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CRITERIA FOR D_4 SINGULARITIES OF WAVE FRONTS

KENTARO SAJI

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Abstract. We give useful and simple criteria for determining $D_4^{\pm}$ singularities of wave fronts. As an application, we investigate behaviors of singular curvatures of cuspidal edges near D_4^{+} singularities.

1. Introduction. In this paper, we study criteria for determining $D_4^{\pm}$ singularities.

A generic classification of singularities of wave fronts was given by Arnol'd and Zakalyukin. They showed that the generic singularities of wave fronts in $\boldsymbol{R}^3$ are cuspidal edges and swallowtails. Moreover, they showed that the generic singularities of one-parameter bifurcation of wave fronts are cuspidal lips, cuspidal beaks, cuspidal butterflies and $D_4^{\pm}$ singularities (see [1]). Classifications of further degenerate singularities have been considered by many authors (see [1, 2, 3, 5, 19] for example).

To state the theorem, we define some terms here. The unit cotangent bundle $T_1^*\boldsymbol{R}^{n+1}$ of $\boldsymbol{R}^{n+1}$ has the canonical contact structure and can be identified with the unit tangent bundle $T_1\boldsymbol{R}^{n+1}$. Let α denote the canonical contact form on it. A map $i : M \to T_1\boldsymbol{R}^{n+1}$ is said to be *isotropic* if $\dim M = n$ and the pull-back $i^*\alpha$ vanishes identically. An isotropic immersion is called a *Legendrian immersion*. We call the image of $\pi \circ i$ the *wave front set* of i, where $\pi : T_1\boldsymbol{R}^{n+1} \to \boldsymbol{R}^{n+1}$ is the canonical projection. We denote by $W(i)$ the wave front set of i. Moreover, i is called the *Legendrian lift* of $W(i)$. With this framework, we define the notion of fronts as follows: A map-germ $f : (\boldsymbol{R}^n, \boldsymbol{0}) \to (\boldsymbol{R}^{n+1}, \boldsymbol{0})$ is called a *wave front* or a *front* if there exists a unit vector field ν of $\boldsymbol{R}^{n+1}$ along f such that $L = (f, \nu) : (\boldsymbol{R}^n, \boldsymbol{0}) \to (T_1\boldsymbol{R}^{n+1}, L(\boldsymbol{0}))$ is a Legendrian immersion (cf. [1], see also [11]).

The main result of this paper is as follows:

THEOREM 1.1. *Let $f(u, v) : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}^3, \boldsymbol{0})$ be a front and (f, ν) its Legendrian lift. The germ f at $\boldsymbol{0}$ is a D_4^{+} singularity* (*resp.* D_4^{-} *singularity*) *if and only if the following two conditions hold.*

(a) *The rank of the differential map $df_{\boldsymbol{0}}$ is equal to zero.*

(b) $\det(\mathrm{Hess}\,\lambda(\boldsymbol{0})) < 0$ (*resp.* $\det(\mathrm{Hess}\,\lambda(\boldsymbol{0})) > 0$),

where $\lambda(u, v) = \det(f_u, f_v, \nu)$, $f_u = df(\partial/\partial u)$, $f_v = df(\partial/\partial v)$ *and* $\det(\mathrm{Hess}\,\lambda(\boldsymbol{0}))$ *means the determinant of the Hessian matrix of λ at $\boldsymbol{0}$.*

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Since our criteria require only the Taylor coefficients of the given germ, this can be useful for identifying the $D_4^{\pm}$ singularities on explicitly parameterized maps.

Criteria for other singularities of fronts were obtained in [4, 8, 9, 11, 17]. Recently, several applications of these criteria were obtained in various situations [6, 9, 10, 12, 14, 15, 16].

This paper is organized as follows. In Section 2 we give fundamental notions and state criteria for 4-dimensional $D_4^{\pm}$ singularities (Theorem 2.3). In Section 3 we prove Theorem 2.3, and in Section 4 we prove Theorem 1.1. In Section 5 we apply Theorem 1.1 to the normal forms of the $D_4^{\pm}$ singularities, which confirms one direction of the conclusion of Theorem 1.1, since the conditions in Theorem 1.1 are independent of the right-left equivalence. In Section 6, as an application of Theorem 1.1, we study the singular curvatures of four cuspidal edges near a "generic" D_4^{+} singularity.

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All maps considered here are of class C^{∞}.

2. Fundamental notions. Let $f(u_1,\dots,u_n):(\boldsymbol{R}^n,\boldsymbol{0})\to(\boldsymbol{R}^{n+1},\boldsymbol{0})$ be a front and $L=(f,\nu):(\boldsymbol{R}^n,\boldsymbol{0})\to(T_1\boldsymbol{R}^{n+1},\boldsymbol{0})$ its Legendrian lift. The isotropicity of L is equivalent to the orthogonality condition

$$\langle df(X_p),\nu(p)\rangle=0\quad(X_p\in T_p\boldsymbol{R}^n,\ p\in\boldsymbol{R}^n),$$

where $\langle\ ,\ \rangle$ is the Euclidean inner product. The vector field ν is called the *unit normal vector field* of the front f. For a front $f:(\boldsymbol{R}^n,\boldsymbol{0})\to(\boldsymbol{R}^{n+1},\boldsymbol{0})$, a function

$$\lambda(u_1,\dots,u_n)=\det(f_{u_1},\dots,f_{u_n},\nu)(u_1,\dots,u_n)\tag{2.1}$$

is called the *signed volume density function* of f, where $f_{u_i}=df(\partial/\partial u_i)$, $(i=1,\dots,n)$. The set of singular points $S(f)$ of f coincides with the zeros of λ. If $n=3$ and the rank of $df_{\boldsymbol{0}}$ is equal to 1, then there exist vector fields τ,ξ,η on $(\boldsymbol{R}^3,\boldsymbol{0})$ such that $\xi_{\boldsymbol{0}}$ and $\eta_{\boldsymbol{0}}$ generate the kernel of $df_{\boldsymbol{0}}$, and $\tau_{\boldsymbol{0}}$ is transverse to $\ker(df_{\boldsymbol{0}})$.

DEFINITION 2.1. Two map-germs $f_1, f_2:(\boldsymbol{R}^n,\boldsymbol{0})\to(\boldsymbol{R}^m,\boldsymbol{0})$ are *right-left equivalent* if there exist diffeomorphisms-germs $S:(\boldsymbol{R}^n,\boldsymbol{0})\to(\boldsymbol{R}^n,\boldsymbol{0})$ and $T:(\boldsymbol{R}^m,\boldsymbol{0})\to(\boldsymbol{R}^m,\boldsymbol{0})$ such that $f_2\circ S=T\circ f_1$ holds. If one can take T to be the identity, the two map-germs are called *right equivalent*.

DEFINITION 2.2. A *cuspidal edge* is a map-germ right-left equivalent to $(u,v)\mapsto(u,v^2,v^3)$ at $\boldsymbol{0}$. A *swallowtail* is a map-germ right-left equivalent to $(u,v)\mapsto(u,3v^4+uv^2,4v^3+2uv)$ at $\boldsymbol{0}$. A map-germ right-left equivalent to $(u,v)\mapsto(uv,u^2+3\varepsilon v^2,u^2v+\varepsilon v^3)$ at $\boldsymbol{0}$ is called a D_4^{+} *singularity* if $\varepsilon=1$ (resp. a D_4^{-} *singularity* if $\varepsilon=-1$) (see Figure 1, where the left figure is the D_4^{+} singularity and the right figure is the D_4^{-} singularity). A map-germ $(\boldsymbol{R}^3,\boldsymbol{0})\to(\boldsymbol{R}^4,\boldsymbol{0})$ right-left equivalent to $(u,v,t)\mapsto(uv,u^2+2tv\pm3v^2,2u^2v+tv^2\pm2v^3,t)$ at $\boldsymbol{0}$ is called a 4-*dimensional* $D_4^{\pm}$ *singularity*, respectively.

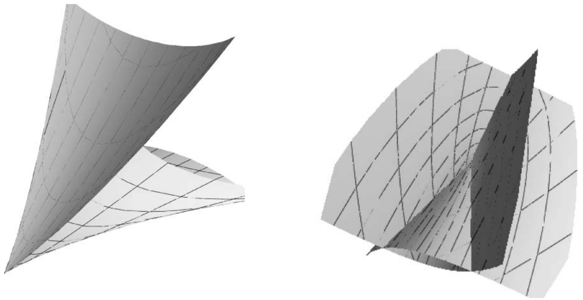

FIGURE 1. The $D_4^{\pm}$ singularities.

Since $D_4^{\pm}$ singularities appear as generic singularities of fronts $(\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^4, \mathbf{0})$, Theorem 1.1 is based on the following theorem:

THEOREM 2.3. *Let $f(u, v, t) : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^4, \mathbf{0})$ be a front and ν its unit normal vector. The germ f at $\mathbf{0}$ is a* 4*-dimensional D_4^+ singularity* (*resp. a* 4*-dimensional D_4^- singularity*) *if and only if the following three conditions hold.*

(1) *The rank of the differential map $df_{\mathbf{0}}$ is equal to one.*

(2) $\det(\mathrm{Hess}_{(\xi,\eta)}\,\lambda(\mathbf{0})) < 0$ (*resp.* $\det(\mathrm{Hess}_{(\xi,\eta)}\,\lambda(\mathbf{0})) > 0$), *where the* 2×2 *matrix* $\mathrm{Hess}_{(\xi,\eta)}\,\lambda$ *is the Hessian matrix with respect to ξ and η of λ. Here, ξ and η are vector fields on $\boldsymbol{R}^3$ that generate the kernel of $df_{\mathbf{0}}$, and λ is the signed volume density function as in* (2.1).

(3) *The map-germ*

$$\big(\langle df(\xi), d\nu(\xi)\rangle, \langle df(\xi), d\nu(\eta)\rangle, \langle df(\eta), d\nu(\eta)\rangle\big) : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$$

is an immersion at $\mathbf{0}$.

Note that $\mathrm{Hess}_{(\xi,\eta)}\,\lambda(\mathbf{0})$ is symmetric, since ξ and η belong to the kernel of $df_{\mathbf{0}}$.

REMARK 2.4. In Theorem 1.1, the condition corresponding to (3) in Theorem 2.3 is not needed, and is not used in the theorem's proof. On the other hand, if a front $f : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ satisfies the conditions (a) and (b) of Theorem 1.1, then the map $(h_{11}, h_{12}, h_{22}) : (\boldsymbol{R}^2, \mathbf{0}) \to (\boldsymbol{R}^3, \mathbf{0})$ is an immersion, where h_{11}, h_{12}, h_{22} are the components of the second fundamental form of f. See [18] for relations between A_k singularities and the second fundamental form of fronts.

3. Proof of Theorem 2.3. In this section, we prove Theorem 2.3. First, we show that the conditions in Theorem 2.3 do not depend on the choices of vector fields and the coordinate systems.

3.1. Well-definedness of the conditions. It is easy to see that the conditions in Theorem 2.3 are independent of the choices of vector fields on the source. We show that they are independent of the choice of coordinate system on the target as well.

We take a diffeomorphism $T : (\boldsymbol{R}^4, \boldsymbol{0}) \to (\boldsymbol{R}^4, \boldsymbol{0})$. The differential map dT can be considered as a $GL(4, \boldsymbol{R})$-valued function $q \mapsto dT_q$. Since

$$A\boldsymbol{x}_1 \wedge A\boldsymbol{x}_2 \wedge A\boldsymbol{x}_3 = (\det A)\, {}^t\!A^{-1}(\boldsymbol{x}_1 \wedge \boldsymbol{x}_2 \wedge \boldsymbol{x}_3)$$

holds for any vectors $\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3 \in \boldsymbol{R}^4$ and any non-singular matrix A, we can take

$$\tilde{\nu} = \frac{1}{\delta}\, {}^t(dT_f)^{-1}\nu\,, \quad \delta = \sqrt{\langle {}^t(dT_f)^{-1}\nu, {}^t(dT_f)^{-1}\nu \rangle} \tag{3.1}$$

as the unit normal vector of $T \circ f$. Using (3.1), we can easily see that the conditions (2) and (3) of Theorem 2.3 are independent of the choice of coordinate system on the target by noticing that $(dT_f)_u = (dT_f)_v = O$ holds at $\boldsymbol{0}$ if $df_{\boldsymbol{0}}(\partial/\partial u) = df_{\boldsymbol{0}}(\partial/\partial v) = \boldsymbol{0}$.

3.2. Criteria for a function to be right equivalent to $u^3 \pm uv^2$. In order to show Theorem 2.3, criteria for a function to be right equivalent to the function $u^3 \pm uv^2$ play the crucial role. For a function-germ $\varphi : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}, 0)$, we define the following number:

$$\begin{aligned} \Delta_\varphi = \big(&(\varphi_{uuu})^2(\varphi_{vvv})^2 - 6\varphi_{uuu}\varphi_{uuv}\varphi_{uvv}\varphi_{vvv} - 3(\varphi_{uuv})^2(\varphi_{uvv})^2 \\ &+ 4(\varphi_{uuv})^3\varphi_{vvv} + 4\varphi_{uuu}(\varphi_{uvv})^3 \big)(0,0)\,. \end{aligned} \tag{3.2}$$

Then the following lemma holds:

LEMMA 3.1. *Let $\varphi : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}, 0)$ be a function satisfying $j^2\varphi = 0$. Then φ at $\boldsymbol{0}$ is right equivalent to $(u^3 + uv^2)$ (resp. $(u^3 - uv^2)$) if and only if $\Delta_\varphi > 0$ (resp. $\Delta_\varphi < 0$). Here, $j^k\varphi$ means the k-jet of φ at $\boldsymbol{0}$.*

PROOF. The number Δ_φ is the discriminant of the cubic equation $j^3\varphi(u, 1) = 0$ with respect to u. The standard method yields that $\Delta_\varphi > 0$ (resp. $\Delta_\varphi < 0$) is the necessary and sufficient condition for $j^3\varphi$ to be right equivalent to $u^3 + uv^2$ (resp. $u^3 - uv^2$) (see [13, page 30], for example). It is known that if $j^3\varphi$ is right equivalent to $u^3 \pm uv^2$, then φ is right equivalent to $u^3 \pm uv^2$ (see also [13, page 30]). Thus we have the lemma. □

3.3. Versal unfoldings and their discriminant sets. Let us define a function

$$\mathcal{V}(u, v, x, y, z, t) = u^3 \pm uv^2 + u^2t + ux + vy + z\,. \tag{3.3}$$

Then $\mathcal{V}$ is a $\mathcal{K}$-versal unfolding of $\mathcal{V}(u, v, 0, 0, 0, 0) = u^3 \pm uv^2$. By definition, any $\mathcal{K}$-versal unfolding of a function which is right equivalent to $u^3 \pm uv^2$ is P-$\mathcal{K}$-equivalent to $\mathcal{V}$. See [1, Section 8] for the definitions of unfoldings, their $\mathcal{K}$-versality and P-$\mathcal{K}$-equivalence between them. See also [9, Section 7]. An unfolding $G(u, v, \boldsymbol{x}) : (\boldsymbol{R}^2 \times \boldsymbol{R}^k, (\boldsymbol{0}, \boldsymbol{0})) \to (\boldsymbol{R}, 0)$ of a function $g : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}, 0)$ is a *Morse family of hypersurfaces* if the map $(G, G_u, G_v) : (\boldsymbol{R}^2 \times \boldsymbol{R}^k, (\boldsymbol{0}, \boldsymbol{0})) \to (\boldsymbol{R}^3, \boldsymbol{0})$ is a submersion. Moreover, the *discriminant set* $\mathcal{D}_G$ of a Morse family of hypersurfaces G is defined by

$$\mathcal{D}_G = \{\boldsymbol{x} \in \boldsymbol{R}^k;\ \text{there exists } (u, v) \in \boldsymbol{R}^2 \text{ such that } G = G_u = G_v = 0 \text{ at } (u, v, \boldsymbol{x})\}\,.$$

If two Morse families of hypersurfaces $G_1, G_2 : (\boldsymbol{R}^2 \times \boldsymbol{R}^k, \boldsymbol{0}) \to (\boldsymbol{R}, 0)$ satisfy that both regular sets of their discriminant sets are dense in $(\boldsymbol{R}^k, \boldsymbol{0})$, then G_1 and G_2 are P-$\mathcal{K}$-equivalent if and only if the discriminant sets $(\mathcal{D}_{G_1}, \boldsymbol{0})$ and $(\mathcal{D}_{G_2}, \boldsymbol{0})$ are diffeomorphic as set-germs (see [20, Section 1.1], see also [11, Appendix A]).

The discriminant set of $\mathcal{V}$ is given by

$$\mathcal{D}_{\mathcal{V}} = \{(x, y, z, t);\, x = -3u^2 \mp v^2 - 2ut,\ y = \mp 2uv,\ z = 2u^3 + u^2 t \pm 2uv^2\}\,.$$

This gives a parameterization of a 4-dimensional $D_4^{\pm}$ singularity.

Thus, in order to show Theorem 2.3, it is sufficient to construct a function φ and an unfolding Φ of φ, which is a Morse family of hypersurfaces, such that the discriminant set coincides with the image of f, and show that φ is right equivalent to $u^3 \pm uv^2$ and Φ is a $\mathcal{K}$-versal unfolding.

3.4. Unfolding of a given front. Let $f : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^4, \mathbf{0})$ be a front and ν its unit normal vector. We assume that conditions (1), (2) and (3) of Theorem 2.3 are satisfied. In particular, $\operatorname{rank} df_{\mathbf{0}} = 1$ holds, and so by the implicit function theorem and by coordinate transformations on the source and target, f can be written as

$$f(u, v, t) = (f_1(u, v, t), f_2(u, v, t), f_3(u, v, t), t)\,, \quad d(f_1, f_2, f_3)_{\mathbf{0}} = \mathbf{0}\,,$$

where $\partial/\partial u$ and $\partial/\partial v$ generate the kernel of $df_{\mathbf{0}}$, and $\nu(\mathbf{0})$, $\nu_u(\mathbf{0})$, $\nu_v(\mathbf{0})$ are orthonormal. Now we define functions Φ and φ by

$$\begin{aligned}\Phi(u, v, x, y, z, t) &= \big\langle (f_1(u, v, t), f_2(u, v, t), f_3(u, v, t), t) - (x, y, z, t)\nu(u, v, t)\big\rangle \\ &= \langle (f_1 - x, f_2 - y, f_3 - z, 0), \nu\rangle\ , \\ \varphi(u, v) &= \Phi(u, v, 0, 0, 0, 0)\,.\end{aligned}$$

LEMMA 3.2. *The discriminant set $\mathcal{D}_\Phi$ of Φ coincides with the image of f.*

PROOF. It is easy to check that the image of f is contained in $\mathcal{D}_\Phi$. We now show image $f \supset \mathcal{D}_\Phi$. Set $\boldsymbol{w} = (x, y, z, t) \in \mathcal{D}_\Phi$. Since $\Phi_u = \langle f_u, \nu\rangle + \langle f - \boldsymbol{w}, \nu_u\rangle = \langle f - \boldsymbol{w}, \nu_u\rangle$ holds, $\boldsymbol{w} \in \mathcal{D}_\Phi$ is equivalent to existence of (u, v) such that $\langle f(u, v, t) - \boldsymbol{w}, \nu(u, v, t)\rangle = 0$, $\langle f(u, v, t) - \boldsymbol{w}, \nu_u(u, v, t)\rangle = 0$ and $\langle f(u, v, t) - \boldsymbol{w}, \nu_v(u, v, t)\rangle = 0$. Moreover, since $\langle f(u, v, t) - \boldsymbol{w}, \boldsymbol{e}_4\rangle = 0$ and $f_t(\mathbf{0}) = \boldsymbol{e}_4$ hold, the four vectors $\{\boldsymbol{e}_4, \nu, \nu_u, \nu_v\}$ are linearly independent near $\mathbf{0}$, where $\boldsymbol{e}_4 = (0, 0, 0, 1)$. Thus it follows that $f(u, v, t) - \boldsymbol{w} = \mathbf{0}$ for some (u, v). Hence we have $\boldsymbol{w} \in$ image f. □

If f is a front, then Φ is a Morse family of hypersurfaces. To see this, it is sufficient to show that the matrix

$$A = \begin{pmatrix} \Phi_x & \Phi_{ux} & \Phi_{vx} \\ \Phi_y & \Phi_{uy} & \Phi_{vy} \\ \Phi_z & \Phi_{uz} & \Phi_{vz} \end{pmatrix}$$

is regular at $\mathbf{0}$. Since each column vectors of $A(\mathbf{0})$ coincide with the first three components of the vectors $\nu(\mathbf{0})$, $\nu_u(\mathbf{0})$, $\nu_v(\mathbf{0}) \in \boldsymbol{R}^4$, and $\{\boldsymbol{e}_4, \nu, \nu_u, \nu_v\}$ are linearly independent near $\mathbf{0}$, we see that $A(\mathbf{0})$ is regular. By the assumption $\det(\operatorname{Hess}_{(\xi,\eta)} \lambda) \neq 0$, the regular points of f are dense in $(\boldsymbol{R}^3, \mathbf{0})$.

3.5. Right equivalence of φ and $u^3 \pm uv^2$. Let $f : (\boldsymbol{R}^3, \mathbf{0}) \to (\boldsymbol{R}^4, \mathbf{0})$ be a front and assume that the conditions (1), (2) and (3) of Theorem 2.3 are satisfied. Let Φ and φ be as above. Here, we prove that if $\det(\mathrm{Hess}\,\lambda(\mathbf{0})) > 0$ (resp. $\det(\mathrm{Hess}\,\lambda(\mathbf{0})) < 0$) then φ is right equivalent to $u^3 - uv^2$ (resp. $u^3 + uv^2$).

Calculating the third order differentials of φ,we have

$$\begin{aligned}&\varphi_u = \varphi_{uu} = \varphi_{uv} = \varphi_v = \varphi_{vv} = 0\,,\\&\varphi_{uuu} = \langle f_{uu}, \nu_u\rangle\,, \quad \varphi_{uuv} = \langle f_{uv}, \nu_u\rangle\,,\ \varphi_{uvv} = \langle f_{uv}, \nu_v\rangle\,,\ \varphi_{vvv} = \langle f_{vv}, \nu_v\rangle\end{aligned}$$

at $\mathbf{0}$. Also, $\langle f_u, \nu\rangle = \langle f_v, \nu\rangle = 0$ holds identically, and taking derivatives of this, we have

$$\begin{aligned}\langle f_{uuu}, \nu\rangle &= -2\,\langle f_{uu}, \nu_u\rangle && = -2\varphi_{uuu},\\ \langle f_{uuv}, \nu\rangle &= -2\,\langle f_{uv}, \nu_u\rangle = -2\,\langle f_{uu}, \nu_v\rangle && = -2\varphi_{uuv},\\ \langle f_{uvv}, \nu\rangle &= -2\,\langle f_{uv}, \nu_v\rangle = -2\,\langle f_{vv}, \nu_u\rangle && = -2\varphi_{uvv},\\ \langle f_{vvv}, \nu\rangle &= -2\,\langle f_{vv}, \nu_v\rangle && = -2\varphi_{vvv}\end{aligned} \tag{3.4}$$

at $\mathbf{0}$. On the other hand, $\{f_t, \nu_u, \nu_v, \nu\}$ are linearly independent near $\mathbf{0}$. So there exist functions a_i, b_i, c_i $(i = 1, 2, 3, 4)$ such that

$$\begin{aligned}f_{uu} &= a_1 f_t + a_2\nu_u + a_3\nu_v + a_4\nu\,,\\ f_{uv} &= b_1 f_t + b_2\nu_u + b_3\nu_v + b_4\nu\,,\\ f_{uu} &= c_1 f_t + c_2\nu_u + c_3\nu_v + c_4\nu\,.\end{aligned}$$

Note that by (3.4), $a_3(\mathbf{0}) = b_2(\mathbf{0})$ and $b_3(\mathbf{0}) = c_2(\mathbf{0})$ hold. By a direct calculation,

$$\lambda_{uu} = a_2b_3 - a_3b_2\,, \quad \lambda_{uv} = a_2c_3 - a_3c_2\,, \quad \lambda_{vv} = b_2c_3 - b_3c_2 \tag{3.5}$$

and

$$\begin{aligned}a_2 = \langle f_{uu}, \nu_u\rangle = \varphi_{uuu}\,, \quad a_3 = b_2 = \langle f_{uu}, \nu_v\rangle = \varphi_{uuv}\,,\\ b_3 = c_2 = \langle f_{uv}, \nu_v\rangle = \varphi_{uvv}\,, \quad c_3 = \langle f_{vv}, \nu_v\rangle = \varphi_{vvv}\end{aligned} \tag{3.6}$$

hold at $\mathbf{0}$. By (3.2), (3.4), (3.5) and (3.6), we have

$$\det(\mathrm{Hess}_{(\partial/\partial u, \partial/\partial v)}\,\lambda(\mathbf{0})) = -\Delta_\varphi\,.$$

This proves the assertion.

3.6. Versality of Φ. Here, we prove that Φ is a $\mathcal{K}$-versal unfolding of φ.

Recall that an unfolding $\Phi(u, v, x, y, z, t)$ of $\varphi(u, v)$ is $\mathcal{K}$-versal if the following equality holds ([13], see also [7, Appendix]):

$$\mathcal{E}_2 = \langle \varphi_u, \varphi_v, \varphi\rangle_{\mathcal{E}_2} + V_\Phi\,, \tag{3.7}$$

where $\mathcal{E}_2$ is the local ring of function-germs $(\boldsymbol{R}^2, \mathbf{0}) \to \boldsymbol{R}$ with the unique maximal ideal $\mathcal{M}_2 = \{h \in \mathcal{E}_2\,; h(0) = 0\}$, and $\langle \varphi_u, \varphi_v, \varphi\rangle_{\mathcal{E}_2}$ is the ideal generated by φ_u, φ_v and φ in $\mathcal{E}_2$. Moreover, V_Φ is the vector subspace of $\mathcal{E}_2$ generated by $\Phi_x(u, v, \mathbf{0})$, $\Phi_y(u, v, \mathbf{0})$, $\Phi_z(u, v, \mathbf{0})$ and $\Phi_t(u, v, \mathbf{0})$ over $\boldsymbol{R}$. We set $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)$. Then we see that $\Phi_x = \nu_1, \Phi_y = \nu_2, \Phi_z = \nu_3$. Since f is a front, $df_{\mathbf{0}}(\partial/\partial u) = df_{\mathbf{0}}(\partial/\partial v) = \mathbf{0}$ and $\nu_4(\mathbf{0}) = \mathbf{0}$, and

$$\langle \nu_1(u, v, 0), \nu_2(u, v, 0), \nu_3(u, v, 0)\rangle_{\boldsymbol{R}} \supset \boldsymbol{R} \oplus u\boldsymbol{R} \oplus v\boldsymbol{R}$$

holds, where $\langle \nu_1, \nu_2, \nu_3\rangle_{\boldsymbol{R}}$ means the vector space generated by ν_1, ν_2, ν_3 over $\boldsymbol{R}$. On the other hand, let we assume that φ is right equivalent to $u^3 \pm uv^2$. Then there exists a diffeomorphism germ s such that $s^*(\varphi) = u^3 \pm uv^2$, where $s^* : \mathcal{E}_2 \to \mathcal{E}_2$ is the pull-back isomorphism defined by $s^*(\varphi) = \varphi \circ s$. Then it holds that $(\mathcal{M}_2)^3 \subset \mathcal{M}_2\, s^*\big(\langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2}\big)$. Since $s^*\mathcal{M}_2 = \mathcal{M}_2$, we have $(\mathcal{M}_2)^3 \subset \mathcal{M}_2 \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2}$. We set

$$\psi_1(u, v) = \Phi_t(u, v, \mathbf{0}), \quad \psi_2(u, v) = \varphi_u(u, v), \quad \psi_1(u, v) = \varphi_v(u, v).$$

Since $\mathcal{M}_2 \langle \varphi_u, \varphi_v \rangle_{\mathcal{E}_2} \subset \langle \varphi_u, \varphi_v, \varphi \rangle_{\mathcal{E}_2}$ holds, for showing (3.7) it suffices to prove that

$$\langle \psi_1(u, v), \psi_2(u, v), \psi_3(u, v) \rangle_{\boldsymbol{R}} \supset u^2\boldsymbol{R} \oplus uv\boldsymbol{R} \oplus v^2\boldsymbol{R}.$$

This is equivalent to

$$\det \begin{pmatrix} (\psi_1)_{uu} & (\psi_1)_{uv} & (\psi_1)_{vv} \\ (\psi_2)_{uu} & (\psi_2)_{uv} & (\psi_2)_{vv} \\ (\psi_3)_{uu} & (\psi_3)_{uv} & (\psi_3)_{vv} \end{pmatrix}(\mathbf{0}) = \det \begin{pmatrix} (\psi_1)_{uu} & (\psi_1)_{uv} & (\psi_1)_{vv} \\ \varphi_{uuu} & \varphi_{uuv} & \varphi_{uvv} \\ \varphi_{uuv} & \varphi_{uvv} & \varphi_{vvv} \end{pmatrix}(\mathbf{0}) \neq 0.$$

Since $\psi_1(u, v) = -\nu_4(u, v, 0) + \langle f, \nu_t \rangle (u, v, 0)$ holds, we have

$$\begin{aligned} (\psi_1)_{uu} &= -\langle \nu_{uu}, f_t \rangle + \langle f_{uu}, \nu_t \rangle, \\ (\psi_1)_{uv} &= -\langle \nu_{uv}, f_t \rangle + \langle f_{uv}, \nu_t \rangle, \\ (\psi_1)_{vv} &= -\langle \nu_{vv}, f_t \rangle + \langle f_{vv}, \nu_t \rangle \end{aligned} \tag{3.8}$$

at $\mathbf{0}$. Taking derivatives of $\langle f_t, \nu \rangle = \langle f_u, \nu \rangle = \langle f_v, \nu \rangle = 0$, we have

$$\begin{aligned} &-\langle \nu_{uu}, f_t \rangle + \langle f_{uu}, \nu_t \rangle = \langle f_{tu}, \nu_u \rangle, \\ &-\langle \nu_{uv}, f_t \rangle + \langle f_{uv}, \nu_t \rangle = (1/2)(\langle f_{tu}, \nu_v \rangle + \langle f_{tv}, \nu_u \rangle) = \langle f_{tu}, \nu_v \rangle = \langle f_{tv}, \nu_u \rangle, \\ &-\langle \nu_{vv}, f_t \rangle + \langle f_{vv}, \nu_t \rangle = \langle f_{tv}, \nu_v \rangle \end{aligned} \tag{3.9}$$

at $\mathbf{0}$. Hence by (3.8) and (3.9), it holds that

$$\begin{pmatrix} (\psi_1)_{uu} & (\psi_1)_{uv} & (\psi_1)_{vv} \\ \varphi_{uuu} & \varphi_{uuv} & \varphi_{uvv} \\ \varphi_{uuv} & \varphi_{uvv} & \varphi_{vvv} \end{pmatrix}(\mathbf{0}) = \begin{pmatrix} \langle f_{tu}, \nu_u \rangle & \langle f_{tu}, \nu_v \rangle & \langle f_{tv}, \nu_v \rangle \\ \langle f_{uu}, \nu_u \rangle & \langle f_{uv}, \nu_u \rangle & \langle f_{uv}, \nu_v \rangle \\ \langle f_{uv}, \nu_u \rangle & \langle f_{uv}, \nu_v \rangle & \langle f_{vv}, \nu_v \rangle \end{pmatrix}(\mathbf{0}).$$

This is the Jacobi matrix of the map

$$(\langle f_u, \nu_u \rangle, \langle f_u, \nu_v \rangle, \langle f_v, \nu_v \rangle)(t, u, v) : \boldsymbol{R}^3 \to \boldsymbol{R}^3.$$

This proves the assertion.

4. Proof of Theorem 1.1. To prove criteria for the three dimensional case, we first show the following lemma.

LEMMA 4.1. *Let $G(u, v, x, y, z) : (\boldsymbol{R}^2 \times \boldsymbol{R}^3, (\mathbf{0}, \mathbf{0})) \to (\boldsymbol{R}, 0)$ be an unfolding of a function $u^3 + uv^2$ (resp. $u^3 - uv^2$). Suppose that G is a Morse family of hypersurfaces and that the regular set of its discriminant set is dense in $(\boldsymbol{R}^3, \mathbf{0})$. Then G is P-$\mathcal{K}$ equivalent to*

$$\mathcal{V}_0(u, v, x, y, z) = u^3 + uv^2 + xu + yv + z, \quad (\textit{resp. } \mathcal{V}_0 = u^3 - uv^2 + xu + yv + z).$$

PROOF. Since the unfolding $\mathcal{V}$ in (3.3) is a $\mathcal{K}$-versal unfolding of $u^3 \pm uv^2$, there exists a map $(g_1, g_2, g_3, g_4) : \boldsymbol{R}^3 \to \boldsymbol{R}^4$ such that G is P-$\mathcal{K}$ equivalent to

$$G_1 = u^3 \pm uv^2 + g_1(x, y, z)u^2 + g_2(x, y, z)u + g_3(x, y, z)v + g_4(x, y, z).$$

Since the condition and the assertion of the lemma do not depend on the P-$\mathcal{K}$ equivalence, we may suppose that G is equal to G_1. Moreover, $(g_2, g_3, g_4) : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ is an immersion, because G is a Morse family of hypersurfaces. Thus G is P-$\mathcal{K}$ equivalent to

$$G_2 = u^3 \pm uv^2 + g_1(x, y, z)u^2 + xu + yv + z\,.$$

Hence we may suppose G is equal to G_2. Now we consider the following function

$$\overline{G}(u, v, t, x, y, z) = u^3 \pm uv^2 + (t - g_1(x, y, z))u^2 + xu + yv + z\,.$$

The following is the special case of Zakalyukin's lemma [20, Theorem 1.4]. Note that we are considering the $\mathcal{K}$-versal unfolding $\mathcal{V}$ defined as (3.3), the neutral subspace of $\mathcal{V}$ is empty (see [20, Section 1.3]).

LEMMA 4.2. *For the $\mathcal{K}$-versal unfolding $\mathcal{V} : (\boldsymbol{R}^2 \times \boldsymbol{R}^4, \boldsymbol{0}) \to (\boldsymbol{R}, 0)$ in* (3.3), *and a function $\sigma : (\boldsymbol{R}^4, \boldsymbol{0}) \to (\boldsymbol{R}, 0)$ satisfying $\partial\sigma/\partial t(\boldsymbol{0}) \neq 0$, there exists a diffeomorphism-germ $\Theta : (\boldsymbol{R}^4, \boldsymbol{0}) \to (\boldsymbol{R}^4, \boldsymbol{0})$ such that $\Theta(\mathcal{D}_{\mathcal{V}}) = \mathcal{D}_{\mathcal{V}}$ and $\sigma \circ \Theta(t, x, y, z) = t$.*

Let us continue the proof of Lemma 4.1. Applying Lemma 4.2 to $\sigma = t - g_1(x, y, z)$, we know there exists a diffeomorphism-germ $\Theta : (\boldsymbol{R}^4, \boldsymbol{0}) \to (\boldsymbol{R}^4, \boldsymbol{0})$ such that $\Theta(\mathcal{D}_{\mathcal{V}}) = \mathcal{D}_{\mathcal{V}}$ and

$$(t - g_1(x, y, z)) \circ \Theta = t\,.$$

Let $\Psi : (\boldsymbol{R}^4, \boldsymbol{0}) \to (\boldsymbol{R}^4, \boldsymbol{0})$ be a diffeomorphism-germ defined by

$$\Psi(t, x, y, z) = (t - g_1(x, y, z), x, y, z)\,.$$

Then we have $\mathcal{V} \circ \Psi = \overline{G}$. We also define a diffeomorphism-germ by $\tilde{\Theta} = \Psi \circ \Theta$, and then we have

$$\tilde{\Theta}(\mathcal{D}_{\mathcal{V}}) = \Psi \circ \Theta(\mathcal{D}_{\mathcal{V}}) = \Psi(\mathcal{D}_{\mathcal{V}}) = \mathcal{D}_{\overline{G}}\,.$$

Hence $\mathcal{D}_{\mathcal{V}}$ and $\mathcal{D}_{\overline{G}}$ are diffeomorphic. On the other hand, defining the projection $\pi : \boldsymbol{R}^4 \to \boldsymbol{R}$ as $\pi(t, x, y, z) = t$, we have

$$\pi \circ \tilde{\Theta} = \pi \circ \Psi \circ \Theta = (t - g_1(x, y, z)) \circ \Theta = t.$$

Thus it holds that $\pi \circ \tilde{\Theta} = \pi$. Hence for each t, it holds that $\mathcal{D}_{\mathcal{V}} \cap \{t = 0\}$ and $\mathcal{D}_{\overline{G}} \cap \{t = 0\}$ are also diffeomorphic. Since $\mathcal{D}_{\overline{G}} \cap \{t = 0\} = \mathcal{D}_G$ and both the regular sets $\mathcal{D}_{\mathcal{V}} \cap \{t = 0\}$ and $\mathcal{D}_G$ are dense in $(\boldsymbol{R}^2, \boldsymbol{0})$, it follows that $\mathcal{V}(u, v, x, y, z, 0) = \mathcal{V}_0(u, v, x, y, z)$ and $G(u, v, x, y, z)$ are P-$\mathcal{K}$ equivalent. □

Here, we calculate the discriminant set of $\mathcal{V}_0$:

$$\mathcal{D}_{\mathcal{V}_0} = \{(x, y, z); x = -3u^2 \mp v^2,\ y = \mp 2uv,\ z = 2u^3 \pm 2uv^2\}\,.$$

This is a parameterization of a $D_4^{\pm}$ singularity of a front. By the same arguments as in Section 3, for proving Theorem 1.1 it suffices that we construct a function φ and an unfolding Φ satisfying the conditions in Lemma 4.1.

4.1. Unfolding of a given front. Let $f : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}^3, \boldsymbol{0})$ be a front and ν its unit normal vector, satisfying the conditions (a) and (b) of Theorem 1.1. Consider the maps

$$\begin{aligned}\Phi(u, v, x, y, z) &= \big\langle \big(f_1(u, v), f_2(u, v), f_3(u, v)\big) - (x, y, z)\nu(u, v)\big\rangle \\ &= \langle (f_1 - x, f_2 - y, f_3 - z), \nu\rangle\,, \\ \varphi(u, v) &= \Phi(u, v, 0, 0, 0).\end{aligned}$$

By the same argument as in the case of $\boldsymbol{R}^4$, we see that the discriminant set $\mathcal{D}_\Phi$ coincides with the image of f. Again by the same calculation as in the case of $\boldsymbol{R}^4$, we can show that φ is right equivalent to $u^3 + uv^2$ (resp. $u^3 - uv^2$) if and only if $\det(\mathrm{Hess}\,\lambda(\boldsymbol{0})) < 0$ (resp. $\det(\mathrm{Hess}\,\lambda(\boldsymbol{0})) > 0$). Since f is a front, an unfolding $\Phi(u, v, x, y, z)$ of φ is a Morse family of hypersurfaces. By the condition for the determinant of the Hessian matrix, the regular set of f is dense in $(\boldsymbol{R}^2, \boldsymbol{0})$. Thus the regular set of $\mathcal{D}_\Phi$ is also dense. Hence by Lemma 4.1, $\Phi(u, v, x, y, z)$ is P-$\mathcal{K}$-equivalent to $\mathcal{V}_0$. This proves Theorem 1.1.

5. Examples. Here we give two examples where the criteria for typical $D_4^\pm$ singularities appear. Let us consider the map $(u, v) \mapsto (uv, u^2 + 3v^2, u^2 v + v^3)$. This has a D_4^+ singularity at $\boldsymbol{0}$. Set $\nu = (2u, v, -2)/\delta$ $(\delta = (4u^2 + v^2 + 4)^{1/2})$. Then by (2.1), we have $\lambda = (4u^2 + 4u^4 - 12v^2 - 11u^2v^2 - 3v^4)/\delta$. Thus we have $d\lambda(\boldsymbol{0}) = \boldsymbol{0}$ and

$$\det(\mathrm{Hess}\,\lambda(\boldsymbol{0})) = \frac{1}{4}\det\begin{pmatrix} 8 & 0 \\ 0 & -24\end{pmatrix} < 0\,.$$

Now consider the map $(u, v) \mapsto (uv, u^2 - 3v^2, u^2 v - v^3)$. This has a D_4^- singularity. Set $\nu = (2u, v, -2)/\delta$. Then we have $\lambda = (4u^2 + 4u^4 + 12v^2 + 13u^2v^2 + 3v^4)/\delta$. Thus we also have $d\lambda(\boldsymbol{0}) = \boldsymbol{0}$ and

$$\det(\mathrm{Hess}\,\lambda(\boldsymbol{0})) = \frac{1}{4}\det\begin{pmatrix} 8 & 0 \\ 0 & 24\end{pmatrix} > 0\,.$$

6. Application. In this section, as an application of Theorem 1.1, we study the singular curvature of cuspidal edges near a D_4^+ singularity in $\boldsymbol{R}^3$. First, we give a brief review of the singular curvature of cuspidal edges, as given in [16]. Let $f : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}^3, \boldsymbol{0})$ be a cuspidal edge and ν its unit normal vector. Then there exists a regular curve $\gamma(t)$ passing through $\boldsymbol{0}$. Furthermore, we have a non-vanishing vector field $\eta(t)$ along $\gamma(t)$ such that $\eta(t) \in \ker df_{\gamma(t)}$ and $\big(\gamma'(t), \eta(t)\big)$ is a positively oriented frame field of $\boldsymbol{R}^2$ along γ. Then the *singular curvature* $\kappa_s(t)$ of the cuspidal edge $\gamma(t)$ is defined as follows [16]:

$$\kappa_s(t) = \mathrm{sgn}(d\lambda(\eta))\frac{\det\big(\hat{\gamma}'(t), \hat{\gamma}''(t), \nu\circ\gamma(t)\big)}{|\hat{\gamma}'(t)|^3}\,, \tag{6.1}$$

where $\hat{\gamma}(t) = f(\gamma(t))$. For the geometric meaning of the singular curvature, see [16].

There are four curves emanating from a D_4^+ singularity, each consisting of a cuspidal edge (see Figure 1). We study the properties of the singular curvatures of these four curves.

Let $f : (\boldsymbol{R}^2, \boldsymbol{0}) \to (\boldsymbol{R}^3, \boldsymbol{0})$ be a D_4^+ singularity. Then by Theorem 1.1, we have a regular curve $\gamma : ((-\varepsilon, \varepsilon), 0) \to (\boldsymbol{R}^2, \boldsymbol{0})$ such that $\mathrm{image}\,\gamma \subset S(f)$. Set $\hat{\gamma}(t) = f(\gamma(t))$. Then the following proposition holds.

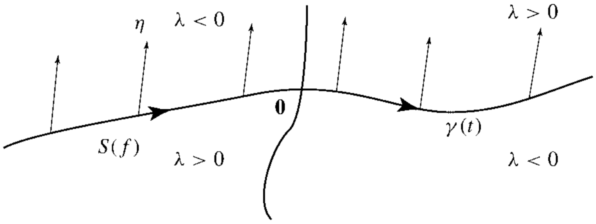

FIGURE 2. Singular set and the null vector field.

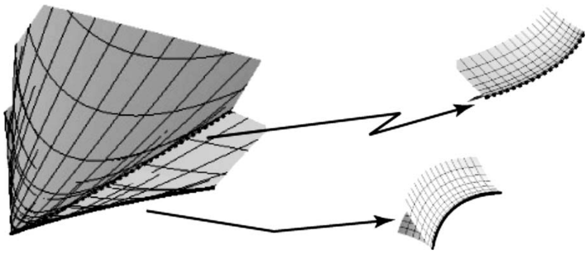

FIGURE 3. The shapes of cuspidal edges nearthe D_4^+ singularity of f.

PROPOSITION 6.1. *If*

$$\det(\hat{\gamma}''(0), \hat{\gamma}'''(0), \nu(\mathbf{0})) \neq 0$$

holds, then the singular curvature $\kappa_s(t)$ of $\gamma(t)$, approaching $\gamma(0)$ from both sides, diverges. Moreover, it diverges with opposite sign on opposite sides.

PROOF. Since $\hat{\gamma}$ at 0 is right-left equivalent to the germ $(t^2, t^3, 0)$ at $t = 0$, the signs of

$$\lim_{t\to+0} \det(\hat{\gamma}', \hat{\gamma}'', \nu(\gamma)), \qquad \lim_{t\to-0} \det(\hat{\gamma}', \hat{\gamma}'', \nu(\gamma))$$

are the same. On the other hand, by Theorem 1.1, $\mathrm{sgn}(d\lambda(\eta))$ when $t > 0$ and when $t < 0$ are opposite to each other (see Figure 2). Applying L'Hospital's rule to the formula (6.1) twice, we have the conclusion. □

Here we give an example applying this proposition:

EXAMPLE 6.2. The front-germ $f(u, v) = (uv, u^2 + 3v^2, u^2(1 + v) + v^2(3 + v))$ at $\mathbf{0}$ has a D_4^+ singularity. The front f has two curves $\gamma_\pm(t) = \big(\pm\sqrt{3}t, t\big)$ passing through $\mathbf{0}$, consisting of cuspidal edges. Set $\gamma(t) = \gamma_+(t)$ and $\hat{\gamma}(t) = f(\gamma(t))$. Then we have $\det(\hat{\gamma}''(0), \hat{\gamma}'''(0), \nu(\mathbf{0})) = -24\sqrt{6} \neq 0$. The singular curvature is calculated as follows:

$$\kappa_s(t) = \mathrm{sgn}(t)\frac{t^2(2 - 11t)}{|t^2(25 + 24t + 12t^2)|^{3/2}}\,.$$

Thus $\lim_{t\to+0} \kappa_s(t) = +\infty$ and $\lim_{t\to-0} \kappa_s(t) = -\infty$ hold (see Figure 3).

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DEPARTMENT OF MATHEMATICS
FACULTY OF EDUCATION
GIFU UNIVERSITY
YANAGIDO 1–1, GIFU, 501–1193
JAPAN

E-mail address: ksaji@gifu-u.ac.jp