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Creeping Flow of Micropolar Fluid Past a Fluid Sphere With Non-Zero Spin Boundary Condition

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Abstract

This paper concerns the problem of creeping flow of an incompressible micropolar fluid past a fluid sphere with non-homogeneous boundary condition for micro rotation vector i.e. the micro rotation on the boundary of the fluid sphere is assumed to be proportional to the rotation rate of the velocity field on the boundary. The stream functions are determined by matching the solution of micropolar field equation for flow outside the fluid sphere with that of the Stokes equation for the flow inside the fluid sphere. The drag force experienced by a fluid sphere is evaluated and its variation is studied with respect to the material parameters. Some well-known results are then deduced.

Keywords: *Micropolar fluid; Gegenbauer functions; Modified Bessel functions; Drag force.*

1 Introduction

Micropolar fluids are fluids with microstructure. Micropolar fluids consist of rigid, randomly oriented particles with their own and spins and microrotations, suspended in a viscous medium. In micropolar fluids, rigid particles contained in a small volume element can rotate about the centre of the volume element described by the micro rotation vector. In micropolar fluid theory, laws of classical continuum mechanics are improved with additional equations that account for conservation of microrotation moments and balance of first stress moments that arises due to consideration of microstructure in a material. Thus new kinematics variables, e.g. gyration tensor and micro inertia moment tensor and the concepts of body moments, stress moments and micro stress are combined with classical continuum mechanics. An interesting feature of this class of fluid is sustenance of couple stress. Some anisotropic fluids animal bloods, liquid crystals (with dumb-bell type molecules) are the examples of micropolar fluids. The micro polar fluid theory is applicable to certain polymer solutions, lubricant fluids, colloidal expansions and complex biological structures. A well accepted theory which accounts for internal structure of micropolar fluid was investigated by Eringen [1, 2]. Here, individual particles can rotate independently from the rotation and movement of the fluid as whole.

Lakshmana Rao and Bhujanga Rao [3] studied the slow stationary flow of a micropolar liquid past a sphere and they found that the drag on the sphere is more than in present case than that in the case of non-polar fluids. Ramkissoon and Majumdar [4] have obtained the solution of a micropolar fluid flow around a sphere and evaluated the drag force exerted on a sphere. Flow of a micropolar fluid past a Newtonian fluid sphere was also studied by Ramkissoon [5] and he obtained the exact solutions for the flow fields characterized by stream functions and then derived formula for drag force and its terminal velocity. Micropolar fluid past a slightly deformed fluid sphere was studied by Ramkissoon [6] and evaluated the drag force experienced by a slightly deformed fluid sphere. Drag on a sphere in micropolar fluids with non-zero spin boundary condition for micro-rotations was studied by Hoffman *et al.*[7] and he extended the Stokes formula for the resistance force exerted on a sphere moving with constant velocity in a fluid for micropolar fluids. He also finds that in spite of the couple stress in the fluid, there is no resultant action by it on the sphere.

This paper concerns the creeping flow of an incompressible micro polar fluid past a fluid sphere with non-homogeneous boundary condition for micro rotation vector i.e. the micro rotation on the boundary of the sphere is assumed to be proportional to the rotation rate of the velocity field on the boundary. The stream functions are determined by matching the solutions of Micropolar field equations for flow outside the fluid sphere with that of the Stokes equation for the flow inside the fluid sphere. The drag force experienced by fluid sphere is evaluated and its variation is studied with respect to the material parameters. Some well-known results are then deduced.

2 Micropolar Field Equations

The field equations of the micropolar fluid dynamics (Eringen [1]) are

$$\frac{\partial \rho}{\partial t} + div \rho \, \mathbf{v} = 0 , \tag{1.1}$$

$$\rho \frac{d\mathbf{v}}{dt} = \rho \mathbf{f} - \nabla p + \kappa \nabla \times \boldsymbol{\omega} - (\mu + \kappa) \nabla \times \nabla \times \mathbf{v} + (\lambda + 2\mu + \kappa) \nabla (div \mathbf{v}) , \tag{1.2}$$

$$\rho J \frac{d\boldsymbol{\omega}}{dt} = \rho \mathbf{l} - 2\kappa \boldsymbol{\omega} + \kappa \nabla \times \mathbf{v} - \gamma \nabla \times \nabla \times \boldsymbol{\omega} + (\alpha + \beta + \gamma) \nabla (div \, \boldsymbol{\omega}) , \tag{1.3}$$

where ρ is the density, $\mathbf{v}$ the velocity field, $\boldsymbol{\omega}$ the micro-rotation field, J the gyration parameter, $\mathbf{f}$ body forces per unit mass, $\mathbf{l}$ micro-rotation driving forces per unit mass, p the hydrostatical pressure, μ the classical viscosity coefficient, κ, λ the vortex viscosity coefficient and α, β, γ are gyroviscosity coefficients satisfying the following inequalities

$$3\alpha + \beta + \gamma \geq 0 , \quad 2\mu + \kappa \geq 0 , \; 3\lambda + 2\mu + \kappa \geq 0 , \; \gamma \geq |\beta| , \quad \kappa \geq 0 , \; \gamma \geq 0 . \tag{1.4}$$

The constitutive equations for the stress tensor T_{ij} and couple stress tensor m_{ij} are expressed by following equations, respectively as

$$T_{ij} = (-p + div \, v) \delta_{ij} + (2\mu + \kappa) d_{ij} + \kappa \varepsilon_{ijm} (\xi_m - \omega_m) \tag{1.5}$$

and

$$m_{ij} = (\alpha \, div \, v) \delta_{ij} + \beta \xi_{i,j} + \gamma \xi_{j,i} , \tag{1.6}$$

where d_{ij} denote the rate of strain components, ω_m and $2\xi_m$ are the components of microrotation vector and vorticity vector, respectively and δ_{ij} is the kronecker delta.

Let us consider a uniform, axi-symetric slow viscous flow of an unbounded incompressible micropolar fluid past a Newtonian fluid sphere. The governing differential equations for creeping flow around and through the fluid sphere written for two regions separated by the interface. The flow of fluid for outside region is assumed to be Stokesian, i.e. it is assumed that the inertial terms in the momentum equation and bilinear terms in balance of first stress moments can be neglected. Further, for outside region (1) we assume that the body force and body couple terms are absent. Therefore, the governing equations for outside flow are given by

$$div\,\mathbf{v}^{(1)} = 0, \tag{1.7}$$

$$-\nabla p^{(1)} + \kappa \nabla \times \boldsymbol{\omega}^{(1)} - (\mu+\kappa)\nabla \times \nabla \times \mathbf{v}^{(1)} = 0, \tag{1.8}$$

$$-2\kappa\boldsymbol{\omega}^{(1)} + \kappa\nabla \times \mathbf{v}^{(1)} - \gamma\nabla \times \nabla \times \boldsymbol{\omega}^{(1)} + (\alpha+\beta+\gamma)\nabla(\nabla.\boldsymbol{\omega}^{(1)}) = 0. \tag{1.9}$$

For the region (2) inside the fluid sphere governed by Stokes equations [Happel and Brenner, (1983)] as

$$\mu\nabla^2 \mathbf{v}^{(2)} = \nabla p^{(2)}, \qquad div\,\mathbf{v}^{(2)} = 0. \tag{1.10}$$

3 Stream Function Formulation

Since, the flow generated is axially symmetric; all the flow functions are independent of ϕ. Hence the velocity and microrotation can be chosen in the spherical polar coordinates (r,θ,ϕ) as

$$\mathbf{v}^{(i)} = v_r^{(i)}(r,\theta)\hat{e}_r + v_\theta^{(i)}(r,\theta)\hat{e}_\theta \tag{2.1}$$

and $$\boldsymbol{\omega}^{(i)} = \upsilon_\phi^{(i)}(r,\theta)\hat{e}_\phi. \tag{2.2}$$

Introducing the stream functions for both regions through

$$v_r^{(i)} = -\frac{1}{r^2 \sin\theta}\frac{\partial \psi^{(i)}}{\partial\theta} \quad ; \qquad v_\theta^{(i)} = \frac{1}{r\ \sin\theta}\frac{\partial\psi^{(i)}}{\partial r}. \quad i=1,2. \tag{2.3}$$

These velocity components satisfy the equations of continuity.

Eliminating pressure from equation (1.8) and using equation (2.3), we get

$$E^4\psi^{(1)} - NE^2(\mathrm{r}\sin\theta\ \upsilon_\phi^{(1)}) = 0. \tag{2.4}$$

Using above equation (2.4) in equation (1.9), we find that

$$\upsilon_\phi^{(1)} = \frac{1}{2r\sin\theta}[E^2\psi^{(1)} + \frac{2-N}{Nm^2}E^4\psi^{(1)}]. \tag{2.5}$$

Therefore, the stream function formulation for the outside flow can be found by eliminating $\upsilon_\phi^{(1)}$ from equations (2.4) and (2.5) as

$$E^4(E^2 - m^2)\psi^{(1)} = 0\,. \tag{2.6}$$

Similarly, eliminating the pressure from equations (1.10) and using equations (2.3), we obtain the following equation as

$$\mathrm{E}^2\,(\mathrm{E}^2\,\psi^{(2)}) = 0, \tag{2.7}$$

where $E^2 = \dfrac{\partial^2}{\partial r^2} + \dfrac{(1-\zeta^2)}{r^2}\dfrac{\partial^2}{\partial \zeta^2}$, $\zeta = \cos\theta$, $m^2 = \dfrac{\kappa(2\mu+\kappa)}{\gamma(\mu+\kappa)}a^2$ and $N = \dfrac{\kappa}{\mu+\kappa}$

being the coupling number $(0 \le N < 1)$.

Using the separation of variables, the general regular solution of equation (2.6) can be expressed as

$$\begin{aligned}\psi^{(1)}(r,\zeta) = \sum_{n=2}^{\infty}[A_n r^n + B_n r^{-n+1} + C_n r^{n+2} + D_n r^{-n+3} + E_n r^{1/2} K_{n-1/2}(mr) + \\ F_n r^{1/2} I_{n-1/2}(mr)]G_n(\zeta).\end{aligned} \tag{2.8}$$

where, $I_\nu(mr)$ and $K_\nu(mr)$ are the modified Bessel functions of first and second kind of non- integer index $\nu = n - \frac{1}{2}$, respectively and G_n (ζ) is the Gegenbauer function of first kind as defined in (Abramowitz and Stegun [9]).Since the modified Bessel functions of first kind $I_{n-1/2}(mr)$ are unbounded at infinity, so the terms involving these functions must be absent and the terms r^n and r^{n+2} must also be absent with the exception of the term including r^2 as it satisfy the condition at infinity. Hence the solution for the region outside the sphere contain only the terms of the order $n = 2$ in the general solution, therefore the above solution (2.8) which satisfies the uniform condition at infinity and requirement for spherical case reduces to

$$\psi^{(1)}(r,\zeta) = [r^2 + A_2 r^{-1} + B_2 r + C_2\sqrt{r}\,K_{3/2}(mr)]G_2(\zeta)\,. \tag{2.9}$$

For the region inside the fluid sphere the general regular solution of Stokes equation (Happel and Brenner [8])

$$\psi^{(2)}(r,\zeta) = \sum_{n=2}^{\infty}[A_n^* r^n + B_n^* r^{-n+1} + C_n^* r^{-n+3} + D_n^* r^{n+2}]G_n(\zeta). \tag{2.10}$$

for spherical case we can take the above expansion (2.10) for $n = 2$ only, i.e.

$$\psi^{(2)}(r,\zeta) = [A_2^* r^2 + B_2^* r^{-1} + C_2^* r + D_2^* r^4]G_2(\zeta)$$

For internal flow, the velocity components shall be regular if we take $C_2^*=0$ and $B_2^*=0$, hence $\psi^{(2)}$ must be of the form

$$\psi^{(2)}(r,\zeta)=[A_2^* r^2 + D_2^* r^4]G_2(\zeta)\,. \tag{2.11}$$

Substituting the value of $\psi^{(1)}$ from equation (2.9) in equation (2.5), we get microrotation components, respectively as

$$\upsilon_\phi^{(1)}(r,\zeta)=\frac{1}{r\sin\theta}[-B_2\, r^{-1}+\frac{m^2(\mu_1+\kappa)}{\kappa}C_2\sqrt{r}K_{3/2}(mr)]G_2(\zeta)\,. \tag{2.12}$$

4 Boundary Conditions

the rotation rate of the velocity field on the boundary i.e. The boundary conditions those are physically realistic and mathematically consistent for this proposed problem can be taken as given below:

The kinematical condition of mutual impenetrability at the surface requires that we take

$$\psi^{(1)}=0 \qquad \text{on} \quad r=1, \tag{3.1}$$

$$\psi^{(2)}=0 \qquad \text{on} \quad r=1. \tag{3.2}$$

We assume that the tangential velocity is continuous across the surface. Hence

$$\frac{\partial\psi^{(1)}}{\partial r}=\frac{\partial\psi^{(2)}}{\partial r} \quad \text{on} \quad r=1. \tag{3.3}$$

Now from the theory of interfacial tension, the presence of interfacial tension only produces a discontinuity in the normal stress T_{rr} and does not in anyway affect the tangential stress $T_{r\theta}$. Therefore, the latter is continuous across the surface and so that we may take

$$T_{r\theta}^{(1)}=T_{r\theta}^{(2)} \quad \text{on} \quad r=1. \tag{3.4}$$

The micro-rotation on the boundary of the sphere is assumed proportional to the rotation rate of the velocity field on the boundary i.e.

$$\boldsymbol{\omega}^{(1)}=\frac{\tau}{2}curl\,\mathbf{v}^{(1)}$$

which on simplification provides

$$\upsilon_{\phi}^{(1)} = \frac{\tau}{2r\sin\theta} E^2 \psi^{(1)}, \qquad \text{on } r = 1. \tag{3.5}$$

Far away from the fluid sphere, the flow is uniform so that condition at infinity implies that

$$\psi^{(2)}(r,\varsigma) = -\frac{1}{2} r^2 \sin^2\theta = -r^2 G_2(\varsigma) \qquad \text{as} \quad r \to \infty. \tag{3.6}$$

Applying these above boundary conditions (3.1)-(3.6), we get the following equations respectively as

$$A_2 + B_2 + C_2 K_{3/2}(m) = -1 \quad , \tag{3.7}$$

$$A_2^* + D_2^* = 0 \qquad , \tag{3.8}$$

$$A_2 - B_2 + C_2(mK_{1/2}(m) + K_{3/2}(m)\} + 2A_2^* + 4D_2^* = 2 \quad , \tag{3.9}$$

$$\begin{aligned} &-(4\mu_1 + 3\kappa)A_2 + (2\mu_1 + \kappa)B_2 - \{m^2(\mu_1 + \kappa)K_{3/2}(m) + (4\mu_1 + 3\kappa) \\ &K_{3/2}(m) + m(2\mu_1 + \kappa)K_{1/2}(m)\}C_2 - 2\mu_2 A_2^* + 4\mu_2 D_2^* = -2\mu_1 \end{aligned}, \tag{3.10}$$

$$B_2(1-\tau) + m^2\left(\frac{\tau}{2} - \frac{(\mu_1 + \kappa)}{\kappa}\right)K_{3/2}(m)C_2 = 0 \ . \tag{3.11}$$

Solving these above equations we get following constants

$$A_2 = \frac{\begin{bmatrix} 2\mu_1[\kappa\{6(-1+\tau) + 6m(-1+\tau) + m^3\tau + m^2(-2+3\tau)\} - 3m^2(1+m)\mu_2\} \\ ] + \kappa[\kappa\{6(-1+\tau) + 6m(-1+\tau) + m^3\tau + m^2(-2+3\tau)\} + 3\{m^3(-2+\tau) \\ + 6(-1+\tau) + 6m(-1+\tau) + m^2(-4+3\tau)\mu_2\}] \end{bmatrix}}{2m^2\Delta}, \tag{3.14}$$

$$B_2 = -\frac{3[(1+m)\{\kappa(-2+\tau) - 2\mu_1\}(\kappa + 2\mu_1 + 3\mu_2)]}{2\Delta}, \tag{3.15}$$

$$C_2 = -\frac{3e^m\sqrt{\frac{2}{\pi}}\kappa(\kappa + 2\mu_1 + 3\mu_2)(-1+\tau)}{\sqrt{m}\Delta/2}, \tag{3.16}$$

$$A_2^* = \frac{3(1+m)(\kappa^2 + 3\kappa\mu_1 + 2\mu_1^2)}{\Delta}, \tag{3.17}$$

$$D_2^* = \frac{-3(1+m)(\kappa^2 + 3\kappa\mu_1 + 2\mu_1^2)}{\Delta}. \tag{3.18}$$

where $$\begin{aligned}\Delta = 2[\{-6(-1+m)\mu_1^2 + \mu_1\{\kappa(-7-9m+2m\tau) - 6(-1+m)\mu_2\} \\ + \kappa\{\kappa(-2+m(-3+\tau)) + 3(-1+m(-2+\tau))\mu_2\}\}]\end{aligned}. \tag{3.19}$$

Thus all the coefficients have been determined and hence, we get explicit expressions for the stream function in both regions given by equations (2.9) and (2.11), respectively.

5 Evaluation Of The Drag Force

The drag force experienced by fluid sphere is given by Ramkissoon and Mazumdar [4] the formula is

$$D = 4\pi(2\mu_e + k)\lim_{r\to\infty} \frac{r(\psi^{(1)} - \psi_\infty)}{\varpi^2}, \tag{4.1}$$

where ψ_∞ is the stream function corresponding to the fluid motion at infinity and ϖ is the cylindrical radius coordinate whose values are

$$\psi_\infty = \frac{1}{2}Ur^2\sin^2\theta, \quad \varpi = r\sin\theta. \tag{4.2}$$

Thus for present case, substituting these above values in (4.1) and taking the limit, we found that

$$\begin{aligned} D &= 2\pi(2\mu_e + k)UaB_2 \\ &= -\frac{6\pi(2\mu_e + k)Ua[(1+m)\{\kappa(-2+\tau) - 2\mu_1\}(\kappa + 2\mu_1 + 3\mu_2)]}{2\Delta}, \end{aligned} \tag{4.3}$$

where Δ is given by equation (3.19).

The following special case can be deduced immediately:

Case I: Drag for no-spin on boundary ($\tau = 0$):

It is interesting to compare the calculated value of drag force by putting $\tau = 0$ i.e. no-spin condition on boundary, we found that

$$F = \frac{12\pi Ua(2\mu_e + k)(1+m)(\kappa + \mu_1)(\kappa + 2\mu_1 + 3\mu_2)}{\Delta'}, \tag{4.4}$$

Where $\begin{aligned}\Delta' = 6(1+m)\mu_1^2 + \mu_1\{7+9m)\kappa + 6(1+m)\mu_2\} \\ + \kappa\{\kappa(2+3m) + (3+6m)\mu_2\}\}\end{aligned}.$

Case II: Drag on a fluid sphere embedded in another fluid:

If $k \to 0$ i.e. $m \to 0$ then micropolar fluid becomes a Newtonian fluid. Therefore, the drag force on the fluid sphere by equation (4.3) reduces to

$$F = -6\pi\mu Ua \frac{(1+\frac{2}{3}\eta)}{(1+\eta)}, \qquad (4.5)$$

which agree with the result reported earlier by Happel and Brenner[8] for the drag force experienced by fluid sphere in a clear fluid.

Case III: If $\eta \to 0$, then fluid sphere behaves like a solid sphere, then the drag force from the equation (4.5) comes out as

$$F = -6\pi\mu Ua \qquad (4.6)$$

which is well-known Stokes result for flow past a rigid sphere in unbounded medium.

Case IV: if $\mu_2 << \mu_1$ *i.e* $\eta \to \infty$ then fluid sphere behaves like a gaseous spherical bubble in this case drag force can be found as

$$F = -4\pi\mu Ua. \qquad (4.7)$$

This result is identical to that previously given for a sphere at whose surface perfect slip occurs (Happel and Brenner[8]).

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