CREDIBILITY PREMIUM PLANS WHICH MAKE ALLOWANCE

FOR BONUS HUNGER

by

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1. Introduction

1 A. For the purpose of fair premium calculation, the insurer will partition his total mass of risks into risk groups according to directly observable risk factors such as type and use of the vehicle in motor insurance. The risks within each risk group will be relatively homogeneous as compared to the variations within the total risk mass, but a risk group will rarely be entirely homogeneous.

Usually, there will be additional unobservable risk factors, such as the temperament and skill of the driver in motor insurance, and these additional factors give rise to accident proneness differentials within the risk group. To the insurer, these differentials will appear only through the claims experience of each risk.

In risk theory, the accident proneness of an individual risk is represented by a risk parameter θ . Within a risk group, each θ is regarded as a realisation of a random variable Θ , whose distribution function $U(\bullet)$ represents the risk structure of the group.

We assume that the premiums are paid at the beginning of each insurance period. By the equivalence principle, the premium for period n is set equal to the pure premium, which is the expected value of the total claim amount X_n of the risk in period n , with an additional loading for security and administration. The probability distribution of X_n depends on the information available

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about the risk. Thus, the pure premium for period n is really the conditional expectation $E(X_n | Z_{n-1})$, where Z_{n-1} represents all the information that the insurer has concerning the risk at the beginning of period n , viz., the claims experience of the risk during the n-1 preceding periods. A premium system which takes account of the claims experience of the risk, is called an (individual) experience rating system. Experience rating is treated by Bühlmann (1970, pp. 85-110) and by Seal (1969, pp. 63-87).

From a practical point of view it is important that experience rating (presumably) stimulates the insured to take loss-preventing measures, by punishing those who have an unfavourable claims experience through a premium increase.

1 B. The previous remark presupposes that the policy holders adapt themselves more or less consciously to the conditions induced by the premium system. This paper treats another manner in which an experience rating plan may influence the risk behaviour of the policy holders, viz., through their bonus hunger. This works as follows.

The insured has the right to claim indemnity for all losses covered by the policy conditions. However, if he finds that a loss is smaller than the future increases of premiums that would follow from a claim, he will probably prefer to defray the expenses (if he can afford it) and say nothing to the insurer. This phenomenon is well known in connection with the bonus systems in motor insurance, which explains why it is called bonus hunger.

Earlier papers by Grenander (1957) and Straub (1968 and 1969) have treated bonus hunger from a game-theoretical point of view. Their interest is focused on the balance of the insurance company,

while less attention is paid to the optimum properties of the resulting experience rating system. It is the purpose of the present paper to work out premium systems which take account of bonus hunger and in some sense is optimal from the experience rating point of view. On the basis of a simple model, we argue that the pure premium $E(X_n|Z_{n-1})$ will depend only on the claim numbers $K_1, \ldots, K_{n-1}$ of the n-1 first periods. Restricting ourselves to linear premium formulas, we can then write the premium for period n in the form of

$$\sum_{j=1}^{n-1} b_{n,j} K_j + a_n .$$

We will call such linear experience rating formulas <u>credibility</u> <u>formulas</u>, in accordance with Bühlmann (1970, pp. 100-102). In this class of linear premium formulas we consider as optimal the one which approximates X_n best, in the sense that it minimizes

$$Q_n = E(X_n - \sum_{j=1}^{n-1} b_{n,j} K_j - a_n)^2 .$$

Following Straub (1968), we will assume that the bonus hunger strategy of the policy holder is to claim indemnity for a loss in period j if the loss amount exceeds a <u>barrier value</u> s_j . In Straub's game-theoretic approach, the choice of s_j-values represents the strategy of the policy holders, while the premium system represents the strategy of the insurer. This setting of the problem seems relevant when the choice of s_j-values is entirely unknown to the insurer. It is the opinion of this author, however, that the insurer will frequently have extensive knowledge of the strategy of the policy holders. Suppose first, as in Straub (1968) that all policy holders choose the same set of barriers s_j , independently

of the premium system. Then the insurer can estimate the s_j-values
from the reported claims, and eventually they will be known. We
consider this case, and we find the optimal set of coefficients
$b_{n,j}$ and a_n (which then depend on the barriers s_j) by minimiz-
ing Q_n. Suppose next that the policy holders base their choice
of barriers on some rational (e.g., economical) deliberations. In
particular, the choice may rest heavily on the premium system. Then
the s_j-values are functions of the coefficients $b_{n,j}$ and a_n,
and the question arises whether these coefficients may be chosen
such as to coincide with the ones which make Q a minimum. This
would be the situation when the policy holder reports a claim when-
ever the loss amount exceeds the present value of all future in-
creases in premiums caused by the claim. We study this special case
in some detail, and we find the premiums in a numerical example.
We also mention some other possible bonus hunger strategies. The
final section of the paper contains a discussion of the usefulness
of our new results and includes a critical examination of the valid-
ity of the mathematical model.

2. A model for the risk process.

2 A. For an individual risk, let M_n be the number of losses
in the n-th time period and let $Y_{n1}, \ldots, Y_{nM_n}$ be the corresponding
loss amounts (when $M_n > 0$). We take the loss amounts to be mutu-
ally independent outcomes of a random variate Y with distribution
function G, and to be independent of $(\Theta, M_1, M_2, \ldots)$. We assume
that $G(0) = 0$, which means that only real losses are considered.
The loss numbers $M_1, M_2, \ldots$ will be taken as mutually independent
outcomes of a random variate M with expectation $E_\Theta M$ and variance

$\text{var}_\theta M$, conditional on $\Theta = \theta$. Applied to automobile insurance, these assumptions mean that we assume individual differences in driving skills to be of importance for the loss incidence but not for the severity of the losses, and also that skills do not change with age and driving experience. More realistic assumptions will be discussed in Section 6.

3. A model for the strategy of the policy holder.

3 A. We will assume that a policy holder will claim indemnity for a loss in period n if and only if the loss amount exceeds a lower barrier s_n . We shall say that he follows the barrier strategy $(s_1, s_2, \ldots)$. Note, incidentally, that a barrier strategy acts in the same way as a system of indemnities with a variable minimum deductable.

During the n-th period, the risk considered produces the random vector $(M_n, Y_{no}, \ldots, Y_{nM_n})$, where for convenience we introduce $Y_{no} \equiv 0$. When there is bonus hunger, the insurer does not observe this vector, however, but rather some other vector $(K_n, Y'_{no}, \ldots, Y'_{nK_n})$, where K_n is the number of claims entered by the policy holder in the n-th period, and where Y'_{nj} is the j-th claim amount in this period for $j \geq 1$, while $Y'_{no} \equiv 0$. Obviously, $0 \leq K_n \leq M_n$, and when $K_n > 0$, $Y'_{n1}, \ldots, Y'_{nK_n}$ are the K_n largest of the M_n loss amounts $Y_{n1}, \ldots, Y_{nM_n}$. Our assumptions imply that the claim amounts Y'_{nj} $(n, j \geq 1)$ are mutually independent and have the same distribution function, namely

$$\{G(y) - G(s_n)\} / \overline{G}(s_n) \quad \text{for } y \geq s_n ,$$

where

$$\overline{G}(y) = 1 - G(y)$$

is the probability that a loss exceeds y .

We introduce

$$\psi(s) = \int\limits_{s}^{\infty} y\,dG(y) \ ,$$

and can then write

$$EY'_{nj} = \psi(s_n)/\overline{G}(s_n) \quad \text{for} \quad n,j \geq 1 \ . \tag{1}$$

The claim amounts are independent of $(\Theta, K_1, K_2, \ldots)$. Given that $M_n = m$, K_n is binomially distributed $(m, \overline{G}(s_n))$, irrespective of the value of Θ . It follows that

$$E_\theta K_n = E_\theta E_\theta(K_n | M_n) = E_\theta \{M_n \overline{G}(s_n)\} = \overline{G}(s_n) E_\theta M \ , \tag{2}$$

and, by integrating the expressions in (2) with respect to the distribution of Θ , we get

$$EK_n = \overline{G}(s_n)EM \ . \tag{3}$$

4. Construction of a linear credibility formula for a fixed barrier strategy.

4 A. At the beginning of period n , the insurer's experience with the risk considered consists of the vector

$$Z_{n-1} = (K_1, K_2, \ldots, K_{n-1}, Y'_{10}, \ldots, Y'_{1K_1}, Y'_{20}, \ldots, Y'_{n-1, K_{n-1}}) \ ,$$

(with the obvious interpretation that Z_0 is empty).
The total claim amount in period n is

$$X_n = \sum_{j=0}^{K_n} Y'_{nj} \ ,$$

and the pure premium for period n , based on the experience Z_{n-1} , is

$$E(\sum_{j=0}^{K_n} Y'_{nj} \,|Z_{n-1}) = E\{ E(\sum_{j=0}^{K_n} Y'_{nj} \,|Z_{n-1}, K_n) \,|Z_{n-1}\}$$

$$= E\{ K_n \circ \psi(s_n)/\bar{G}(s_n) \,|Z_{n-1}\}$$

$$= E(K_n |K_1, \ldots, K_{n-1}, Y'_{10}, \ldots, Y'_{n-1}, K_{n-1}) \circ \psi(s_n)/\bar{G}(s_n)$$

$$= E(K_n |K_1, \ldots, K_{n-1}) \circ \psi(s_n)/\bar{G}(s_n) \ .$$

The second equality follows from (1), and the fourth from the independence between the claim amounts and the claim numbers. We see that the pure premium depends on Z_{n-1} only through the claim numbers. The corresponding linear credibility formula should therefore be written in the form of

$$\sum_{j=1}^{n-1} b_{n,j} K_j + a_n \ , \qquad n = 1,2,\ldots \ . \qquad (4)$$

We adopt the convention $\sum\limits_{j=1}^{o} x_j = 0$. (When the barriers s_n depend on n , the claim numbers $K_1, K_2, \ldots$ are not identically distributed. It then seems hard to justify a credibility formula which depends on the claim numbers $K_1, \ldots, K_{n-1}$ only through their mean, as is usual in "classical" credibility theory.)

The expected squared deviation between the total claim amount of period n on the one hand and/the premium paid by the policy holder
the
for this period on the other hand is

$$Q_n = E(X_n - \sum_{j=1}^{n-1} b_{n,j} K_j - a_n)^2 \ . \qquad (5)$$

In the class of linear premium formulas we choose the one which
minimizes Q_n .

(To follow Bühlmann's (1970) now classical approach to credibility
theory would mean to define the credibility premium as
$E(E_\Theta X_n | Z_{n-1})$. The credibility formula could then be defined as
the linear formula of the form (4) which minimizes
$E\{E_\Theta(X_n | Z_{n-1}) - \sum\limits_{j=1}^{n-1} b_{n,j} K_j - a_n\}^2$. It is easy to show that this is
equivalent to minimizing Q_n defined in (5) under the assumptions of
the model of Section 2.)

Theorem. Q_n attains its absolute minimum at the point
$(\tilde{b}_{n,1}, \ldots, \tilde{b}_{n,n-1}, \tilde{a}_n)$ given for $n = 1, 2, \ldots$ by

$$\tilde{a}_n = \psi(s_n) \frac{k \circ EM}{\sum\limits_{i=1}^{n-1} \Phi(s_i) \overline{G}(s_i) + k} \tag{6}$$

and

$$\tilde{b}_{n,j} = \psi(s_n) \frac{\Phi(s_j)}{\sum\limits_{i=1}^{n-1} \Phi(s_i) \overline{G}(s_i) + k} \quad \text{for} \quad j = 1, \ldots, n-1 . \tag{7}$$

Here Φ and k are defined by

$$\Phi(s) = \{1 + (EM/E \, var_\Theta M - 1) G(s)\}^{-1} \tag{8}$$

and

$$k = E \, var_\Theta M / var \, E_\Theta M . \tag{9}$$

Proof: For any set of fixed coefficients $b_{n,1}, \ldots, b_{n,n-1}$, Q_n is minimized by the choice

$$a_n = E(X_n - \sum_{j=1}^{n-1} b_{n,j} K_j) \ .$$

We have

$$EX_n = EE(\sum_{j=0}^{K_n} Y'_{n,j} | K_n) = E\{K_n \circ \psi(s_n)/\overline{G}(s_n)\} = EM \cdot \psi(s_n) \ , \tag{10}$$

by (1) and (3). Hence

$$a_n = \{\psi(s_n) - \sum_{j=1}^{n-1} b_{n,j} \overline{G}(s_j)\} \cdot EM \ . \tag{11}$$

With this choice of a_n, Q_n becomes

$$Q_n^* = \text{var}(X_n - \sum_{j=1}^{n-1} b_{n,j} K_j) = \text{var} \, E_\Theta(X_n - \sum_{j=1}^{n-1} b_{n,j} K_j) + E\text{var}_\Theta(X_n - \sum_{j=1}^{n-1} b_{n,j} K_j) \ .$$

Relation (10) will still be valid if we replace E by E_θ there. We use this to rewrite the first term above. We similarly use the conditional mutual independence of $(X_n, K_1, \ldots, K_{n-1})$ to rewrite the second term, and get

$$Q_n^* = \text{var}\{E_\Theta M \cdot \psi(s_n) - \sum_{j=1}^{n-1} b_{n,j} E_\Theta K_j\} + E\text{var}_\Theta X_n + \sum_{j=1}^{n-1} b_{n,j}^2 E\text{var}_\Theta K_j \ . \tag{12}$$

Furthermore,

$$\text{var}_\theta K_j = \text{var}_\theta E_\theta(K_j | M_j) + E_\theta \text{var}_\theta(K_j | M_j)$$

$$= \text{var}_\theta\{M_j \overline{G}(s_j)\} + E_\theta[M_j \overline{G}(s_j)\{1 - \overline{G}(s_j)\}]$$

$$= \overline{G}^2(s_j)\mathrm{var}_\theta M + \overline{G}(s_j)G(s_j)E_\theta M \ ,$$

and hence

$$E \ \mathrm{var}_\theta K_j = \overline{G}^2(s_j)E \ \mathrm{var}_\theta M + \overline{G}(s_j)G(s_j)EM$$

$$= E \ \mathrm{var}_\theta M \cdot \overline{G}(s_j)/\Phi(s_j) \ , \tag{13}$$

where $\Phi(s)$ is defined by (8). Substituting (2) and (13) in (12), we finally get

$$Q_n^* = \mathrm{var}[\{\psi(s_n) - \sum_{j=1}^{n-1} b_{n,j}\overline{G}(s_j)\} E_\theta M]$$

$$+ E \ \mathrm{var}_\theta X_n + \sum_{j=1}^{n-1} b_{n,j}^2 E \ \mathrm{var}_\theta M \cdot \overline{G}(s_j)/\Phi(s_j)$$

$$= \mathrm{var} \ E_\theta M \cdot \{\psi(s_n) - \sum_{j=1}^{n-1} b_{n,j}\overline{G}(s_j)\}^2$$

$$+ E \ \mathrm{var}_\theta M \sum_{j=1}^{n-1} b_{n,j}^2 \overline{G}(s_j)/\Phi(s_j) + E \ \mathrm{var}_\theta X_n \tag{14}$$

Since $\Phi(s)$ is positive, Q_n^* is a positive definite quadratic form in $(b_{n,1}, \ldots, b_{n,n-1})$. It therefore attains its absolute minimum in the unique point $(\tilde{b}_{n,1}, \ldots, \tilde{b}_{n,n-1})$ which satisfies the first order conditions of an extremum, viz.,

$$\frac{\partial Q_n^*}{\partial b_{n,j}} \Big|_{b_{n,i} = \tilde{b}_{n,i}; i=1,\ldots,n-1} = 0 \quad \text{for} \quad j = 1,\ldots,n-1 \ . \tag{15}$$

From (14) we find that the derivatives of Q_n^* are

$$\frac{\delta Q_n^*}{\delta b_{n,j}} = \text{var } E_\Theta M \cdot 2\{\psi(s_n) - \sum_{i=1}^{n-1} b_{n,i}\overline{G}(s_i)\}\{-\overline{G}(s_j)\}$$

$$+ E \text{ var}_\Theta M \cdot 2b_{n,j} \frac{\overline{G}(s_j)}{\overline{\Phi}(s_j)} \ ,$$

and thus (15) is equivalent to

$$k \cdot \tilde{b}_{n,j} = \Phi(s_j)\{\psi(s_n) - \sum_{i=1}^{n-1} \tilde{b}_{n,i}\overline{G}(s_i)\} \quad \text{for} \quad j = 1,\ldots,n-1 \ . \quad (16)$$

(Remember that $k = E \text{ var}_\Theta M/\text{var } E_\Theta M$.) Multiplication with $\overline{G}(s_j)$ on both sides of (16) and summation over all j gives the equation

$$k \sum_{j=1}^{n-1} \tilde{b}_{n,j}\overline{G}(s_j) = \{\sum_{j=1}^{n-1} \Phi(s_j)\overline{G}(s_j)\}\{\psi(s_n) - \sum_{i=1}^{n-1} \tilde{b}_{n,i}\overline{G}(s_i)\} \ ,$$

which is equivalent to

$$\sum_{i=1}^{n-1} \tilde{b}_{n,i}\overline{G}(s_i) = \frac{\psi(s_n) \sum_{i=1}^{n-1} \Phi(s_i)\overline{G}(s_i)}{\sum_{i=1}^{n-1} \Phi(s_i)\overline{G}(s_i) + k} \ .$$

Combining this with (16), we obtain formula (7). Formula (6) then follows if we replace $b_{n,j}$ in (11) with $\tilde{b}_{n,j}$ from (7) . $\quad\square$

4 B. Setting $a_n = \tilde{a}_n$ and $b_{n,j} = \tilde{b}_{n,j}$ in (4) we finally find that the optimal linear credibility premium in period n for a policy holder who follows the barrier strategy $(s_1,s_2,\ldots)$ is

$$\tilde{\mu}_n(K_1,\ldots,K_{n-1}) = \frac{\psi(s_n)}{\sum\limits_{j=1}^{n-1} \Phi(s_j)\bar{G}(s_j)+k} \{ \sum\limits_{j=1}^{n-1} \Phi(s_j)K_j+kEM\} . \qquad (17)$$

Straub (1968 and 1969) $\overset{\text{suggests}}{/}$ this formula in the special case when the individual risk process for a given value of Θ is compound poisson.

It is noteworthy that the expected premium per policy in period n by formula (17) is equal to

$$\frac{\psi(s_n)}{\sum\limits_{j=1}^{n-1} \Phi(s_j)\bar{G}(s_j)+k} \{ \sum\limits_{j=1}^{n-1} \Phi(s_j)\bar{G}(s_j)EM+kEM\} = \psi(s_n)EM$$

$$= \int_{s_n}^{\infty} y \frac{dG(y)}{\bar{G}(s_n)} \cdot \bar{G}(s_n)EM = E(Y|Y > s_n)\cdot EK_n ,$$

which exactly is the expected indemnification per policy in the same period. The premium system defined by (17) thus ensures that premium incomes and indemnity payments balance "on the average".

5. Application of the constructed credibility formula to some bonus hunger strategies.

5 A. The simplest possible situation is the one where the policy holder fixes the barriers $(s_1,s_2,\ldots)$ independently of the premium system, and where these barriers are known by the insurer. An optimal premium system is then simply given by formula (17) alone. In the special case when the policy holder shows no bonus hunger, all s_j are equal to 0 , and the premium in (17) then becomes

$$EY\{\frac{n-1}{n-1+k} \bar{K}_{n-1} + \frac{k}{n-1+k} EM\} \ , \qquad (18)$$

where $\bar{K}_{n-1} = \frac{1}{n-1} \sum_{j=1}^{n-1} K_j$. This is the well known credibility premium/from the now classical theory which disregards bonus hunger. (See, e.g. Bühlmann, 1970, p. 108.)

5 B. When bonus hunger is present, however, the premium system probably is one of the most important factors behind the strategy selected by the policy holder.

Let us assume that the policy holder claims indemnity for a loss if and only if the loss amount exceeds the present value of all future premium increases which will be the consequence of a claim. If the insurer applies the linear premium formula (4), then any claim entered in period j will result in an increase of $b_{n,j}$ in the premium for each successive period n . The present value of all future premium increases is

$$\sum_{n=j+1}^{N} v^{n-j-1} b_{n,j} \ ,$$

where N is the total number of insurance periods under the policy, while $v = (1+i)^{-1}$ is the (one-period) discount factor corresponding to a rate of interest i per period. (For convenience, we assume that any indemnification or self-insurance payment is made at the end of the period in which the loss occurred.) Under these assumptions, therefore, the policy holder will follow a barrier strategy with barriers which are functions of N and the $b_{n,j}$s , and which are given by

$$s_j = \sum_{n=j+1}^{N} v^{n-j-1} b_{n,j} \quad \text{for} \quad j = 1,\dots,N-1 \qquad (19)$$

and

$$s_N = 0 . \qquad (20)$$

Relation (20) follows since no premiums will be paid after period N , and the policy holder will claim indemnity for all losses in this period. In this situation, optimal experience rating is achieved if, for all n and j , $b_{n,j}$ coincides with the $\tilde{b}_{n,j}$ of our Theorem. If we make the substitution $b_{n,j} = \tilde{b}_{n,j}$ for all n and j and express $\tilde{b}_{n,j}$ by (7), we get the equations

$$\frac{s_j}{\Phi(s_j)} = \sum_{n=j+1}^{N} v^{n-j-1} \frac{\psi(s_n)}{\sum\limits_{i=1}^{n-1} \Phi(s_i)\bar{G}(s_i)+k} \quad \text{for} \quad j = 1,\dots,N-1 .$$

These equations together with equation (20) are equivalent to

$$\frac{s_{j-1}}{\Phi(s_{j-1})} -v \frac{s_j}{\Phi(s_j)} = \frac{\psi(s_j)}{\sum\limits_{i=1}^{j-1} \Phi(s_i)\bar{G}(s_i)+k} \quad \text{for} \quad j = 2,\dots,N ,$$

$$\qquad (21)$$

$$s_N = 0 .$$

Thus, the premium system is constructed by first finding the solution $s_1^{(N)},\dots,s_N^{(N)}$ of (21) (if a solution exists), and then calculating the corresponding coefficients $\tilde{a}_n^{(N)}$ and $\tilde{b}_{n,j}^{(N)}$ from (6) and (7).

No attempt will be made in this paper to investigate condit-
ions which ensure the existence and uniqueness of the solution of
(21). Such an investigation is considered to be of limited interest
since in practice a computer must be employed to find the solution,
and any algorithm is easily extended to trace all possible solutions.
The author believes that there is a single solution in most practi-
cal situations. When no solution exists, the present theory does
not provide the wanted premium system which accounts for bonus
hunger. If there is more than one solution, the choice between the
solutions must be based on deliberations other than those considered
so far, e.g., by comparing the different systems of deductibles
$(s_1,\ldots,s_N)$.

5 C. We now apply the premium system defined by (6), (7)
and (21) to a simple example. We assume that M has a Poisson
distribution with parameter θ when $\Theta = \theta$, and that Y has the
exponential distribution $G(y) = 1-e^{-y}$ for $y > 0$. (This implies
that EY is chosen as the monetary unit.) Then

$$\psi(s) = \int_s^\infty y e^{-y} dy = (s+1)e^{-s} ,$$

and

$$E_\theta M = \mathrm{var}_\theta M = \theta .$$

Hence $EM/E\,\mathrm{var}_\Theta M = 1$, and $\wp(s) \equiv 1$ by (8). The premium for
period n is given by (17) as

$$\tilde{\mu}_n = \frac{(s_n+1)e^{-s_n}}{\sum\limits_{j=1}^{n-1} e^{-s_j}+k} \left\{ \sum\limits_{j=1}^{n-1} K_j + kEM \right\}$$

$$= \frac{(n-1)(s_n+1)e^{-s_n}}{\sum\limits_{j=1}^{n-1} e^{-s_j}+k} \overline{K}_{n-1} + \frac{k(s_n+1)e^{-s_n}}{\sum\limits_{j=1}^{n-1} e^{-s_j}+k} EM \ . \qquad (22)$$

Table 1 below shows the barriers $s_n^{(N)}$ and the credibility factors of the mean claim numbers $\overline{K}_{n-1}$ in formula (22) i.e.,

$$\tilde{b}_n^{(N)} = \frac{(n-1)(s_n^{(N)}+1)e^{-s_n^{(N)}}}{\sum\limits_{j=1}^{n-1} e^{-s_j^{(N)}}+k} \qquad (= \sum\limits_{j=1}^{n-1} \tilde{b}_{n,j}^{(N)}) \ ,$$

when $N = 43$, $k = 10$ and $i = 4\%$. The final column of the table gives the "ordinary" credibility factors which correspond to the case of no bonus hunger, i.e.,

$$\tilde{b}_n = \frac{n-1}{n-1+k}$$

obtained from formula (18) with $EY = 1$.

Table 1

n	$s_n^{(43)}$	$\tilde{b}_n^{(43)}$	$\tilde{b}_n$
1	1.1000	-	-
2	1.0726	0.0686	0.0909
3	1.0455	0.1348	0.1666
4	1.0186	0.1983	0.2307
5	0.9918	0.2596	0.2856
6	0.9653	0.3185	0.3335
7	0.9390	0.3750	0.3750
8	0.9128	0.4291	0.4116
9	0.8868	0.4808	0.4448
10	0.8609	0.5310	0.4734
11	0.8352	0.5780	0.5000
12	0.8097	0.6237	0.5236
13	0.7842	0.6672	0.5460
14	0.7589	0.7085	0.5655
15	0.7336	0.7490	0.5838
.	.	.	.
.	.	.	.
20	0.6083	0.9196	0.6555
.	.	.	.
.	.	.	.
25	0.4834	1.0464	0.7056
.	.	.	.
.	.	.	.
30	0.3569	1.1368	0.7424
.	.	.	.
.	.	.	.
35	0.2270	1.1934	0.7718
.	.	.	.
.	.	.	.
40	0.0920	1.2090	0.7956
.	.	.	.
43	0.0000	1.2012	0.8064

We se that the calculated barriers $s_n^{(43)}$ decrease monotonically
to 0 . This might have been expected since the added future prem-
ium costs per claim will decrease towards the end of the total
period of insurance. It is interesting to note that the credibili-
ty factor $\tilde{b}_n^{(43)}$ is smaller than the ordinary credibility factor
$\tilde{b}_n$ during the first six periods, while it is larger than $\tilde{b}_n$ from
the eighth period onwards. Here is a reasonable explanation: The
barriers $s_n^{(43)}$ of the earliest periods are comparatively large,
since a large amount of future premium expences can be saved by
self-insuring a loss. Therefore few claims will occur and the in-
surer demands a correspondingly low premium. This effect will de-
crease as time increases. On the other hand, one can
expect a policy holder who has made $\sum_{j=1}^{n-1} k_j$ claims during the first
n-1 periods and who has bonus hunger to be a worse risk than some-
one who has made equally many claims during the same period without
bonus hunger, since the latter has reported all losses while the
first one probably has not. The importance of this effect will in-
crease with the number of periods passed.

5 D. The procedure described above may be adjusted to cover
more general barrier strategies if we reinterpret the powers
v^{n-j-1} of $v = (1+i)^{-1}$ in (19) as "subjective" discounting factors.
For an example, suppose that the policy holder claim indemnity for
a loss if and only if the loss amount exceeds the next-period
premium increase following from a claim. In this case, $v^0 = 1$
and $v^j = 0$ for all $j \geq 1$. The barriers are then $s_j = b_{j+1,j}$,
and the optimal choice of coefficients $b_{n,j}$ and a_n is deter-
mined by the equations (6), (7) and

$$s_j = \psi(s_{j+1}) \; \frac{\varphi(s_j)}{\sum\limits_{i=1}^{j} \varphi(s_i)\overline{G}(s_i)+k} \qquad \text{for} \quad j = 1,\ldots,N-1 , \qquad (23)$$

$$s_N = 0 .$$

5 E. In our previous arguments we have taken N to be finite. The same arguments hold for the case $N = \infty$, which corresponds to an infinite planning horizon, except that the relation $s_N = 0$ should be deleted everywhere.

To see what happens to s_j as $j \to \infty$, consider first the strategy described in Subsection 5 D. Let $\alpha = \min\{1,\mathrm{Evar}_\oplus M/\mathrm{EM}\}$ and $\beta = \max\{1,\mathrm{Evar}_\oplus M/\mathrm{EM}\}$. Then $\alpha \leq \varphi(s) \leq \beta$ for all s . Now assume that $\overline{G}(s_j) \to 0$ as $j \to \infty$. Then $\psi(s_j) \to 0$ by its definition, and hence $s_j \to 0$ by (23). This implies that $\overline{G}(s_j) \to 1$, however, and we have produced a contradiction. It follows that there exists a subsequence $\{s_{j_\nu}\}$ and an $\varepsilon > 0$ such that $\overline{G}(s_{j_\nu}) \geq \varepsilon$ for all ν . Let m(j) be the number of $j_\nu \leq j$, and note that $\psi(s) \leq EY$. By (23), therefore,

$$s_j \leq \frac{\beta \cdot EY}{\alpha \sum\limits_{i=1}^{j} \overline{G}(s_i)+k} \leq \frac{\beta \cdot EY}{\alpha \varepsilon m(j)+k}$$

Since $m(j) \to \infty$ as $j \to \infty$, it follows that

$$\lim_{j \to \infty} s_j = 0 .$$

This is in accordance with the intuitive idea that when n goes to infinity, $\oplus$ eventually is determined exactly by the experience

Z_{n-1} . Since the individual accident proneness is known by then, experience rating, which motivates bonus hunger, is no longer needed.

The strategy described in Subsection 5 B. leads to the barriers s_j given by the first equation in (21), for $j = 1,2,...$. These equations are compatible with s_j tending to zero with increasing j . To show the necessity of $\lim s_j = 0$ as $j \to \infty$, it seems that further conditions must be placed on the distributions of Y and Θ .

6. Discussion.

6 A. In this section we will discuss the practical applicability and relevance of the theoretical results obtained in this paper, including a critical examination of the mathematical model and a scetch of how it could be generalized within the present mathematical framework.

6 B. The results of Subsection 5 B may seem to be mainly of theoretical interest. For one thing, the bonus hunger strategy will not be known in detail, not even by the policy holder himself. And even if the policy holder wanted to follow the strategy described in Subsection 5 B, he would not have the necessary theoretical and computational skill to find the barriers s_j . In addition, there is the difficulty that N usually is more or less unknown both to the insurer and to the policy holder.

These objections are probably not so serious as they might seem at first sight, however. The insurer can influence the bonus hunger strategy by supplying the policy holder with a certain type of information. It should be possible to persuade the policy holder to adapt the strategy described in Subsection 5 B. When the policy

holder has decided that this advice is sound, the insurer can publish the optimal barriers $s_j^{(N)}$ for various N , e.g. simply by letting the $s_j^{(N)}$-values be deductibles which depend on N and are stated in each individual contract.

The fact that N cannot really be determined at the outset is not so important. Even if N is not predicted perfectly when the contract is settled, the policy holder will consider the barriers $s_j^{(N)}$ to be the best possible ones from his own point of view, at least to the extent that he believes in his own estimate of N . The possibility of cheating the insurer by deliberately overstating the value of N may be prevented by extensive rules of recovery. The crucial property of the proposed premium system is that the policy holders can be brought to follow a barrier strategy which is known to the insurer. Then the premium system will be optimal from an experience rating point of view, and the premium incomes will balance the indemnity payments on the average, as is demonstrated in Subsection 4 B. It is also worth mentioning that when the barriers s_j are unknown, they will act as nuisance parameters which complicate statistical inference about the risk process.

6 C. Some of the assumption in Section 2 were made primarily for mathematical convenience. The model can easily be extended to take into consideration the possibility that the accident proneness changes over time. Instead of assuming the claim numbers M_n to be identically distributed, we can permit the conditional distribution of M_n , given $\Theta = \theta$, to depend on n . The extension will only result in trivial changes in Sections 4 and 5.

The assumption that the loss amounts Y_{nj} are independent of Θ is more crucial. It implies that the credibility premium depends on the claim numbers only, and this is the justification of the linear premium formula (4). If the distribution of the loss amounts is permitted to depend on the value of Θ , the credibility formula should also take into account the information provided by the claim amounts. Then the optimal bonus hunger strategy would no longer be of the simple type considered here. We shall not pursue these ideas.

Among other possible considerations which we have left aside, liquidity constraints on the policy holder should be mentioned. We have assumed that his strategy is based on a straightforward comparison of the loss amount with the discounted value of the future premium increases. If the policy holder can raise only the amount s_j^* for the purpose of selfinsurance in period j , and $s_j^* < s_j^{(N)}$, then s_j^* will be his real barrier in this period. s_j^* may be considered as a random variate, and taking account of it would complicate the analysis considerably. The insurer could meet this problem by offering credit to policy holders without sufficient funds.

References.

[1] Bühlmann, Hans (1970): "Mathematical Methods in Risk Theory".
     Springer-Verlag.

[2] Grenander, Ulf (1957): "Some remarks on bonus systems in
     automobile insurance". Skandinavisk Aktuarietid-
     skrift 40 : 180-197.

[3] Seal, Hilary (1969): "Stochastic Theory of a Risk Business".
     John Wiley & Sons, Inc.

[4] Straub, Erwin (1968): "Zur Theorie der Prämienstufensysteme".
     Doctoral dissertation No. 4250, Eidgennössische
     Technische Hochschule, Zürich.

[5] Straub, Erwin (1969): "Zur Theorie der Prämienstufensysteme".
     Mitteilungen der Verein. schweizer. versich. math.
     69 : 75-85.