structurally similar, prior knowledge is activated immediately before new concepts are introduced will result in positive transfer to the target domain.

In the current study, we investigate *spontaneous* transfer from children's prior knowledge of familiar concepts to structurally related, but more difficult concepts. Analogical transfer in problem solving is often construed as an explicit, analytical process during which people identify similar previously-solved problems, map across corresponding parts between problems, and make inferences to the target problem (e.g., Gentner, 1983; Gick & Holyoak, 1980; Novick & Holyoak, 1991; Richland, 2011). However, several studies suggest that analogical transfer can also occur without explicit effort or even without awareness of analogical reasoning (Bassok, Pedigo, & Oskarsson, 2008; Day & Gentner, 2007; Day & Goldstone, 2011; Estes & Jones, 2006; Green, Fugelsang, & Dunbar, 2006; Leo & Greene, 2008; Popov & Hristova, 2015; Schunn & Dunbar, 1996; Spellman, Holyoak, & Morrison, 2001). For example, Day and Goldstone (2011) demonstrated that adults benefit from problem-solving experience with a structurally similar, but perceptually dissimilar problem when learning about the causal structure of a target problem, even when they do not report drawing on the first problem to make sense of the target problem. These studies suggest that adults need not notice an analogy in order for previously-solved analogous problems to support performance on new problems; instead, learners' prior knowledge shapes their construal of target material, thereby constraining their representation of target material and the strategies they use to engage with the material (Day & Goldstone, 2012). We suggest that *implicit* analogies are important to consider in theories of analogical reasoning during problem solving, because teachers often set up analogies across domains in mathematics without fully guiding the mapping and adaptation of those analogies during instruction (Richland, Zur, & Holyoak, 2007).

Research on implicit analogies has largely been conducted with adults, and the timing of the analogies varies across studies. In some studies, the initial problem is solved immediately before the target problem (e.g., Bassok et al., 2008; Day & Goldstone, 2011) and in other studies it is solved hours or even days before the target problem (e.g., Schunn & Dunbar, 1996). In some studies, there is intervening content (e.g., Day & Gentner, 2007) and in others there is not (e.g., Spellman et al., 2001). It remains an empirical question how the timing of the analogy affects adult problem solving.

The timing of analogies during instruction is likely to matter for children's ability to effectively use implicit analogies. Children may be especially likely to rely on very recently activated concepts or procedures (e.g., Chen & Daehler, 1989), regardless of structural similarity. Indeed, implicit analogies have been shown to support children's learning of difficult target material when children's knowledge of familiar concepts is activated immediately before learning (Sidney & Alibali, 2015), but not when knowledge of familiar concepts is activated after the lesson (Richland & Hansen, 2013). These studies suggest that activating children's most relevant knowledge immediately before tackling difficult material, without any intervening content that could activate irrelevant knowledge, would best facilitate children's correct construal of target problems. In the current study, we manipulated the timing of the analogy by varying whether children's relevant prior knowledge was activated immediately before the target problems or several trials before the target problems, with intervening mathematics content in between. In this way, we test whether recency matters for an implicit analogy.

Understanding Fraction Division

The current study is focused on conceptual understanding of fraction division—a topic that poses great challenges for many children (Sidney & Alibali, 2015; Siegler & Lortie-Forgues, 2015; Siegler & Pyke, 2013). We examined children's understanding of the conceptual structure of fraction division – the relationships among the dividend, divisor, and quotient.

The analogical transfer framework makes predictions regarding what kinds of prior knowledge will be most useful for understanding fraction division, as it implies *specific* links across two categories of arithmetic problems, rather than *broad* links across broad swaths of arithmetic knowledge. Based on this framework (e.g., Gentner, 1983), we predict that whole number division problems should be a useful analogue, because they are structurally similar to fraction division problems, in that they share the same conceptual structure of division, even though they differ in surface features. For example, in both whole number division problems and fraction division problems, the quotient (the answer) can be thought of as the number of groups the size of the second operand (the divisor) that can be made from the first operand (the dividend). However, whole number and fraction division problems differ in the surface features of the numbers, as fractions "look different" and are more perceptually complex than whole numbers. Furthermore, classroom instruction on whole number division and instruction on fraction division are often separated by many months, if not years.

In contrast, other operations on fractions (i.e., fraction addition, subtraction, and multiplication) should be less useful analogues for fraction division, because they differ from fraction division in conceptual structure, despite surface similarities. Fraction addition and fraction subtraction involve additive relationships, which are distinct from the multiplicative relationship inherent in fraction division. Fraction multiplication can be construed as iterating a group, whose size is specified by one operand, the number of times specified by the second operand. Although multiplication and division are fundamentally related, as both can be grounded in a

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multiplicative grouping structure, the relationship between the operands is not the same across operations. Despite these differences in conceptual structure, other operations on fractions are perceptually similar to fraction division, as they share the representation of quantity as a fraction symbol. Fraction division is also similar to other operations on fractions in terms of timing within curricula; fraction arithmetic problems of various sorts often occur within the same instructional unit.

Of course, the conceptual similarity between whole number operations and fraction operations is true for each operation, not only for fraction division. In this research, we focus on fraction division because children's conceptual understanding of fraction division has been shown to be weak (Sidney & Alibali, 2015; Siegler & Lortie-Forgues, 2015; Siegler & Pyke, 2013; Richland & Hansen, 2013) and because difficulties with fraction division persist into adulthood (e.g., Lo & Leu, 2012; Ma, 1999; Sidney, Hattikudur, & Alibali, 2015). For example, Ma (1999) interviewed 23 American pre-service teachers about their fraction division understanding. Although many could implement an invert-and-multiply procedure to find the quotient, only one could generate a story problem scenario that accurately represented fraction division. To test whether recent activation of children's whole number division concepts would support accurate conceptual understanding of fraction division more so than recent activation of other fraction operation concepts, we targeted participants whom we expected to have substantial knowledge of fraction addition, subtraction and multiplication, but little knowledge of fraction division.

Two previous studies have shown that analogies between whole number division and fraction division *do* support children's learning from a lesson about fraction division (Sidney & Alibali, 2015; Richland & Hansen, 2013). However, these studies did not examine whether analogies can affect children's construal of difficult problems in the absence of an explicit lesson. In the current study, we examine the effect of a whole number division analogue on children's conceptual understanding of fraction division more directly by testing whether activating children's whole number division knowledge affects how they approach fraction division problems, without providing any additional instruction.

The Current Study

Building on this prior work, the present study was designed to address two primary aims. First, we aim to contribute to a resolution of the whole number bias debate. Specifically, we aim to replicate and extend previous findings (e.g., Sidney & Alibali, 2015) in a direct test of the hypothesis that children's whole number division knowledge will positively support children's fraction division understanding. We hypothesize that children's whole number knowledge will be most useful when links between whole number and fraction concepts are made specifically across problems with shared conceptual structure (i.e., whole number division and fraction division), rather than when children's whole number knowledge is activated more generally. Second, we aim to demonstrate that analogical links between whole number understanding and fraction understanding can be activated implicitly, by presenting conceptually analogical problems close in time, and that the timing matters. Given previous findings that learners are sensitive to analogies that are made implicitly through temporal juxtaposition (e.g., Day & Goldstone, 2011), we hypothesize that activating children's knowledge of whole number division just before they encounter fraction division will be beneficial, even when they are unaware of the analogical link.

To address these aims, we measured children's conceptual understanding of fraction division under two experimental conditions, one in which children's knowledge of a structurally similar domain (whole number division) was activated immediately before they were asked to demonstrate fraction division and another in which children's knowledge of a perceptually similar, but structurally dissimilar, domain (fraction multiplication) was activated immediately before they were asked to demonstrate fraction division. Importantly, we activated children's knowledge of both whole number and fraction addition, subtraction, multiplication, and division concepts in both conditions. The critical differences between the two experimental conditions were in the *timing* of this activation and the presence of intervening trials. If activating children's knowledge of whole numbers is generally detrimental to reasoning about fractions (as suggested by the whole number bias literature), then children should experience interference from their whole number knowledge when reasoning about fraction operations in both conditions. However, if activating children's prior knowledge of specific, structurally similar concepts supports reasoning about more difficult concepts (as suggested by the analogical transfer literature), consistent with the hypotheses stated above, then children's ability to demonstrate fraction division should be better when fraction division problems immediately follow whole number division problems than when they immediately follow fraction multiplication problems.

Method

Participants

Participants were recruited from a public school district in a mid-sized Midwestern city through invitations distributed to students in their 5th and 6th grade classes. Participants were recruited at these grade levels because fraction operations are often covered in 5th and 6th grade within this district. We expected students to be proficient in modeling all of the whole number problems and in modeling addition, subtraction, and multiplication with fractions. A total of 63 children participated in the study in the summer after their 5th or 6th grade year. One child was excluded due to not finishing the session in the available time. We asked parents or guardians to report children's gender and race/ethnicity. The final sample was made up of 28 children who had completed grade 5, and 34 children who had completed grade 6. The sample included 24 girls and 38 boys, and race and ethnicity were distributed as follows: 79% White, 3% Hispanic, 6% Asian, 2% African-American and 10% multiple races or ethnicities.

Task and Materials

We activated and measured children's conceptual understanding of arithmetic operations using a modeling task in which children were asked to represent equations using physical manipulatives (unlabeled fraction bars). We chose this task rather than the picture generation and story generation tasks used in prior work (e.g., Ball, 1990; Ma, 1999; Richland & Hansen, 2013; Sidney & Alibali, 2015; Sidney, Hattikudur, & Alibali, 2015) for several reasons. Although story generation and picture generation tasks have been used to successfully measure children's conceptual understanding of fraction division, they have limitations that we aimed to avoid. First, the story generation task requires coming up with a story scenario that affords dividing by a fraction. In contrast, our modeling task does not rely on children's storytelling creativity in order to demonstrate division understanding. The picture generation task also does not include this verbal demand, however, it does require children to correctly represent fraction magnitudes (e.g., correctly draw 1/5 in order to draw 4 ÷ 1/5). We sought to measure children's knowledge of fraction division separately from their ability to represent fraction magnitudes. Therefore, we adapted the picture generation task to a concrete context in which we provided unlabeled fraction bars to support children's representations of fraction magnitudes. Similar object-based tasks have been used in prior work to assess children's understanding of fraction division (Bulgar, 2009) and whole number division (Mulligan & Mitchelmore, 1997; Sidney, Chan, & Alibali, 2013).

Fraction Bars

During the study session, participants were asked to use unlabeled fraction bars to model a series of arithmetic expressions. The bars were made up of interlocking blocks of various sizes. Such blocks are sometimes called "fraction blocks" or "fraction towers", because the size of the fractional pieces is proportional to the size of the whole (e.g., two halves is the same size as three thirds or one whole), and because the pieces can be snapped together and stacked. Children were provided with six kinds of bars: wholes, halves, thirds, fourths, fifths, and sixths. Bars with different sized fractional pieces were different colors, and they were presented in separate containers and connected to make wholes (e.g., the sixths pieces were all in bars of 6/6ths; see Figure 1). There were 20 wholes in each container, and children used these pieces to model both the whole number and the fraction expressions.

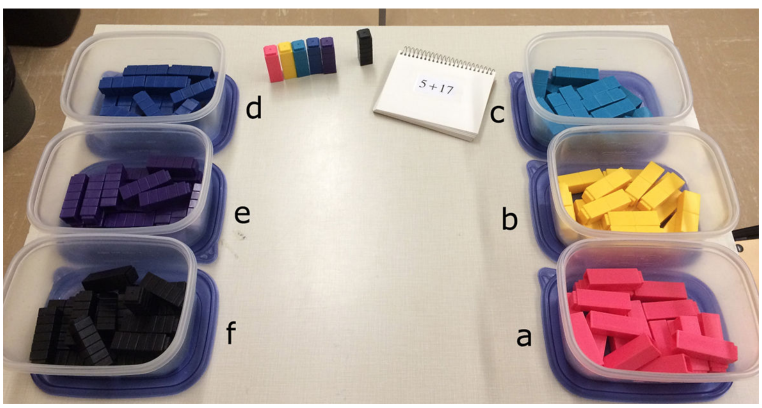

Figure 1. To model the expressions, children were provided with unlabeled fraction bars. The pink bars (a) could not be further divided. The yellow bars (b) could be divided into halves. The light blue bars (c) could be divided into thirds. The dark blue bars (d) could be divided into fourths. The purple bars (e) could be divided into fifths. The black bars (f) could be divided into sixths.

The Problems

During the study session, children were shown a set of problems, one at a time. The set contained 16 whole number arithmetic problems and 16 fraction arithmetic problems (see Appendix A). Within each number type (i.e., whole number or fraction), there were four problems for each of the four arithmetic operations. All the operands in the whole number arithmetic problems were between 1 and 32. All fraction arithmetic problems included a whole number as the first operand and a fraction as the second operand. Within the problems for each fraction operation (e.g., the four fraction addition problems), two out of four problems included a unit fraction (i.e., a fraction with 1 in the numerator), and the others were non-unit fractions less than 1. No mixed or improper fractions were used.

Procedure

Each session took place in a quiet room on a university campus, and lasted approximately 45 minutes. First, the experimenter introduced the task of using the bars to model the equations, and asked children to say aloud what they were thinking throughout the experiment. Next, the experimenter introduced the unlabeled fraction bars and demonstrated that the differently colored bars could be used to represent different fraction magnitudes. After giving children the opportunity to ask questions about the fraction bars, the experimenter asked the child to model each problem using the bars on the table. Children were asked to start by modeling the first number in the given problem, and then "show what it looks like to *do the operation* using the second number." The experimenter used the first problem as an example of how to model the first number. Finally, the experimenter emphasized that the goal of the task was not to solve the problems in their minds or to obtain a solution for the expression, but to demonstrate the expression with the bars. Full instructions are provided in Appendix B. These instructions were identical across conditions.

Children were randomly assigned to one of two conditions: blocked by *number type* (n = 31) or blocked by *operation* (n = 31). In the *number type* condition, participants were asked to demonstrate all 16 whole number problems (in the order addition, subtraction, multiplication, division) before being asked to demonstrate all 16 fraction problems (in the order addition, subtraction, multiplication, division). In this problem sequence, children modeled fraction multiplication problems immediately before fraction division problems; thus, children's knowledge of fraction multiplication was activated immediately prior to their demonstrating fraction division. In the *operation* condition, participants were asked to demonstrate all the addition problems first, followed by all the subtraction, multiplication, and division problems. Within each operation block, all the whole number problems were completed before the fraction problems. In this problem sequence, children modeled each whole number operation immediately before the analogous fraction operation (e.g., four whole number addition problems, followed by four fraction addition problems). Critically, in this condition, children's knowledge of whole number division was activated immediately prior to their modeling of fraction division.

Data Coding

During the experimental session, the experimenter (the first author) coded the strategies children used to model each problem, based on children's speech and actions. Children's strategies were classified as correct or incorrect based on whether they accurately represented the *conceptual structure* of the operation. As we were focused on children's strategies for representing the conceptual structure for each operation, we did not require children to state the answer to the problems, though some did. It is important to note that the correct strategies for each operation do not depend on whether the second operand is a whole number or a fraction, as whole number operations and their corresponding fraction operations have the same conceptual structure (e.g., whole number addition and fraction addition).

Addition strategies were coded as correct if children represented the joining of two sets, one as big as the first operand and one as big as the second operand. For example, for 9 + 3/4, children often took out nine whole bars, either nine whole bars or nine conjoined bars that were each equivalent to a whole bar (e.g., nine conjoined bars each made up of two half bars), from the container and put them in one group on the table, then made a second group of three one-fourths bars on the table, and finally pushed these groups together to represent addition of the two operands. Subtraction strategies were coded as correct if children represented taking away or cancelling the second operand from the first operand. For example, for 5 – 2/3, children often made a group of five conjoined wholes from the container of thirds bars, and then disconnected two thirds bars from one of these wholes, separating the two thirds bars from the remaining four and one-third bars. Multiplication strategies were coded as correct if children represented one operand as the size of a group and the other operand as the number of groups. For example, for 6 * 2/3, some children made six groups with two thirds bars in each group. Other children counted out six conjoined wholes using the thirds bars, disconnected two thirds from each whole bar, and put all the two-thirds bars together in one group as their answer.

Strategies on division problems were coded as correct if they reflected either a *partitive* model of division (i.e., the divisor represents the *number* of groups) or a *quotative* model of division (i.e., the divisor represents the *size* of each group). The specific model that children used on each trial was also noted.

Children who used a correct, quotative strategy modeled the divisor as the size of the groups. For example, for the problem, 15 ÷ 5, a quotative strategy might involve separating five bars at a time from the initial group, yielding three equal groups of five bars each (see Figure 2A). In their think-aloud protocols, children who used a quotative strategy often explicitly mentioned making groups the size of the divisor (e.g., "so I need groups of 5"). Within the quotative model, the quotient (e.g., 3), is the number of groups. Again, we did not require children to name the answer as long as the quotative strategy was reflected in their behavior or speech.

Children who used a correct, partitive strategy modeled the divisor as the number of groups. For example, for the problem 15 ÷ 5, a partitive strategy might involve making a group of 15 whole bars and then dividing them into five groups (see Figure 2B). Children employed the partitive strategy in two ways; they either separated three bars at a time from the initial group, yielding five groups of three, or they separated one bar at a time, distributing each bar to one of five groups, in order to distribute the 15 bars evenly across five groups. Within the partitive model, the quotient (e.g., 3), is the number of bars in each group. We did not require children to name the answer as long as the partitive strategy was reflected in their behavior or speech.

Importantly, although US learners often use both partitive and quotative models to represent *whole number division* (e.g., Sidney & Alibali, 2013), they tend to prefer quotative strategies for *division by a fraction* (Fischbein, Deri, Nello, & Marino, 1985; see Ma, 1999 for a cross-cultural comparison). Consistent with this data, the children in our sample used both partitive and quotative strategies on whole number division, and they used only quotative strategies for fraction division. As an example of a quotative strategy for fraction division, for the problem 8 ÷ 3/4, some children counted out 8 whole bars from the container of wholes that could be divided into one-fourths. Then, they split those bars into segments of three-fourths of a whole (i.e., bars of size three-fourths), modeling the divisor as the *size* of the groups. As with quotative whole number division, the quotient is the number of groups of three-fourths, in this case, 10 groups of three-fourths, with two fourths bars remaining, or 10-2/3 groups (see Figure 2C).

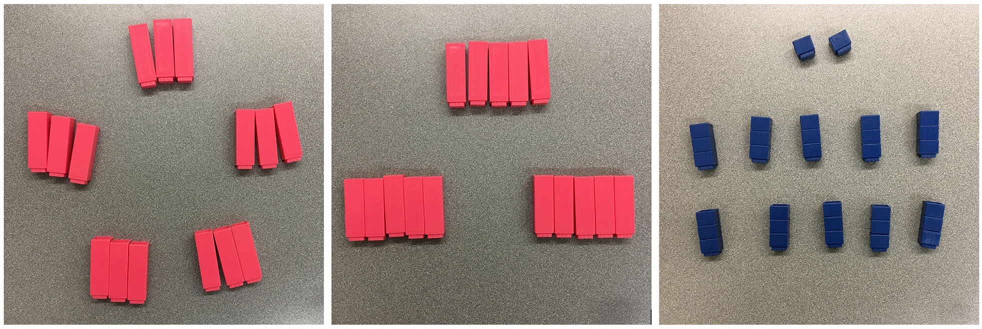

Figure 2. Sample end states for the modeling task for partitive division of 15 ÷ 5 (Panel A), quotative division of 15 ÷ 5 (Panel B), and quotative division of 8 ÷ 3/4 (Panel C).

No child in our dataset used a partitive strategy for modeling fraction division. In this regard, it is worth noting that partitive division is *possible* for fraction division. In a pilot study, two adults (out of 27) used a partitive strategy for problems that involved division by a unit fraction (e.g., 5 ÷ 1/4). These adults represented the division with 5 whole bars, and *named* this initial group as 1/4 of the total group. Then, they added three more groups of 5 (1/4 of the total, three times), in order to get the size of the total group. One potential issue is that a partitive strategy of this sort may be difficult to distinguish from an attempt to represent the invert-and-multiply procedure (e.g., translating 5 ÷ 1/4 into 5 * 4; see Table 1). Thus, in coding our data, we relied on children's *speech* to disambiguate partitive division by a fraction (e.g., for 5 ÷ 1/4, speaking about *1/4 of a group*) from modeling the invert-and-multiply procedure (e.g., for 5 ÷ 1/4, speaking about about *multiplying by 4*). One child in our dataset briefly expressed the beginning of a partitive strategy in his speech (for the problem 5 ÷ 1/4, he said, in a questioning tone, "1/4 piles?"), but he quickly dismissed this idea and used a different approach; this case is discussed below in the section on strategy switching.

One child generated a unique, hybrid strategy for division by a non-unit fraction that combined elements of quotative division by a unit fraction and partitive whole number division. For the problem 7 ÷ 3/5, this child first represented 7 ÷ 1/5 using a quotative approach, creating 35 groups of size 1/5. She then further divided these pieces into 3 groups using partitive whole number division, yielding 3 groups of 11 with a remainder of 2. This unexpected "hybrid" strategy was classified in a separate category, and was used only by this particular child on two problems that involved division by a non-unit fraction.

On two whole number division problems and two fraction division problems, the quotient included a remainder. We did not require children to correctly name the remainder as a fraction of a group (e.g., on 8 ÷ 3/4, a remainder of 2/4 as 2/3 of one group) in order to code their behavior and speech as reflecting a partitive or quotative division strategy. Instead, we focused on the nature of the grouping structure. Many children ignored the remainder, some children correctly named the remainder (either as a fraction of a group or as a remainder), and some children incorrectly named the remainder.

Table 1

Strategies for Modeling Division Operations

Strategy	Whole Number Division Examples (15 ÷ 5)	Fraction Division Examples (8 ÷ 3/4)
Quotative (Correct)	Set out 15 bars, divide into three groups of 5	Set out 8 cojoined bars of fourths, divide into ten groups of 3/4, set aside remainder
Partitive (Correct)	Set out 15 bars, divide into five groups of 3	Not observed
Quotative and Partitive Hybrid (Correct)	Not applicable	Set out 8 bars, divide into 32 groups of 1/4, divide those groups into three groups of 10, representing (8 ÷ 1/4) ÷ 3, set aside remainder
Invert-and-Multiply[a]	Not observed	Make 8 bars, each made of up 4 one-third pieces, representing 8 * (4/3) *or* Set out 8 groups of 4 bars, then divide into groups of 3, representing (8 * 4) ÷ 3
Non-Division Operation(s)	Not observed	Set out eight 3/4 bars (8 * 3/4) *or* Set out 8 conjoined bars of fourths, remove 3/4 from each one (representing 8 – 8 * 3/4)
Division of Whole Number Components	Not applicable	Set out 8 bars, divide into groups of 4, representing (8 ÷ 4)
Model Operands Only	Set out one set of 15 bars and one set of 5 bars	Set out one set of 8 bars and one 3/4 bar

[a]Although this strategy yields a correct *solution* for the expression, it does not accurately represent the *division structure* of the expression, so it was considered an incorrect response.

Children also sometimes implemented incorrect strategies for modeling fraction division; these strategies were classified into several subcategories (see Table 1), including: (1) representing the invert-and-multiply procedure (which yields a correct solution but which does not model the operation of division, so was treated as incorrect), (2) representing non-division fraction operations, (3) representing division by a whole number component, or (4) modeling only the operands. Trials on which children refused to model the expression or represented only the solution to the arithmetic expression were also considered incorrect.

One common variant of modeling non-division fraction operations involved a combination of fraction subtraction and fraction multiplication. For example, for 8 ÷ 3/4, this strategy involved first taking eight whole bars made of fourths bars, disconnecting three fourths bars from each whole bar, separating these eight three-fourths bars from the original group, and indicating the remaining set of 8 one-fourth bars as the answer. Given the similarity of the first part of this strategy (i.e., disconnecting three-fourths from each bar) to children's fraction multiplication strategies, and the similarity of the second part of the strategy (i.e., separating the disconnected bars from the original group) to children's subtraction strategies, we considered this strategy to represent 8 – (8 * 3/4), a combination of non-division fraction operations.

It is important to note that all of the “incorrect” strategies resulted in incorrect solutions for the arithmetic expression, with the exception of the invert-and-multiply strategy. However, the task put to the participant was to “show” what the expression means. In this context, the invert-and-multiply strategy was considered to be incorrect, because it does not reflect the grouping relationship between the dividend and divisor.

Strategy Switching

On some trials, a child began to use one strategy, abandoned it, and then used a different strategy. For example, for the problem 5 ÷ 1/4, one child set out five whole bars, and then said, “you have to have 1/4 piles?”. Interestingly, this language mirrored this participant’s language in the preceding whole number division trial (for 17 ÷ 6, “you have to have 6 piles”), and reflects an attempt to apply a partitive model of division to fraction division. Then, he began to divide a whole bar into groups of one-fourth, reflecting a quotative model of fraction division. However, after making two groups of one-fourth, he decided to start again, and said, “it would be like taking one fourth away from each one”, and then demonstrated 5 – (5 * 1/4) instead. In such cases of strategy switching, the child’s final strategy was coded for analyses of accuracy and strategy use. In our analyses of children’s division strategies, we also report evidence of strategy switching on the critical, fraction division trials.

Reliability

To evaluate the reliability of the on-line coding of accuracy, strategy use, and strategy switching carried out by the experimenter during data collection, the second author independently coded the data from the video recordings of the sessions for a subset of the participants. The two coders agreed on whether the child’s modeling of division was correct or incorrect on 100% of 80 trials (4 whole number and 4 fraction division trials from each of 10 participants). Moreover, the two coders agreed on the specific strategy code on 99% (79) of these trials. The two coders agreed on the presence or absence of strategy switching on 99% (71) of 72 trials (4 fraction division trials from each of 18 participants).

Data Availability

The coded data and the corresponding analyses on which the results are based can be found in the Supplementary Material.

Results

Overall Performance

Table 2 presents the average number of trials (out of 4) on which children correctly modeled the expression for each problem type and in each condition. Overall, participants were more successful on whole number problems than on problems including fractions. Also, they were more successful on addition and subtraction problems than on multiplication and division problems. As expected, children had the greatest difficulty modeling fraction division.

Table 2

Average Number of Correctly Modeled Trials, out of 4, by Condition.

Problem Type, Condition	Addition	Subtraction	Multiplication	Division
Whole Number Problems				
Blocked by Number Type	4.00 (0.00)	4.00 (0.00)	3.61 (1.02)	3.77 (0.50)
Blocked by Operation	4.00 (0.00)	3.87 (0.72)	3.84 (0.73)	3.68 (0.65)
Fraction Problems				
Blocked by Number Type	3.90 (0.55)	3.60 (1.13)	3.00 (1.67)	1.26 (1.86)
Blocked by Operation	3.94 (0.36)	3.77 (0.88)	3.29 (1.42)	1.97 (1.96)

Note. Standard deviation given in parentheses.

Conceptual Understanding of Fraction Operations

We examined children's conceptual understanding of each fraction operation under the two experimental conditions. We hypothesized that children whose whole number division concepts were activated immediately before modeling fraction division (i.e., children in the *operation* condition) would be more likely to successfully model fraction division than those children whose knowledge of other operations on fractions was activated immediately before modeling fraction division (i.e., children in the *number type* condition). We expected this to be the case because whole number division shares conceptual structure with fraction division, whereas other operations on fractions share surface similarity but not conceptual structure with fraction division. Though our focus in this study was children's fraction division understanding, each fraction operation is analogous to its corresponding whole number operation. Therefore, if activation of structurally similar prior knowledge supports understanding, we should see an advantage across *each* fraction operation for students in the *operation* condition. Thus, we examined performance on each operation, separately, as a function of condition.

As our goal was to investigate whether children's whole number knowledge would support their fraction understanding, for each analysis, we restricted our sample to only those participants who accurately modeled the analogous whole number problems. Thus, the sample size for each analysis varied slightly, depending on the number of participants who were successful with the whole number operation in question. The restricted sample sizes are as follows: n_O = 31, n_{NT} = 30 for the addition analysis (one missing due to missing data on the fraction addition outcome), n_O = 30, n_{NT} = 30 for subtraction (one missing due to missing data on the fraction subtraction outcome), n_O = 29, n_{NT} = 26 for multiplication, and n_O = 24, n_{NT} = 24 for division.

Fraction Addition, Subtraction, and Multiplication

To test whether condition affected children's ability to model addition, subtraction, or multiplication, we regressed the number of fraction addition, subtraction, and multiplication problems modeled correctly (in three separate regression models) on condition, controlling for child's grade level. As shown in Table 2, participants in our sample were highly proficient at demonstrating fraction addition, subtraction, and multiplication. Given this close-to-ceiling performance on these operations, there was little variability to be explained by experimental condition, so it is not surprising that condition did not significantly predict differences in fraction addition, b = 0.04, $t(58)$ = 0.31, p = .76, $\eta_p^2 < .01$, fraction subtraction, b = 0.28, $t(57)$ = 1.21, p = .23, η_p^2 = .03, or fraction multiplication, b = 0.10, $t(52)$ = 0.26, p = .80, $\eta_p^2 < .01$. However, in each case, there was a similar trend, in that participants who received the problems blocked by operation modeled more fraction problems correctly than those who received the problems blocked by number type (see Figure 3).

Fraction Division

To test our main hypothesis, we regressed the number of fraction division equations modeled correctly (0 - 4) on condition, controlling for child's grade level. Condition significantly predicted fraction division performance, b = 1.30, $t(45)$ = 2.32, p = .02, η_p^2 = .11. Participants in the *operation* condition, who solved whole number division problems immediately before modeling fraction division problems, had greater success in modeling fraction division, M_O = 2.45, SE_O = 0.40, than did participants in the *number type* condition, M_{NT} = 1.15, SE_{NT} = 0.39 (see Figure 3). Grade level was not a significant predictor of fraction division modeling scores, b = -0.43, $t(45)$ = -0.76, p = .45, η_p^2 = .01. The effect remained significant when also controlling for performance on the other fraction problems, b = 1.28, $t(44)$ = 2.28, p = .03, η_p^2 = .11.

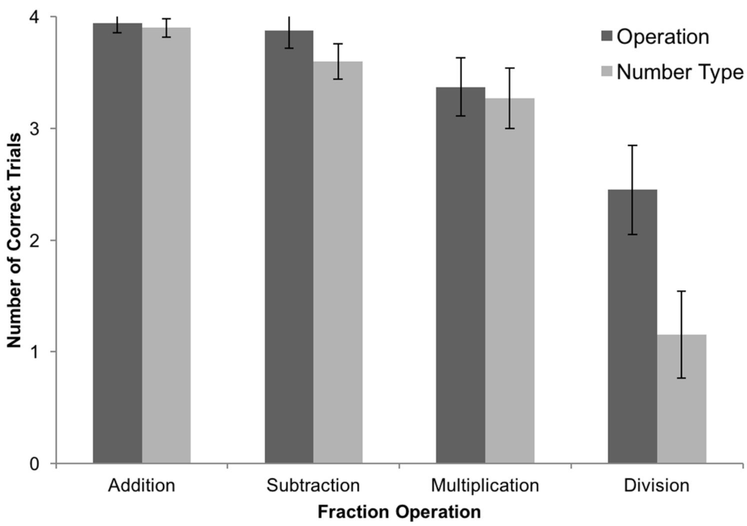

Figure 3. The average number of correctly modeled trials for each fraction operation, out of four, by experimental condition, controlling for grade level. Error bars represent the standard errors of each point estimate from each model.

Children's Division Strategies

To more closely examine transfer from children's knowledge of whole number division or other operations on fractions to children's strategies on fraction division trials, we analyzed the nature of children's strategy use on fraction division trials in both conditions.

Correct Strategies

We first examined the consistency of children's strategy use on whole number and fraction division, and whether the degree of consistency varied across conditions. To address this issue, we examined whether children in the *operation* condition who used quotative strategies on whole number division problems were more likely to use quotative strategies on fraction division than children in the *number type* condition who used quotative strategies on whole number division. As in the earlier analysis of children's fraction division understanding, we limited these analyses to children who successfully modeled whole number division: n_O = 24, n_{NT} = 24.

We coded whether children ever used a quotative strategy on the whole number division trials. Among the 48 children who used either a correct partitive or quotative strategy on all four whole number division trials, 22 children used quotative strategies on all four trials, 7 used a mixture of quotative and partitive strategies across

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trials, and 19 used partitive strategies on all four trials; see Table 3. Children's strategies on the fraction division trials were more variable; 33 out of 48 children used a single strategy across all four trials, and 15 children varied in their strategy use. Therefore, we coded children's *dominant* strategies for the fraction division trials, defined as the strategy used most often across the four fraction division trials.

The distributions of dominant fraction division strategies by whole number division strategy use and by experimental condition are shown in Table 3. Among children who used the quotative strategy on at least one whole number division problem (n = 29), those in the *operation* condition were more likely than those in the number type condition to use the quotative strategy as their dominant strategy on fraction division trials (12 out of 16, 75.0%, vs. 4 out of 13, 30.8%), $\chi^2(1, N = 29) = 5.67$, $p = .02$, odds ratio (OR) = 6.75. In contrast, among children who used solely partitive strategies on whole number division problems (n = 19), those in the *operation* condition were *not* significantly more likely than those in the number type condition to use the quotative strategy as their dominant strategy on fraction division trials (3 out of 8, 37.5% vs. 3 out of 11, 27.3%), Fisher's exact probability test, N = 19, p = .62, OR = 1.60. The pattern of data suggests that children were more likely to consistently use quotative strategies for fraction division if they had successfully used the quotative strategy for modeling whole number division in the immediately preceding problems.

Incorrect Strategies

We also examined whether grouping problems by number type caused children to draw on their recently activated *fraction operation* knowledge when modeling fraction division. To do so, we coded whether children incorrectly modeled non-division operations with fractions (e.g., fraction multiplication, fraction subtraction, or their combination) on at least one fraction division trial.

Table 3

Dominant Fraction Division Strategies by Whole Number Strategy

Whole Number Division Strategies	Quotative[a]	Non-Division Operations	Invert-and-Multiply	Division of Whole Number Components	No Model	N
Whole Number Division: Quotative Only						
Blocked by Number Type	3 (33.3%)	4 (44.4%)	2 (22.2%)	0 (0.0%)	0 (0.0%)	9
Blocked by Operation	10 (76.9%)	2 (15.4%)	1 (7.7%)	0 (0.0%)	0 (0.0%)	13
Whole Number: Mixed Quotative & Partitive						
Blocked by Number Type	1 (25.0%)	0 (0.0%)	3 (75.0%)	0 (0.0%)	0 (0.0%)	4
Blocked by Operation	2 (66.7%)	0 (0.0%)	0 (0.0%)	0 (0.0%)	1 (33.3%)	3
Whole Number: Partitive Only						
Blocked by Number Type	3 (27.3%)	4 (36.4%)	3 (27.3%)	0 (0.0%)	1 (9.1%)	11
Blocked by Operation	3 (37.5%)	2 (25.0%)	2 (25.0%)	0 (0.0%)	1 (12.5%)	8

Note. Percentage of participants given in parentheses.

[a]One child used quotative division on two fraction division trials and the hybrid strategy on the other two trials. This child's dominant strategy was coded as quotative division.

Such strategies representing fraction multiplication, fraction subtraction, or a mix of incorrect operations with the given divisor were coded as *non-division operation errors*. Among children whose problems were blocked by number type, 10 of 24 children (41.7%) made non-division operation errors on at least one trial. Among children whose problems were blocked by operation, seven of 24 children (29.2%) made non-division operation errors on at least one trial. However, this difference was not statistically significant, $\chi^2(1, N = 48) = 0.82$, $p = .37$, $OR = 1.73$.

We also considered the frequency with which children used non-division operations as a *dominant* strategy (see Table 3). Children in the *number type* condition did not use other operations as their dominant strategy on fraction division problems (8 of 24, 33.3%) at a greater rate than those in the operation condition (4 of 24, 16.7%), $\chi^2(1, N = 48) = 1.78$, $p = .18$, $OR = 2.50$. In brief, it was the case that a greater proportion of the children who demonstrated all the fraction problems back-to-back made non-division operation errors on fraction division trials than those who demonstrated problems with the same operation back-to-back, but these differences were not statistically reliable.

Strategy Switching

Finally, we examined the frequency with which children switched from one strategy to another mid-trial on any of the fraction division trials. Given that students in the United States tend to prefer quotative models of division for fraction division problems, we expected that children who had used only partitive strategies on whole number division might have more difficulty modeling fraction division problems, and that this difficulty might manifest in unsuccessful attempts to model fraction division.

The rate of strategy switching was low in the sample as a whole; only five children were coded as using more than one strategy in the course of at least one fraction division trial. However, all five of these children had used only partitive strategies for whole number division. Thus, as expected, children who *never* used a quotative strategy on whole number division were more likely to have difficulty representing fraction division, as manifested in overt strategy switches (5 of 19), than those who had used a quotative strategy on whole number division at least once (0 of 29), Fisher's exact probability test, $N = 48$, $p < .01$, OR is undefined.

Discussion

Children readily apply their prior knowledge to make sense of problem-solving challenges, and this spontaneous transfer can bias them in either positive or negative ways. In the case of children's fraction learning, there are many ways in which children's prior knowledge of whole number arithmetic negatively biases fraction reasoning (e.g., Ni & Zhou, 2005). However, children's whole number concepts also have the potential to support reasoning about fraction operations (e.g., Siegler et al., 2011; Vukovic et al., 2014). In this study, we applied an analogical transfer framework to predict which aspects of prior knowledge would best support conceptual understanding of fraction operations. Since whole number operations share the same conceptual structure as the corresponding fraction operations (e.g., whole number division and fraction division), we hypothesized that children's knowledge of whole number operations would support their understanding of analogous fraction operations. In this study, we focused on fraction division. Thus, we construed whole number division as structurally similar to fraction division, and other kinds of fraction operations, namely fraction multiplication, as perceptually but not structurally similar to fraction division.

We found that children who demonstrated whole number division immediately before modeling fraction division were significantly more successful at modeling fraction division than those who had demonstrated fraction multiplication and other operations on fractions immediately before modeling fraction division. This finding is in line with other research that shows the beneficial effect of analogies between whole number division and fraction division (Sidney & Alibali, 2015; Richland & Hansen, 2013). Importantly, all children modeled all of the same problems before modeling fraction division—it was only the timing that differed. Thus, the *recency* of activation of children's whole number division knowledge was critical for supporting children's fraction division concepts.

Facilitation or Interference?

Our study leaves open the question of the specific mechanism underlying this recency effect. One possibility is that recency is beneficial in this context because the recently activated knowledge facilitates performance, but it does not remain activated for very long. Another possibility is that recency is beneficial because activation is subject to interference from intervening content. To better understand the mechanism underlying the recency effect, we examined the nature of children's strategies in each condition.

Among children who used the quotative strategy for whole number division, those who demonstrated fraction division immediately after whole number division were more likely to use quotative strategies on fraction division than children who demonstrated fraction division after other operations on fractions. This finding suggests that the increased success of children in the *operation* condition was likely due to spontaneous analogical transfer of the quotative strategy. This finding lends support to the idea that whole number division facilitates modeling of fraction division, rather than the idea that other operations on fractions interfere with fraction division.

We also examined children's incorrect strategies in each condition for evidence of negative transfer from other operations on fractions. Many children whose knowledge of fraction multiplication (and other fraction operations) was recently activated made errors reflecting negative transfer from other operations on fractions. However, several children whose knowledge of whole number division was recently activated also made such errors. Thus, we did not find a significant difference across conditions in non-division errors.

One reason why children whose knowledge of whole number division was recently activated may have made other operation errors at similar rates to those children whose fraction multiplication knowledge was recently activated may be that such errors reflect robust misconceptions about fraction division. Other studies have revealed that other-operations errors are quite common on fraction division problems (e.g., Ma, 1999; Siegler & Pyke, 2013), and some students may have strong misconceptions about fraction division that prevent them from approaching it in new ways. For these students, activating their conceptually related prior knowledge may not be sufficiently helpful to overcome this misconception. It remains an open question whether students with highly entrenched misconceptions learn from analogies, either implicit or explicit, or whether analogies are only beneficial for students with fewer or less entrenched misconceptions.

We did not include a condition in which children modeled fraction division without any targeted activation of their prior knowledge. Thus, without knowing what strategies are typical in this task, absent any experimental manipulation, we are unable to conclude whether such interference is truly occurring. Future research is needed to establish whether implicit analogies can lead to *negative* transfer in school-aged children, and whether children's knowledge of other operations on fractions interferes with their fraction division understanding in a lasting way.

In sum, our strategy data suggest that *recent* activation of children's quotative whole number division models increases the likelihood that they successfully apply the quotative model to fraction division. However, our data do not definitively rule out the possibility that recent activation of other operations on fractions also interferes with modeling fraction division.

Whole Number Biases

In many cases, children's reasoning about fractions appears to be hindered by their strongly activated whole number knowledge. Children's, and even adults', reliance on whole number knowledge when reasoning about fractions often appears automatic and difficult to inhibit (e.g., DeWolf & Vosniadou, 2015; Meert et al., 2010; Vamvakoussi, Van Dooren, & Verschaffel, 2012), particularly when their representations of fraction magnitudes are less precise or less robust (Alibali & Sidney, 2015). This phenomenon may lead some educators to emphasize differences between whole number concepts and fraction concepts, in order to foster a coherent understanding of fraction arithmetic that does not suffer from the negative effects of the whole number bias. However, much of what children know about difficult mathematics concepts in the context of whole numbers, such as the structure of the division relationship and the notion of infinite divisibility, can extend to fractions.

Though many previous studies have found evidence for detrimental effects of the whole number bias, we found an advantage for activating children's knowledge of a specific whole number concept, whole number division, to support their reasoning about an analogous fraction concept, fraction division. These results fit well with the integrated theory of numerical development, posited by Siegler and colleagues (2011), and are consistent with longitudinal studies of children's whole number and fraction understanding (e.g., Bailey, Siegler, & Geary, 2014; Vukovic et al., 2014). The current study suggests that children's whole number operation knowledge can be directly leveraged to support their understanding of fraction operations, and fraction division in particular.

Given the mixed findings on the benefits of drawing on children's whole number knowledge for understanding fractions, the specificity of the instructional analogy is presumably critical. Had we activated children's whole number division knowledge more generally, children might have drawn on other aspects of that knowledge that do not apply to fraction division, such as the idea that "*division makes smaller*" (e.g., Fischbein et al., 1985). This general, unguided transfer from prior knowledge may be the cause of the detrimental whole number bias that often occurs in reasoning about fractions. In this study, *general* transfer was precluded by activating *specific* whole number operation concepts immediately before analogous fraction operations in the operation condition. This repeated juxtaposition may have supported a focus on conceptual structure, leading children to use this similarity as the basis for transfer. Our study extends previous work demonstrating that children's whole number arithmetic knowledge can support fraction reasoning (Richland & Hansen, 2013; Sidney & Alibali, 2015), as long as analogies are made across specific structural similarities or shared concepts.

Furthermore, studies that show a negative whole number bias on children's operation understanding (e.g., Van Hoof, Vandewalle, et al., 2015) tend to use different tasks than studies that show positive transfer from children's whole number knowledge, such as this one. The current study examined analogies between children's conceptual understanding of whole number and fraction operations, rather than understanding of symbolic equations or problem-solving procedures. Much of the evidence for the whole number bias in children's understanding of fraction operations occurs in the context of children's understanding of symbolic equations. For example, Van Hoof, Vandewalle, Verschaffel, and Van Dooren (2015) found that when evaluating whether algebraic expressions (e.g., $x/4 < x$; $x/4 > x$) *can be true* or *can not be true*, 8th, 10th, and 12th grade students were likely to use natural whole number reasoning to evaluate these statements, especially when these statements included division operations. These students were likely to report that division *always* results in a smaller number. In contrast, studies that demonstrate a benefit for drawing on students' knowledge of whole number operations have focused on the conceptual structure of those operations (e.g., Richland & Hansen, 2013; Sidney & Alibali, 2015).

It remains unclear the extent to which children's whole number division knowledge supports children's *symbolic* fraction division understanding. The procedures used for whole number division (e.g., long division) are quite different from those used for fraction division (e.g., invert-and-multiply), suggesting that activating children's procedural understanding of whole number division would not necessarily benefit their procedural understanding of fraction division. However, Sidney and Alibali (2015) demonstrated that children whose conceptual *and* procedural whole number division knowledge was activated before a fraction division lesson showed improvements, not only in conceptual understanding, but also in their ability to transfer the newly-learned, symbolic invert-and-multiply procedure to solve more difficult fraction division problems. One open question is whether supporting children's fraction division understanding as we have in the current study (i.e., in a non-symbolic modeling task) would similarly prepare students to learn about procedures for fraction division.

Implicit Analogies in Instruction

In studies of analogies in mathematics instruction, analogies involve explicitly supporting children's comparisons across structurally similar problems (e.g., Richland & Hansen, 2013; Thompson & Opfer, 2010). However, instructional analogies can vary in the extent to which the mappings between corresponding elements across problems are highlighted. Richland and colleagues (2007) found that while explicit analogies were commonly used during mathematics instruction in several countries, American teachers provided much less support for processing and understanding those analogies than did teachers in other high-achieving countries. These explicit, but less-scaffolded, analogies in mathematics do not promote student learning as much as explicit analogies that are highly scaffolded (Richland & Hansen, 2013; Thompson & Opfer, 2010).

In contrast, several studies show that adults need not notice an analogy in order for it to affect their understanding of a target concept (e.g., Bassok et al., 2008; Day & Goldstone, 2011; Schunn & Dunbar, 1996), and our work shows that this is true of children's analogical reasoning in mathematics, as well. In the current study, the analogies between whole number problems and fraction problems with the same operation were never explicitly cued by the experimenter; they were cued only implicitly by temporal juxtaposition. The findings reveal that children may benefit from implicit analogies when engaging in difficult tasks such as representing fraction division. In short, what children are thinking about when they approach new problems matters, even when they are not explicitly attempting to apply prior knowledge.

Further work is needed to better understand the mechanisms by which implicit analogies support children's thinking, how long the advantages conferred by implicit analogies last, and how implicit analogies might be capitalized on in instruction. Given previous studies demonstrating that more cognitive support for explicit analogies leads to better learning, future studies are needed to directly compare implicit analogies to explicit, highly-cued analogies. It may be the case that implicit analogies and explicit, highly-cued analogies have different effects on learning, or that highly-cued analogies have longer-lasting effects on learning.

It is also unclear whether implicit analogies may be more useful for some types of analogical transfer than for others. In the current study, we considered analogies from earlier-learned to later-learned problems, and this type of analogy could be construed as *vertical* transfer from lower-order problems to more complex problems of a similar nature (Gagne, 1965). Spontaneous transfer across implicit analogies may be more frequent in this context than in the context of *horizontal* transfer across domains (e.g., from science relationships to mathematics relationships).

Implications for Educational Practice

We have argued that children's recently activated knowledge sets the stage for their understanding of what comes next. This general claim has obvious relevance for the organization of classroom instruction. Teachers often begin lessons by reviewing recently learned concepts or posing "warm-up" questions, and it may be useful to think of these activities as implicit analogies. These analogies will support new learning to the extent that the concepts or procedures activated by the analogies are relevant to the new concepts or procedures that follow. Thus, there are likely to be advantages to identifying structurally similar analogues from students' own prior knowledge before beginning a new lesson, and practicing the most relevant previously-mastered concept to "warm up".

For fraction learning specifically, instruction on fraction division often follows instruction on other fraction operations in classroom lessons. However, activating students' knowledge of whole number division may be a better way to start such lessons. This implication is particularly important given the difficulties that children have with fraction division (e.g., Sidney & Alibali, 2015; Siegler & Pyke, 2013) that persist into adulthood (e.g., Ma, 1999). More generally, the conceptual structure of each operation (not only division) is shared across number types; thus, our findings suggest that students' understanding of operations with fractions may be enhanced by making analogies to the corresponding operations on whole numbers. Given close-to-ceiling performance on fraction addition, subtraction, and multiplication in our sample, we were unable to demonstrate transfer for all operations here; however, this hypothesis could be tested with younger children who have yet to learn about these fraction operations, or with children who struggle with these operations. We further predict that students' understanding of operations with negative numbers may also be enhanced by making analogies to the corresponding operations on natural numbers; this conjecture also remains to be tested.

Finally, our strategy data suggest an unexpected avenue for future research that has potential to influence educational practice: examining the role of the *remainder* in children's fraction and division reasoning. Although we did not focus on children's understanding of remainders in this study, we observed that many children ignored remainders, perhaps indicating uncertainty about what the remainder represents, and many children reported remainders as fractions of a whole, rather than as a fraction of the unit specified by the divisor. Children's learning of rational number concepts should support their reasoning about remainders as fractions of a unit specified by the divisor. However, children may have difficulty representing fractions of units other than one (e.g., if 3/4 is one unit, then 2/4 is 2/3 of a unit). Future research addressing the relationship between children's understanding of remainders in division and children's understanding of fractions could shed further light on how children reason about fractions of units other than one.

Conclusion

Though children's whole number concepts sometimes negatively bias their understanding of fraction concepts, we believe that it is also important to examine the conditions under which children's whole number knowledge supports fraction learning. In this research, we found that activating children's most relevant, structurally similar whole number concept (i.e., whole number division) immediately before asking them to model an analogous fraction concept (i.e., fraction division) increased children's likelihood of demonstrating the fraction concept correctly. Thus, analogies made across number domains that are often thought to cause interference (such as whole numbers and fractions) can be fruitful, as long as targeted links—even implicit ones—are made across specific, analogous concepts.

The analogy between whole number division and fraction division was not made explicit in this study. Instead, the temporal proximity of analogous problems implicitly cued the analogy. These findings add to a growing body of work on the importance of implicit cues in problem solving and learning, and they document that analogical transfer can be cued implicitly as well as explicitly. Moreover, they show that implicit cuing of analogy occurs among children and in mathematics tasks.

Our findings underscore the importance of sequencing in mathematics curricula, and they demonstrate that order matters, even at the level of individual problem types. What children are thinking about immediately before solving difficult problems shapes their problem-solving approaches and affects their likelihood of solving those problems correctly. By applying an analogical transfer perspective to the task of creating contexts for learning, we can gain insight into effective ways to actively draw on students' prior knowledge for new learning.

Supplementary Materials

Coded data and analyses. doi:10.17605/OSF.IO/KEVMR

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Competing Interests

The authors have declared that no competing interests exist.

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Appendices

Appendix A

Problems by Number Type and Operation

Operation	Whole Numbers	Fractions
Addition	5 + 17	$6 + \frac{1}{2}$
	23 + 8	$9 + \frac{3}{4}$
	7 + 11	$4 + \frac{2}{5}$
	19 + 4	$3 + \frac{1}{6}$
Subtraction	20 – 8	$5 - \frac{2}{3}$
	14 – 3	$2 - \frac{4}{5}$
	22 – 15	$6 - \frac{1}{4}$
	13 – 9	$7 - \frac{1}{2}$
Multiplication	7 × 10	$3 \times \frac{5}{6}$
	9 × 8	$5 \times \frac{1}{3}$
	11 × 5	$4 \times \frac{1}{5}$
	3 × 12	$6 \times \frac{2}{3}$
Division	15 ÷ 5	$5 \div \frac{1}{4}$
	32 ÷ 8	$7 \div \frac{3}{5}$
	18 ÷ 4	$4 \div \frac{1}{6}$
	17 ÷ 6	$8 \div \frac{3}{4}$

Appendix B

Verbal Instructions

Note: Scripted gesture are embedded in the text [*in brackets*].

Today, I'm going to ask you to show me what a bunch of math sentences like this one [*point to first equation*] mean by using these blocks on this table. For each expression, you should show what it means to do, for example, 5 + 17.

This is a study about what kids think about math. So, to know what you're thinking, I'm going to ask you to talk out loud the whole time. What I mean by talk out loud is that I want you to say out loud everything that you say to yourself silently. Don't worry about whether or not your thoughts make sense to anyone else, just say whatever you're thinking.

Before we get started, I'm going to tell you a bit about these blocks. When you use these blocks, consider something that's this big [*take out one pink block*] equals one. So one pink block equals one, and each of these stacks is equal to one [*take out one of each other color*], because they are the same amount as one pink block. But, you can see that by using different types of blocks, you can split wholes into different fractions. So, because two yellow blocks equal one, [*split the yellow block* *into halves*] each is one half [*put the yellow bar back together*]. Each of these individual blocks equals one third [*point to it*], these are fourths [*point to it*], these are fifths [*point to it*], and these are sixths [*point to it*]. Any questions about the blocks?

For each of these equations, start by showing the first number and then show what it looks like to do the operation using the second number. For example, if I want to model 5 + 17, I should start by showing 5 [*model 5 using differently colored blocks*]. Then I'd show what it looks like to *add* 17 [*point to +17 in the expression*]. Does that make sense?

[*If participant doesn't understand, repeat relevant directions.*]

Great, one last thing. Try not to solve the problem in your mind, but instead show how you can use the blocks to solve the problem. Some problems will be easy, so you will want to solve it in your mind, but please try to show it with the blocks instead. Some problems will be harder. Even if you're not sure how to show what the problem means, please try anyways, because even your mistakes can tell us about how kids like you think about math.

Responding to Participant Questions

If participant asks, "How do I do it?" or "Can I do ____?"

Experimenter: "Demonstrate whatever you think the operation means. Make sure you use the exact quantities in the equation and just demonstrate how to do the problem."

If participant asks, "What if I don't know the answer?" or "I don't know the answer."

Experimenter: "That's okay, we're not interested in whether or not you know the right answer. We just want to know what you think."

During the Experiment

If participant only models each number in the equation:

Experimenter: Can you show me what it looks like to (operation) (number) & (number)?

[*For example: Can you show me what it looks like to multiply 7 by 11?*, *with emphasis on the operation*.]

If participant's action is unclear:

Experimenter: Can you show me which part is the answer?

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