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Critical Behavior in Anisotropic Cubic Systems with Short-Range Interaction*)

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Critical behavior in anisotropic cubic systems with the short-range interaction is studied by means of the Callan-Symanzik equations. As the static critical behavior, the stability of fixed points, the critical exponents η^C, γ^C, ϕ_s^C and ϕ_c^C, etc. are investigated. As the dynamic critical behavior, the dynamic critical exponent z_ϕ is derived on the basis of the time dependent Ginzburg-Landau stochastic model. New results are a correction term of order ε^3 in η^C, the crossover index ϕ_s^C and the dynamic critical exponent z_ϕ.

§ 1. Introduction

Critical behavior in the second order phase transition has been investigated considerably in detail.[1] One of the most interesting problems in this phase transition is what effects the anisotropies in (magnetic) materials affect the critical behavior. In this article effects of the cubic anisotropy, which many magnetic materials have as a reflection of the lattice symmetry, to the critical behavior will be studied. For a cubic-symmetry lattice the lowest-order single-ion terms have the form

$$\mathcal{H}_c(x) = (4!)^{-1} g_{0c} \sum_{\alpha=1}^{N} S_{0\alpha}^4 \tag{1·1}$$

in the N-component spin system. The total Hamiltonian of the system is specified by

$$\mathcal{H}(x) = 2^{-1}[(\nabla \mathbf{S}_0)^2 + m_0^2 \mathbf{S}_0^2] + (4!)^{-1} g_{0S}(\mathbf{S}^2)^2 + \mathcal{H}_c(x) \tag{1·2}$$

with $\mathbf{S}_0^2 \equiv \sum_{\alpha=1}^N S_{0\alpha}^2$ and $g_{0s}>0$. Since higher order terms in S do not contribute to the critical behavior in the second order phase transition in the three-dimensional space (the S^6-terms give at most them logarithmic contributions) and the S^4-terms with different-space dependences ($S^2(x)S^2(y)$, etc.) are not most dominant their terms will be neglected. The constants g_{0s} and g_{0c} have also no space-dependence as the most dominant term. By the sign of g_{0c} the direction in which the spins tend to align, i.e., the easy axis is determined; the diagonal directions $(\pm 1, \pm 1, \cdots)$ for $g_{0c}>0$ and the cubic axes $(0, \cdots, \pm 1, 0, \cdots)$ for $g_{0c}<0$. In case of $g_{0c}<0$ a first order transition is shown in zero field.[2] In this article the case $g_{0c}>0$ is discussed.

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The case $g_{0c}>0$ has been studied by several authors.[3),4)] Wilson and Fisher considered the system for $N=2$ and concluded the existence of an additional Ising-like fixed point. Cowley and Bruce (for $N=3$) and Wegner (for general N) studied the stability of the Heisenberg fixed point ($g_{0c}=0$) and concluded that its fixed point is stable for $N=3$. Aharony[4)] studied the stability of the anisotropic cubic system with general N with the aid of Wilson's renormalization-group approach and concluded that its fixed point is stable for $N=3$ and $\varepsilon=1$, and derived the critical exponents η^C, ν^C, crossover index $\phi_c{}^C$ to order ε^2 and the equation of state to order ε.

In this article the static and dynamic critical behavior of the anisotropic cubic system with general N is studied by means of the Callan-Symanzik equations. In § 2 the static properties are investigated; the stability of the fixed points is studied to order ε^2 and the critical exponents η^C to *order* ε^3, γ^C, crossover indices $\phi_S{}^C$ and $\phi_c{}^C$ to order ε^2 and the equation of state to order ε. In § 3 the dynamic properties are studied on the basis of the time dependent Ginzburg-Landau (TDGL) stochastic model and the dynamic critical exponent z_ϕ is derived to order ε^2. The new results are the ε^3-correction term for η, $\phi_s{}^C$ and z_ϕ.

§ 2. Static critical behavior

At first let us consider the case without an external field. The effective Hamiltonian given in (1·2) can be rewritten by the use of the renormalized quantities[1),5)] (i.e., fields S, coupling constants g_s, g_c (>0) and mass m^2) and the renormalization constants (i.e., Z_{1s}, Z_{1c} and Z_3 for the coupling constants g_s, g_c and the field S), as

$$\begin{aligned}\mathscr{H}(x) &= 2^{-1}[(\nabla \boldsymbol{S})^2+m^2\boldsymbol{S}^2]+(4!)^{-1}[g_s(\boldsymbol{S}^2)^2+g_c\sum_{\alpha=1}^{N}S_\alpha{}^4]\\ &\quad +2^{-1}[(Z_3-1)(\nabla \boldsymbol{S})^2+(Z_3m_0{}^2-m^2)\boldsymbol{S}^2]+(4!)^{-1}[g_s(Z_{1s}-1)(\boldsymbol{S}^2)^2\\ &\quad +g_c(Z_{1c}-1)\sum_{\alpha=1}^{N}S_\alpha{}^4],\end{aligned} \tag{2·1}$$

where the last two terms are counter terms determined by the following normalization conditions for the amputated two- and four-point one particle irreducible (1-P.I.) vertex functions:

$$\begin{aligned}&\Gamma^{(2)}(p)|_{p^2=0}=m^2,\ \partial\Gamma^{(2)}(p)/\partial p^2|_{p^2=\mu^2}=1\,,\\ &\Gamma_w{}^{(4)}(0000)_{sp(\mu)}=\mu^\varepsilon g_w\,,\quad w=s,\ c,\\ &\Gamma^{(1,\ 2)}(0;00)=1\,,\quad \Gamma^{(2)}_{\phi\phi}(0;00)=1\ \text{with}\ \varepsilon\equiv 4-d\,.\end{aligned} \tag{2·2}$$

By expanding the two- and four-point vertex functions in a power series of g_s and g_c according to the Feynman rule and matching them with the normalization conditions (2·2), the renormalization constants Z_{1s}, Z_{1c} and Z_3 are obtained in a power expansion form as

$$Z_{1w} \equiv 1 + Z_{1w}^{(2)} + Z_{1w}^{(3)} + O(g^3); \quad Z_{1w}^{(2)} \equiv z_{1w}^{(2)}(1,0)\hat{g}_s + z_{1w}^{(2)}(0,1)\hat{g}_c\,,$$
$$Z_{1w}^{(3)} \equiv z_{1w}^{(3)}(2,0)\hat{g}_s{}^2 + z_{1w}^{(3)}(1,1)\hat{g}_s\hat{g}_c + z_{1w}^{(3)}(0,2)\hat{g}_c{}^2 \quad \text{for } w=s,\ c \qquad (2\cdot3)$$

and

$$Z_3 \equiv 1 + Z_3{}^{(2)} + Z_3{}^{(3)} + O(g^4);$$
$$Z_3{}^{(2)} \equiv z_3{}^{(2)}(2,0)\hat{g}_s{}^2 + z_3{}^{(2)}(1,1)\hat{g}_s\hat{g}_c + z_3{}^{(2)}(0,2)\hat{g}_c{}^2\,,$$
$$Z_3{}^{(3)} \equiv z_3{}^{(3)}(3,0)\hat{g}_s{}^3 + z_3{}^{(3)}(2,1)\hat{g}_s{}^2\hat{g}_c + z_3{}^{(3)}(1,2)\hat{g}_s\hat{g}_c{}^2 + z_3{}^{(3)}(0,3)\hat{g}_c{}^3\,, \qquad (2\cdot4)$$

where

$$z_{1w}^{(2)}(1,0) \equiv (N+8)a/6,\ 2a; \quad z_{1w}^{(2)}(0,1) \equiv a,\ 3a/2;$$
$$z_{1w}^{(3)}(2,0) \equiv (N+8)^2a^2/36 - (5N+22)b/9,\ (N+20)a^2/6 - (N+14)b/3;$$
$$z_{1w}^{(3)}(1,1) \equiv (N+12)a^2/4 - 4b,\ 11a^2/2 - 8b;$$
$$z_{1w}^{(3)}(0,2) \equiv 5a^2/4 - b,\ 9a^2/4 - 3b; \quad \text{for } w=s,c, \text{ respectively},$$

and

$$z_3{}^{(2)}(2,0) \equiv (N+2)c/18;\ z_3{}^{(2)}(1,1) \equiv c/3;\ z_3{}^{(2)}(0,2) \equiv c/6;$$
$$z_3{}^{(3)}(3,0) \equiv (N+2)(N+8)d/54;\ z_3{}^{(3)}(2,1) \equiv (N+8)d/6;$$
$$z_3{}^{(3)}(1,2) \equiv 3d/2;\ z_3{}^{(3)}(0,3) \equiv d/2\,. \qquad (2\cdot5)$$

Here $\hat{g}_w \equiv \hat{s}g_w$ for $w=s,c$ with $\hat{s} \equiv 2\pi^{d/2}/\{\Gamma(d/2)(2\pi)^d\}$. The constants a, b, c and d in (2·5) describe the contributions of the Feynman diagrams expanded in powers of ε as

$$a \equiv (2\pi)^{-d}\int d^dk(k^2+1)^{-2}/\hat{s} = (1+\varepsilon/2)/\varepsilon + O(\varepsilon),$$
$$b \equiv (2\pi)^{-2d}\int d^dk_1d^dk_2\{(k_1{}^2+1)^2(k_2{}^2+1)[(\mathrm{k}_1+k_2)^2+1]\}^{-1}/\hat{s}^2 - a^2/2$$
$$= 1/(4\varepsilon) + O(1),$$
$$c \equiv (2\pi)^{-2d}d/dp^2\int d^dk_1d^dk_2\{(k_1{}^2+1)(k_2{}^2+1)[(p+k_1+k_2)^2+1]\}^{-1}|_{p^2=\mu^2}/\hat{s}^2$$
$$= -[1+\varepsilon(5/4-l)]/(8\varepsilon) + O(1),$$
$$d \equiv ab - 1/2(2\pi)^{-3d}d/dp^2\int d^dk_1[(p+k_1)^2+1]^{-1}\left\{\int d^dk_2(k_2{}^2+1)^{-1}\right.$$
$$\left.\times[(k_1+k_2)^2+1]^{-1}\right\}^2|_{p^2=\mu^2}/\hat{s}^3 = -(1+\varepsilon 5/4)/(24\varepsilon^2) + O(1). \qquad (2\cdot6)$$

To study the critical behavior the amputated 1-P.I. vertex functions $\Gamma^{(n)}(\{p\}; g_s, g_c, \mu)$ without insertion of any composite field, and $\Gamma^{(1,n)}(q; \{p\}; g_s, g_c, \mu)$ and $\Gamma^{(n)}_{\phi\phi}(q; \{p\}; g_s, g_c, \mu)$ with insertion of a composite field $S^2(x)$ and $S_\alpha(x)S_\beta(x)$ $(\alpha\neq\beta)$, respectively, are needed. The Callan-Symanzik equations for these vertex functions are described as

$$[\mu\partial/\partial\mu+\beta_s\partial/\partial g_s+\beta_c\partial/\partial g_c-\gamma_3 n/2]\Gamma^{(n)}=\alpha m^2\Gamma^{(1,n)},$$
$$[\mu\partial/\partial\mu+\beta_s\partial/\partial g_s+\beta_c\partial/\partial g_c-\gamma_3(n/2-1)-\gamma_4]\Gamma^{(1,n)}=\alpha m^2\Gamma^{(2,n)},$$
$$[\mu\partial/\partial\mu+\beta_s\partial/\partial g_s+\beta_c\partial/\partial g_c-\gamma_3(n/2-1)-\gamma_5]\Gamma^{(n)}_{\phi\phi}=\alpha m^2\Gamma^{(1,n)}_{\phi\phi}, \quad (2\cdot7)$$

where

$$\beta_s(g_s,g_c)=\mu\partial g_s/\partial\mu|_{g_{0s},g_{0c},\Lambda}=-\varepsilon[\partial\ln(g_sZ_{1s}/Z_3^2)/\partial g_s]^{-1},$$
$$\beta_c(g_s,g_c)=\mu\partial g_c/\partial\mu|_{g_{0s},g_{0c},\Lambda}=-\varepsilon[\partial\ln(g_cZ_{1c}/Z_3^2)/\partial g_c]^{-1},$$
$$\gamma_j(g_s,g_c)=\mu\partial\ln Z_j(g_s,g_c)/\partial\mu|_{g_{0s},g_{0c},\Lambda}=\beta_s\partial\ln Z_j/\partial g_s+\beta_c\partial\ln Z_j/\partial g_c,$$
$$\text{for}\quad j=3,4,5. \quad (2\cdot8)$$

From the two coupled equations

$$\beta_s\partial/\partial g_s\ln g_sZ_s+\beta_c\partial/\partial g_c\ln g_sZ_s=-(\varepsilon-2\gamma_3),$$
$$\beta_s\partial/\partial g_s\ln g_cZ_c+\beta_c\partial/\partial g_c\ln g_cZ_c=-(\varepsilon-2\gamma_3), \quad (2\cdot9)$$

the coefficient functions β_s and β_c can be determined to order $g_s{}^p g_c{}^q\varepsilon^r$ with $p+q+r\leq3$ as

$$\beta_w(g_s,g_c)=-\varepsilon\hat{g}_w\{1-[\hat{g}_sz^{(2)}_{\beta w}(1,0)+\hat{g}_cz^{(2)}_{\beta w}(0,1)]$$
$$-[\hat{g}_s{}^2z^{(3)}_{\beta w}(2,0)+\hat{g}_s\hat{g}_cz^{(3)}_{\beta w}(1,1)+\hat{g}_c{}^2z^{(3)}_{\beta w}(0,2)]\}+O(g^4), \quad (2\cdot10)$$

where

$$z^{(2)}_{\beta w}(1,0)\equiv(N+8)a/6,\ 2a;\quad z^{(2)}_{\beta w}(0,1)\equiv a,\ 3a/2;$$
$$z^{(3)}_{\beta w}(2,0)\equiv-2(5N+22)b/9-2(N+2)c/9,\ -2(N+14)b/3-2(N+2)c/9;$$
$$z^{(3)}_{\beta w}(1,1)\equiv-8b-4c/3,\ -16b-4c/3;\quad z^{(3)}_{\beta w}(0,2)\equiv-2b-2c/3,\ -6b-2c/3$$
$$\text{for}\quad w=s,c,\ \text{respectively.} \quad (2\cdot11)$$

The renormalization constants Z_4 and Z_5 associated with the vertex functions with insertion of a composite operator S^2 and $S_\alpha S_\beta(\alpha\neq\beta)$, respectively, are obtained by the Feynman rule and the last two normalization conditions in (2·2) in the same way as for Z_{1s}, Z_{1c} and Z_3:

$$Z_j{}^{-1}\equiv1+\tilde{Z}_j{}^{(2)}+\tilde{Z}_j{}^{(3)}+O(g^3);\quad \tilde{Z}_j{}^{(2)}\equiv\tilde{z}_j{}^{(2)}(1,0)\hat{g}_s+\tilde{z}_j{}^{(2)}(0,1)\hat{g}_c,$$
$$\tilde{Z}_j{}^{(3)}\equiv\tilde{z}_j{}^{(3)}(2,0)\hat{g}_s{}^2+\tilde{z}_j{}^{(3)}(1,1)\hat{g}_s\hat{g}_c+\tilde{z}_j{}^{(3)}(0,2)\hat{g}_c{}^2, \quad (2\cdot12)$$

where

$$\tilde{z}_j{}^{(2)}(1,0)\equiv-(N+2)a/6,\ -a/3;\ \tilde{z}_j{}^{(2)}(0,1)\equiv-a/2,\ 0;$$
$$\tilde{z}_j{}^{(3)}(2,0)\equiv(b-a^2/2)(N+2)/6,\ (b-a^2/2)(N+6)/18;$$
$$\tilde{z}_j{}^{(3)}(1,1)\equiv(b-a^2/2),\ (b-a^2/2)/3;\quad \tilde{z}_j{}^{(3)}(0,2)\equiv(b-a^2/2)/2,\ 0$$
$$\text{for } j=4,5,\ \text{respectively.} \quad (2\cdot13)$$

The zeros $\hat{g}_{\infty s}$ and $\hat{g}_{\infty c}$ of order ε of these $\beta_w(\hat{g}_s, \hat{g}_c)=0$ for $w=s$, c are calculated as

(i) Gaussian: $\hat{g}^G_{\infty s}=\hat{g}^G_{\infty c}=0$;

(ii) Ising: $\hat{g}^I_{\infty s}=0$, $\hat{g}^I_{\infty c}=\varepsilon 2/3[1+\varepsilon(17/27-1/2)]+O(\varepsilon^3)$;

(iii) Heisenberg: $\hat{g}^H_{\infty s}=\varepsilon 6(N+8)^{-1}[1+\varepsilon\{3(3N+14)(N+8)^{-2}-1/2\}]$
$+O(\varepsilon^3)$, $\hat{g}^H_{\infty c}=0$;

(iv) Cubic: $\hat{g}^C_{\infty s}=\varepsilon 2/N[1+\varepsilon\{(N-1)(106-19N)/(27N^2)-1/2\}]+O(\varepsilon^3)$,

$$\hat{g}^C_{\infty c}=\varepsilon 2/(3N)[N-4+\varepsilon\{(N-1)(17N^2+110N-424)/(27N^2) - (N-4)/2\}]+O(\varepsilon^3). \qquad (2\cdot 14)$$

The eigenvalues for the matrix defined by $B_{ww'}\equiv\partial\beta_w/\partial g_{w'}|_{g=g_\infty}$ are obtained by the use of the linearized relations (2·10) about their fixed points as

(i) Gaussian: $\lambda_s{}^G=\lambda_c{}^G=-\varepsilon$;

(ii) Ising: $\lambda_s{}^I=-\varepsilon/3+\varepsilon^2 19/81+O(\varepsilon^3)$, $\lambda_c{}^I=\varepsilon-\varepsilon^2 17/27+O(\varepsilon^3)$;

(iii) Heisenberg: $\lambda_s{}^H=\varepsilon-\varepsilon^2(9N+42)(N+8)^{-2}+O(\varepsilon^3)$, $\lambda_c{}^H=\varepsilon(4-N)/(N+8)$
$-\varepsilon^2(5N^2+14N+152)(N+8)^{-3}+O(\varepsilon^3)$;

(iv) Cubic: $\lambda_1{}^C=\varepsilon-\varepsilon^2(N-1)(17N^2-4N+212)/[27N^2(N+2)]+O(\varepsilon^3)$,

$$\lambda_2{}^C=\varepsilon(N-4)/(3N)-\varepsilon^2(N-1)(19N^3-72N^2-660N+848) /[81N^3(N+2)]+O(\varepsilon^3). \qquad (2\cdot 15)$$

The fixed points and their eigenvalues for the Gaussian and Ising systems are of course independent of the spin component N. Since any fixed point with positive eigenvalues for the matrix $B_{ww'}$ is stable in the critical limit, the Gaussian system is stable for $\varepsilon<0$ and unstable for $\varepsilon\geq 0$. Let us discuss the stability of the fixed points to order ε^2. The Ising system is stable for $1.421<\varepsilon<1.588$ whose ε satisfies the relations $\lambda_s{}^I>0$ and $\lambda_c{}^I>0$. The Heisenberg system is stable for ε satisfying $0<\varepsilon<(4-N)(N+8)^2/(5N^2+14N+152)$ for $4/7<N<4$ or $0<\varepsilon<(N+8)^2/(9N+42)$ for $-2\leq N<4/7$. The stable regions for the typical cases are as follows: *For* $N=1$, $0<\varepsilon<1.421$; *for* $N=2$, $0<\varepsilon<1$; and *for* $N=3$, $0<\varepsilon<0.506$. The anisotropic cubic system is stable for $0<\varepsilon<y^{-1}$ for $N>N_{c2}$, $0<\varepsilon<x^{-1}$ for $4<N<N_{c2}$, $y^{-1}<\varepsilon<x^{-1}$ for $N_{c1}<N<4$ and $\varepsilon>y^{-1}$ for $0<N<1$ where $x\equiv(N-1)(17N^2-4N+212)/[27N^2(N+2)]$, $y\equiv(N-1)(19N^3-72N^2-660N+848)/[27N^2\times(N+2)(N-4)]$, and N_{c1} and N_{c2} are the values satisfying $x=y$ with $1.9<N_{c1}<2.0$ and $20<N_{c2}<20.1$. The typical-numerical results are as follows: *For* $N=2$, $1.421<\varepsilon<1.636$; *for* $N=3$, $0.479<\varepsilon<1.721$; and *for* $N=\infty$, $0<\varepsilon<1.421$. Let us summarize the important points in these results to order ε^2. The stability of the Gaussian fixed point is of course rigorous in order ε. There is no region of the stable fixed point for the Ising system in order ε but the Ising fixed point is stabilized by the second order terms in ε. The Heisenberg and anisotropic cubic systems are stable in order ε for $N<4$ and $N>4$, respectively, and for any positive ε(i.e., ε is not a function of N), but their stable regions are modified in order ε^2 and described by the inequalities depending on N and ε. That is, their stable regions depend on the order of the ε-expansion and the number of spin component N. Although ε-expansions usually give reasonable results on truncation at order ε^2, this approach is not always clearly justified. Let us consider the most realistic dimension $d=3$, i.e., $\varepsilon=1$ in order ε^2. The Gaussian and Ising systems are unstable. The Heisenberg system with $N\leq 2$ is stable but that with $N>2$ becomes unstable because the instability due to the anisotropic cubic interaction grows and the behavior of the cubic interaction dominates over the Heisenberg behavior. The cubic system is stable for the region mentioned above but otherwise unstable since the instability due to the isotropic interaction arises and suppresses the anisotropic cubic behavior. The cubic fixed point is degenerate with the Gaussian fixed point for $N=1$ and with the Ising fixed point for $N=2$, as pointed by Wilson and Fisher. It is also stable for $N=3$ whose region is contained in the third inequality ($N>N_{c3}$, $2.2<N_{c3}<2.3$).

Near the Heisenberg fixed point, the correlation length $\xi(t, g_c)$ scales with $g_c/t^{\phi_c{}^c}$ since the cubic-symmetric behavior is expected to appear characteristically for $t\equiv(T-T_c)/T_c\leq g_c{}^{1/\phi_c{}^c}$ where the crossover exponent $\phi_c{}^c$ is $\phi_c{}^c\equiv-\nu^H\lambda_c{}^H$. That is, the system under consideration shows cubic symmetric behavior only for small values of t(if $0<g_c<1$ is assumed) in case of $\lambda_c{}^H<0$ and approaches the Heisenberg behavior only for very small values of t in case of $\lambda_c{}^H>0$.

The other coefficient functions γ_3, γ_4 and γ_5 in (2·7) are obtained by the relations $\gamma_j(g_s, g_c)=\sum_{w=s,c}\beta_w(g_s, g_c)\partial\ln Z_j(g_s, g_c)/\partial g_w$ for $j=3, 4, 5$ as

$$\gamma_3(g_{\infty s}, g_{\infty c})=\varepsilon^2(N-1)(N+2)/(54N^2)+\varepsilon^3(N-1)(109N^3-222N^2+1728N-1696)/(5832N^4)+O(\varepsilon^4),$$

$$\gamma_4(g_{\infty s}, g_{\infty c})=-\varepsilon 2(N-1)/(3N)-\varepsilon^2(N-1)(-11N^2+160N-212)/(81N^3)+O(\varepsilon^3),$$

$$\gamma_5(g_{\infty s}, g_{\infty c})=-\varepsilon 2/(3N)+\varepsilon^2 2(65N^2-268N+212)/(162N^3)+O(\varepsilon^3). \quad (2\cdot16)$$

From these results the main critical exponents can be derived by the relations $\eta=\gamma_3(g_{\infty s}, g_{\infty c})$, $\gamma=(2-\eta)/[2-\eta+\gamma_4(g_{\infty s}, g_{\infty c})]$, $\phi_s{}^c=\gamma(d-d_{\phi\phi})/(2-\eta)$ with $d_{\phi\phi}\equiv d-2+\eta-\gamma_5(g_{\infty s}, g_{\infty c})$ and $\phi_c{}^c=-\nu^H\lambda_c{}^H$, as

$$\eta^C=\varepsilon^2(N-1)(N+2)/(54N^2)+\varepsilon^3(N-1)(109N^3-222N^2+1728N-1696)/(5832N^4)+O(\varepsilon^4),$$

$$\gamma^C=1+\varepsilon(N-1)/(3N)+\varepsilon^2(N-1)(7N^2+142N-212)/(162N^3)+O(\varepsilon^3),$$

$$\phi_s{}^C=1+\varepsilon(N-2)/(3N)+\varepsilon^2(N-2)(7N^2+196N-212)/(162N^3)+O(\varepsilon^3),$$

$$\phi_c{}^C=\varepsilon(N-4)/[2(N+8)]+\varepsilon^2(N^3+16N^2+4N+240)/[4(N+8)^3]+O(\varepsilon^3). \tag{2·17}$$

Let us show the numerical values for these critical exponents in the system with $N=3$ in the three-dimensional space ($\varepsilon=1$). The critical exponent η^C has the value 0.0206 and 0.0465 to order ε^2 and ε^3, respectively. The exponents γ^C, $\phi_s{}^C$ and $\phi_c{}^C$ to order ε and ε^2 are 1.2222, 1.3489; 1.1111, 1.2115; and −0.0455, 0.0340, respectively. The eigenvalues $\lambda_1{}^C$ and $\lambda_2{}^C$ have the values 1, 0.4189; −0.1111, 0.1206 to order ε and ε^2, respectively. In the spherical model ($N\to\infty$) in the three-dimensional space η^C is 0.0185 and 0.0372 to order ε^2 and ε^3, respectively. The exponents γ^C, $\phi_s{}^C$ and $\phi_c{}^C$ to order ε and ε^2 are 1.3333, 1.3765; 1.3333, 1.3765; and 0.5, 0.75, respectively. The eigenvalues $\lambda_1{}^C$ and $\lambda_2{}^C$ to order ε and ε^2 are 1, 0.3704; 0.3333, 0.0988, respectively. These may be compared with the critical exponents for the isotropic (Heisenberg-like) N-component spin system:[6] In case $N=3$ and $d=3$, $\eta=0.039$, $\gamma=1.347$ and $\phi_s=1.22$. In case $N=\infty$ and $d=3$, $\eta=0$, $\gamma=1.750$ and $\phi_s=1.75$. From these results it is very difficult to distinguish for $N=3$ between the exponents in the anisotropic cubic system and those in the isotropic N-component spin system. For $N=\infty$ it seems to be possible. In the former system the crossover phenomena arise.

The results for the cubic system obtained in (2·14) and (2·15) are essentially equivalent to those obtained by Aharony[4] except for the differences based on the used method. In the results of (2·17) γ^C and $\phi_c{}^C$ coincide with those obtained by Aharony, and η^C in order ε^3 and $\phi_s{}^C$ are new results.

Now let us consider the case in a magnetic field h with $g_s, g_c>0$:

$$\mathscr{H}=2^{-1}[(1+b)(\nabla \boldsymbol{S})^2+(m^2-a)\boldsymbol{S}^2]+(4!)^{-1}(g_s-c_s)(\boldsymbol{S}^2)^2 + (4!)^{-1}(g_c-c_c)\sum_{\alpha=1}^{N}S_\alpha{}^4-\sum_{\alpha=1}^{N}h_\alpha S_\alpha\,, \tag{2·18}$$

where $a\equiv m^2-Z_3m_0{}^2$, $b\equiv Z_3-1$, $c_w\equiv(1-Z_{1w})g_w$ for $w=s,c$. The aim to study this system is to derive the equation of state near the critical point. The equation of state in order ε has been derived by Aharony[4] by means of a method similar to the one used by Brezin, Wallace and Wilson[7] when this author derived it and attacked it in second order in ε by means of the soft-renormalization procedure.[5] The equation of state obtained in order ε in a similar way to the one used for the quenched random systems[8] coincides with that by Aharony:

$$\begin{aligned} h/M^\delta=&x+1+\varepsilon/(3N^2)[2(N-1)(x+3)\ln(x+3)+(N-1)(N-2)\\ &\times\{x+3(N-2)/[2(N-1)]\}\ln\{x+3(N-2)/[2(N-1)]\}-\{6(N-1)\ln 3\\ &+3/2(N-2)^2\ln(3(N-2)/[2(N-1)])\}(x+1)+\{4(N-1)\ln 2+1/2(N-2)\\ &\times(N-4)\ln((N-4)/[2(N-1)])\}x]+O(\varepsilon^2) \end{aligned} \tag{2·19}$$

by choosing the standard normalizations: $h/M^\delta=1$ at $t=0$ and $t/M^{1/\beta}\equiv x=-1$ at

$h=0$, $T<T_c$ with $t\equiv m_0{}^2-m_c{}^2$ ($m_c{}^2$: critical mass), $\delta=3+\varepsilon+0(\varepsilon^2)$ and $1/\beta=2+\varepsilon\times(N+2)/(3N)+O(\varepsilon^2)$. This expression reflects the stability of the anisotropic cubic fixed point in order ε, i.e., (2·19) is valid for $N>4$ but otherwise the negative logarithmic term arises on the right-hand side of it. For the typical cases $N=5(\infty)$ the numerical results are as follows: $f(5)=6+\varepsilon 1.845$ (13.352), $f(4)=5+\varepsilon 1.354$ (10.731), $f(3)=4+\varepsilon 0.910$ (8.266), $f(2)=3+\varepsilon 0.522$ (5.987), $f(1)=2+\varepsilon 0.209$ (3.939), $f(0)=1+\varepsilon 0$ (0), $f(-1)=\varepsilon 0$ (0), respectively. For the Heisenberg system with $N=5$ (∞), $f(5)=6+\varepsilon 2.091$ (5.375), $f(4)=5+\varepsilon 1.548$ (4.024), $f(3)=4+\varepsilon 1.052$ (2.773), $f(2)=3+\varepsilon 0.615$ (1.648), $f(1)=2+\varepsilon 0.252$ (0.693), $f(0)=1+\varepsilon 0$ (0), $f(-1)=\varepsilon 0$ (0), respectively. In case $N=5$ the behavior in both the systems is qualitatively the same but in case $N=\infty$ apparently different because the number of easy axes, i.e., the number of the degenerate lowest states rapidly increases with N in the anisotropic cubic system. The static critical exponents to the next order will be published in the near future.

§ 3. Dynamic critical behavior

Let us study the dynamic critical behavior in the anisotropic cubic system based on the time dependent Ginzburg-Landau (TDGL) stochastic model:

$$\partial\phi_{0p}^{\alpha}/\partial t=-\Gamma_0(1+ib_0)\left[\delta\mathcal{H}(\phi_0,\phi_0{}^*)/\delta\phi_{0p}^{*\alpha}(t)-h_p{}^{\alpha}(t)\right]\eta_p{}^{\alpha}(t),$$

$$\mathcal{H}(\phi_0,\phi_0{}^*)\equiv\sum_p(r_0+p^2)\phi_{0p}^{\alpha}\phi_{0p}^{*\alpha}+(3!)^{-1}\mu^{\varepsilon}\Big[g_{0s}\int d^dx\,\phi_0{}^{\alpha}(x)\phi_0^{*\alpha}(x)\phi_0{}^{\beta}(x)\phi_0^{*\beta}(x)$$
$$+g_{0c}\int d^dx\sum_{\alpha=1}^{N}\phi_0{}^{\alpha}(x)\phi_0^{*\alpha}(x)\phi_0{}^{\alpha}(x)\phi_0^{*\alpha}(x)\Big],\tag{3·1}$$

where $\phi_{0p}^{\alpha}(t)$ and $h_p{}^{\alpha}$ stand for the α-th component ($\alpha=1,\cdots,N$) of the unrenormalized complex order parameters and of the space- and time-varying external fields, respectively. $\eta_p{}^{\alpha}$ is a Langevin noise source governed by a Gaussian probability distribution $\langle\eta^{\alpha}(x,t)\eta^{\beta}(y,t')\rangle=2\Gamma_0\delta_{\alpha\beta}\delta(x-y)\delta(t-t')$ and $\Gamma_0(1+ib_0)$ is an inverse complex time scale. The subscript 0 describes the unrenormalized quantity, the repeated subscripts α, β are summed over 1 to N. The space-time correlation function of the order parameter $G_{0p}^{\alpha\beta}(t)$, averaged over all $\eta_p{}^{\alpha}$ with the Gaussian distribution,[9] can be described in the classical limit by the use of renormalized quantities (without the subscript 0), as

$$G_p{}^{\alpha\beta}(\omega)\equiv\Xi^{-1}\int\{d\phi\}\{d\phi^*\}\exp[\mathcal{L}(\phi,\phi^*)]\phi_{p\omega}^{\alpha}\phi_{p\omega}^{*\alpha},$$

$$\Xi=\int\{d\phi\}\{d\phi^*\}\exp[\mathcal{L}(\phi,\phi^*)],$$

$$-\mathcal{L}\equiv\sum_{p,\omega}[-i\omega Z_{\phi}\{\Gamma_0 Z_{\Gamma}(1+ib)\}^{-1}+r+p^2]\phi_{p\omega}^{\alpha}\phi_{p\omega}^{*\alpha}+\sum_{p,\omega}[(r_0Z_{\phi}-r)$$
$$+p^2(Z_{\phi}-1)]\phi_{p\omega}^{\alpha}\phi_{p\omega}^{*\alpha}+(3!)^{-1}\mu^{\varepsilon}\sum_{\omega}[(g_s-c_s)\int\phi_{\omega_1}^{\alpha}(x)\phi_{\omega_2}^{*\alpha}(x)\phi_{\omega_3}^{\beta}(x)\phi_{\omega_1-\omega_2+\omega_3}^{*\beta}(x)$$

$$+ (g_c - c_c)\int\sum_{\alpha=1}^{N}\phi^{\alpha}_{\omega_1}(x)\phi^{*\alpha}_{\omega_2}(x)\phi^{\alpha}_{\omega_3}(x)\phi^{*\alpha}_{\omega_1-\omega_2+\omega_3}(x)] - \sum_{p,\omega}\{i\omega Z_\phi[\Gamma_0 Z_\Gamma(1+ib)]^{-1}$$
$$\times [Z_\Gamma(1+ib)/(1+ib_0) - 1]\}\phi^{\alpha}_{p\omega}\phi^{*\alpha}_{p\omega}\,, \qquad (3\cdot2)$$

where i abbreviates $is(\omega)\equiv i\,\mathrm{sgn}\,\mathrm{Im}(\omega)$ and the sum over ω takes the values $\omega=i2\pi l$, $l=0$, $\pm1,\cdots$. The last three counter terms are determined by the normalization conditions at criticality:

$$\Gamma^{(2)}(0;0)=0\,,\quad \partial\Gamma^{(2)}(0;p)/\partial p^2|_{p^2=\mu^2}=1\,,$$
$$\Gamma_w^{(4)}(0;0000)_{sp(\mu)}=\mu^{\varepsilon}g_w\,,\quad w=s,c,$$
$$\partial\Gamma^{(2)}(-i\zeta;p=0)/\partial(-i\zeta)=(1+ib)^{-1}\ \text{with}\ \zeta\equiv zZ_\phi(\Gamma_0 Z_\Gamma)^{-1}. \qquad (3\cdot3)$$

Let us assume $\partial\Gamma_0/\partial t$ constant and take into consideration the renormalization constants $Z_\phi(g_s, g_c)$ and $Z_\Gamma(g_s, g_c, \check{b})$. Further let us apply the normalization conditions to the R-G equation for $\Gamma^{(2)}(-i\zeta;p;ib,g_s,g_c,\mu)$

$$[\mu\partial/\partial\mu+W_s\partial/\partial g_s+W_c\partial/\partial g_c+iW_b\partial/\partial(ib)-\gamma_\phi-(\gamma_\Gamma-\gamma_\phi)\zeta\partial/\partial\zeta]\Gamma^{(2)}=0\,. \qquad (3\cdot4)$$

Then the asymptotic behavior of $\Gamma^{(2)}$ at the infrared stable fixed point $(g_{\infty s}, g_{\infty c}, b_\infty)$ can be expressed in the form $\Gamma^{(2)}(-i\zeta;p)\sim\mu^2(p/\mu)^{2-\gamma_\phi{}^*}\Phi(-i\zeta/\mu^2(\mu/p)^{2+\gamma_\Gamma{}^*-\gamma_\phi{}^*})$, where the quantities with the superscript * stand for the value at criticality. The dynamic critical exponent z_ϕ defined by $\omega_\phi\equiv(p/\mu)^{z_\phi}$ satisfies the scaling law $z_\phi=2+\gamma_\Gamma{}^*-\eta$ and is calculated by using the static fixed point values in the dynamic part of $\Gamma^{(2)}$ as

$$z_\phi-2=\eta[6\ln 4/3-1]+O(\varepsilon^3) \qquad (3\cdot5)$$

at the stable fixed point $(g_{\infty s}, g_{\infty c}, b_\infty)=(g_{\infty s}, g_{\infty c}, 0)$. In the nonconserved case the anisotropic cubic system has also the same behavior as Halperin, Hohenberg and Ma pointed out in the TDGL stochastic model with the isotropic N-component spin. The difference between the anisotropic cubic and the (Heisenberg like) isotropic systems in the dynamic critical behavior is very small for the system with $N=3$ in $d=3$ and increases with N in $d=3$.

§ 4. Concluding remarks

Critical behavior in the anisotropic cubic system with N-component spin has been studied by means of the Callan-Symanzik equations and the conclusions are as follows: The stability of the fixed points can be discussed by the positivity of the eigenvalues for the matrix $B_{ww'}$ and the stable regions for any fixed point depend on the truncated order in ε and the value of N. In three dimensions the anisotropic cubic system is stable for $N>N_{c3}(2.2<N_{c3}<2.3)$. The static critical exponents in $d=3$ are $\eta^C=0.0465$ (0.0372), $\gamma^C=1.3489$ (1.3765), $\phi_s{}^C=1.2115$ (1.3765) and $\phi_c{}^C=0.0340$ (0.75) for $N=3$ (∞, respectively). The equation of state in order ε coincides with that obtained by Aharony. The dynamic critical exponent z_ϕ in order ε^2 has the form $z_\phi - 2 = \eta[6\ln 4/3 - 1] + O(\varepsilon^3)$ which is the same form derived in the TDGL stochastic model with the isotropic N-component spin by Halperin, Hohenberg and Ma.

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