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Critical behavior in spherical and hyperbolic spaces

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Abstract

We study the effects of curved background geometries on the critical behavior of scalar field theory. In particular we concentrate on two maximally symmetric spaces: d-dimensional spheres and hyperboloids. In the first part of the paper, by applying the Ginzburg criterion, we find that for large correlation length the Gaussian approximation is valid on the hyperboloid for any dimension $d \geq 2$, while it is not trustable on the sphere for any dimension. This is understood in terms of various notions of effective dimension, such as the spectral and Hausdorff dimension. In the second part of the paper, we apply functional renormalization group methods to develop a different perspective on such phenomena, and to deduce them from a renormalization group analysis. By making use of the local potential approximation, we discuss the consequences of having a fixed scale in the renormalization group equations. In particular, we show that in the case of spheres there is no true phase transition, as symmetry restoration always occurs at large scales. In the case of hyperboloids, the phase transition is still present, but as the only true fixed point is the Gaussian one, mean field exponents are valid also in dimensions lower than four.

Contents

1 Introduction

Curved spaces in physics are typically associated to the setting of general relativity and cosmology. However, their relevance is of course much more general, and they appear for example in the classical mechanics of constrained systems, as well as in the study of membranes and interfaces in condensed matter. A brief review of theoretical and experimental motivations for studying the effects of curved geometry in condensed matter can be found in [1]. The search and study of condensed matter systems characterized by an actual or effective curved geometry is also stimulated by the idea of analogue gravity [2].

In this paper we will be interested primarily in the case in which the geometry is non-dynamical. Such situation is commonly considered in the cosmological setting as a first approximation in which the gravitational degrees of freedom are frozen, and one studies just a quantum field theory in curved spacetime. From the condensed matter perspective this can also be seen as a first approximation, or alternatively as the primary case of interest in situations where the curvature is introduced as a technical device (e.g. [3]) or for theoretical modeling (e.g. [4]).

The presence of curvature in the background geometry can have drastic effects on the infrared behavior of a model [5], and in particular on its phase transitions and critical behavior. Much work has gone in this direction for the case of constant negative curvature, that is, for the case of statistical models in hyperbolic space. The differences between models in the usual flat background and in the hyperbolic one have been studied in the context of liquids [6, 7], percolation [8, 9], Ising model [10, 11, 12, 13, 14, 15], XY model [16], self-avoiding walks [17] and more. Besides the hyperbolic case, it is worth mentioning also that curved spaces appear in the study of finite size effects [18], curvature defects [19], topological effects [20] and of course in the presence of compactified dimensions [21].

Despite the many relevant works, many directions appear to be unexplored. In particular, a renormalization group approach to this kind of problems seems to be lacking. Of course the situation is quite different in the high-energy context, where renormalization group investigations on curved backgrounds are quite common. However, the focus there is typically on ultraviolet properties, at least until recently. Over the past few years there has been an increased interest on IR effects in the cosmological setting of de Sitter spacetime (e.g. [22, 23, 24, 25]). In particular, it has been noticed how nonperturbative renormalization group techniques can be applied to this context and it was showed that spontaneously broken symmetries are radiatively restored in de Sitter spacetime in any dimension [26]. The de Sitter case, because of the Lorentzian signature of the metric, presents a number of technical challenges, and one would expect the situation to be somewhat easier in Euclidean signature. Surprisingly, as far as we know, there has not been a thorough study of this sort in Euclidean signature.

The purpose of this paper is to in part bridge such gap. We will study scalar field theory on two standard types of curved Riemannian geometry, d-dimensional spheres and hyperboloids. Our goal will be to gain a detailed understanding of how the background curvature affects the critical behavior of the model at large distances. In Sec. 2 we will use the Ginzburg criterion in order to test when and whether we should expect that mean field gives trustable results. In this way we confirm general expectations based on effective dimensionality arguments, which we expand upon at the end of the section. In the second part of the paper, Sec. 3, we will explore more in detail the effects brought in by the curvature of space, making use of functional renormalization group techniques in the local potential approximation. We will in particular study the question of symmetry restoration (or existence of a phase transition), and more in general we will discuss how the presence of a dimensional external scale affects the usual renormalization group picture. In order to keep the treatment as self-contained as possible, we include four appendices detailing the geometry of spheres and hyperboloids (App. A), the spectra of their respective Laplace-Beltrami operators (App. B), the associated heat kernels (App. C) and propagators (App. D).

2 The Gaussian approximation and effective dimensionality

Let $(\mathcal{M}, g_{\mu\nu})$ be a d-dimensional Riemannian manifold, that is, a differentiable manifold $\mathcal{M}$ equipped with a positive-definite metric $g_{\mu\nu}$ (in a given coordinate basis x^μ, $\mu = 1, \ldots, d$). The metric can be defined by the associated line element,[1]

$$ds^2_{(\mathcal{M})} = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{2.1}$$

In this work we will restrict to homogenous spaces, and in particular we will consider only three types of d-dimensional Riemannian spaces: the flat space (as a benchmark), the sphere and the hyperboloid. The respective geometries are briefly reviewed in App. A.

We are interested on the statistical properties of scalar field theories on such backgrounds. The statistical field theory of the field $\phi = \phi(x)$ is characterized as usual by the generating functional

$$Z[J] \equiv e^{W[J]} = \int D\phi \, e^{-S[\phi] + \int d^d x \sqrt{g} J \phi} \,, \tag{2.2}$$

and by the bare action

$$S[\phi] = \int d^d x \sqrt{g} \left[\frac{Z}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi + V(\phi) \right] \,, \tag{2.3}$$

[1] We use the Einstein convention, according to which repeated indices imply a summation.

where g is the metric determinant, and $g^{\mu\nu}$ the inverse metric. For the purpose of this Section, we choose

$$V(\phi) = \frac{1}{2}(m^2 + \xi R)\phi^2 + \frac{u}{4!}\phi^4 \,, \tag{2.4}$$

where R is the Ricci scalar of the background, and ξ a dimensionless coupling. The mean-field approximation is obtained evaluating the partition function $Z[0]$ by saddle point method. For constant field the classical solution satisfies $V'(\phi_0) = 0$, that is,

$$\phi_0 = \begin{cases} 0 & \text{for } \; m^2 + \xi R > 0 \,, \\ \pm\sqrt{-6\frac{m^2+\xi R}{u}} & \text{for } \; m^2 + \xi R < 0 \,. \end{cases} \tag{2.5}$$

The transition from zero to non-zero mean field is a text-book example of second-order phase transition.

In the Gaussian approximation we keep also the quadratic fluctuations around the minimum of the potential, their covariance being given by the inverse of the second functional derivative (Hessian) of the action, evaluated at ϕ_0:

$$S^{(2)} = \begin{cases} -Z\,\nabla^2 + m^2 + \xi R & \text{for } \; m^2 + \xi R > 0 \,, \\ -Z\,\nabla^2 - 2\,(m^2 + \xi R) & \text{for } \; m^2 + \xi R < 0 \,. \end{cases} \tag{2.6}$$

Here ∇^2 is the Laplace-Beltrami operator (or simply the Laplacian) on the curved background, see (B.1), and from the structure of the Hessian we read off the correlation length ℓ_c,

$$\ell_c^{-2} = \begin{cases} \frac{m^2+\xi R}{Z} & \text{for } \; m^2 + \xi R > 0 \,, \\ -2\frac{m^2+\xi R}{Z} & \text{for } \; m^2 + \xi R < 0 \,. \end{cases} \tag{2.7}$$

A simple test for the validity of the Gaussian approximation is given by the Ginzburg criterion (e.g. [27]), obtained by computing (in the broken phase) the quantity

$$Q = \frac{\int_{\ell_c} d^d x \sqrt{g}\, G(\sigma; \ell_c^{-2})}{\int_{\ell_c} d^d x \sqrt{g}\, \phi_0^2} \,, \tag{2.8}$$

where the integrals extend over a region of radius ℓ_c. Here $G(\sigma; \ell_c^{-2})$ is the correlation function,

$$G(\sigma; \ell_c^{-2}) = \frac{\delta^2 W[J]}{\delta J(x) \delta J(x')}\bigg|_{J=0} \,, \tag{2.9}$$

which on homogeneous spaces depends on the space points x and x' only via their geodesic distance $\sigma(x, x')$, and in the Gaussian approximation it is given by the inverse of (2.6). The correlation functions, or propagators, on curved backgrounds are reviewed in App.D.

If $Q \ll 1$, the fluctuations are small and the Gaussian approximation provides a good approximation. On the other hand, if $Q \gg 1$, fluctuations are large, the Gaussian approximation breaks down and we need a nonperturbative treatment. At a second order phase transition, the correlation length diverges, hence we are interested in checking what happens to Q in such limit.

In flat space, approximating the integral in the numerator with an integral over the whole space (exploiting the fact that the correlation function cuts off the integration at about a radius ℓ_c, see (D.5)), one finds

$$Q \sim \frac{\ell_c^{4-d} u}{3Z^2}\,, \tag{2.10}$$

from which in the large-ℓ_c limit one deduces the well-known critical dimension $d_c = 4$, below which the Gaussian approximation does not provide a valid description of the phase transition.

2.1 The hyperboloid

On an hyperboloid H^d, we can use the results of Appendix D. In odd dimensions, we need the expression (D.17), together with the volume integral, which is (see Appendix A)

$$\int_{\ell_c} d^d x \sqrt{g} = a^d\, \Omega_{d-1} \int_0^{\ell_c/a} dy \sinh(y)^{d-1}\,. \tag{2.11}$$

For $\ell_c/a \gg 1$, we thus find

$$Q_{(H^d)} \sim \frac{e^{(\rho-\omega_+)\ell_c/a}\omega_+^\rho}{\omega_+ - \rho} \frac{d-1}{e^{(d-1)\ell_c/a}} \frac{u\ell_c^2 a^{2-d}}{3Z^2}\,, \tag{2.12}$$

and since $\omega_+ \to \rho + \frac{1}{2\rho}\frac{a^2}{\ell_c^2}$ for $\ell_c \to \infty$, we obtain

$$Q_{(H^d)} \sim 2(d-1)\rho^{\rho+1} e^{-(d-1)\ell_c/a} \frac{u\ell_c^4}{3Z^2 a^d} \to 0\,, \tag{2.13}$$

for any odd $d > 1$. On the other hand, for $\ell_c/a \ll 1$, we recover (2.10) as expected.

In even dimensions the calculation is complicated by the integral nature of the fractional derivative in (D.16). However, for $d \geq 4$ even on flat space we know that mean field gives the correct critical exponents, and it is quite clear that the hyperbolic space will not change that situation. The most interesting even dimensional case is thus $d = 2$. For the latter, (D.16) reduces to

$$G_{(H^2)}(y; \ell_c^{-2}) = \frac{\sqrt{2}}{4\pi} \int_y^{+\infty} dx\, \frac{e^{-\frac{\sqrt{4a^2+\ell_c^2}}{2\ell_c}x}}{(\cosh x - \cosh y)^{1/2}}\,. \tag{2.14}$$

For $\ell_c/a \gg 1$, we obtain $G_{(H^2)}(\ell_c/a) \sim \frac{1}{\pi} e^{-\ell_c/a}$, and again we find an exponentially decaying $Q_{(H^2)}$.

We conclude that on H^d the Gaussian approximation provides a trustable description of the phase transition for any $d > 1$.

2.2 The sphere

On the sphere the dominant contribution to IR physics comes from the presence of a zero mode. From (D.11) we find that in the limit in which we approach a phase transition and the correlation length diverges, the propagator is dominated by the zero mode contribution $G_{(S^d)}(y; \ell_c^{-2}) \sim \ell_c^2$, and as a consequence

$$Q_{(S^d)} \sim \frac{\ell_c^4 a^{-d}}{3Z} u\,, \tag{2.15}$$

and comparing to (2.10) we conclude that, for large ℓ_c, the effective behavior on S^d is that of a zero-dimensional space, and in particular the Gaussian approximation is expected to be insufficient at large correlation length for any d.

2.3 Interpretation in terms of effective dimension

The conclusions we have reached with the Ginzburg criterion could have also been guessed by a heuristic in terms of effective dimensionality. We are going to illustrate such an argument for two different notions of effective dimension, that is, the spectral and the Hausdorff dimension. The former is defined as

$$d_s \equiv -2\frac{\partial \log \mathrm{Tr}[K(s)]}{\partial \log s}\,, \tag{2.16}$$

where $K(s)$ is the heat kernel for the Laplace-Beltrami operator ($b = 0$, see App. C). On flat space $d_s = d$, which justifies the definition, while on a general space it is in the limit of $s \to 0$ that we always have $d_s \to d$. A simple interpretation of such property is that, s being the diffusion time, small s means that only a small neighborhood of a point is being explored by the diffusion process, hence the space looks flat at those scales.

For large s, curvature effects become important, and for $s \to +\infty$ (at $\ell_c^2 \gg s$) we find that $d_s \to 0$ on S^d, while $d_s \to +\infty$ on H^d. Such limits are easily found. For the sphere we use the spectral sum representation of the heat trace, which is convergent in the large-s domain,

$$\mathrm{Tr}[K_{(S^d)}(s)] = \frac{1}{\Omega_d a^d}\sum_n D_n e^{-s\omega_n}\,, \tag{2.17}$$

from which we see that $\mathrm{Tr}[K_{(S^d)}(s)] \to \frac{1}{\Omega_d a^d}$ for $s \to +\infty$ (again due to the zero mode), and hence $d_s \to 0$. This can be heuristically understood as the statement that the sphere looks like a point when observed from a very large distance.

For the hyperboloid, we can use (C.7) to find that for d odd

$$\mathrm{Tr}[K_{(H^d)}(s)] \propto \frac{e^{-s\rho}}{(4\pi s)^{d/2}}\,, \tag{2.18}$$

where essentially the exponential decay is due to the presence of a “mass gap” in the spectrum and the absence of a zero mode. Plugging (2.18) into (2.16) we find $d_s = d + 2s\rho$, that is, the spectral dimension grows linearly with s. For d even, the expression for the heat trace is complicated by the integral nature of the pseudo-differential operator, however it is not hard to see that the same exponential damping is in place, hence the same result is obtained for the spectral dimension at large s.

We can also use a different notion of effective dimension, the Hausdorff dimension

$$d_H = \frac{\partial \log \int_L d^d x \sqrt{g}}{\partial \log L}\,, \tag{2.19}$$

where the integral extends over the set of points for which $\sigma(x, 0) \leq L$. For the sphere such integral reaches a plateau at $L = \pi a$, hence the Hausdorff dimension is zero at large L. On the contrary, for the hyperboloid the integral keeps growing exponentially, that is, faster than any power of L, and the Hausdorff dimension diverges.

It is well known that mean field theory becomes exact at large number of dimensions, hence the infinite effective dimensionality of the hyperboloid at large scales provides a heuristic explanation of the result we obtained from the Ginzburg criterion. At the same time, we know that in the Ising universality class there is no phase transition below $d = 2$, hence we might expect a failure of the Gaussian approximation for the sphere.

We should stress however that even though such arguments based on the effective dimension give a correct picture of the underlying physical mechanism, the Ginzburg criterion is more trustable as it involves directly the correlation function.

3 A functional renormalization group perspective

In this Section we want to analyze more in detail the effects induced by the curvature of the background geometry. To that end, we will use the method known as *functional renormalization group* (FRG).[2] There are many reviews on the FRG [28, 29, 30, 31, 32, 27], to which we refer for an introduction to the topic. We will employ here the FRG version that deals with the so-called effective average action $\Gamma_k[\phi]$ [33], which is an IR-regulated version (k being the running RG scale associated with the IR cutoff) of the Legendre transform of the functional $W[J]$, introduced in (2.2). It satisfies the general equation

$$k\partial_k\Gamma_k = \frac{1}{2}\mathrm{Tr}\left[\left(\Gamma_k^{(2)} + \mathcal{R}_k\right)^{-1} k\partial_k\mathcal{R}_k\right] , \tag{3.1}$$

where $\mathcal{R}_k$ denotes the IR cutoff, and $\Gamma_k^{(2)}$ the Hessian of Γ_k. The equation is amenable to several approximation schemes, one of the most common being the derivative expansion, in which Γ_k is expanded in invariants containing an increasing number of derivatives. The lowest order of the derivative expansion is known as local potential approximation (LPA), and it is the one we will consider here.

The ansatz for the average effective action in the LPA is

$$\Gamma_k[\phi] = \int d^dx\sqrt{g}\left[\frac{Z_k}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + V_k(\phi)\right] . \tag{3.2}$$

Strictly speaking, LPA usually stands for the case $Z_k = 1$, otherwise (3.2) is typically referred to as LPA′ (e.g. [34]). The functional RG equation (3.1) for the potential reads

$$k\partial_k V_k(\phi) = \frac{1}{2}\mathrm{Tr}_{(\mathcal{M})}\left[\frac{k\partial_k\mathcal{R}_k(-\nabla^2/k^2)}{-Z_k\nabla^2 + V_k''(\phi) + \mathcal{R}_k(-\nabla^2/k^2)}\right]_{|\phi=\mathrm{const.}} , \tag{3.3}$$

where we have redefined the trace on the Riemannian manifold $\mathcal{M}$ by dividing out the volume of $\mathcal{M}$, and we have projected the equation onto constant field configurations.

[2] Sometimes referred to also as *exact* or *nonperturbative* renormalization group.

The equation (3.3) describes the evolution of the potential under (continuous) coarse graining. The latter is the first of the two standard steps of the renormalization group [35], the second consisting in a rescaling of lengths and momenta such as to bring back the cutoff to its original value, and in a field redefinition that restores the normalization of the kinetical term. In the LPA, the second step is taken care of by the introduction of dimensionless variables

$$\tilde{\phi} = Z_k^{1/2} k^{(2-d)/2} \phi \,, \quad \tilde{V}(\tilde{\phi}) = k^{-d} V(\phi(\tilde{\phi}))\,. \tag{3.4}$$

We also define $\tilde{\Delta} = -\nabla^2/k^2$, and we write the cutoff as $\mathcal{R}_k(z) = Z_k k^2 r(z)$, for some dimensionless function $r(z)$ constrained only by standard requirements [36]. In dimensionless variables, (3.3) reads

$$k\partial_k \tilde{V}_k(\tilde{\phi}) + d\,\tilde{V}_k(\tilde{\phi}) - \frac{d-2+\eta_k}{2}\tilde{\phi}\tilde{V}_k'(\tilde{\phi}) = \widetilde{\mathrm{Tr}}_{(\mathcal{M})}\left[\frac{(1-\eta_k/2)r(\tilde{\Delta}) - \tilde{\Delta}\, r'(\tilde{\Delta})}{\tilde{\Delta} + \tilde{V}_k''(\tilde{\phi}) + r(\tilde{\Delta})}\right]_{|_{\tilde{\phi}=\mathrm{const.}}} \,, \tag{3.5}$$

where $\eta_k = -k\partial_k \ln Z_k$ is the scale-dependent anomalous dimension, and $\widetilde{\mathrm{Tr}}_{(\mathcal{M})} = k^{-d}\mathrm{Tr}_{(\mathcal{M})}$. The LPA$'$ needs an additional equation for the flow of Z_k, which can be expressed as a relation between η_k and (the derivatives of) the potential evaluated at its minimum (e.g. [30, 37, 38]). In what follows, we will actually set $\eta_k = 0$ in any practical calculation (i.e. we will only perform calculations within the strict LPA), so we will not need such expression.

A cutoff that leads to a very simple expression for the righ-hand-side of (3.3) or (3.5) is Litim's optimized cutoff [39, 36]

$$r(z) = (1-z)\theta(1-z)\,, \tag{3.6}$$

with which we obtain

$$k\partial_k \tilde{V}_k(\tilde{\phi}) + d\,\tilde{V}_k(\tilde{\phi}) - \frac{d-2+\eta_k}{2}\tilde{\phi}\tilde{V}_k'(\tilde{\phi}) = \frac{1}{1+\tilde{V}_k''(\tilde{\phi})}F_{(\mathcal{M})}(\tilde{a},\eta_k)\,, \tag{3.7}$$

where

$$F_{(\mathcal{M})}(\tilde{a},\eta_k) = \widetilde{\mathrm{Tr}}_{(\mathcal{M})}[\theta(1-\tilde{\Delta})] - \frac{\eta_k}{2}\,\widetilde{\mathrm{Tr}}_{(\mathcal{M})}[(1-\tilde{\Delta})\theta(1-\tilde{\Delta})]\,, \tag{3.8}$$

and we have introduced

$$\tilde{a} = ak\,. \tag{3.9}$$

In order to explicitly perform the traces, we need to fix the dimension d. It is instructive to consider the case of $d = 3$, for which computations are easiest, and where a nontrivial critical behavior is known to occur in the flat case. In flat space, using Fourier transform we find

$$F_{(E^3)}(\infty,\eta_k) = \frac{\Omega_{d-1}}{d\,(2\pi)^d}\left(1-\frac{\eta_k}{d+2}\right)\Big|_{d=3} = \frac{1}{6\pi^2}\left(1-\frac{\eta_k}{5}\right)\,, \tag{3.10}$$

and the analysis of the equation is standard (see e.g. [40, 41, 37, 38]): one finds a non-trivial (Wilson-Fisher) fixed point, at which the critical exponents differ from their mean field value, and are in good agreement with the observed values.

On the hyperboloid, using the results collected in Appendix B, we find

$$F_{(H^3)}(\tilde{a},\eta_k) = \frac{1}{6\pi^2}\left(1-\frac{1}{a^2k^2}\right)^{\frac{3}{2}}\theta\left(1-\frac{1}{a^2k^2}\right)\left(1-\frac{\eta_k}{5}\left(1-\frac{1}{a^2k^2}\right)\right). \tag{3.11}$$

Finally, on the sphere we find

$$F_{(S^3)}(\tilde{a},\eta_k) = \frac{1}{a^3k^3\Omega_3}\mathcal{P}(\lfloor N_3\rfloor)\left(1-\frac{\eta_k}{2}\mathcal{Q}(\lfloor N_3\rfloor)\right), \tag{3.12}$$

where $\lfloor x\rfloor$ is the floor function,

$$\mathcal{P}(N) = \sum_{n=0}^{N} D_n = \frac{1}{6}(1+N)(2+N)(3+2N), \tag{3.13}$$

$$\mathcal{Q}(N) = \frac{1}{\mathcal{P}(N)}\sum_{n=0}^{N} D_n(1-\tilde{\omega}_n) = \frac{5a^2k^2-9N-3N^2}{5a^2k^2}, \tag{3.14}$$

being $\tilde{\omega}_n$ the eigenvalues (B.2) in units of k, and

$$N_3 = -1+\sqrt{1+a^2k^2}. \tag{3.15}$$

The spherical case gives rise to a staircase function, as a combined effect of the discrete spectrum and the use of a step function in the cutoff, a phenomenon already known in the literature (e.g. [42, 43, 44]).

We notice a crucial difference between the flat and the curved cases: in the curved backgrounds the FRG equation is a non-autonomous equation, in the sense that there is an explicit dependence upon k on the rhs. On flat space, it is the introduction of dimensionless variables that leads to an autonomous equation. In the curved background case, the existence of a fixed external scale implies that we cannot in general achieve an autonomous equation. The same thing generically happens if any non-running scale is present, for example in quantum field theory at finite temperature [30], or on a non-commutative spacetime [45].

We thus immediately realize that true fixed points are unlikely, the potential will always retain a dependence on k via its dimensionless product with a. In special cases such dependence can be harmless, as in the case of the massless free theory. The latter is given by a $\tilde{\phi}$-independent potential $\tilde{V}_k(\tilde{\phi}) = v_k$, with v_k satisfying ($\eta_k = 0$)

$$k\partial_k v_k + d v_k = \widetilde{\mathrm{Tr}}_{(\mathcal{M})}\left[\frac{r(\tilde{\Delta})-\tilde{\Delta}\,r'(\tilde{\Delta})}{\tilde{\Delta}+r(\tilde{\Delta})}\right]. \tag{3.16}$$

Note that also on flat space the Gaussian solution to (3.7) has a non-zero vacuum term, $\tilde{V}(\tilde{\phi}) = 1/(d\,6\,\pi^2)$. We could eliminate such running vacuum terms, and obtain a proper Gaussian fixed point with $\tilde{V}_k(\tilde{\phi}) = 0$, by a modified equation in which vacuum contributions are appropriately subtracted (see for example [27] or [46]).

Alternatively, we can introduce the concept of *floating-points* [45], i.e. solutions of the FRG equation which are independent of k, up to dependence on $\tilde{a} = ak$. In other words, we can introduce, and keep track of, an explicit dependence on $\tilde{a}$, as if it was another field, an external field. Clearly, in the present case such procedure can be seen as a first step towards the treatment of cases in which the geometry is dynamical, and the curvature is indeed treated as on a par with other fields.

3.1 Scaling dimension and symmetry restoration

We will first discuss the consequences of the non-autonomy of the equation, taking the explicit formulas for $d = 3$ with optimized cutoff as a guidance.

On the hyperboloid we observe that the loop contributions on the rhs of the FRG equation (i.e. the functional trace (3.11)), vanish as soon as $k < 1/a$, thus leaving us with the classical (tree level) part of the equation. Although we have not computed explicitly the functional traces needed to evaluate the anomalous dimension, it is not hard to check that a similar phenomenon occurs also in such traces, and hence the anomalous dimension also vanishes in the deep IR. As a consequence, IR fixed points coincide with classical scale invariant theories, and thus mean field behavior is recovered at large distances, confirming our conclusions from Sec. 2.1. It should be stressed that the use of a cutoff with step function, such as (3.6), provides us with an extreme version of the general case: with a generic cutoff the approach to zero will be smooth, but in general fast enough for $k < 1/a$, thus leading to the same conclusion.

On the sphere, we see that $\mathcal{P}(\lfloor N_3 \rfloor) \to 1$ for $k \to 0$, or more precisely as soon as $k^2 < 3/a^2$, that is, only the zero mode remains unsuppressed. However, the dimensionless volume of the 3-sphere goes to zero, making the rhs of the FRG equation divergent. The presence of a singularity for $k \to 0$ is a general consequence of the presence of compact dimensions, with the d-sphere behaving as k^{-d} because all its dimensions are compact. In order to absorb such divergence we should rescale the potential and the field such that the lhs be as divergent as the rhs. This is achieved by introducing the new variables

$$\bar{\phi} = (ka)^{d/2} \tilde{\phi} = a^{d/2} k \phi \,, \tag{3.17}$$

$$\bar{V}(\bar{\phi}) = (ka)^d \tilde{V}(k^{-d/2} a^{-d/2} \bar{\phi}) = a^d V(a^{-d/2} k^{-1} \bar{\phi}) \,. \tag{3.18}$$

The scaling of $\bar{\phi}$ with k has been chosen so that $1 + \tilde{V}_k''(\tilde{\phi}) \to 1 + \bar{V}_k''(\bar{\phi})$. The resulting equation for $k^2 < d/a^2$ is

$$k\partial_k \bar{V}_k(\bar{\phi}) + \bar{\phi} \bar{V}_k'(\bar{\phi}) = \frac{1}{\Omega_d} \frac{1}{1 + \bar{V}_k''(\bar{\phi})} \,, \tag{3.19}$$

which can be recognized as the flat FRG equation for $d = 0$, apart from the Ω_d factor which could anyway be removed with a k-independent rescaling of field and potential. We thus expect that the IR properties of scalar field theory on a spherical background will resemble those the same theory in zero dimensions. In particular, we expect no phase transition to be present.

Such expectation can be directly tested by studying the flow of the dimensionful potential, and looking for a transition in the IR between a potential with spontaneous symmetry breaking and one without. On flat space, this is a standard analysis (see for example [30, 27, 38]), and it proceeds as following: one solves numerically the flow equation for the dimensionful potential with an initial condition at $k = \Lambda$ corresponding to a potential with spontaneous symmetry breaking, i.e. $V_\Lambda(\phi) = \lambda_\Lambda(\phi^2 - \rho_\Lambda)^2$ with $\rho_\Lambda > 0$. Integrating down towards $k = 0$ one observes in general that the local maximum at $\phi = 0$ flattens out, and two different behaviors can arise depending on the initial condition ρ_Λ, namely in one case the potential becomes flat in a finite interval around the origin at $k = 0$, corresponding to the effective potential of a broken phase, while in the other case the lowering of the maximum continues until we obtain at finite $k > 0$ a global minimum at $\phi = 0$, i.e. we obtain a symmetry restoration. On a flat background we know

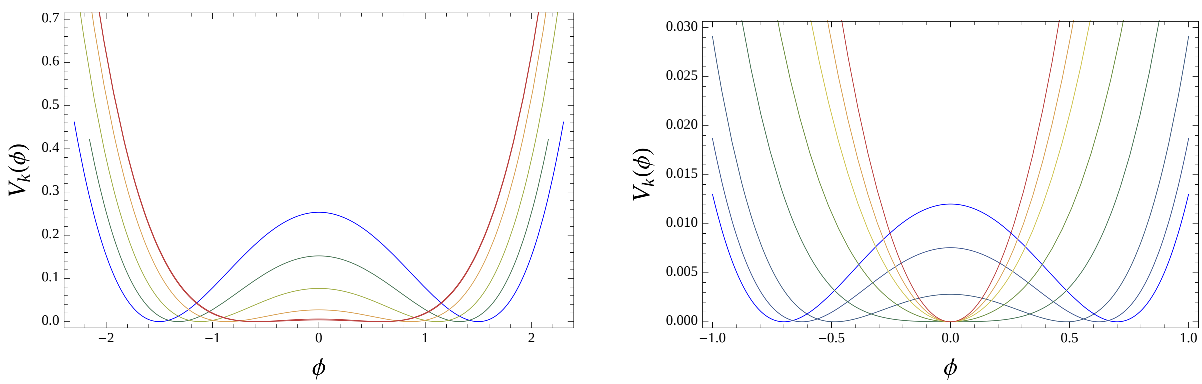

Figure 1: The flow of the potential (with the minimum subtracted for graphical purposes) on a flat background. The blue curve is the initial condition $V_\Lambda(\phi) = \lambda_\Lambda(\phi^2 - \rho_\Lambda)^2$, with $\Lambda = 20$, $\lambda_\Lambda = .05$, $\rho_\Lambda = 2.25$ (left) and $\rho_\Lambda = 0.49$ (right). The red curve is at $k = 0.1$, smaller values of k being indistinguishable on the scale of the plot. Symmetry restoration is evident in the plot on the right. The phase transition occurs near $\rho_\Lambda \simeq 1.82$.

that both phases are present for $d \geq 2$ (or $d > 2$ in the $O(N)$ model with $N > 1$ [30, 47]), and a continuous phase transition separates them at some critical value $\rho_\Lambda = \rho_c > 0$. The behaviors characteristic of the two phases are depicted in Fig. 1. On the hyperboloid, the phase diagram is qualitatively similar to the flat case, i.e. there exists a broken phase, and the plots look very similar to those in Fig. 1. In the spherical case the situation is instead quite different, as it turns out that symmetry is always restored at some finite $k > 0$, i.e. we do not find the broken phase for any value of ρ_Λ. An example of symmetry restoration is shown in Fig. 2. Interestingly we find that, for large enough ρ_Λ, at some intermediate scale ($k \sim 0.6$ in the specific case of Fig. 2) the potential is basically that of a broken phase, but as we keep decreasing the scale the symmetry is restored by the development of a minimum at $\phi = 0$.

3.2 Floating points

We can introduce an explicit dependence on $\tilde{a} = ak$ in the potential, so as to transform the flow equation into an autonomous equation, but with an additional independent variable. In order to highlight the presence of an additional argument in the potential, we denote the potentail as

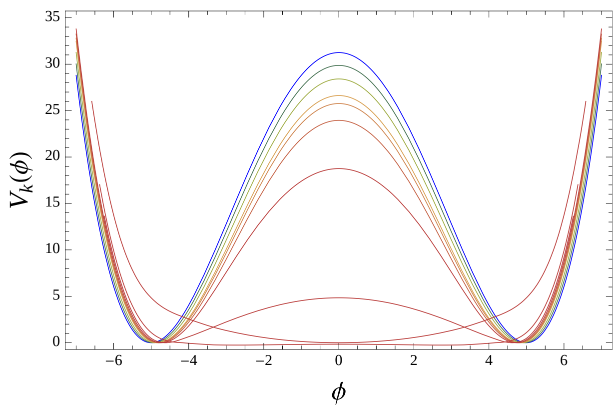

Figure 2: The flow of the potential (again with the minimum subtracted) on a spherical background with $a = 1/5$. The blue curve is the initial condition $V_\Lambda(\phi) = \lambda_\Lambda(\phi^2 - \rho_\Lambda)^2$, with $\Lambda = 20$, $\lambda_\Lambda = .05$, $\rho_\Lambda = 25$, while the red curve is at $k = 0.1$. Despite the large value of the initial symmetry breaking parameter, it is evident that symmetry restoration still takes place.

$U_k(\phi, a)$. We then introduce the dimensionless potential

$$\tilde{U}_k(\tilde{\phi}, \tilde{a}) = k^{-d}\, U_k(\phi(\tilde{\phi}), \tilde{a}/k)\,, \tag{3.20}$$

for which we obtain the equation

$$k\partial_k \tilde{U}_k(\tilde{\phi}, \tilde{a}) + d\,\tilde{U}_k(\tilde{\phi}, \tilde{a}) - \frac{d-2+\eta_k}{2}\tilde{\phi}\,\partial_{\tilde{\phi}}\tilde{U}_k(\tilde{\phi}, \tilde{a}) + \tilde{a}\,\partial_{\tilde{a}}\tilde{U}_k(\tilde{\phi}, \tilde{a}) = \frac{1}{1+\partial^2_{\tilde{\phi}}\tilde{U}_k(\tilde{\phi}, \tilde{a})} F_{(\mathcal{M})}(\tilde{a}, \eta_k)\,. \tag{3.21}$$

This is simply a rewriting of (3.7), with $k\partial_k \tilde{V}_k(\tilde{\phi}) = k\partial_k \tilde{U}_k(\tilde{\phi}, \tilde{a}) + \tilde{a}\,\partial_{\tilde{a}}\tilde{U}_k(\tilde{\phi}, \tilde{a})$. Now the fixed point (or floating point) equation is provided by the PDE obtained by setting $k\partial_k \tilde{U}_k(\tilde{\phi}, \tilde{a}) = 0$.

In order to understand the meaning of (3.21) we can first consider the case in which we discard the loop contribution on the rhs. That is, we study the tree level equation (with $\eta_k = 0$)

$$d\,\tilde{U}_k(\tilde{\phi}, \tilde{a}) - \frac{d-2}{2}\tilde{\phi}\,\partial_{\tilde{\phi}}\tilde{U}_k(\tilde{\phi}, \tilde{a}) + \tilde{a}\,\partial_{\tilde{a}}\tilde{U}_k(\tilde{\phi}, \tilde{a}) = 0\,. \tag{3.22}$$

Such equation, in the absence of boundary conditions simply constraints the dependence on the variables to

$$\tilde{U}_k(\tilde{\phi}, \tilde{a}) = \tilde{a}^{-d}\, Y\left(\tilde{a}^{\frac{d-2}{2}}\,\tilde{\phi}\right)\,, \quad \text{or} \quad U(\phi, a) = a^{-d}\, Y\left(a^{\frac{d-2}{2}}\,\phi\right)\,, \tag{3.23}$$

or in other words, the floating point potential is effectively a fixed point potential, where dimensional quantities (the potential and the field) are expressed in units of a. This was to be expected as in the absence of the rhs, the FRG equation is simply a statement of classical scale invariance (or k-independence), and thus dimensional analysis is enough to fix the potential. If in addition we require analyticity in both ϕ and a^{-1} (the latter in order to recover the flat space limit), that is, if we require regular behavior at $a^{-1} = \phi = 0$, and also $\mathbb{Z}_2$ symmetry, we find

$$U(\phi, a) = \sum_{n=0}^{\lfloor \frac{d}{d-2} \rfloor} c_n\, a^{-d+n(d-2)}\, \phi^{2n}\,, \tag{3.24}$$

with free dimensionless coefficients c_n. Because of the presence of the dimensionful parameter a, we find a more general potential than the usual $\phi^{\frac{2d}{d-2}}$ required by scale invariance in flat space.

As another consequence of the dimensional scale given by the curvature, we can straightforwardly see that the Gaussian fixed point has a massive generalization that would be forbidden on flat space: we can solve the full floating point equation with an ansatz of the type

$$\tilde{U}(\tilde{\phi},\tilde{a}) = u(\tilde{a}) + c\,\frac{\tilde{\phi}^2}{\tilde{a}^2}\,. \tag{3.25}$$

When plugged into (3.21), the second term disappears from the linear (or tree level) part of the equation, because it is a solution of (3.22). On the other hand, the trace part becomes $\tilde{\phi}$-independent, with $\partial^2_{\tilde{\phi}}\tilde{U}_k(\tilde{\phi},\tilde{a}) = 2c/\tilde{a}^2$. We are then left with an inhomogeneous linear ODE for $u(\tilde{a})$, for which the solution of the associated homogeneous equation is $u_{\rm hom}(\tilde{a}) = u_0/\tilde{a}^d$, for an arbitrary constant u_0, while the special solution of the inhomogeneous equation is scheme and dimension dependent.

Non-trivial floating points are much harder to study without truncations, as they require the flat case solution as boundary condition at $\tilde{a}^{-1} = 0$. For this reason, we will now resort to a polynomial truncation.

3.3 A simple truncation

Truncations of the potential to polynomial form are a very useful approximation even on flat space. The lowest order truncations can serve as a playground to understand qualitative features of the theory under examination (e.g. [29, 30]), while their recursive extension can serve even as a quantitative method for the extraction of precise critical exponents (e.g. [48, 41]). Here, since we focus on the qualitative picture rather than on precise quantitative estimates, we will consider the simplest possible truncation, that is, a simple quartic potential, and again $Z_k = 1$. We distinguish two cases, corresponding to equation (3.7) and (3.21) respectively:

$$\tilde{V}_k(\tilde{\phi}) = v_0(k) + v_2(k)\,\tilde{\phi}^2 + v_4(k)\,\tilde{\phi}^4\,, \tag{3.26}$$

and

$$\tilde{U}_k(\tilde{\phi},\tilde{a}) = u_0(k,\tilde{a}) + u_2(k,\tilde{a})\,\tilde{\phi}^2 + u_4(k,\tilde{a})\,\tilde{\phi}^4\,. \tag{3.27}$$

As observed previously, the first case (3.26) does not lead to nontrivial fixed points. We obtain the system of beta functions

$$k\partial_k v_0 = -d\,v_0 + \frac{F_{(\mathcal{M})}(\tilde{a},0)}{1+2\,v_2}\,, \tag{3.28}$$

$$k\partial_k v_2 = -2\,v_2 - 12\,v_4\,\frac{F_{(\mathcal{M})}(\tilde{a},0)}{(1+2\,v_2)^2}\,, \tag{3.29}$$

$$k\partial_k v_4 = (d-4)\,v_4 + 144\,v_4^2\,\frac{F_{(\mathcal{M})}(\tilde{a},0)}{(1+2\,v_2)^3}\,. \tag{3.30}$$

It is easy to check that setting the left-hand-sides to zero, if $F_{(\mathcal{M})}(\tilde{a},0)$ has a nontrivial dependence on $\tilde{a}$ (i.e. in the non-flat case), then the only fixed point is at $v_2 = v_4 = 0$ (as already discussed, in order to fix also v_0 we would need to modify the equation). The nontrivial solution is

$$v_2^* = \frac{4-d}{2d-32}\,, \quad v_4^* = \frac{12(d-4)}{(d-16)^3 F_{(\mathcal{M})}(\tilde{a},0)}\,. \tag{3.31}$$

In the flat case $F_{(\mathcal{M})}(\tilde{a},0)$ is a constant, and for $d < 4$ this is the Wilson Fisher fixed point in the simplest truncation. In the curved case this solution changes with k (at fixed a), and we cannot interpret it as a fixed point. From the known behavior of $F_{(\mathcal{M})}(\tilde{a},0)$, we find that as $k \to 0$, $v_4^* \to +\infty$ in the hyperbolic case (actually as $k \to 1/a$ because of the optimized cutoff), while $v_4^* \to 0$ in the spherical case. In the spherical case, the nontrivial solution merges with the Gaussian fixed point, but as we already know, there is no phase transition in this case. In the hyperbolic case, the nontrivial fixed point is pushed to infinity, leaving us with only the Gaussian fixed point, thus explaining why the Gaussian approximation is valid in this case. We can also study the linear perturbations around (3.31) in H^3. We find that the stability eigenvalues are k-independent and equal to $\nu_\pm = \frac{1}{6}(2 \pm \sqrt{82})$, but the eigendirections are k-dependent and become degenerate at $k = 1/a$, both reducing to the vector $(0, 1)$. As a consequence, critical exponents have to be taken from the Gaussian fixed point, and thus they trivially coincide with the results from the Gaussian approximation (i.e. in $d = 3$ they are $\nu_2 = 2$ and $\nu_4 = 1$).

In the parametrization (3.27) we obtain instead

$$k\partial_k u_0 = -\tilde{a}\partial_{\tilde{a}} u_0 - d\, u_0 + \frac{F_{(\mathcal{M})}(\tilde{a},0)}{1+2\,u_2}\,, \tag{3.32}$$

$$k\partial_k u_2 = -\tilde{a}\partial_{\tilde{a}} u_2 - 2\,u_2 - 12\,u_4\,\frac{F_{(\mathcal{M})}(\tilde{a},0)}{(1+2\,u_2)^2}\,, \tag{3.33}$$

$$k\partial_k u_4 = -\tilde{a}\partial_{\tilde{a}} u_4 + (d-4)\,u_4 + 144\,u_4^2\,\frac{F_{(\mathcal{M})}(\tilde{a},0)}{(1+2\,u_2)^3}\,. \tag{3.34}$$

Imposing again the vanishing of the left-hand-sides, we obtain this time a system of ordinary differential equations. The interesting case is the hyperboloid, which we can study once more in $d = 3$. The equation for $\tilde{a} > 1$ is not easily integrated analytically but can of course be integrated numerically. However, whatever the solution is in that range, this has to be matched with the solution for $\tilde{a} < 1$. The latter is trivial because of the vanishing of $F_{(\mathcal{M})}(\tilde{a},0)$, and we are left with a set of equations that is essentially the same of the usual tree-level flow equations but with k replaced by $\tilde{a}$. We thus obtain

$$u_0^* = \frac{c_0}{\tilde{a}^3}\,, \quad u_2^* = \frac{c_1}{\tilde{a}^2}\,, \quad u_4^* = \frac{c_2}{\tilde{a}}\,, \tag{3.35}$$

corresponding to the potential

$$U^*(\phi, a) = \frac{c_0}{a^3} + \frac{c_1}{a^2}\phi^2 + \frac{c_2}{a}\phi^4\,. \tag{3.36}$$

Of course from (3.35) we obtain the same result as from (3.31), i.e. that the dimensionless couplings go to infinity as $\tilde{a} \to 0$. However, (3.36) gives a different point of view on what is going on: due to the dimensionful scale a, we obtain a mean field k-independent potential.

4 Conclusions

In this paper we have studied the effects of curvature on the critical behavior of a scalar field, concentrating on spherical and hyperbolic spaces. By applying the Ginzburg criterion we have deduced that on a d-dimensional sphere the Gaussian approximation is never trustable when the correlation length becomes large, while on a d-dimensional hyperboloid it is trustable for any $d \geq 2$. We have interpreted this in terms of effective dimensions, such as the spectral and Hausdorff dimension: in the far IR both notions of dimension indicate that spheres are effectively zero-dimensional (they look like a point) while hyperboloids have an infinite effective dimension. In view of the known dependence of the Ising universality class on the dimension, one would then expect to find no phase transition on the sphere, and to find a phase transition well described by mean field theory on the hyperboloid. Such expectations were confirmed in Sec. 3, where we applied functional renormalization group techniques to the analysis of the scalar model on curved backgrounds. After discussing the general new features of the FRG equation in the presence of an external scale, we have shown by numerical integration in the local potential approximation that on the sphere there is only the symmetric phase. Finally, with the help of a simple truncation, we have shown how the Wilson-Fisher fixed point is pushed to infinity in H^3, thus leaving us with the sole Gaussian fixed point, with trivial critical exponents.

The main purpose of this paper was to show how the FRG can help us understanding the effects of geometry on critical phenomena. To that end, we studied the simplest model, and tried to keep things simple, but many other extension and applications are of course possible. On the technical level, it would be desirable to consider smooth cutoffs, thus avoiding the nonanalytic staircase effects encountered with (3.6), and to study the LPA$'$ more in detail, as well as the full next-to-leading order of the derivative expansion. A natural and simple extension of this work would be to study the $O(N)$ model, something to which we hope to come back in the near future. It would also be interesting to study what happens on different spaces, and in particular whether some space can be found in which a nontrivial behavior persists at the phase transition, but with different exponents than in the flat case.

Acknowledgements

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A Geometry of backgrounds

The most trivial homogeneous space, which we consider in this work as reference case, is flat Euclidean space E^d, with metric element $ds^2_{(E^d)} = \delta_{\mu\nu} dx^\mu dx^\nu$. The other two spaces we study here are d-dimensional spheres and hyperboloids.

The d-sphere can be defined in an intrinsic way as the quotient $S^d \simeq SO(d+1)/SO(d)$, or in an extrinsic way via its embedding in E^{d+1}

$$\sum_{A=1}^{d+1} (X^A)^2 = a^2 \,, \tag{A.1}$$

where X^A are the Cartesian coordinates in $\mathbb{R}^{d+1}$, and a is the radius of the sphere. Its metric element can be written as

$$ds^2_{(S^d)} = a^2 d\Omega_d \equiv a^2 \sum_{i=1}^{d} d\theta_i^2 \prod_{j=i+1}^{d} \sin^2(\theta_j) = a^2 d\theta_d^2 + a^2 \sin^2(\theta_d) d\Omega_{d-1} \,, \tag{A.2}$$

where the product is omitted for $i = d$. The angles θ_i take values in $[0, \pi]$, except for $\theta_1 \in [0, 2\pi]$. As any homogeneous space, the d-sphere is maximally symmetric, which implies that it is an Einstein space, i.e. $R_{\mu\nu} = \frac{1}{d} g_{\mu\nu} R$ with constant scalar Ricci curvature R, and that it has zero Weyl tensor. In other words, the Riemann tensor reduces to

$$R_{\mu\nu\rho\sigma} = \frac{R}{d(d-1)} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \,. \tag{A.3}$$

On S^d the scalar Ricci curvature is given by

$$R_{(S^d)} = \frac{d(d-1)}{a^2} \,. \tag{A.4}$$

We often use the volume of the unit d-sphere, which is

$$\Omega_d \equiv a^{-d} \int_{(S^d)} d^d x \sqrt{g} = \frac{\Gamma(d/2)}{\Gamma(d)} (4\pi)^{d/2} \,. \tag{A.5}$$

The d-dimensional hyperboloid is defined intrinsically as the quotient $H^d \simeq SO(d,1)/SO(d)$, or extrinsically as the upper sheet ($X^{d+1} > 0$) of the hypersurface

$$\sum_{A=1}^{d} (X^A)^2 - (X^{d+1})^2 = -a^2 \,, \tag{A.6}$$

embedded in Minkowski space $M^{d,1}$, i.e. $\mathbb{R}^{d+1}$ with flat metric of signature $(+, \ldots, +, -)$. Its metric element can be written as

$$ds^2_{(H^d)} = d\tau^2 + a^2 \sinh^2(\tau/a) d\Omega_{d-1} \,, \tag{A.7}$$

where $d\Omega_{d-1}$ is the metric element on the unit $(d-1)$-sphere, defined above, and $\tau \in [0, +\infty)$ is the geodesic distance from the origin. The dimensional parameter a is the characteristic length or "radius" of the hyperboloid, in terms of which the scalar Ricci curvature is

$$R_{(H^d)} = -\frac{d(d-1)}{a^2} \,. \tag{A.8}$$

B Spectra of Laplacian operators

On flat space the eigenfunctions of the Laplacian are of course the plane waves, and the functional traces are evaluated via Fourier transform. On curved backgrounds we lack a Fourier transform, however we are on a comparable situation whenever we know the spectrum of the Laplacian, as in the case of the spaces we consider in this work.

On generic Riemannian manifold with metric $g_{\mu\nu}$ the Laplace-Beltrami operator acting on a scalar field $\phi(x)$ is given by

$$\nabla^2 \phi(x) = \frac{1}{\sqrt{g}} \partial_\mu (\sqrt{g} g^{\mu\nu} \partial_\nu \phi(x)) \,. \tag{B.1}$$

The Laplacian spectrum on the sphere is well known [49], the scalar eigenmodes satisfying

$$- \nabla^2 \psi_{n,j} = \frac{n(n+d-1)}{a^2} \psi_{n,j} \equiv \omega_n \psi_{n,j} \,, \tag{B.2}$$

with multiplicity $D_n = \frac{(n+d-2)!\,(2n+d-1)}{n!(d-1)!}$, $j = 1, 2, ... D_n$, and $n = 0, 1, 2, ... + \infty$. Eigenmodes (whose explicit expression we do not need here) are orthonormal, that is,

$$\int_{S^d} d^d x \sqrt{g} \psi^*_{m,j}(x) \psi_{m',j'}(x) = \delta_{mm'} \delta_{jj'} \,. \tag{B.3}$$

For the scalar Laplacian on the hyperboloid we follow [50, 51, 52]. The eigenmodes of the Laplacian on H^d satisfy

$$- \nabla^2 \phi_{\lambda,l} = \frac{1}{a^2} (\lambda^2 + \rho^2) \phi_{\lambda,l} \equiv \nu_\lambda \psi_{\lambda,l} \,, \tag{B.4}$$

where

$$\rho = (d-1)/2 \,, \tag{B.5}$$

$\lambda \in [0, +\infty)$, and $l = 0, 1, 2, ... + \infty$. Eigenmodes are normalized as

$$\int_{H^d} d^d x \sqrt{g} \phi^*_{\lambda,l}(x) \phi_{\lambda',l'}(x) = \delta_{jj'} \, \delta(\lambda - \lambda') \,. \tag{B.6}$$

The analogue of the multiplicty for the continuum spectrum is the spectral function, or Plancherel measure, which is defined by

$$\mu(\lambda) \equiv \frac{\pi \Omega_{d-1} a^d}{2^{d-2}} \sum_l \phi^*_{\lambda,l}(0) \phi_{\lambda,l}(0) \,, \tag{B.7}$$

and explicitly given by

$$\mu(\lambda) = \frac{\pi}{2^{2(d-2)} \Gamma(d/2)^2} \prod_{j=0}^{(d-3)/2} (\lambda^2 + j^2) \tag{B.8}$$

for odd $d \geq 3$, and by

$$\mu(\lambda) = \frac{\pi \lambda \tanh(\pi\lambda)}{2^{2(d-2)} \Gamma(d/2)^2} \prod_{j=1/2}^{(d-3)/2} (\lambda^2 + j^2) \tag{B.9}$$

for even $d \geq 2$ (for $d = 2$ the product is omitted).

Functional traces (which we define divided by the volume) reduce to

$$\mathrm{Tr}_{(S^d)}[W(-\nabla^2)] = \frac{1}{\Omega_d a^d} \sum_n D_n W(\omega_n) \tag{B.10}$$

for the sphere, and to

$$\mathrm{Tr}_{(H^d)}[W(-\nabla^2)] = \frac{2^{d-2}}{\pi \Omega_{d-1} a^d} \int_0^\infty d\lambda\, \mu(\lambda) W(\nu_\lambda) \tag{B.11}$$

for the hyperboloid.

C Heat kernel

By definition the heat kernel is the solution of the heat equation

$$(\partial_s - \nabla_x^2 + b) K(x, s; x_0, b) = 0\,, \tag{C.1}$$

with initial condition

$$\lim_{s \to 0} K(x, s; x_0, b) = \frac{\delta(x - x_0)}{\sqrt{g}}\,. \tag{C.2}$$

On a homogeneous space the heat kernel depends only on the geodesic distance between x and x_0, which we denote by $\sigma(x, x_0)$. We thus write $K(x, x_0, s; b) = K(\sigma, s; b)$.

Knowing the spectrum of the Laplacian we can write the general solution in the form

$$K(\sigma, s; b) = \sum_u e^{-s(\lambda_u + b)} \chi_u(x) \chi_u^*(x_0)\,, \tag{C.3}$$

where $-\nabla^2 \chi_u(x) = \lambda_u \chi_u(x)$, and u labels the whole set of eigenmodes.

On flat space we have

$$K_{(E^d)}(x, s; b) = \frac{e^{-\frac{|x|^2}{4s} - sb}}{(4\pi s)^{\frac{d}{2}}}\,. \tag{C.4}$$

For spheres and hyperboloids, the nearest we can get to a closed expression for the heat kernel on these spaces is probably in terms of fractional derivatives [50]. We introduce the dimensionless variable $y = \sigma/a$, rescale $s \to a^2 s$, and define

$$\omega_\pm = \sqrt{\rho^2 \pm a^2 b}\,. \tag{C.5}$$

On the sphere one finds

$$K_{(S^d)}(y, s; b) = \frac{1}{a^d} \frac{e^{s\omega_-^2}}{(4\pi s)^{\frac{1}{2}}} \left(\frac{1}{2\pi} \frac{\partial}{\partial(\cos(y) + 1)} \right)^{\frac{d-1}{2}} \sum_{n=-\infty}^{+\infty} (\pm 1)^n e^{-\frac{(y + 2\pi n)^2}{4s}}\,, \tag{C.6}$$

where the plus and minus sign are for d odd and even respectively. As $y = \theta_d$ (see (A.2)), we have that $y \in [0, \pi]$, however geodesics can wrap several times around the sphere, and such "indirect paths" precisely give rise to the sum over n in (C.6).

On the hyperboloid we have

$$K_{(H^d)}(y,s;b) = \frac{1}{a^d}\frac{e^{-s\omega_+^2}}{(4\pi s)^{\frac{1}{2}}}\left(-\frac{1}{2\pi}\frac{\partial}{\partial\cosh(y)}\right)^{\frac{d-1}{2}} e^{-\frac{y^2}{4s}}\,. \tag{C.7}$$

Note that the fractional derivatives have different definitions for the cases of the sphere and the hyperboloid [50], but always reduce to ordinary derivatives for d odd. Note also that the absence of indirect paths for the geodesics makes the expression for the hyperboloid simpler than that for the sphere.

D Propagators

By definition the propagator $G(x,x_0;b)$ is the solution to the equation

$$(-\nabla_x^2 + b)G(x,x_0;b) = \frac{\delta(x-x_0)}{\sqrt{g}}\,. \tag{D.1}$$

Again, due to homogeneity of space the propagator depends only on $y = \sigma(x,x_0)/a$, hence we will simply write $G(y;b)$ for the propagator. Its relation to the heat kernel is provided by the Schwinger proper time integral,

$$G(y;b) = a^2\int_0^\infty ds K(y,s;b)\,, \tag{D.2}$$

which, upon using (C.3), gives (assuming that $\lambda_u + b > 0$, $\forall u$)

$$G(y;b) = \sum_u \frac{1}{\lambda_u + b}\chi_u(x)\chi_u^*(x_0)\,. \tag{D.3}$$

On flat space the propagator is well known, and it takes the form (e.g. using (C.4) and (D.2))

$$G_{(E^d)}(x;b) = \frac{b^{d-2}}{(2\pi)^{d/2}}(\sqrt{b}\,|x|)^{1-\frac{d}{2}}K_{\frac{d-2}{2}}\left(\sqrt{b}\,|x|\right)\,, \tag{D.4}$$

where $K_\nu(x)$ modified Bessel function of the second kind, leading to the asymptotic behavior

$$G_{(E^d)}(x;b) \sim \frac{b^{d-2}}{(2\pi)^{d/2}}(\sqrt{b}\,|x|)^{\frac{1-d}{2}}\sqrt{\frac{\pi}{2}}\,e^{-\sqrt{b}\,|x|} \tag{D.5}$$

for $\sqrt{b}\,|x| \gg 1$, and

$$G_{(E^d)}(x;b) \sim \frac{2^{\frac{d-4}{2}}}{(2\pi)^{d/2}}\Gamma(\frac{d-2}{2})\,|x|^{2-d} \tag{D.6}$$

for $\sqrt{b}\,|x| \ll 1$. This justifies the definition (2.7) of correlation length $\ell_c = b^{-1/2}$.

The propagators for both S^d and H^d have been computed in [53] directly solving (D.1), or from an explicit mode sum in [54] for the sphere and in [55, 51] for the hyperboloid. Define

$$\alpha_\pm = \rho + \omega_\pm\,, \tag{D.7}$$

$$\beta_\pm = \rho - \omega_\pm \,. \tag{D.8}$$

The propagator on S^d is given by

$$G_{(S^d)}(y;b) = a^{2-d} \frac{\Gamma(\alpha_-)\Gamma(\beta_-)}{\Gamma(\gamma)\, 2^d\, \pi^{d/2}}\, F(\alpha_-,\beta_-;d/2;z)\,, \tag{D.9}$$

where $F(\alpha,\beta;\gamma;z)$ is the hypergeometric function, and

$$z = \cos^2(y/2)\,. \tag{D.10}$$

For small b the propagator has the following Laurent expansion

$$G_{(S^d)}(y;b) = \frac{1}{b\, a^d\, \Omega_d} + f(y) + O(b)\,, \tag{D.11}$$

where the b-independent part $f(y)$ contains the UV divergent ($y=0$) contributions. For d odd it takes the form

$$f(y) = \frac{a^{2-d}}{2\,(4\pi)^{(d-1)/2}} \left((1-z)^{1-\frac{d}{2}} f_1(1-z) + f_2(1-z) \right)\,, \tag{D.12}$$

with $f_1(1-z)$ and $f_2(1-z)$ two analytic functions at $z=1$, while for d even we find

$$f(y) = \frac{a^{2-d}}{(4\pi)^{d/2}} \left(\sum_{n=1}^{d/2} \frac{a_n}{(1-z)^{\frac{d}{2}-1}} - \frac{(d-2)!}{(\frac{d}{2}-1)!} \log(1-z) \right)\,, \tag{D.13}$$

where for example $a_0 = -1$ for $d=2$, and $a_0 = -7/3$, $a_1 = 1/2$ for $d=4$. Note that for $b \to 0$ the divergent part of $G_{(S^d)}(y;b)$ arises from the constant mode $-\nabla^2 \psi_{0,0} = 0$, which by the normalization condition (B.3) is $\psi_{0,0}(x) = a^{-d/2}\Omega_d^{-1/2}$ (compare (D.3) with (D.11)).

The propagator on H^d is given by

$$G_{(H^d)}(y;b) = a^{2-d} \frac{\Gamma(\alpha_+)\Gamma(\alpha_+ - d/2+1)}{\Gamma(\alpha_+ - \beta_+ + 1) 2^d \pi^{d/2}} z^{-\alpha_+}\, F(\alpha_+, \alpha_+ - d/2+1; \alpha_+ - \beta_+ + 1; z^{-1})\,, \tag{D.14}$$

where

$$z = \cosh^2(y/2)\,. \tag{D.15}$$

A more compact expression can be obtained for the propagator on H^d by plugging (C.7) into (D.2). Exchanging integral and (fractional) derivative, we find

$$G_{(H^d)}(y) = \frac{a^{2-d}}{2\omega_+ Z} \left(-\frac{1}{2\pi} \frac{\partial}{\partial \cosh(y)} \right)^{\frac{d-1}{2}} e^{-\omega_+ y}\,. \tag{D.16}$$

In odd dimensions, the evaluation is trivial as the derivatives are ordinary ones, and we find

$$G_{(H^d)}(y) = \frac{a^{2-d}}{2\omega_+ Z} \left(\frac{\omega_+}{2\pi \sinh(y)} \right)^{\frac{d-1}{2}} e^{-\omega_+ y}\,. \tag{D.17}$$

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