

C R I T I C A L D Y N A M I C S

THE EXPANSION OF THE MASTER EQUATION INCLUDING A CRITICAL POINT

CRITICAL DYNAMICS

the expansion of the master equation
including a critical point

PROEFSCHRIFT

ter verkrijging van de graad van doctor in de wiskunde en natuurwetenschappen aan de Rijksuniversiteit te Utrecht, op gezag van de Rector Magnificus Prof. Dr. A. Verhoeff, volgens besluit van het College van Decanen in het openbaar te verdedigen op maandag 21 april 1980 te 14.45 uur

door

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geboren op 18 januari 1947 te Amsterdam

*'- the great thing is not to be free of theoretical prejudices,
but to have the right theoretical prejudices.'*

*Steven Weinberg in
'The First Three Minutes'*

*To my mother
To the memory of my father*

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Acknowledgements/Curriculum vitae

SAMENVATTING

Statistische methodes in de fysica (evenals in de chemie of bijv. de sociologie) voor de beschrijving van 'macroscopische' systemen zijn vaak nodig ten gevolge van de onmogelijkheid of onwenselijkheid van een gedetailleerde 'microscopische' beschrijving. Afgezien van de experimentele of fundamentele onmogelijkheid, is niemand geïnteresseerd in bijvoorbeeld de precieze posities en snelheden van alle 10^{26} molekulen in zijn huiskamer, terwijl aan de andere kant het aantal microscopische objecten (zoals molekulen, photonen of mensen) vaak toch niet voldoende groot is om tevreden te zijn met een zuiver deterministische (d.w.z. niet-stochastische) beschrijving van het macroscopische systeem (bijv. een reactie vat, laser of populatie).

Statistische modellen zijn zelden exact oplosbaar. Daarom is het, zeker in de fysica, zinvol te zoeken naar niet-exacte oplossingen waarvan gegarandeerd kan worden dat ze de werkelijke oplossingen steeds dichter zullen naderen naarmate het systeem meer macroscopisch wordt. Dit laatste wordt uitgedrukt in een *kleine parameter*, i.h.a. aangeduid als het inverse volume van het systeem.

Laat nu een systeem stochastisch beschreven zijn d.m.v. de zogenaamde Markovse master vergelijking. En laat ons beginnen te veronderstellen dat de toestand van het systeem in de macroscopische limiet stabiel is in de lineaire benadering, d.w.z. onder zeer kleine verstoringen. Zo'n toestand heet normaal. Indien er slechts één normale deterministische toestand bestaat, spreekt men van een *monostabiele* fase van het systeem. Indien er evenwel, als gevolg van het wijzigen van een externe grootte (de zogenaamde *pomp-parameter*, bijv. temperatuur, elektrische stroom, chemische concentratie of mechanische kracht) twee normale deterministische toestanden mogelijk zijn, spreekt men van een *bistabiele* fase. De tot dusver bekende ontwikkeling van de master vergelijking is toepasbaar op zowel de monostabiele toestand als op de twee separate toestanden van de bistabiele fase. Ze is evenwel ongeldig in het *critische gebied* van de pomp-parameter, dat de normale monostabiliteit scheidt van de normale bistabiliteit. De in dit proefschrift gegeven ontwikkeling is wél geldig in het kritische gebied.

Het wezenlijke idee van de theorie is de master vergelijking te separeren in een niet verder te reduceren *nulde orde* deel en een restant, dat steeds kleiner wordt naarmate het systeem groter wordt. Deze separatie is mogelijk na een precieze quantificering van het

critische gebied. De nulde orde stochastische benadering is een tamelijk eenvoudige Fokker-Planck vergelijking, die de essentiële aspecten van het proces bevat. De oplossingen (eigenfuncties) van deze diffusie vergelijking worden besproken. Als zij eenmaal bekend zijn kunnen hogere orde *correcties* in de oorspronkelijke master vergelijking op systematische wijze in rekening worden gebracht.

In het eerste deel van dit proefschrift wordt het probleem van diffusie in een externe potentiaal behandeld. De appendix van deel I bevat een gedetailleerde discussie van de oplossingen van de irreducibele Fokker-Planck vergelijking. In het tweede deel van dit proefschrift wordt de ontwikkeling gegeven voor het meer algemene Markov proces. De appendix van deel II is ter illustratie gewijd aan het magnetische 'mean field' Ising model.

SUMMARY

In this thesis it is shown how to solve the master equation for a Markov process including a *critical point* by means of successive approximations in terms of a *small parameter*. A critical point occurs if, by adjusting an externally controlled quantity, the system shows a transition from normal *monostable* to *bistable* behaviour. Examples of the external quantity (the *pump parameter*) are temperature, electric discharge current, chemical concentrations and mechanical force. The appropriate small parameter may be either the diffusion coefficient or the inverse size of the system. The latter is usually given by the volume or by the total number of constituents such as spins, photons or molecules.

The fundamental idea of the theory is to separate the master equation into its proper irreducible part and a corrective remainder. The *irreducible or zeroth order* stochastic approximation will be a relatively simple Fokker-Planck equation that contains the essential features of the process. Once the solution of this irreducible equation is known, the *higher order corrections* in the original master equation can be incorporated in a systematic manner.

In part I of this thesis we consider the problem of diffusion in an externally applied potential showing a monostable to bistable transition. The appendix of part I presents a discussion of the irreducible solutions. In part II we examine the general Markov process. The appendix of part II is devoted to an example, namely the magnetic mean field Ising model.

PART I: DIFFUSION PROCESSES

C R I T I C A L D Y N A M I C S

THE EXPANSION OF THE MASTER EQUATION INCLUDING A CRITICAL POINT

PART I: DIFFUSION PROCESSES

ABSTRACT

The master equation for a diffusion process that takes place in an external potential will be evaluated systematically in terms of a small parameter, namely the diffusion coefficient. Contrary to the known expansion the present solution is not only uniformly valid in the normal *monostable* and *bistable* cases, but also applies at the *critical point*. This has been achieved by using in zeroth order approximation the complete set of eigenfunctions belonging to the appropriate irreducible description of the process. Successive higher order corrections are evaluated explicitly.

1. INTRODUCTION

One of the outstanding problems in theoretical physics remained the proper description of the connection between a system's normal stochastic features and its critical dynamical behaviour. In brief, the formidable difficulties encountered are due to the occurrence of widely separated scales in both space and time. This statement might well deserve some further elaboration.

For that purpose we shall consider a diffusion process in one dimension z with diffusion coefficient ν , for example of ions in a membrane, subject to an external potential $U(z)$. The probability density $P(z,t)$ obeys the master equation [1, 2]

$$\frac{\partial P(z,t)}{\partial t} = \frac{\partial}{\partial z} U'(z)P + \nu \frac{\partial^2 P}{\partial z^2}, \quad (1.1)$$

where the prime denotes differentiation with respect to the argument. Totally disregarding fluctuations would mean setting $\nu=0$. In that case (1.1) reduces to the Liouville equation associated with the deterministic equation of motion

$$\dot{z} = -U'(z). \quad (1.2)$$

Normally, in order to find the correct stochastic description in terms of ν one should return to (1.1) and set [3] $z = \varphi(t) + \nu^{1/2} \xi$; $\varphi(t)$ is a

solution of the deterministic equation (1.2). The leading part of (1.1) becomes a linear Fokker-Planck equation of order ν^0 in terms of ξ [3, 4]:

$$\frac{\partial P(\xi, t)}{\partial t} = U''(\varphi) \frac{\partial}{\partial \xi} \xi P + \frac{\partial^2 P}{\partial \xi^2} + o(\nu^{\frac{1}{2}}) \quad (1.3)$$

Higher order corrections can in principle be calculated systematically in powers of $\nu^{\frac{1}{2}}$. Let us for clarity confine ourselves to the simple potential

$$U(z) = \frac{1}{2}\alpha z^2 + \frac{1}{4}z^4, \quad (1.4)$$

which is shown in figure 1. This potential involves a so-called Landau critical point for $\alpha=0$ [5]. If $\alpha > 0$, it has only one minimum, at $z=0$, corresponding to the steady state solution of (1.2). Any solution $\varphi(t)$ ends up in that equilibrium state, which is both locally and globally (asymptotically) stable in the linear approximation (see e.g. [6]). In this monostable case the zeroth order approximation for the equilibrium fluctuations is therefore indeed given by (1.3) with $U''(\varphi) = U''(0) = \alpha > 0$. So the fluctuations are of order $\nu^{\frac{1}{2}}$ and there is one relevant time scale of order unity.

The fundamental solution of (1.3) is a Gaussian. The Gaussian property of the zeroth order propagates here over finite time intervals because of the local linear stability of the deterministic state. Such situations will be called *normal*. Although the Gaussian propagation property remains true for diffusion processes over sufficiently small time lapses, it is lost for finite time intervals if the stability properties of the system become more involved due to nonlinearities [7].

If $\alpha < 0$, $U(z)$ develops one maximum at $z=0$, and two minima at $z = \pm(-\alpha)^{\frac{1}{2}}$. See figure 1, and e.g. [8]. That is, the previously stable equilibrium state $\varphi_0=0$ becomes unstable. Clearly, the linear noise approximation (1.3) at this unstable state breaks down after a time period of order $\ln \nu^{-\frac{1}{2}}$, although several more or less ingenious attempts have been made to extend its validity [9-12]. After such a time period the initial local fluctuations (of order $\nu^{\frac{1}{2}}$) grow to global size (of order ν^0) and the anharmonicity of $U(z)$ becomes essential.

In the deterministic approximation the process terminates in one of the two locally stable steady states φ_{\pm} , completely predestined by the initial condition [8]. In a stochastic description (disregarding initial values in a small neighbourhood of order $\nu^{\frac{1}{2}}$ of the unstable state $\varphi_0=0$) the system also tends first on the time scale t (of order ν^0) to one of these two minima. However, in the end it diffuses further partly into the other well. This sluggish process takes place on a time scale which

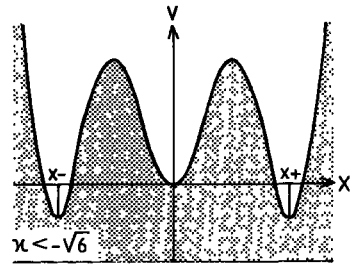
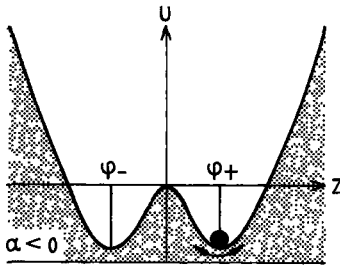
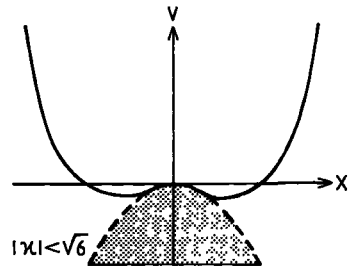
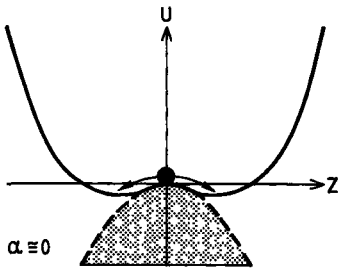
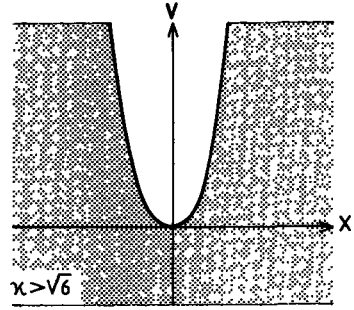
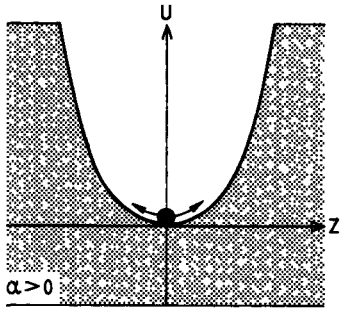


Figure 1. The potential $U=U_{\alpha}(z)$ according to (1.4). Note that $U_{\alpha}(z) \equiv U_{\kappa}(x)$, the irreducible potential (3.1). If $\alpha < 0$: depth of wells (i.e. barrier height) is $\frac{1}{4}|\kappa|^2$.

Figure 2. The Schrödinger potential $V(x)$ according to (3.11). The x -axis has been drawn at $V = -\frac{1}{2}\kappa$. If $\kappa \rightarrow -\infty$: depth of outer wells is $|\kappa|$; barriers height becomes $\frac{1}{27}|\kappa|^3$.

is essentially determined by the Arrhenius-Boltzmann factor $e^{-\alpha^2/4\nu}$ connected with the potential barrier height [8, 13-16]. If α is negative of order ν^0 , and one neglects this so-called Kramers' diffusion rate [13], the system may be approximately described by two isolated local Gaussians [10, 17] which are solutions of (1.3) with $U''(\varphi)=U''(\varphi_{\pm})=-\alpha>0$.

The above-critical *monostable* case ($\alpha>0$) and the below-critical *bistable* case ($\alpha<0$) are separated by the *critical* point $\alpha=0$. Here the potential (1.4) has one single minimum at $z=0$, which is very broad. Again consult figure 1. The potential is essentially anharmonic. Hence, although $\varphi_0=0$ still is asymptotically stable, there is no stability in the linear approximation. Obviously, (1.3) ceases to be valid as zeroth order approximation: initial fluctuations would grow beyond bound. If we set [4] $z=\nu^{\frac{1}{2}}\eta$, we obtain the nonlinear Fokker-Planck equation

$$\frac{\partial P(\eta, \tau)}{\partial \tau} = \frac{\partial}{\partial \eta} (\Delta \eta + \eta^3) P + \frac{\partial^2 P}{\partial \eta^2} \quad , \quad (1.5)$$

where $\Delta = \nu^{-\frac{1}{2}}\alpha$. Clearly, in a range where the *pump parameter* α is small of order $\nu^{\frac{1}{2}}$, Δ is of order unity and (1.5) presents a correct zeroth order. Note that this range of α includes the critical point $\alpha=0$. The time scale $\tau = \nu^{\frac{1}{2}}t$ reflects the slowing down of the critical dynamics. The critical fluctuations are of order $\nu^{\frac{1}{2}}$ and therefore are much larger than the normal fluctuations.

In summary, it should be clear that the normal Ansatz $z = \varphi + \nu^{\frac{1}{2}}\xi$ cannot lead to a systematic evaluation of the diffusion process (1.1) with (1.4) for small ν , that will be uniformly valid for all α . The reasons can be traced to the quite diverse scales both in space ($z \sim \nu^{\frac{1}{2}}$ if $\alpha>0$; $\nu^{\frac{1}{2}}$ if $\alpha=0$; $\nu^{\frac{1}{2}}$ and ν^0 if $\alpha<0$) and time ($t \sim \nu^0$ if $\alpha>0$; $\nu^{-\frac{1}{2}}$ if $\alpha=0$; ν^0 and $e^{1/\nu}$ if $\alpha<0$), which will be subtly intertwined in a unified treatment. In this article it will be shown how such a unified treatment of monostable, critical and bistable behaviour can be achieved for more general potentials than (1.4).

The essential point is the recognition of the proper critical range, where α is of order $\nu^{\frac{1}{2}}$, z is of order $\nu^{\frac{1}{2}}$ and t is of order $\nu^{-\frac{1}{2}}$. Using the appropriate scale transformations one then separates the master equation into its *irreducible zeroth order* part and a corrective remainder (section 2). The irreducible part will be of the form (1.5). Having obtained complete knowledge of the eigensolutions of the irreducible problem (section 3 & appendix), one knows its propagator (Green's function) and can proceed to the inclusion of successive *higher order corrections* (section 4).

2. REDUCTION OF THE PROBLEM

Let us investigate a diffusion process described by (1.1). Consider the general symmetric potential

$$U(z) = \frac{1}{2}\alpha^{(1)}z^2 + \frac{1}{4}\alpha^{(3)}z^4 + \frac{1}{6}\alpha^{(5)}z^6 + \dots \quad (2.1)$$

$\alpha^{(1)}$ changes sign at the critical point and $\alpha^{(3)}$ is assumed to be positive. The coefficients do not depend on ν . Obviously, the potential (2.1) has one minimum at $z=0$ if $\alpha^{(1)} > 0$. It changes into a maximum when $\alpha^{(1)} < 0$. In that case $U(z)$ has two minima at $z = \varphi_{\pm}$, where $\varphi_- = -\varphi_+$. Throughout it is presumed that there exist no other minima along the real z -axis. For example, the deterministic equation (1.2) with (2.1) has the following stable stationary states if $\alpha^{(1)} = \alpha, \alpha^{(3)} = \alpha^{(5)} = 1$ and all other coefficients are zero:

$$\varphi_{\pm}^{(\infty)} = \begin{cases} \varphi_0 = 0 & \text{if } \alpha > 0 \\ \varphi_{\pm} = \pm \sqrt{\frac{1}{2}(\sqrt{1-4\alpha}-1)} & \text{if } \alpha < 0 \end{cases} \quad (2.2)$$

2.1 The critical region and above: monostability

Here $\alpha^{(1)} \geq 0$ or negative of order $\nu^{\frac{1}{2}}$. If one introduces the scale transformations

$$z = \nu^{\frac{1}{4}}\eta, \quad t = \nu^{-\frac{1}{2}}\tau, \quad \alpha^{(1)}/\alpha^{(3)} = \nu^{\frac{1}{2}}\Delta \quad (2.3)$$

into (1.1) with the potential (2.1) one finds

$$\frac{\partial P(\eta, \tau)}{\partial \tau} = \frac{\partial}{\partial \eta} [\alpha^{(3)}\eta(\Delta + \eta^2) + \nu^{\frac{1}{2}}\alpha^{(5)}\eta^5 + \dots] P + \frac{\partial^2 P}{\partial \eta^2} \quad (2.4)$$

Considering formally η, τ and Δ as quantities of order unity, and disregarding terms in (2.4) that vanish as $\nu \rightarrow 0$, one obtains the irreducible part of the equation, namely

$$\frac{\partial P(x, s)}{\partial s} = \frac{\partial}{\partial x} (\kappa x + x^3) P + \frac{\partial^2 P}{\partial x^2} \quad (2.5)$$

where we have set

$$x = \eta[\alpha^{(3)}]^{-\frac{1}{4}}, \quad s = \tau[\alpha^{(3)}]^{-\frac{1}{2}}, \quad \kappa = \Delta[\alpha^{(3)}]^{-\frac{1}{2}} \quad (2.6)$$

in order to have the equation available in standard notation; κ is the standard pump parameter. Eq. (2.5) shows that the irreducible part represents in fact a one parameter problem. Higher order corrections come in successive powers of $\nu^{\frac{1}{2}}$. In section 4 we will show that these corrections indeed remain small in terms of ν for all $\kappa > 0$, that is for all $\alpha^{(1)} \geq 0$.

2.2 The critical point and below: bistability

Here ($\alpha^{(1)} < 0$) the situation is complicated by the existence of two stable states. In order to find the correct irreducible description of the equilibrium fluctuations we shall need a certain reordering of coefficients in the potential, as will be shown below. For that purpose we must explicitly introduce the deterministic states φ_{\pm} , which correspond to the precise positions of the minima of $U(z)$, that means to the zeroes of $U'(z)$. Since the force also has the unstable zero $\varphi_0=0$, we may formally write it as

$$U'(z) = z[\alpha^{(3)}(z-\varphi_+)(z-\varphi_-) + \dots + a^{(2m+1)}(z-\varphi_+)^m(z-\varphi_-)^m + \dots] \quad (2.7)$$

Using $\varphi_- = -\varphi_+$ and the identity $z^2 = (z^2 - \varphi_{\pm}^2) + \varphi_{\pm}^2$ in the original form (2.1) of $U(z)$, and invoking the binomial theorem, one obtains the relation

$$a^{(2m+1)} = \sum_{k=m}^{\infty} \binom{k}{m} \alpha^{(2k+1)} \varphi_{\pm}^{2(k-m)} \quad ; \quad m=1,2,\dots \quad (2.8)$$

between the new and old coefficients. Incidentally, $a^{(3)}$ can be easily cast into closed form, either from an algebraic manipulation with (2.8) or directly from (2.7). This yields

$$a^{(3)} = U''(\varphi_{\pm}) / 2\varphi_{\pm}^2 \quad (2.9)$$

For the special case (2.2) $\alpha^{(1)} = \alpha, \alpha^{(3)} = \alpha^{(5)} = 1$, while all other coefficients are zero, so that the only nonzero renormalized coefficients may be given explicitly as

$$\alpha^{(3)} = 1 + 2\varphi_{\pm}^2 = \sqrt{1-4\alpha^1} \quad , \quad \alpha^{(5)} = 1 \quad (2.10)$$

In general, if one now introduces the scale transformations

$$z = v^{\frac{1}{4}} \eta, \quad t = v^{-\frac{1}{2}} \tau, \quad \varphi_{\pm}^2 = -v^{\frac{1}{2}} \nabla \quad (2.11)$$

into (1.1) with the renormalized potential according to (2.7) and (2.8), one finds

$$\frac{\partial P(\eta, \tau)}{\partial \tau} = \frac{\partial}{\partial \eta} [a^{(3)} \eta (\nabla + \eta^2) + v^{\frac{1}{2}} a^{(5)} \eta (\nabla + \eta^2)^2 + \dots] + \frac{\partial^2 P}{\partial \eta^2} \quad (2.12)$$

Considering formally η, τ and ∇ as quantities of order unity implies by virtue of (2.11) that φ_{\pm}^2 is of order $v^{\frac{1}{2}}$. In view of (2.8) the $a^{(2m+1)} = \alpha^{(2m+1)} + o(v^{\frac{1}{2}})$ remain of order v^0 . Disregarding then in (2.12) terms that explicitly vanish if $v \rightarrow 0$, one obtains the irreducible part of the equation below the critical point. This result is readily transformed into the standard form (2.5) if we set

$$x = \eta [a^{(3)}]^{-\frac{1}{4}}, \quad s = \tau [a^{(3)}]^{-\frac{1}{2}}, \quad \kappa = \nabla [a^{(3)}]^{-\frac{1}{2}} \quad (2.13)$$

In section 4 it will be shown how the above renormalization guarantees the higher order corrections in (2.12) to remain small in terms of ν for all $\kappa \leq 0$, that is for all $\alpha^{(1)} \leq 0$.

2.3 Some comments

We have seen how the master equation (1.1) with the potential (2.1) reduces in the limit $\nu \rightarrow 0$ to the standard form (2.5). Note that the transformations relating the original variables and coefficients to x, s and κ are different above and below the critical point (although (2.3) applies to the whole critical range). Nevertheless, the irreducible part of $U(z)$ is given for any $\alpha^{(1)}$ by a quartic form. This is the form of potential used in the introduction to illustrate the essential features of monostable, bistable and critical dynamics.

Well above the critical point in the normal regime $\alpha^{(1)}$ is positive and of order unity, so that by (2.6) and (2.3) the pump parameter $\kappa \equiv \Delta \approx \nu^{-\frac{1}{2}} \alpha^{(1)}$ tends to plus infinity. Rescaling then in (2.5) according to $x = [\kappa / \alpha^{(1)}]^{-\frac{1}{2}} \xi$, which in effect amounts to $z = \nu^{\frac{1}{2}} \xi$, one easily retrieves the normal linear noise approximation (1.3) with $\varphi = \varphi_0 = 0$ as the leading part of (2.5).

On the other hand, in the normal regime well below the critical point $\alpha^{(1)}$ is negative and of order unity, so that φ_{\pm} are of order ν^0 . Hence, by (2.13) and (2.11) $\kappa \equiv \nabla = -\nu^{-\frac{1}{2}} \varphi_{\pm}^2$ tends to minus infinity. Setting then $x [\alpha^{(3)}]^{-\frac{1}{2}} = \pm (-\nabla)^{\frac{1}{2}} + \nu^{\frac{1}{2}} \xi$, which in effect corresponds to $z = \varphi_{\pm} + \nu^{\frac{1}{2}} \xi$, one readily obtains in leading order (1.3) with $\varphi = \varphi_{\pm}$, which is the correct Gaussian linear noise description locally around these stable steady states. Further it is not difficult to see that the above-critical and below critical formulae indeed smoothly connect at the critical point $\alpha^{(1)} = 0$.

In contrast with the normal linear noise approximation [3, 4], the irreducible problem (2.5) does not allow the computation of moments explicitly in successive higher orders of $\nu^{\frac{1}{2}}$ directly from the differential equation due to its nonlinear drift term (see also [18]). Therefore one must know the solution of (2.5). Of course, it suffices to know its Green's function, also called transition probability or propagator.

3. THE IRREDUCIBLE PROBLEM

Let us repeat (2.5) here as

$$\left. \begin{aligned} \frac{\partial P^{(0)}(x,s)}{\partial s} &= \frac{\partial}{\partial x} U_{\kappa}^{(0)}(x) P^{(0)} + \frac{\partial^2 P^{(0)}}{\partial x^2} , \\ U_{\kappa}(x) &= \frac{1}{2}\kappa x^2 + \frac{1}{4}x^4 . \end{aligned} \right\} \quad (3.1)$$

Eq. (3.1) has been used occasionally as a model for the single mode laser (see e.g. [19]), although a physically realistic description would involve not only intensity but also phase diffusion [20-23].

The propagator of (3.1) will be written in terms of the eigen-solutions of the equation, which has the natural boundary condition $P^{(0)}(x,s) \rightarrow 0$ if $x \rightarrow \pm\infty$, as follows (also see [8, 15, 24-29]):

$$P_{\kappa}^{(0)}(x,s|x_0,0) = \sum_{n=0}^{\infty} P_n^{(0)}(x) Q_n^{(0)}(x_0) e^{-\mu_n^{(0)} s} . \quad (3.2)$$

Here $Q_n^{(0)}(x)$ is the adjoint eigenfunction of $P_n^{(0)}(x)$ defined by

$$P_n^{(0)}(x) = P_0^{(0)}(x) Q_n^{(0)}(x) , \quad (3.3)$$

$$P_0^{(0)}(x) = N_0 \exp[-U_{\kappa}(x)] . \quad (3.4)$$

N_0 is the normalization factor of the stationary solution, corresponding to the lowest (zerth) eigenfunction (because it has no zeroes). The $Q_n^{(0)}(x)$ obey

$$\frac{d^2 Q^{(0)}}{dx^2} - U_{\kappa}'(x) \frac{dQ^{(0)}}{dx} + \mu^{(0)} Q^{(0)}(x) = 0 . \quad (3.5)$$

Clearly, if $\mu^{(0)} = \mu_0^{(0)} = 0$ we have $Q_0^{(0)}(x) = 1$, which by (3.3) indeed belongs to the stationary solution. The $P_n^{(0)}(x)$ and the $Q_n^{(0)}(x)$ form a biorthogonal and presumably complete set (see e.g. [24]):

$$\int_{-\infty}^{\infty} P_n^{(0)}(x) Q_m^{(0)}(x) dx = \delta_{nm} , \quad (3.6)$$

$$\sum_{n=0}^{\infty} P_n^{(0)}(x) Q_n^{(0)}(x_0) = \delta(x-x_0) . \quad (3.7)$$

Equilibrium quantities now take on relatively simple forms. For example, the correlation function becomes

$$\Gamma(s) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx_0 x x_0 P_{\kappa}^{(0)}(x,s|x_0,0) P_0^{(0)}(x_0) \quad (3.8)$$

$$= \sum_{n=0}^{\infty} \langle x \rangle_n^2 e^{-\mu_n^{(0)} s} , \quad (3.9)$$

where

$$\langle x \rangle_n = \int_{-\infty}^{\infty} P_n^{(0)}(x) dx \quad (3.10)$$

represents the first moment of the n-th probability eigenfunction.

Rapid insight into the eigenspectrum can be obtained by transforming the original eigenvalue problem to its selfadjoint representation. Putting $P_n = P_0^{1/2} S_n$ one obtains the Schrödinger-like equation (see also [8, 24, 26-28, 30-33])

$$\left. \begin{aligned} \frac{d^2 S_n^{(0)}}{dx^2} + [\mu^{(0)} - V(x)] S_n^{(0)} &= 0, \\ V(x) &= \frac{1}{4} U_{\kappa}^{1/2} - \frac{1}{2} U_{\kappa}'' = -\frac{1}{2} \kappa + \frac{1}{2} (\frac{1}{2} \kappa^2 - 3) x^2 + \frac{1}{2} \kappa x^4 + \frac{1}{4} x^6. \end{aligned} \right\} \quad (3.11)$$

The potential $V(x)$ is shown in figure 2. As $V(x) \rightarrow +\infty$ if $x \rightarrow \pm\infty$ there will be a pure point spectrum (see also [24, 34-36]). The properties of the eigenfunctions $S_n^{(0)}$ are useful to gain insight in the higher order corrections in the full master equation ((2.4) or (2.12)), because in perturbation theory the pertinent matrix elements involve in effect the products $P_n^{(0)} Q_m^{(0)} \equiv S_n^{(0)} S_m^{(0)}$. See section 4. However, as can be seen for example from (3.10), the original probability eigenfunctions (and their higher order corrections) will be the ultimate relevant ones in the equilibrium quantities. Presently merely a description of the properties of the eigensolutions will be given, sufficient to proceed to higher orders in the master equation. More details can be found in the appendix. See also figure 3, and [27, 33].

3.1 Well above the critical point

Here $\kappa \rightarrow \infty$; both $U_{\kappa}(x)$ and $V(x)$ have only one minimum at $x=0$. Rescaling to the asymptotic local variable $\rho = x\kappa^{1/2}$, one readily shows that the eigenfunctions $S^{(0)}(\rho)$ neatly tend to the quantal harmonic oscillator eigenfunctions, the Weber-Hermite functions. This implies that x will be of order $\kappa^{-1/2}$. The eigenvalues are $\mu_n^{(0)} \approx n\kappa$, with $n=0, 1, 2, \dots$.

3.2 The critical region

Here $\kappa \approx 0$; both $U_{\kappa}(x)$ and $V(x)$ become relatively broad, which reflects the large critical fluctuations, so that x will be of the order of one. If $\kappa > 6^{1/2} \approx 2.45$ $V(x)$ still has one minimum; if $|\kappa| < 6^{1/2}$ it has one maximum at $x=0$ and two shallow minima (at $x = \pm 2^{1/2} \approx 1.19$ if $\kappa=0$); and if $\kappa < -6^{1/2}$ it attains its typical below-critical structure with three minima separated by two barriers. At the critical point no known special functions of

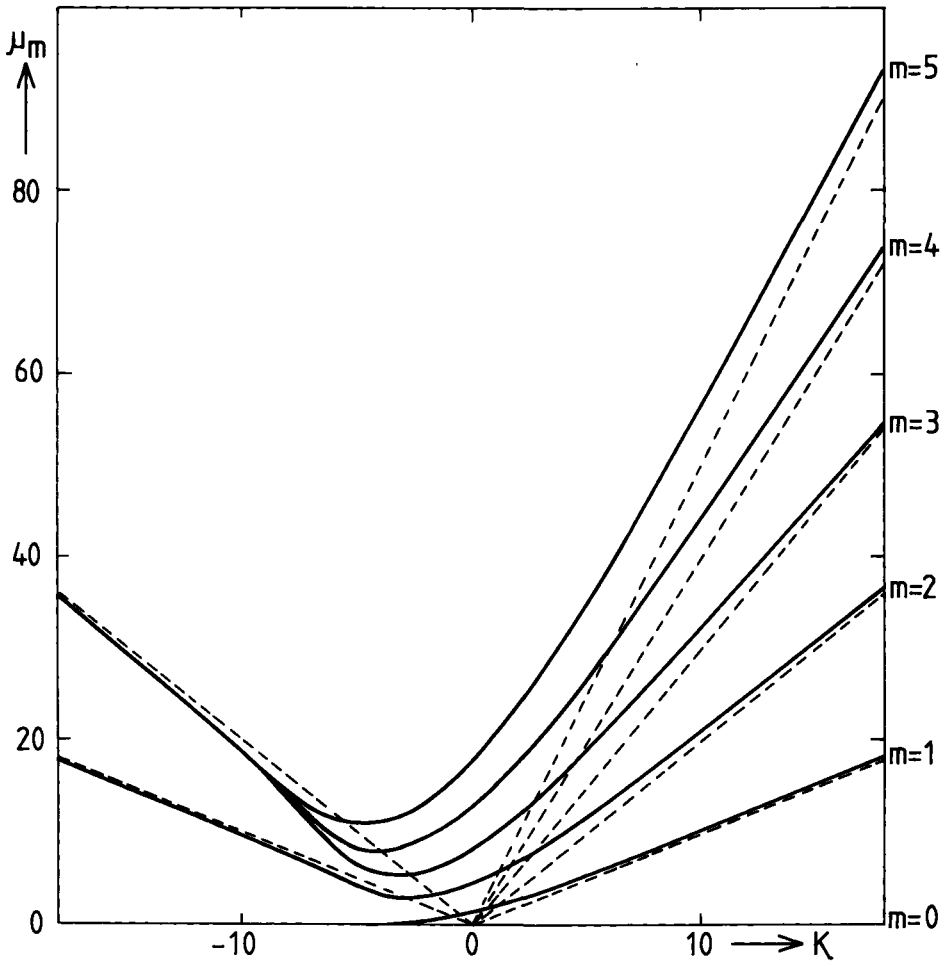


Figure 3. Eigenvalues μ_m of irreducible problem for $m=0$ through 5.

mathematical physics exist to describe this system. Even a WKB-analysis will not be very accurate, in particular not for the dynamically important low lying eigenvalues. Such analysis for $\kappa=0$ results in $\mu_n^{(0)} = 2^{-\frac{1}{2}}(n\pi^{\frac{1}{2}})^{\frac{3}{2}} \cong 1.66n^{\frac{3}{2}}$ (see also [37]). Further, using the simplest polynomial trial functions for the lower $Q^{(0)}(x)$ in the pertinent variational principle (see the appendix, and e.g. [24, 27, 34, 38], one obtains for example $\mu_1^{(0)} = 2\Gamma(\frac{5}{4})/\Gamma(\frac{3}{4}) \cong 1.48$. Detailed numerical analysis [33] reveals that the correct values for some of the lower eigensolutions at $\kappa=0$ are $\mu_1^{(0)} = 1.37$; $\mu_2^{(0)} = 4.45$; and $\mu_3^{(0)} = 8.26$. The important facet here is that the eigenvalues are of the order of one, in contrast with being very large (of order κ) outside the critical range. This represents the critical slowing down.

3.3 Well below the critical point

Here $\kappa \rightarrow -\infty$; $V(x)$ takes on the typical shape shown in figure 2. Three deep local minima, one inner at $x=0$ and two outer at $x_{\pm} \cong (-\kappa)^{\frac{1}{2}}$, are separated by two high barriers. The outer minima almost coincide with those of $U_{\kappa}(x)$. See also figure 1. Transforming in each of the local wells of $V(x)$ to the appropriate local variable ($\rho = x(-\kappa)^{\frac{1}{2}}$ or $\rho = (x - x_{\pm})(-2\kappa)^{\frac{1}{2}}$), one sees that there exist asymptotic eigensolutions $S^{(0)}(\rho)$ which are locally given again by the harmonic oscillator Weber-Hermite functions. Therefore, although x is globally of order $(-\kappa)^{\frac{1}{2}}$, it will locally be of order $(-\kappa)^{-\frac{1}{2}}$. The local eigenvalues become $\mu^{(0)} \cong (n+1)|\kappa|$ for the inner well and $\mu^{(0)} \cong 2n|\kappa|$ for the outer wells, with $n=0, 1, 2, \dots$. In view of the global symmetry of $V(x)$ the outer-solutions of course combine in even and odd pairs. The spectrum may then be described as follows.

First comes one *doublet*, formed by the even and odd paired outer functions with $\mu^{(0)} \cong 0$. The even function with $\mu_0^{(0)} = 0$ of course is the stationary solution. It can be shown [8, 14, 15, 26-28] that the odd one has an extremely small but nonzero eigenvalue $\mu_1^{(0)} \cong e^{-\kappa^2/4}$, where $\kappa^2/4 = U_{\kappa}(0) - U_{\kappa}(x_{\pm})$ represents the height of the potential barrier in (3.1). This corresponds to Kramers' diffusion rate between the two metastable states $x_{\pm} = (-\kappa)^{\frac{1}{2}}$. Further, due to the normalization (3.6) one has in (3.4) that $N_0 \cong e^{-\kappa^2/4}$.

Second, there exists a set of *singlets*, namely the even inner solutions with $\mu_{4n-2}^{(0)} \cong (2n-1)|\kappa|$, where $n=1, 2, \dots$. In view of the normalization (3.6) the inner functions $S^{(0)}$ are of order unity in terms of $e^{-\kappa^2/4}$. Regarding their definition this implies, however, that the

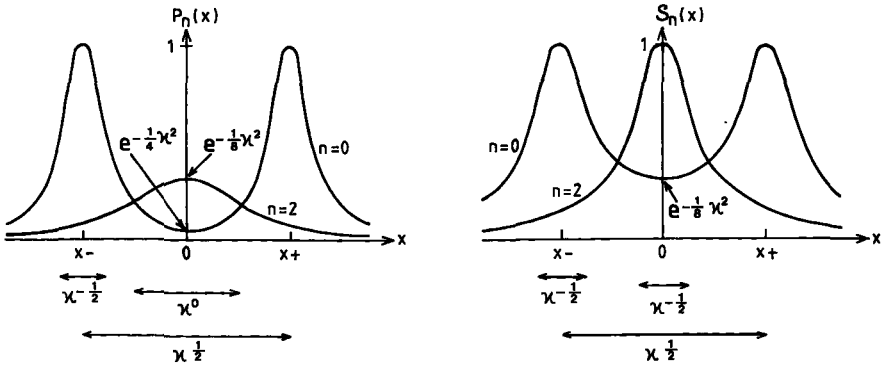


Figure 4. Irreducible eigenfunctions far below the critical point; $n=0$ represents the stationary solution; $n=2$ is the lowest singlet solution.

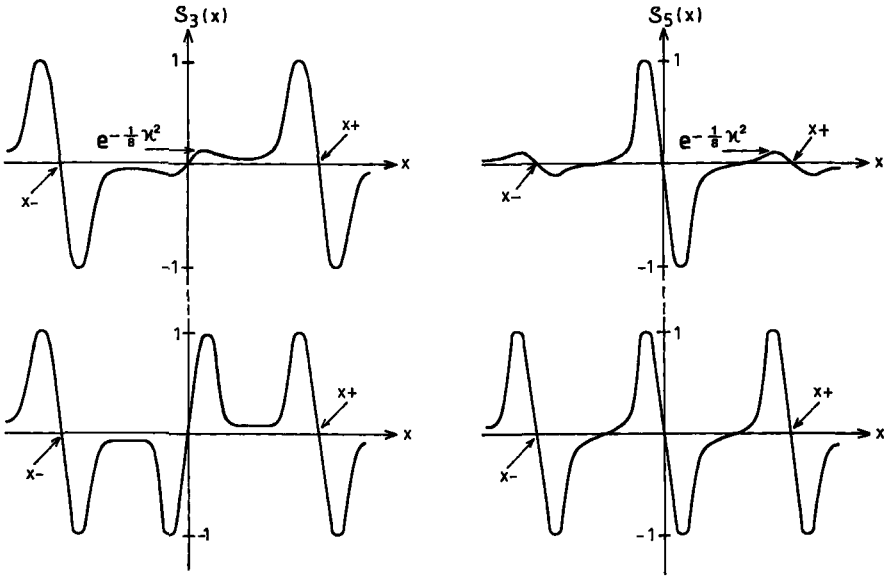


Figure 5. Asymptotic ($\kappa \rightarrow \infty$) odd eigenfunctions $S_3(x)$ and $S_5(x)$ of lowest triplet. Upper sketches: weak degeneracy (weak coupling). Lower sketches: strong degeneracy (strong coupling).

inner probability eigenfunctions $P^{(0)}$ are of the order of $[P_0^{(0)}]^{1/2} \cong N_0^{1/2}$ which means very small of order $e^{-\kappa^2/8}$. See figure 4.

Third, one observed a set of *triplets*, consisting of an odd inner solution and two (even and odd paired) outer solutions with $\mu_{4n-1}^{(0)} \cong \mu_{4n}^{(0)} \cong \mu_{4n+1}^{(0)} \cong 2n|\kappa|$, where $n=1, 2, \dots$. The odd members of the asymptotic triplets may mix, of course. There are two possibilities. Either the higher order corrections in terms of κ^{-1} to the local eigenvalues are different for the inner and outer wells. In case of such *weak* degeneracy it can be shown that the true eigenfunctions are indeed essentially confined either to the inner or to the outer well (see the appendix). Or the degeneracy is lifted only by ever present exponentially small differences (due to the finiteness of the exponential barriers). In case of such *strong* degeneracy it can likewise be shown that the inner and outer odd triplet members combine on equal footing. See figure 5. Curiously, so far we have not been able to show which case actually occurs. Nevertheless, in any case the probability eigenfunctions $P^{(0)}$ near $x=0$ will be asymptotically small of order $e^{-\kappa^2/8}$ or $e^{-\kappa^2/4}$ if $\kappa \rightarrow \infty$. Therefore, with reference to (3.10), ultimately the actual inner functions do not contribute anymore in equilibrium quantities.

Combining the above mentioned features of the eigenspectrum of (3.1) one arrives at figure 3, that has been obtained from extensive numerical calculations [33].

4. HIGHER ORDER CORRECTIONS

Equations (2.4) and (2.12) may be written in a single formula,

$$\frac{\partial P(x,s)}{\partial s} = \frac{\partial}{\partial x}(\kappa x + x^3)P + \frac{\partial^2 P}{\partial x^2} - v^{\frac{1}{2}}V\left(\frac{\partial}{\partial x}, x\right)P, \quad (4.1)$$

where

$$V\left(\frac{\partial}{\partial x}, x\right) = V^{(1)}\left(\frac{\partial}{\partial x}, x\right) + v^{\frac{1}{2}}V^{(2)}\left(\frac{\partial}{\partial x}, x\right) + \dots \quad (4.2)$$

Below the critical point the perturbational operators are given by

$$\left. \begin{aligned} V^{(1)}\left(\frac{\partial}{\partial x}, x\right) &= -\alpha^{(5)} [a^{(3)}]^{-3/2} \frac{\partial}{\partial x} x (\kappa + x^2)^2, \\ V^{(2)}\left(\frac{\partial}{\partial x}, x\right) &= -\alpha^{(7)} [a^{(3)}]^{-4/2} \frac{\partial}{\partial x} x (\kappa + x^2)^3, \end{aligned} \right\} \quad (4.3)$$

and so forth. Above the critical point the perturbational operators are obtained from (4.3) simply setting $\kappa=0$ and replacing the renormalized $\alpha^{(k)}$ by the original $\alpha^{(k)}$. One obtains the general eigenvalue problem as

$$\frac{d^2 P}{dx^2} + \frac{d}{dx} \mu_k^{(1)}(x) P + \mu P(x) = v^{\frac{1}{2}} V\left(\frac{d}{dx}, x\right) P(x). \quad (4.4)$$

It will be convenient to cast (4.4) in matrix form. Let

$$P(x) = \sum_{n=0}^{\infty} c_n P_n^{(0)}(x), \quad (4.5)$$

where the $P_n^{(0)}(x)$ are the eigenfunctions of the irreducible problem.

Inserting (4.5) into (4.4), multiplying with the adjoint eigenfunction $Q_k^{(0)}(x)$ and integrating over all x , leads to

$$c_k (\mu_k^{(0)} - \mu) + v^{\frac{1}{2}} \sum_{\ell} V_{k\ell} c_{\ell} = 0, \quad (4.6)$$

where we have used (3.6) and introduced the matrix elements

$$V_{k\ell} = \int_{-\infty}^{\infty} Q_k^{(0)}(x) V\left(\frac{d}{dx}, x\right) P_{\ell}^{(0)}(x) dx. \quad (4.7)$$

Eq. (4.6) is equivalent to (4.4). It is the starting point for our approximation scheme.

4.1 The even solutions

We insert (4.2) and

$$c_k = \delta_{kn} + v^{\frac{1}{2}} c_{kn}^{(1)} (1 - \delta_{kn}) + v c_{kn}^{(2)} (1 - \delta_{kn}) + \dots, \quad (4.8)$$

$$\mu = \mu_n^{(0)} + v^{\frac{1}{2}} \mu_n^{(1)} + v \mu_n^{(2)} + \dots, \quad (4.9)$$

into (4.6) and collect coefficients of equal powers of $v^{\frac{1}{2}}$. Defining

$\mu_{nk}^{(0)} = \mu_n^{(0)} - \mu_k^{(0)}$, this yields:

$$\mu_n^{(1)} = V_{nn}^{(1)}, \quad (4.10)$$

$$c_{kn}^{(1)} = V_{kn}^{(1)} / \mu_{nk}^{(0)}, \quad (4.11)$$

$$\mu_n^{(2)} = V_{nn}^{(2)} + \sum_{\ell \neq n} V_{n\ell}^{(1)} c_{\ell n}^{(1)}, \quad (4.12)$$

$$c_{kn}^{(2)} = [V_{kn}^{(2)} + \sum_{\ell \neq n} V_{k\ell}^{(1)} c_{\ell n}^{(1)} - \mu_n^{(1)} c_{kn}^{(1)}] / \mu_{nk}^{(0)}, \quad (4.13)$$

and so on. From section 3.2 one infers that both x and $\mu_{nk}^{(0)}$ are of order v^0 at the critical point, so that the validity of the above scheme is obvious in that case. It remains to demonstrate that (4.8) and (4.9) retain their significance as systematic perturbation series even in the normal regimes well above and well below the critical point.

4.1.1 Well above the critical point

From section 3.1 it is seen that here $x \sim v^{-\frac{1}{2}}$, which will be of order $v^{\frac{1}{4}}$, while $\mu_n^{(0)} \sim v^{-\frac{1}{2}}$. Hence, the matrix element $V_{kn}^{(p)}$, with $p=1, 2, \dots$, defined by (4.7) and (4.2), becomes of order $v^{\frac{1}{2}(p+1)}$. See below (4.3). Then it is not difficult to see that $\mu_n^{(p)}$ becomes of order $v^{\frac{1}{2}(p+1)}$, and $c_{kn}^{(p)}$ of order $v^{\frac{1}{2}(p+2)}$. So, if we define $\mu_n^{(p)} = v^{\frac{1}{2}(p+1)} \lambda_n^{(p)}$ and $c_{kn}^{(p)} = v^{\frac{1}{2}(p+2)} a_k^{(p)}$, the $\lambda_n^{(p)}$ and $a_k^{(p)}$ will be of order v^0 . Inserting this into (4.8) and (4.9), one finds

$$a_k = \delta_{kn} + v^2 a_{kn}^{(2)} (1 - \delta_{kn}) + v^3 a_{kn}^{(3)} (1 - \delta_{kn}) + \dots, \quad (4.14)$$

$$\lambda = \lambda_n^{(0)} + v^2 \lambda_n^{(2)} + v^3 \lambda_n^{(3)} + \dots, \quad (4.15)$$

where we have set $a_k = c_k$, $\lambda = \mu v^{\frac{1}{2}}$ and $\lambda_n^{(0)} = \mu_n^{(0)} v^{\frac{1}{2}}$, which are of order v^0 here. Clearly, also well above the critical point the higher order corrections come in successive powers of v . The actual perturbation parameter becomes here v itself, instead of the critical $v^{\frac{1}{2}}$. Note further that the expansion is not a true power series as both the zeroth order and the matrix elements also contain higher orders as well. Finally, one easily convinces one self of the correctness of the absence of the first order term in (4.14) and (4.15). This correction of order v is here fully contained already in the irreducible solution.

4.1.2 Well below the critical point

We infer from section 3.3 that due to (3.6) the inner solutions are of the very small order $e^{-\kappa^2/8}$ near $x=0$ as the outer solutions are of the order of unity near x_{\pm} . Also see again figure 4. Hence, in any

pertinent equilibrium quantity, that means moments like (3.10), we can dispense with the even inner solutions*. In order to investigate the behaviour of the $V_{k\ell}$ for the outer eigensolutions we set according to section 3.3 again $x = \pm(-\kappa)^{\frac{1}{2} + \rho}(-2\kappa)^{-\frac{1}{2}}$ into (4.7) with (4.2) and (4.3). This yields that $V_{k\ell}^{(p)}$ will scale up to order $v^{-\frac{1}{2}}$, independent of $p=1, 2, \dots$. One then easily observes that $\mu_n^{(p)}$ becomes of order $v^{-\frac{1}{2}}$, while $c_k^{(p)}$ remains of order v^0 . Therefore, setting $\mu_n^{(p)} = v^{-\frac{1}{2}} \lambda_n^{(p)}$ and introducing again the order unity quantities $\lambda = \mu v^{\frac{1}{2}}$ and $\lambda_n^{(0)} = \mu_n^{(0)} v^{\frac{1}{2}}$ one obtains here:

$$c_k = \delta_{kn} + v^{\frac{1}{2}} c_{kn}^{(1)} (1 - \delta_{kn}) + v c_{kn}^{(2)} (1 - \delta_{kn}) + \dots, \quad (4.16)$$

$$\lambda = \lambda_n^{(0)} + v^{\frac{1}{2}} \lambda_n^{(1)} + v \lambda_n^{(2)} + \dots. \quad (4.17)$$

Hence, also well below the critical point the scheme retains its validity. The corrections come in successive powers of $v^{\frac{1}{2}}$, instead of v as above the critical point, because the original potential $U(z)$, ie. (2.1), is symmetrical about $z=0$ but not about $z=\varphi_{\pm}$. As in section 4.1.1, note that the expansion is not a genuine power series.

4.2 The odd solutions

The preceding nondegenerate calculation scheme for the even solutions also applies to the lowest odd solution P_1 , that is the odd member of the below-critical doublet. See section 3.3. However, in view of the symmetry of the original problem and the presence of two odd eigensolutions of the irreducible problem within each triplet far below the critical point, we must now turn to pseudo-degenerate perturbation theory for a unified treatment (i.e. valid for any $\alpha^{(1)}$). Following Davydov [34] we now first set

$$c_k = \bar{c}_n^{(0)} \delta_{nk} + \bar{c}_m^{(0)} \delta_{mk} + \dots; \quad \mu = \bar{\mu}^{(0)} + \dots. \quad (4.18)$$

Here n and m are the indices of two odd triplet solutions. Inserting (4.18) into (4.6) and collecting terms up to and including order $v^{\frac{1}{2}}$ leads to a set of two homogeneous linear equations, which in matrix notation takes the form

*Of course, if the system were far out of equilibrium near the unstable state $\varphi_0=0$, truly in the inner Schrödinger-well, it would be convenient to expand locally in terms of the inner functions. See also [28] and figure 4.

$$\begin{bmatrix} \mu_n^{(0)} - \bar{\mu}^{(0)} + v^{\frac{1}{2}} V_{nn}^{(1)} & v^{\frac{1}{2}} V_{nm}^{(1)} \\ v^{\frac{1}{2}} V_{mn}^{(1)} & \mu_m^{(0)} - \bar{\mu}^{(0)} + v^{\frac{1}{2}} V_{mm}^{(1)} \end{bmatrix} \begin{bmatrix} \bar{c}_n^{(0)} \\ \bar{c}_m^{(0)} \end{bmatrix} = 0 \quad (4.19)$$

The solvability condition of (4.19) yields

$$\bar{\mu}^{(0)} = \frac{1}{2} [\mu_n^{(0)} + \mu_m^{(0)} + v^{\frac{1}{2}} (V_{nn}^{(1)} + V_{mm}^{(1)})] \pm \sqrt{\frac{1}{4} [\mu_n^{(0)} - \mu_m^{(0)} + v^{\frac{1}{2}} (V_{nn}^{(1)} - V_{mm}^{(1)})]^2 + v V_{nm}^{(1)} V_{mn}^{(1)}} \quad (4.20)$$

Substitution of these values into (4.19) gives the required relations between $\bar{c}_n^{(0)}$ and $\bar{c}_m^{(0)}$.

4.2.1 Well above the critical point

Here $\mu_n^{(0)} - \mu_m^{(0)} = \mu_{nm}^{(0)}$ is of order $\kappa \sim v^{-\frac{1}{2}}$, while the first order matrix elements are of order v , as can be inferred from section 4.1.1. Hence, in (4.20) one may safely expand the square root so as to give

$$\begin{aligned} \bar{\mu}_n^{(0)} &= \mu_n^{(0)} + v^{\frac{1}{2}} V_{nn}^{(1)} + \dots, \\ \bar{c}_{nn}^{(0)} &= 1, \quad \bar{c}_{mn}^{(0)} = v^{\frac{1}{2}} V_{mn}^{(1)} / \mu_{nm}^{(0)} + \dots, \end{aligned} \quad (4.21)$$

and a similar result with m and n interchanged. As expected, this agrees with the purely nondegenerate case discussed for the even solutions.

4.2.2 The critical region

Since here both $\mu_{nm}^{(0)}$ and the first order matrix elements are of order v^0 , (4.20) again leads to the nondegenerate results, as it should.

4.2.3 Well below the critical point

As noted at the end of section 3.3 there are two possibilities of asymptotic degeneracy. Consequently, we distinguish these two cases here. See also again figure 5, and the appendix.

i) *Weak degeneracy*: $\mu_{nm}^{(0)}$ is of the order of $v^{\frac{1}{2}}$ to some power*, while the irreducible odd eigensolutions of the triplet almost completely separate spatially, i.e. for example $S_n \cong S_{\text{outer}}$ and $S_m \cong S_{\text{inner}}$. Hence, the off-

*We know that if such a difference does exist, it must be at least of order $v^{\frac{1}{2}}$. See the appendix.

diagonal elements $V_{nm}^{(1)}$ and $V_{mn}^{(1)}$ are exponentially small, so that (4.20) directly reduces to the nondegenerate case. Clearly, if one could prove that the irreducible problem is really only weakly degenerate well below the critical point, then the calculation of the higher order corrections in the original master equation (1.1) with (2.1) could proceed entirely be nondegenerate theory for any $\alpha^{(1)}$. Unfortunately, such a proof has not been found yet. Therefore, we must also consider:

ii) *Strong degeneracy*: $\mu_{nm}^{(0)}$ is exponentially small and the irreducible odd triplet members do not separate spatially. In stead, they are almost identical everywhere except for a change of sign in either the inner or the outer well of the Schrödinger potential $V(x)$, i.e. for example $S_n = S_{\text{outer}} + S_{\text{inner}}$ and $S_m = S_{\text{outer}} - S_{\text{inner}}$. Hence, $V_{nn}^{(1)}$ and $V_{mm}^{(1)}$, as well as $V_{nm}^{(1)}$ and $V_{mn}^{(1)}$, only differ by exponentially small terms. Therefore, (4.20) leads to:

$$\bar{\mu}^{(0)} = \mu^{(0)} + v^{\frac{1}{2}} (V_{nn}^{(1)} \pm V_{nm}^{(1)}) \quad , \quad (4.22)$$

which clearly lifts the degeneracy. The $\bar{c}^{(0)}$ are ± 1 , which simply implies that the thus obtained zeroth order eigenfunctions, say \bar{S}_n and \bar{S}_m , separate spatially. As usual [34], using now these new nondegenerate odd solutions one may further treat the problem by the standard perturbation theory as discussed in section 4.1.

5. SOME FINAL REMARKS

In conclusion, we have shown how to set up a systematic expansion for the equilibrium solutions of a Fokker-Planck master equation describing a diffusion process with a small diffusion coefficient ν in a potential $U(x)$ showing a transition from mono to bistability. The expansion is valid uniformly for all values of the pertinent pump parameter $\alpha^{(1)}$, including the critical point $\alpha^{(1)}=0$.

In the introduction we have briefly indicated the features of mono-stability, bistability and critical behaviour. In section 2 we have shown how to separate the general problem into its irreducible part and a corrective remainder, which is small in terms of ν . Section 3 has been devoted to a description of the relevant properties of the solutions of the irreducible problem, while in section 4 it has been shown explicitly that the higher order corrections indeed remain small for any $\alpha^{(1)}$.

The unified description becomes possible for the following reasons. First of all, it depends on a proper recognition of the critical region. Second, because the critical fluctuations are larger than the normal ones (but see e.g. [39] for an exception), at least the connection with the above-critical monostable regime presents no serious difficulties. Third, the extension to the below-critical bistable case, however, could be effectuated only after renormalising the original potential so as to collect all relevant information for the local normal equilibrium fluctuations into the irreducible potential. This renormalization makes the global large scale fluctuations (of size ν^0) ineffective in spoiling the expansion. In more technical terms, it prevents the upscaling of the perturbational matrix elements.

If in any formula in section 3 we drop the superscript (0) and normalize the corrected eigenfunctions according to (3.6)*, the propagator

* There is an alternative where one can get around the explicit normalization procedure (3.6). Namely, if one directly uses the corrected eigenfunctions $P_n(x)$ as obtained in section 4, i.e. with $c_{nn}^{(p)}=0$ for $p>1$, one should in (3.2) drop the superscript (0) on the P-function only. However, since $P_n \equiv P_0 Q_n \neq P_0 Q^{(0)}$, the correlation function for example will not be given by the elegant formula (3.9) in this case.

will be represented by (3.2) while for example the correlation function remains given by (3.9) . Finally, in terms of the original variables one explicitly has $P(z,t|z_0,0) \leftrightarrow P_K(x,s|x_0,0)$, where:

| | above-critical $\alpha^{(1)} \geq 0$ | below-critical $\alpha^{(1)} \leq 0$ |
|------------|---|---|
| $x =$ | $v^{-\frac{1}{2}}[\alpha^{(3)}]^{\frac{1}{2}}z$ | $v^{-\frac{1}{2}}[a^{(3)}]^{\frac{1}{2}}z$ |
| $s =$ | $v^{\frac{1}{2}}[\alpha^{(3)}]^{\frac{1}{2}}t$ | $v^{\frac{1}{2}}[a^{(3)}]^{\frac{1}{2}}t$ |
| $\kappa =$ | $v^{-\frac{1}{2}}[\alpha^{(3)}]^{-\frac{1}{2}}\alpha^{(1)}$ | $-v^{-\frac{1}{2}}[a^{(3)}]^{\frac{1}{2}}\varphi_{\pm}^2$ |

APPENDIX: THE IRREDUCIBLE SOLUTIONS

In this appendix we examine the eigensolutions of the irreducible problem (3.1) in some more detail. The three regions, well above the critical point (monostable case; $\kappa \rightarrow \infty$), near the critical point ($\kappa \approx 0$) and well below the critical point (bistable case; $\kappa \rightarrow -\infty$) will be discussed separately. For a numerical analysis we refer to [33]. The spectrum is shown in figure 3.

A1. Well above the critical point

In this case one applies perturbation theory in terms of κ^{-2} to the adjoint equation of (3.1), obeyed by the $Q(x)$. Introducing the appropriately scaled variable $\rho = x(\kappa/2)^{\frac{1}{2}}$ and $\nu = \mu/\kappa$ (μ being the standard eigenvalue), one finds

$$Q''(\rho) - 2\rho Q'(\rho) + 2\nu Q(\rho) = 4\kappa^{-2} \rho^3 Q'(\rho) \quad (A1.1)$$

Throughout the appendix we suppress the superscript (0) for typographical reasons. If $\kappa = \infty$, (A1.1) is satisfied by Hermite polynomials $Q(\rho) = H_n(\rho)$, i.e. $\nu = n = 0, 1, \dots$. Therefore, for large κ we set

$$Q(\rho) = \sum_{n=0}^{\infty} c_n H_n(\rho) \quad (A1.2)$$

and cast (A1.1) as usual into matrix form

$$c_k (k-\nu) + \sum_{\ell} B_{k\ell} c_{\ell} = 0 \quad (A1.3)$$

where $B_{k\ell} = \kappa^{-2} B_{k\ell}^{(1)}$, with

$$B_{k\ell}^{(1)} = \frac{1}{2}(k-2)\delta_{k-2,\ell} + 3k^2\delta_{k,\ell} + 6(k+2)(k+1)\delta_{k+2,\ell} + 4(k+4)(k+3)(k+2)(k+1)\delta_{k+4,\ell} \quad (A1.4)$$

Using the "modified iteration-perturbation" formulae of Morse and Feshbach [24] (which have an improved radius of convergence over the usual Rayleigh-Schrödinger formulae) one finds for the eigenvalues

$$\mu_n = n\kappa \left\{ 1 + \frac{3n}{\kappa^2} - \frac{3}{2\kappa^4} \left[\frac{(n+1)^2(n+2)}{1 + \frac{3}{2}(n+2)^2/\kappa^2} - \frac{(n-1)^2(n-2)}{1 - \frac{3}{2}(n-2)^2/\kappa^2} \right] + \dots \right\} \quad (A1.5)$$

For example,

$$\mu_0 = 0; \mu_1 = \kappa + 3\kappa^{-1} + \dots; \mu_2 = 2\kappa + 12\kappa^{-1} + \dots; \mu_3 = 3\kappa + 27\kappa^{-1} + \dots \quad (A1.6)$$

Note that all eigenvalues lie above the asymptote $\mu_n = n\kappa$. The correlation function $\Gamma(s)$, given in (3.9), reduces effectively to the normal single exponential, expressing ordinary Debye-relaxation.

A2. The critical region

A2.1 Simple variational considerations

Insight into the dynamically important low lying eigenvalues may be obtained from the variational principle (see e.g. [27, 34, 38, 40, 41]). That is, the stationary value of the functional

$$\mu\{Q(x)\} = \int_{-\infty}^{\infty} P_0(x) Q'(x)^2 dx \quad (A2.1)$$

should be a good approximation to the true eigenvalue. In (A2.1) we implicitly assume normalization according to (3.6). Obviously, $Q_0(x) \equiv 1$ corresponds to be stationary solution $\mu_0 = 0$. Let us further consider

$$\left. \begin{aligned} Q_1(x) &= N_1^{\frac{1}{2}} x, \\ Q_2(x) &= N_2^{\frac{1}{2}} (1 - c_2 x^2), \\ Q_3(x) &= N_3^{\frac{1}{2}} (x - c_3 x^3). \end{aligned} \right\} \quad (A2.2)$$

Using e.g. Gradshteyn's [42] formula (3.462.1) in normalizing $Q_1(x)$, one obtains

$$N_1 = \sqrt{2} D_{-\frac{1}{2}}\left(\frac{\kappa}{\sqrt{2}}\right) / D_{-\frac{3}{2}}\left(\frac{\kappa}{\sqrt{2}}\right), \quad (A2.3)$$

where $D_p(z)$ is a Weber-Hermite (parabolic cylinder) function. The factors N_2 and N_3 are readily expressed in terms of N_1 , once the coefficients $c_2 = N_1$ and $c_3 = (N_1 - \kappa)^{-1}$ have been determined by orthogonality according to (3.6). Substituting the obtained functions into (A2.1) results in

$$\mu_1 = N_1; \quad \mu_2 = \frac{4\mu_1}{\mu_1(\mu_1 - \kappa) - 1}; \quad \mu_3 = (\mu_1 - \kappa) \frac{1 + \frac{1}{3}\mu_1(\mu_1 - \kappa)}{1 - \frac{1}{3}\mu_1(\mu_1 - \kappa)}. \quad (A2.4)$$

Hence, at the critical point $\kappa = 0$ the values are $\mu_1(0) = 2\Gamma(\frac{5}{4})/\Gamma(\frac{3}{4}) \approx 1.48$; $\mu_2(0) \approx 4.97$ and $\mu_3(0) \approx 9.49$. For comparison, the numerical results from [33] read $\mu_1(0) = 1.37$; $\mu_2(0) = 4.45$ and $\mu_3(0) = 8.26$. Incidentally, using the appropriate asymptotic formula for the parabolic cylinder functions as $\kappa \rightarrow \infty$ (see e.g. [35]; or [42], form. (9.246.1)), one obtains from (A2.4) the eigenvalues (A1.6) well above the critical point, including the proper first order correction. Well below the critical point (A2.2) and (A2.4) are in error, as will become clear in due course.

A2.2 Turning points analysis

A turning points or improved WKB-analysis is relevant in particular for the higher eigenvalues which follow from (see e.g. [24, 34, 43, 44])

$$\frac{1}{\pi} \int_{x_1}^{x_2} [\mu - V(x)]^{\frac{1}{2}} dx = n + \frac{1}{2} + o\left(\frac{1}{n}\right). \quad (A2.5)$$

The potential $V(x)$ is given in (3.11) and is shown in figure 2. The range of integration in (A2.5) is determined by the real turning points which follow from the equation $\mu=V(x_1)=V(x_2)$. We only consider the case of two turning points* and define $x_0 \equiv x_2 (= -x_1)$. The potential (3.11) is now inserted into (A2.5) and a new variable $z=(x/x_0)^2$ is introduced. Defining $\nu=\mu+\frac{1}{2}\kappa$, one obtains

$$\frac{1}{2\pi} \nu^{2/3} \int_0^1 z^{-1/2} (1-z^3)^{1/2} \left[1 + \beta \frac{1-z^2}{1-z^3} + \gamma \frac{1-z}{1-z^3} \right]^{1/2} dz = n + \frac{1}{2} + o\left(\frac{1}{n}\right), \quad (\text{A2.6})$$

where $\nu_0 = x_0^6$, $\beta = 2\kappa \nu_0^{-1/3}$ and $\gamma = (\kappa^2 - 6)\nu_0^{-2/3}$. In view of the nature of the WKB-approximation, we expand (A2.6) in terms of $\nu_0^{-1/3}$ (see also [45, 46], and**). The leading terms yield $\nu_0 \sim n^{3/2}$, which is assumed to be large. As the r.h.s. of (A2.6) is in fact correct only through order n^0 [44, 45], the expansion of the integral is stopped at order $\nu_0^{-2/3}$. In the evaluation one encounters several integrals which can all be related to the beta function. Next one determines the expansion of ν_0 , that means of the turning point x_0 , in terms of $\nu^{-1/3}$ to sufficient accuracy. The result reads

$$a_2(4\nu)^{2/3} + a_1(4\nu)^{1/3} = 2\pi n, \quad (\text{A2.7})$$

where $a_1 = -\kappa(\pi^{3/2}/4a_2)$, $a_2 = \frac{1}{3}B(1/6, 3/2) = \frac{1}{3}\pi^{1/2}\Gamma(1/6)/\Gamma(2/3) \approx \pi^{1/2}$. This is a quadratic equation in $(4\nu)^{1/3}$. It leads to

$$\mu_n \approx -\frac{1}{2}\kappa + \frac{1}{\sqrt{2}} \left(\frac{3}{128}\right)^{3/2} \left[\kappa + \sqrt{\kappa^2 + \frac{128}{3}n\sqrt{\pi}} \right]^3. \quad (\text{A2.8})$$

Hence, at the critical point these eigenvalues become:

$$\mu_n(0) \approx \frac{1}{\sqrt{2}}(n\sqrt{\pi})^{3/2} = 1.66 n^{3/2}, \quad (\text{A2.9})$$

so that $\mu_1(0) \approx 1.66$; $\mu_2(0) \approx 4.70$ and $\mu_3(0) \approx 8.63$. We mention again that the numerical results from [33] are resp. 1.37; 4.45 and 8.26. Clearly, while the variational method is superior for μ_1 , the WKB-method becomes rapidly better at the higher eigenvalues. Nevertheless, in particular

* This is correct for all $n=0, 1, \dots$ if $\kappa > -6^{1/2}$ (V has one or two minima). If $\kappa < -6^{1/2}$ (V has three minima), it holds if $n\pi^{1/2} \geq (\kappa/2)^2$.

** The convergence will be reasonable under the approximate condition $\max |\beta, \gamma| < 1$. Using $\nu_0^{2/3} \approx 2n\pi^{1/2}$, this yields roughly the same restrictions as in the previous footnote*: only if $\kappa \gg 1$ it adds the requirement $n \geq \kappa^2$.

for the important lower eigenvalues these results are quantitatively insufficient and one must resort to numerical analysis [33]. Finally it is noted that both (A2.4) and (A2.8) confirm the earlier finding that the critical slowing down continues somewhat below $\kappa=0$ [33].

A3. Well below the critical point

Rapid insight into the spectrum as $\kappa \rightarrow -\infty$ is obtained from the Schrödinger-like equation (3.11). In this limit $V(x)$ develops three minima separated by high barriers (of order $|\kappa|^3$). Asymptotically the three wells become isolated local harmonic oscillator potentials with eigenvalues $\mu=(m+1)|\kappa|$ resp. $2n|\kappa|$ for the inner resp. outer minima ($n, m=0, 1, 2, \dots$). The corresponding asymptotic local eigenfunctions $S_n(x)$ are the usual Weber-Hermite or parabolic cylinder functions. Clearly, there is a twofold degeneracy for $n=0$ (globally even and odd combinations of outer solutions) and a triple degeneracy for odd m , viz. $m=2n-1$. See also [27, 28, 33].

A3.1 Inner solutions

Introducing the appropriate local variable $\rho=x(-\kappa/2)^{\frac{1}{2}}$ and $v=(-\mu/\kappa)-1$ into the equation for the adjoint eigenfunctions, and setting $Q(\rho)=e^{-\rho^2}R(\rho)$, one obtains

$$R''(\rho) - 2\rho R'(\rho) + 2vR(\rho) = 4\kappa^{-2} \rho^3 e^{\rho^2} [e^{-\rho^2} R(\rho)]' \quad (A3.1)$$

If $\kappa \rightarrow -\infty$, (A3.1) is solved by $R(\rho) = H_m(\rho)$, i.e. $v=m=0, 1, \dots$. For large negative κ we expand $R(\rho)$ in terms of these Hermite polynomials like in (A1.2) and cast (A3.1) into matrix form as usual. This leads to (A1.3) with

$$B_{k\ell}^{(1)} = -\frac{1}{4} [\delta_{k-\ell}, \ell + 6k\delta_{k-2, \ell} + 12(k+1)^2 \delta_{k, \ell} + 8(k+3)(k+2)(k+1)\delta_{k+2, \ell}] \quad (A3.2)$$

Using standard perturbation theory (see e.g. section 4.1) through third order in κ^{-2} , we have obtained

$$\mu = (m+1)|\kappa| \left\{ 1 - \frac{3}{\kappa^2}(m+1) - \frac{6}{\kappa^4} [2(m+1)^2 + 1] - \frac{3}{\kappa^6} (m+1) [35(m+1)^2 + 52] + \dots \right\} \quad (A3.3)$$

A3.2 The doublet

If $\kappa \rightarrow -\infty$ and $v=-1$, (A3.1) allows for solutions corresponding to $Q_0(\rho) \equiv 1$ and $Q_1(\rho) = \text{erf}(\rho)$. Although they are not contained in the Hilbert space of local inner solutions, these functions do belong to the Hilbert

space of global solutions. Note that they asymptotically connect with the exact stationary solution at the outer wells and beyond. Using the above $Q_1(\rho)$ for the first excited state into the original irreducible adjoint equation and invoking the global Hilbert norm (3.6), one may calculate the first order correction to $v=-1$ (corresponding to $\mu_1^{(0)}=0$). It leads to

$$\mu_1^{(1)} = \frac{2}{\pi} |\kappa| D_{-\frac{1}{2}}\left(-\frac{\kappa}{\sqrt{2}}\right) / D_{-\frac{1}{2}}\left(\frac{\kappa}{\sqrt{2}}\right) . \quad (A3.4)$$

By means of the appropriate asymptotic expression for the parabolic cylinder function this gives

$$\mu_1 \approx \frac{\sqrt{2}}{\pi} |\kappa| e^{-\frac{1}{2}\kappa^2} \left(1 - \frac{3}{2\kappa^2} + \dots\right) . \quad (A3.5)$$

This corresponds to Kramer's diffusion rate [13, 47]; $\kappa^2/4$ is the height of the barrier in the irreducible potential $U_\kappa(x)$. The result (A3.5) agrees with that obtained from using $Q_1(\rho)=\text{erf}(\rho)$ into the variational principle and has been confirmed numerically [27]. It also agrees with Kramers' original analysis [13, 15] if the pertinent integrals are handled carefully [16]. Finally, the leading part in (A3.5) coincides with the result from a WKB-type analysis [28].

A3.3 Outer solutions

Introducing $x=(-\kappa)^{\frac{1}{2}}+\rho(-\kappa)^{-\frac{1}{2}}$ and $v=(-\mu/2\kappa)$ into the irreducible problem, one finds for the adjoint eigensolutions

$$Q''(\rho) - 2\rho Q'(\rho) + 2vQ(\rho) = (-3\kappa^{-1}\rho^2 + \kappa^{-2}\rho^3)Q'(\rho) . \quad (A3.6)$$

If $\kappa=-\infty$, (A3.6) yields $Q(\rho)=H_n(\rho)$, i.e. $v=n=0, 1, \dots$. Again expanding $Q(\rho)$ in general in terms of these polynomials like in (A1.2), the matrix form of (A3.6) leads to (A1.3), where presently $B_{k\ell} = \kappa^{-1}B_{k\ell}^{(1)} + \kappa^{-2}B_{k\ell}^{(2)}$ with

$$B_{k\ell}^{(1)} = -\frac{3}{2}\ell[\delta_{k-1,\ell} + 2(2k+1)\delta_{k+1,\ell} + 4(k+2)(k+1)\delta_{k+3,\ell}] , \quad (A3.7)$$

$$B_{k\ell}^{(2)} = \frac{1}{8}\ell[\delta_{k-2,\ell} + 6k\delta_{k\ell} + 12(k+1)^2\delta_{k+2,\ell} + 8(k+3)(k+2)(k+1)\delta_{k+4,\ell}] .$$

Using standard perturbation theory through sixth order in κ^{-1} , we have found

$$\mu = 2n|\kappa| \left\{ 1 - \frac{6n}{\kappa^2} - \frac{6}{\kappa^4}(8n+1) - \frac{24n}{\kappa^6}(35n^2+13) + \dots \right\} . \quad (A3.8)$$

It is surprising to see that this is identical to the result for the odd inner solutions, viz. (A3.3) with $m=2n-1$. It leads to the conclusion that the degeneracy within each triplet is not lifted by power series terms, at least not up to the shown order in κ^{-2} . Of course, there are always

exponentially small differences between the degenerate eigenvalues due to the finite potential barrier (sections A3.2&4, and [28]).*

A3.4 Triplet degeneracy

In view of the outcome for the local eigenvalues, we briefly examine the relation between the nature of the degeneracy and the pertinent odd triplet eigenfunctions. Suppose we have determined the local solutions to any desired accuracy (in terms of κ^{-1}). Within one triplet, let $Q_{out}(x)$ denote the odd combination of the left and right outer solutions, and let $Q_{in}(x)$ be the corresponding odd inner solution. For convenience, let these functions be appropriately normalized locally. Write the global solutions as $Q(x) = c_{out}Q_{out}(x) + c_{in}Q_{in}(x)$ and insert this into the variational expression (A2.1). Taking derivatives with respect to c_{out} resp. c_{in} according to [41], one obtains two linear homogeneous equations, which may be presented as

$$\begin{bmatrix} \mu_{out} - \mu & \theta' - \mu\theta \\ \theta' - \mu\theta & \mu_{in} - \mu \end{bmatrix} \begin{bmatrix} c_{out} \\ c_{in} \end{bmatrix} = 0 \quad (A3.9)$$

Disregarding terms of order $e^{-\kappa^2/4}$, the μ_{in} and μ_{out} are the local eigenvalues (A3.3) and (A3.8). The offdiagonal elements θ, θ' are related to overlap integrals** and found to be of order $e^{-\kappa^2/8}$. The solvability condition of (A3.9) gives

$$\mu = \frac{1}{2}(\mu_{out} + \mu_{in}) \pm \sqrt{\frac{1}{4}(\mu_{out} - \mu_{in})^2 + [\theta' - \frac{1}{2}(\mu_{out} + \mu_{in})\theta]^2} \quad (A3.10)$$

where we have disregarded some irrelevant small terms for ease of exposition.

Weak degeneracy

Let the observed equality of local eigenvalues be violated in higher orders of κ^{-2} . That is, $\mu_{out} - \mu_{in}$ is of order θ^0 , with $\theta \sim e^{-\kappa^2/8}$. In this case (A3.10) yields: (i) $\mu = \mu_{in}$; setting $c_{in} = 1$ one finds c_{out} to be of order $\theta \sim e^{-\kappa^2/8}$; (ii) $\mu = \mu_{out}$; with $c_{out} = 1$ one finds c_{in} to be of order

*For example, such lifting would occur immediately once we replace the quartic term in the irreducible potential by x^{2p} , with $p=3, 4, \dots$.

**Strictly speaking, the local eigenfunctions are not valid in the overlap region. Nonetheless, $\theta \sim \theta' \sim e^{-\kappa^2/8}$ agrees with results that can be obtained from a global WKB-type analysis [28].

$\theta \sim e^{-\kappa^2/8}$. Hence, in case of weak degeneracy there exists only an exponentially weak coupling between the local eigenfunctions. See figure 5.

Strong degeneracy

Let the local eigenvalues be exactly equal, that is $\mu_{out} = \mu_{in}$. In this case (A3.10) leads to $\mu = \mu_{local} \pm |\theta' - \mu_{local} \theta|$. Inserting this back into (A3.9) gives $c_{in} = \pm c_{out}$. Hence, in the case of exponentially strong degeneracy there exists a strong one-one coupling between the local eigenfunctions. See figure 5.

These conclusions have been used in section 4.2.3. As yet it remains an open problem which type of degeneracy actually occurs.

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PART II: GENERAL MARKOV PROCESSES

C R I T I C A L D Y N A M I C S

THE EXPANSION OF THE MASTER EQUATION INCLUDING A CRITICAL POINT

PART II: GENERAL MARKOV PROCESSES

ABSTRACT

The master equation for a general Markov process that shows a transition from *monostable* to *bistable* behaviour will be evaluated systematically in terms of a small parameter, namely the reciprocal size of the system. The expansion is uniformly valid also at the *critical point*. The fundamental idea is to separate the master equation into its irreducible part and a corrective remainder. The irreducible or zeroth order approximation is a relatively simple Fokker-Planck equation containing the essential features of the process. Having achieved complete knowledge of the eigensolutions of the irreducible equation the higher order corrections are computed explicitly.

1. INTRODUCTION

In a previous article [1] we have shown how to solve the problem of a simple diffusion process involving a mono-to bistable transition by means of a systematic approximation method valid also in the transition (or critical) region. Thereto it is crucial to recognize the correct irreducible stochastic description. Having solved this relatively easy irreducible problem, one can calculate higher order corrections in terms of successive (fractional) powers of a small parameter, namely the diffusion coefficient ν .

Presently we embark on the problem of solving the (integral) Markovian master equation by essentially the same techniques. The appropriate small parameter will be the reciprocal system size $\epsilon=1/\Omega$. The irreducible description is again given by the Fokker-Planck equation describing in effect a diffusion process in a quartic potential. Well above the critical point (monostable case) and well below it (bistable case), it involves the linear noise approximation at the stable macroscopic states [2], where the fluctuations are of relative order $\epsilon^{\frac{1}{2}}$. Near the critical point the linear noise description fails. Here the fluctuations scale up to order $\epsilon^{\frac{1}{3}}$ [3, 4]. Nevertheless, the corrections

to the irreducible approximation always remain small in terms of ϵ and can be incorporated into the solution in a systematic manner.

In section 2 the irreducible part will be extracted from the general master equation. In section 3 we briefly outline the properties of the eigensolutions of this Fokker-Planck equation. The calculation of the higher order corrections is discussed in section 4, while section 5 contains some final remarks. In the appendix we examine the dynamical mean field Ising model as a typical example.

2. REDUCTION OF THE PROBLEM

The probability density $P(n,t)$ of a general Markov process obeys the master equation [2-5]

$$\dot{P}(n,t) = \sum_{n'} [W(n|n')P(n',t) - W(n'|n)P(n,t)] \quad (2.1)$$

where n is the physical extensive variable; n may be continuous or discrete, $\sum_{n'}$, representing integration (with the appropriate measure) or summation respectively. In terms of the intensive variable $z = n/\Omega = \epsilon n$, the transition kernels in (2.1) are assumed to have the property of extensivity [3, 4]

$$W(n'|n) = \Omega w(z, n' - n) \quad (2.2)$$

Setting $v = n - n'$ and introducing the operator [6]

$$E = \exp \epsilon \frac{\partial}{\partial z} = 1 + \epsilon \frac{\partial}{\partial z} + \frac{1}{2} \epsilon^2 \frac{\partial^2}{\partial z^2} + \dots \quad (2.3)$$

one transforms (2.1) into

$$\frac{\partial P(z,t)}{\partial t} = \Omega \sum_v [E^{-v} - 1] w(z,v) P(z,t) \quad (2.4)$$

Defining the intensive jump moments ($n=1, 2, \dots$)

$$\alpha_n(z) = \sum_v v^n w(z,v) \quad (2.5)$$

one obtains [2-4, 7-9]

$$\frac{\partial P(z,t)}{\partial t} = \sum_{n=1}^{\infty} \frac{1}{n!} \epsilon^{n-1} \left(-\frac{\partial}{\partial z}\right)^n \alpha_n(z) P(z,t) \quad (2.6)$$

In order to find its expansion one normally sets $z = \varphi(t) + \epsilon^{\frac{1}{2}} \xi$ [2, 3, 10], which transforms (2.6) into an equation in terms of ξ . Terms of order $\epsilon^{-\frac{1}{2}}$ vanish as the macroscopic part $\varphi(t)$ is a solution of the deterministic equation of motion

$$\dot{z} = \alpha_1(z) \equiv -U'(z) \quad (2.7)$$

Here we have defined a macroscopic potential $U(z)$, the prime indicating differentiation with respect to the argument. The part of order ϵ^0 of the master equation becomes a linear Fokker-Planck equation in terms of ξ . Higher order corrections can be calculated systematically in powers of $\epsilon^{\frac{1}{2}}$. However, this method fails at critical points where the macroscopic solution lacks stability in the linear approximation.

Let us examine the macroscopic equation (2.7) in some more detail. Suitably choosing the origin, we consider the general symmetric macroscopic potential

$$U(z) = -\frac{1}{2} \alpha_1^{(1)} z^2 - \frac{1}{4} \alpha_1^{(3)} z^4 - \frac{1}{6} \alpha_1^{(5)} z^6 \dots \quad (2.8)$$

so that $n! \alpha_1^{(n)} = d^n \alpha_1(z) / dz^n$ at $z=0$. See figure 1 of [1]. It is assumed

that $\alpha_1^{(1)}$ changes sign at the critical point, while $\alpha_1^{(3)}$ is negative (at least at the critical point). The coefficients do not depend on ϵ . Along the real line (2.7) has one stable macroscopic steady state solution $\varphi_0=0$ if $\alpha_1^{(1)} < 0$, and two stable stationary solutions $\varphi_{\pm} \neq 0$ if $\alpha_1^{(1)} > 0$. Furthermore, the symmetry property $\alpha_n(-z) = (-1)^n \alpha_n(z)$ supposedly holds (about the appropriately chosen origin) for any $n=1, 2, \dots$, so that

$$\begin{aligned} \alpha_{2n-1}(z) &= \alpha_{2n-1}^{(1)} z + \alpha_{2n-1}^{(3)} z^3 + \alpha_{2n-1}^{(5)} z^5 + \dots, \\ \alpha_{2n}(z) &= \alpha_{2n}^{(0)} + \alpha_{2n}^{(2)} z^2 + \alpha_{2n}^{(4)} z^4 + \dots. \end{aligned} \quad (2.9)$$

Throughout we take $\alpha_2^{(0)}$ to be nonzero [11].

2.1 Above and at the critical point

In this monostable case ($\alpha_1^{(1)} \leq 0$) one introduces the scale transformations

$$z = \epsilon^{\frac{1}{2}} \eta, \quad t = \epsilon^{\frac{1}{2}} \tau, \quad \alpha^{(1)}/\alpha^{(3)} = \epsilon^{\frac{1}{2}} \Delta \quad (2.10)$$

into (2.6). Using (2.9) one finds

$$\begin{aligned} \frac{\partial P(\eta, \tau)}{\partial \tau} &= -\alpha_1^{(3)} \frac{\partial}{\partial \eta} (\Delta + \eta^2) P + \frac{1}{2} \alpha_2^{(0)} \frac{\partial^2 P}{\partial \eta^2} \\ &\quad - \epsilon^{\frac{1}{2}} [\alpha_1^{(5)} \frac{\partial}{\partial \eta} \eta^5 P - \frac{1}{2!} \alpha_2^{(2)} \frac{\partial^2}{\partial \eta^2} \eta^2 P] \\ &\quad - \epsilon [\alpha_1^{(7)} \frac{\partial}{\partial \eta} \eta^7 P - \frac{1}{2!} \alpha_2^{(4)} \frac{\partial^2}{\partial \eta^2} \eta^4 P + \frac{1}{3!} \alpha_3^{(1)} \frac{\partial^3}{\partial \eta^3} \eta P] \\ &\quad - \epsilon^{\frac{3}{2}} [\alpha_1^{(9)} \frac{\partial}{\partial \eta} \eta^9 P - \frac{1}{2!} \alpha_2^{(6)} \frac{\partial^2}{\partial \eta^2} \eta^6 P + \frac{1}{3!} \alpha_3^{(3)} \frac{\partial^3}{\partial \eta^3} \eta^3 P - \frac{1}{4!} \alpha_4^{(0)} \frac{\partial^4}{\partial \eta^4} P] \\ &\quad + \dots. \end{aligned} \quad (2.11)$$

Formally treating η, τ and Δ as quantities of order unity and omitting terms in (2.11) that vanish as $\epsilon \rightarrow 0$, one extracts the irreducible part

$$\frac{\partial P(x, s)}{\partial s} = \frac{\partial}{\partial x} x(\kappa + x^2) P + \frac{\partial^2 P}{\partial x^2}, \quad (2.12)$$

where we have set

$$x = \frac{\eta}{\sqrt{\frac{1}{2} \alpha_2^{(0)}}} [\alpha_1^{(3)}]^{-\frac{1}{4}}, \quad s = \tau [\alpha_1^{(3)}]^{-\frac{1}{2}}, \quad \kappa = \frac{\Delta}{\frac{1}{2} \alpha_2^{(0)}} [\alpha_1^{(3)}]^{-\frac{1}{2}}, \quad (2.13)$$

with $\alpha^{(3)} = -\frac{1}{2} \alpha_1^{(3)} \alpha_2^{(0)}$, in order to have (2.12) in standard notation; κ is the pump parameter, here ≥ 0 . In section 4 it will be seen that the higher order corrections in (2.11) do remain small in terms of ϵ for all $\kappa > 0$, i.e. for all $\alpha_1^{(1)} \leq 0$.

2.2 Below and at the critical point

In this case ($\alpha_1^{(1)} > 0$) one must explicitly account for the existence of the two stable macroscopic steady states φ_{\pm} , which are the nontrivial solutions of $\dot{z} = \alpha_1(z) = 0$. That is,

$$\alpha_1^{(1)} + \alpha_1^{(3)} \varphi_{\pm}^2 + \alpha_1^{(5)} \varphi_{\pm}^4 + \dots = 0 \quad (2.14)$$

The correct below-critical irreducible description is obtained only after a certain reordering of coefficients in the master equation (2.6). The procedure is a generalized version of the one presented in [1]. The jump moments (2.9) are rewritten as

$$\begin{aligned} \alpha_{2n-1}(z) &= z [\alpha_{2n-1}^{(1)} + \alpha_{2n-1}^{(3)} (z^2 - \varphi_{\pm}^2) + \dots + \alpha_{2n-1}^{(2m+1)} (z^2 - \varphi_{\pm}^2)^m + \dots] \quad , \\ \alpha_{2n}(z) &= \alpha_{2n}^{(0)} + \alpha_{2n}^{(2)} (z^2 - \varphi_{\pm}^2) + \dots + \alpha_{2n}^{(2m)} (z^2 - \varphi_{\pm}^2)^m + \dots \quad . \end{aligned} \quad (2.15)$$

Comparison of (2.15) and (2.9) yields

$$\left. \begin{aligned} \alpha_{2n-1}^{(2m+1)} &= \sum_{k=m}^{\infty} \binom{k}{m} \alpha_{2n-1}^{(2k+1)} \varphi_{\pm}^{2(k-m)} \quad , \\ \alpha_{2n}^{(2m)} &= \sum_{k=m}^{\infty} \binom{k}{m} \alpha_{2n}^{(2k)} \varphi_{\pm}^{2(k-m)} \quad . \end{aligned} \right\} \quad (2.16)$$

By (2.14) one readily checks that $\alpha_1^{(1)} = 0^*$, while $\alpha_1^{(3)}$ may be cast into the closed form

$$\alpha_1^{(3)} = \alpha_1'(\varphi_{\pm}) / 2\varphi_{\pm}^2 \quad (2.17)$$

Clearly, $\alpha_1^{(3)}$ is connected with the local stability in the linear approximation at φ_{\pm} . Further, one notes that $\alpha_2^{(0)} = \alpha_2(\varphi_{\pm})$. Inserting now the scale transformations

$$z = \varepsilon^{\frac{1}{2}} \eta, \quad t = \Omega^{\frac{1}{2}} \tau, \quad \varphi_{\pm}^2 = -\varepsilon^{\frac{1}{2}} \nabla \quad (2.18)$$

into (2.6) with the renormalized jump moments according to (2.15) and (2.16), one obtains

$$\begin{aligned} \frac{\partial P(\eta, \tau)}{\partial \tau} &= -\alpha_1^{(3)} \frac{\partial}{\partial \eta} (\nabla + \eta^2) P + \frac{1}{2} \alpha_2^{(0)} \frac{\partial^2 P}{\partial \eta^2} \\ &\quad - \varepsilon^{\frac{1}{2}} \left[\alpha_1^{(5)} \frac{\partial}{\partial \eta} (\nabla + \eta^2)^2 P - \frac{1}{2!} \alpha_2^{(2)} \frac{\partial^2}{\partial \eta^2} (\nabla + \eta^2) P \right] \\ &\quad - \varepsilon \left[\alpha_1^{(7)} \frac{\partial}{\partial \eta} (\nabla + \eta^2)^3 P - \frac{1}{2!} \alpha_2^{(4)} \frac{\partial^2}{\partial \eta^2} (\nabla + \eta^2)^2 P + \frac{1}{3!} \alpha_3^{(1)} \frac{\partial^3}{\partial \eta^3} \eta P \right] \\ &\quad - \varepsilon^{\frac{3}{2}} \left[\alpha_1^{(9)} \frac{\partial}{\partial \eta} (\nabla + \eta^2)^4 P - \frac{1}{2!} \alpha_2^{(6)} \frac{\partial^2}{\partial \eta^2} (\nabla + \eta^2)^3 P \right. \\ &\quad \left. + \frac{1}{3!} \alpha_3^{(3)} \frac{\partial^3}{\partial \eta^3} \eta (\nabla + \eta^2) P - \frac{1}{4!} \alpha_4^{(0)} \frac{\partial^4 P}{\partial \eta^4} \right] + \dots \quad (2.19) \end{aligned}$$

* For one step processes $\alpha_{2n-1}^{(1)} = 0$ for all $n=1, 2, \dots$

Considering now formally η, τ and ∇ as quantities of order unity and omitting terms in (2.19) that vanish explicitly as $\epsilon \rightarrow 0$, one gets the irreducible part of the master equation below the critical point. Setting

$$x = \frac{\eta}{\sqrt{\frac{1}{2}a_2^{(0)}}} [a_2^{(3)}]^{1/2}, \quad s = \tau [a_2^{(3)}]^{1/2}, \quad \kappa = \frac{\nabla}{\frac{1}{2}a_2^{(0)}} [a_2^{(3)}]^{1/2}, \quad (2.20)$$

where $a_2^{(3)} = -\frac{1}{2}a_1^{(3)}a_2^{(0)}$, it is readily written in the standard form (2.12) with $\kappa < 0$. In section 4 we will show how the above renormalization helps in keeping the higher order corrections (2.19) small in terms of ϵ for all $\kappa < 0$, i.e. for all $\alpha_1^{(1)} > 0$.

2.3 Some comments

Thus it has been shown how the irreducible standard form (2.12) arises from the master equation in the limit $\zeta \rightarrow \infty$. Note that the relations between the standard variables x, s and κ and the original ones z, t and $\alpha_1^{(1)}$ are essentially different above and below the critical point $\alpha_1^{(1)} = 0$.

Well above the critical point in the normal regime $\alpha_1^{(1)}$ is negative and of order unity, so that by (2.10) and (2.13) κ tends to plus infinity as $\Omega^{1/2}$. Appropriately rescaling in (2.12) such that effectively $z = \epsilon^{1/2} \xi$, one easily obtains

$$\frac{\partial P(\xi, t)}{\partial t} = -\alpha_1^{(1)} \frac{\partial}{\partial \xi} \xi P + \frac{1}{2} \alpha_2^{(0)} \frac{\partial^2 P}{\partial \xi^2} \quad (2.21)$$

as the dominant part of order ϵ^0 , which provides the normal linear noise approximation at the single stable state $\varphi_0 = 0$.

In the normal regime well below the critical point $\alpha_1^{(1)}$ is positive and of order unity, so that by (2.18) and (2.20) κ here tends to minus infinity as $\Omega^{1/2}$. Appropriately rescaling in (2.12) about $x = \pm (-\kappa)^{1/2}$, such that effectively $z = \varphi_{\pm} + \epsilon^{1/2} \xi$, and using (2.17) and $a_2^{(0)} = \alpha_2(\varphi_{\pm})$, one easily retrieves the linear noise descriptions at the two stable states φ_{\pm} as the leading parts of order ϵ^0 .

In the critical range, where by definition $\alpha_1^{(1)}$ is of order $\epsilon^{1/2}$, κ is of order unity and the irreducible nonlinear Fokker-Planck equation (2.12) can not be simplified any further. One should note that (2.12) does not allow the computation of moments explicitly in successive orders of $\epsilon^{1/2}$ directly from the differential equation itself due to the nonlinear drift function [2, 3, 9, 12, 13]. Therefore, contrary to the normal cases where the linear noise approximation always holds, we must know the solutions of (2.12) explicitly.

3. THE IRREDUCIBLE PROBLEM

Let us repeat (2.12) here as

$$\left. \begin{aligned} \frac{\partial P^{(0)}(x,s)}{\partial s} &= \frac{\partial}{\partial x} U_{\kappa}^{(0)}(x) P^{(0)} + \frac{\partial^2 P^{(0)}}{\partial x^2} , \\ U_{\kappa}^{(0)}(x) &= \frac{1}{2}\kappa x^2 + \frac{1}{4}x^4 , \end{aligned} \right\} \quad (3.1)$$

which formally represents a diffusion process in a simple potential showing the mono- to bistable transition (see e.g. [14-21]). As such it has been mentioned as a model for the single mode laser [22], disregarding phase diffusion [23-27].

The Green's function solution or propagator of (3.1) will be written as [1]

$$P_{\kappa}^{(0)}(x,s|x_0,0) = \sum_{n=0}^{\infty} P_n^{(0)}(x) Q_n^{(0)}(x_0) e^{-\mu_n^{(0)} s} , \quad (3.2)$$

where $Q_n^{(0)}(x)$ is the adjoint eigenfunction of $P_n^{(0)}(x)$,

$$P_n^{(0)}(x) = P_0^{(0)}(x) Q_n^{(0)}(x) . \quad (3.3)$$

The $P_n^{(0)}(x)$ and $Q_n^{(0)}(x)$ form a presumably complete biorthogonal set:

$$\int_{-\infty}^{\infty} P_n^{(0)}(x) Q_m^{(0)}(x) dx = \delta_{nm} . \quad (3.4)$$

Equilibrium (or rather, steady state) quantities become relatively simple in this format. For example, the dynamical susceptibility becomes [28, 29] .

$$\chi(\omega) = -\frac{\chi(0)}{\Gamma(0)} \int_0^{\infty} \Gamma(s) e^{-i\omega s} ds = \frac{\chi(0)}{\Gamma(0)} \sum_{n=0}^{\infty} \frac{\mu_n^{(0)}}{\mu_n^{(0)} + i\omega} \langle x \rangle_n^2 , \quad (3.5)$$

where we have invoked the pertinent expression [1] for the correlation function $\Gamma(s)$; $\langle x \rangle_n$ represents the first moment of the n-th probability eigenfunction, and $\chi(0)$ can in principle be computed (at least at zero bias of the external 'force' [28]) from the stationary distribution $P_0^{(0)}(x)$.

The eigenvalue spectrum of (3.1) is shown in figure 3 of [1]. More details may be found in [1, 16, 21].

Well above the critical point $\kappa \rightarrow \infty$; the adjoint eigenfunctions tend to become Hermite polynomials about $x=0$ (note that the Gaussian propagator of the linear noise approximation (2.21) can be expanded in terms of these polynomials ; see e.g. [30] form. 10.13(22)); x is effectively of order $\kappa^{-\frac{1}{2}}$; the eigenvalues are $\mu_n^{(0)} \cong n\kappa$, with $n=0, 1, 2, \dots$.

In the critical region $\kappa \approx 0$; both x and $\mu_n^{(0)}$ are of order one, e.g. $\mu_1^{(0)} = 1.37$ (see also [4]).

Well below the critical point $\kappa \rightarrow -\infty$; x is globally of order $(-\kappa)^{\frac{1}{2}}$, but is locally (near the origin and $x = \pm(-\kappa)^{\frac{1}{2}}$) of order $(-\kappa)^{-\frac{1}{2}}$; there is one low lying doublet, consisting of the stationary solution with $\mu_0^{(0)} = 0$ and the first excited state with $\mu_1^{(0)} \cong e^{-\kappa^2/4}$ (corresponding to Kramers' diffusion rate [31-33]); further there is a set of singlets with $\mu_{4n-2}^{(0)} \cong (2n-1)|\kappa|$, where $n=1, 2, \dots$ (and where all eigenfunctions are even and completely confined to the region $x \sim (-\kappa)^{-\frac{1}{2}}$ near the origin); finally, there is a set of triplets with $\mu_{4n-1}^{(0)} \cong \mu_{4n}^{(0)} \cong \mu_{4n+1}^{(0)} \cong 2n|\kappa|$, again with $n=1, 2, \dots$ (hence, each triplet consists of one even and two odd eigenfunctions); for a discussion of the possible nature of this degeneracy the reader is referred to [1].

4. HIGHER ORDER CORRECTIONS

The higher order corrections in the original master equation will be calculated in terms of the standard variables x, s and κ . If one were interested only in the true critical region, the calculation could proceed by definition directly from the standard version of (2.11) and (2.19) as they stand. However, in a systematic unified treatment of above-critical, critical and below-critical cases, one must define the perturbational terms still more carefully. Fortunately it is not too difficult to arrive at the following scheme handling (2.11) and (2.19) simultaneously:

$$\frac{\partial P(x,s)}{\partial s} = \frac{\partial}{\partial x} U'(x)P + \frac{\partial^2 P}{\partial x^2} - \epsilon^{\frac{1}{2}} V\left(\frac{\partial}{\partial x}, x\right)P, \quad (4.1)$$

where

$$V\left(\frac{\partial}{\partial x}, x\right) = V^{(1)}\left(\frac{\partial}{\partial x}, x\right) + \epsilon^{\frac{1}{2}} V^{(2)}\left(\frac{\partial}{\partial x}, x\right) + \dots \quad (4.2)$$

The correct below-critical perturbational operators are

$$\left. \begin{aligned} V^{(1)}\left(\frac{\partial}{\partial x}, x\right) &= \left[\frac{1}{2}a_2^{(0)}\right]^{\frac{1}{2}} \left[-a_1^{(3)}\right]^{-\frac{3}{2}} \left\{ a_1^{(5)} \frac{\partial}{\partial x} - x(\kappa+x^2)^2 - \frac{1}{2!} a_2^{(2)} A \frac{\partial^2}{\partial x^2} (\kappa+x^2) \right. \\ &\quad \left. + \frac{1}{3!} \epsilon^{\frac{1}{2}} a_3^{(1)} A^{\frac{3}{2}} \frac{\partial^3}{\partial x^3} x - \frac{1}{4!} \epsilon a_4^{(0)} A^2 \frac{\partial^4}{\partial x^4} \right\}, \\ V^{(2)}\left(\frac{\partial}{\partial x}, x\right) &= \left[\frac{1}{2}a_2^{(0)}\right] \left[-a_1^{(3)}\right]^{-2} \left\{ a_1^{(7)} \frac{\partial}{\partial x} - x(\kappa+x^2)^3 - \frac{1}{2!} a_2^{(4)} A \frac{\partial^2}{\partial x^2} (\kappa+x^2)^2 \right. \\ &\quad \left. + \frac{1}{3!} \epsilon^{\frac{1}{2}} a_3^{(3)} A^{\frac{3}{2}} \frac{\partial^3}{\partial x^3} x(\kappa+x^2) - \frac{1}{4!} \epsilon a_4^{(2)} A^2 \frac{\partial^4}{\partial x^4} (\kappa+x^2) \right. \\ &\quad \left. + \frac{1}{5!} \epsilon^{\frac{3}{2}} a_5^{(1)} A^{\frac{5}{2}} \frac{\partial^5}{\partial x^5} x - \frac{1}{6!} \epsilon^2 a_6^{(0)} A^3 \frac{\partial^6}{\partial x^6} \right\}, \end{aligned} \right\} \quad (4.3)$$

and so on, where $A = -a_1^{(3)} / \frac{1}{2}a_2^{(0)}$. Above the critical point the perturbational operators are obtained from (4.3) simply setting $\kappa=0$ and replacing the renormalized $a_n^{(m)}$ by the original $\alpha_n^{(m)}$. The terms (4.3) may be gleaned from (2.11) and (2.19) as follows. In order to construct $V^{(p)}$, one takes the first and second derivative terms of critical order $\epsilon^{p/2}$ and adds the third derivative term of the next order and the fourth derivative term of the second next order, and so forth until the series terminates.

As has been done in [1] we cast the eigenvalue problem of (4.1) into matrix form. With

$$P(x) = \sum_{n=0}^{\infty} c_n P_n^{(0)}(x), \quad (4.4)$$

this leads to

$$c_k (\mu_k^{(0)} - \mu) + \epsilon^{\frac{1}{2}} \sum_{\ell} V_{k\ell} c_{\ell} = 0, \quad (4.5)$$

where

$$V_{k\ell} = \int_{-\infty}^{\infty} Q_k^{(0)}(x) V\left(\frac{d}{dx}, x\right) P_{\ell}^{(0)}(x) dx \quad (4.6)$$

4.1 The even solutions

In view of the degeneracies in the irreducible eigenspectrum well below the critical point (see e.g. [1], figure 3), a unified treatment valid for all values of the pump parameter calls for a separate examination of even and odd solutions. Even solutions are either singlets or the single even member of triplets. They can be treated by nondegenerate perturbation theory, which also applies to the Kramers' doublet. Putting

$$\begin{aligned} c_k &= \delta_{kn} + \epsilon^{\frac{1}{2}} c_{kn}^{(1)} (1 - \delta_{kn}) + \epsilon c_{kn}^{(2)} (1 - \delta_{kn}) + \dots, \\ \mu &= \mu_n^{(0)} + \epsilon^{\frac{1}{2}} \mu_n^{(1)} + \epsilon \mu_n^{(2)} + \dots, \end{aligned} \quad (4.7)$$

and defining $\mu_{nk}^{(0)} = \mu_n^{(0)} - \mu_k^{(0)}$, one obtains (see [1]):

$$\left. \begin{aligned} \mu_n^{(1)} &= V_{nn}^{(1)}, \\ c_{kn}^{(1)} &= V_{kn}^{(1)} / \mu_{nk}^{(0)}, \\ \mu_n^{(2)} &= V_{nn}^{(2)} + \sum_{\ell \neq n} V_{n\ell}^{(1)} c_{\ell n}^{(1)}, \\ c_{kn}^{(2)} &= [V_{kn}^{(2)} + \sum_{\ell \neq n} V_{k\ell}^{(1)} c_{\ell n}^{(1)} - \mu_n^{(1)} c_{kn}^{(1)}] / \mu_{nk}^{(0)}, \end{aligned} \right\} \quad (4.8)$$

etcetera. In the critical region both x and $\mu_{nk}^{(0)}$, and hence each $V_{k\ell}^{(p)}$ are of order ϵ^0 . The above scheme is obviously correct in that case.

4.1.1 Well above the critical point

Here $x \sim \kappa^{-\frac{1}{2}}$ which will be of order $\epsilon^{\frac{1}{2}}$; $\mu_n^{(0)} \sim \kappa$ is of order $\Omega^{\frac{1}{2}}$. Using (4.6) and (4.3) with $\kappa=0$, it is not so difficult to see that the matrix element* $V_{k\ell}^{(p)}$ (with $p=1, 2, \dots$) becomes of order $\epsilon^{(p-1)/2}$, and that $c_k^{(p)}$ is of order $\epsilon^{p/2}$. Therefore, if we define $\mu_n^{(p)} = \epsilon^{(p-1)/2} \lambda_n^{(p)}$ and $c_k^{(p)} = \epsilon^{p/2} b_k^{(p)}$, the $\lambda_n^{(p)}$ and $b_k^{(p)}$ are of order unity. Inserting this into (4.7) and defining the order ϵ^0 quantities $b_k = c_k$, $\lambda_n^{(0)} = \epsilon^{\frac{1}{2}} \mu_n^{(0)}$ and $\lambda = \epsilon^{\frac{1}{2}} \mu$, one gets

* To be explicit, the operator $V^{(p)}$ may be given as:

$$V^{(p)}\left(\frac{d}{d\eta}, \eta\right) = \alpha_1^{(2p+3)} \frac{d}{d\eta} \eta^{2p+3} - \sum_{n=2}^{2p+2} \frac{1}{n!} \alpha_n^{(2p+2-n)} \epsilon^{(n-2)/2} \left(-\frac{d}{d\eta}\right)^n \eta^{2p+2-n}.$$

$$\begin{aligned}
 b_k &= \delta_{kn} + \epsilon b_{kn}^{(1)} (1 - \delta_{kn}) + \epsilon^2 b_{kn}^{(2)} (1 - \delta_{kn}) + \dots, \\
 \lambda &= \lambda_n^{(0)} + \epsilon \lambda_n^{(1)} + \epsilon^2 \lambda_n^{(2)} + \dots.
 \end{aligned}
 \tag{4.9}$$

Clearly the presented scheme remains valid. However, the higher order corrections now come in successive powers of ϵ instead of the critical $\epsilon^{\frac{1}{2}}$. Note that, contrary to the diffusion process treated in [1], presently the first order normal correction does occur in (4.9): it is not fully contained here in the irreducible part of the solution.

4.1.2 Well below the critical point

It has been discussed in detail in [1] that the singlet solutions lead to exponentially small contributions (of order $e^{-\Omega}$) in any equilibrium quantity. Hence, they can not be seen in an asymptotic power series expansion. On the other hand, the even triplet solutions are significant precisely near the macroscopic steady states φ_{\pm} . Shifting to $x = \pm(\kappa)^{\frac{1}{2}}$ and rescaling to the appropriate local variable, one then finds that $V_{kl}^{(p)}$ scales up to order $\Omega^{\frac{1}{2}}$, independent of p . Consequently $\mu_n^{(p)}$ becomes of order $\Omega^{\frac{1}{2}}$, while $c_k^{(p)}$ remains of order unity. Therefore, setting $\mu_n^{(p)} = \Omega^{\frac{1}{2}} \lambda_n^{(p)}$ one finds

$$\begin{aligned}
 c_k &= \delta_{kn} + \epsilon^{\frac{1}{2}} c_{kn}^{(1)} (1 - \delta_{kn}) + \epsilon c_{kn}^{(2)} (1 - \delta_{kn}) + \dots, \\
 \lambda &= \lambda_n^{(0)} + \epsilon^{\frac{1}{2}} \lambda_n^{(1)} + \epsilon \lambda_n^{(2)} + \dots,
 \end{aligned}
 \tag{4.10}$$

where we have once more introduced the order ϵ^0 quantities $\lambda_n^{(0)} = \epsilon^{\frac{1}{2}} \mu_n^{(0)}$ and $\lambda = \epsilon^{\frac{1}{2}} \mu$. Eq. (4.10) shows that the presented scheme remains valid also in the normal regime well below the critical point.

4.2 The odd solutions

Apart from the first excited state, the odd eigensolutions of the irreducible problem combine within triplets well below the critical point. Their treatment in the higher order corrections in the master equation requires pseudo-degenerate perturbation theory in order to sufficiently lift the degeneracy. The appropriate lowest order coefficients and eigenvalues follow from [1, 34]

$$\begin{bmatrix}
 \mu_n^{(0)} - \bar{\mu}^{(0)} + \epsilon^{\frac{1}{2}} V_{nn}^{(1)} & \epsilon^{\frac{1}{2}} V_{nm}^{(1)} \\
 \epsilon^{\frac{1}{2}} V_{mn}^{(1)} & \mu_m^{(0)} - \bar{\mu}^{(0)} + \epsilon^{\frac{1}{2}} V_{mm}^{(1)}
 \end{bmatrix}
 \begin{bmatrix}
 \bar{c}_n^{(0)} \\
 \bar{c}_m^{(0)}
 \end{bmatrix}
 = 0.
 \tag{4.11}$$

Hence, the eigenvalues read

$$\bar{\mu}^{(0)} = \frac{1}{2} [\mu_n^{(0)} + \mu_m^{(0)} + \epsilon^{\frac{1}{2}} (V_{nn}^{(1)} + V_{mm}^{(1)})] \pm \sqrt{\frac{1}{4} [\mu_n^{(0)} - \mu_m^{(0)} + \epsilon^{\frac{1}{2}} (V_{nn}^{(1)} - V_{mm}^{(1)})]^2 + \epsilon V_{nm}^{(1)} V_{mn}^{(1)}}. \quad (4.12)$$

Substitution of these values into (4.11) yields the sought linear combinations of $P_n^{(0)}(x)$ and $P_m^{(0)}(x)$.

4.2.1 Well above the critical point

In this case the first order matrix elements remain of order unity, while $\mu_{nm}^{(0)}$ will be of order $\kappa \sim \Omega^{\frac{1}{2}}$ (see section 4.1.1). Expanding then the square root in (4.12), one finds

$$\begin{aligned} \bar{\mu}_n^{(0)} &= \mu_n^{(0)} + \epsilon^{\frac{1}{2}} V_{nn}^{(1)} + \dots, \\ \bar{C}_{nn}^{(0)} &= 1, \quad \bar{C}_{mn}^{(0)} = \epsilon^{\frac{1}{2}} V_{mn}^{(1)} / \mu_{nm}^{(0)} + \dots, \end{aligned} \quad (4.13)$$

and a similar result for the m-th eigensolution. As it should, this reproduces the purely nondegenerate outcome through first order.

4.2.2 The critical region

Here both the first order matrix elements and $\mu_{nm}^{(0)}$ are of order ϵ^0 , so that (4.12) again properly leads to the nondegenerate results.

4.2.3 Well below the critical point

As discussed in more detail in [1], we must in this case examine two possibilities for the asymptotic degeneracy. The somewhat subtle considerations are identical to those presented in [1] and will therefore be omitted here. The simple outcome is that in either case (4.11) provides us with two linear combinations of the original eigenfunctions, such that the new solutions are nondegenerate and significant only near the steady states ϕ_{\pm} . As usual, subsequent higher order corrections can be handled further along the lines of the nondegenerate theory of section 4.1.

5. SOME FINAL REMARKS

In conclusion, we have shown how to set up a systematic expansion for the equilibrium solutions of the master equation describing a Markov process that shows a transition from monostable to bistable behaviour. The expansion proceeds in successive fractional powers of the inverse size of the system ($\epsilon=1/\Omega$) and is valid uniformly for all values of the pertinent parameter (e.g. temperature) $\alpha_1^{(1)}$, including the critical point $\alpha_1^{(1)}=0$.

In the introduction we briefly stated the problem and outlined its solution, making reference to our similar treatment of diffusion processes [1]. In section 2 we have been concerned with the separation of the master equation into its irreducible part and a corrective remainder. The irreducible part is a relatively simple nonlinear Fokker-Planck equation. Some relevant properties of its eigensolutions have been mentioned in section 3. The corrective remainder can be written as a sum of terms that remain small in successive powers of $\epsilon^{\frac{1}{2}}$ for any $\alpha_1^{(1)}$, as has been shown in section 4.

The unified treatment heavily leans on the following notions. To begin with, it crucially depends on a proper recognition of the critical region ($\alpha_1^{(1)}$ of order $\epsilon^{\frac{1}{2}}$). This allows for the separation of irreducible (zeroth order) part and corrective remainder. Further, a certain reshuffling of the critical corrective terms is required for the successive corrections to remain small of the correct order outside the critical region. At least, this suffices in the normal (monostable) regime well above the critical point. Last, but not least, the extension into the normal (bistable) regime well below the critical point could be realized only after a certain renormalization of the original jump moments. This procedure effectively eliminates the global large scale fluctuations (n of order Ω) that would spoil the expansion.

Dropping the superscript (o) and normalizing the corrected eigenfunctions following (3.4), the propagator remains given by (3.2). For processes with the property of detailed balance [3, 35] (or with an equivalent property, such as all one-step processes), the relation (3.3) between the probability eigensolutions and their adjoints persists through all orders of the corrections*. In that case equilibrium quanti-

*For processes without detailed balance there still exist adjoint solutions, but (3.3) ceases to be valid beyond the irreducible approximation. For such processes one can not even always guarantee the

ties like the dynamical susceptibility remain given by such simple formulae as (3.5). Finally, in terms of the original variables one has explicitly $P(n,t|n_0,0) \leftrightarrow P_\kappa(x,s|x_0,0)$, where:

| | above-critical $\alpha_1^{(1)} \leq 0$ | below-critical $\alpha_1^{(1)} \geq 0$ |
|------------|--|--|
| $x =$ | $\varepsilon^{\frac{3}{2}} [-\alpha_1^{(3)} / \frac{1}{2} \alpha_2^{(0)}]^{\frac{1}{2}} n$ | $\varepsilon^{\frac{3}{2}} [-a_1^{(3)} / \frac{1}{2} a_2^{(0)}]^{\frac{1}{2}} n$ |
| $s =$ | $\varepsilon^{\frac{1}{2}} [-\frac{1}{2} \alpha_1^{(3)} \alpha_2^{(0)}]^{\frac{1}{2}} t$ | $\varepsilon^{\frac{1}{2}} [-\frac{1}{2} a_1^{(3)} a_2^{(0)}]^{\frac{1}{2}} t$ |
| $\kappa =$ | $\Omega^{\frac{1}{2}} [-\frac{1}{2} \alpha_1^{(3)} \alpha_2^{(0)}]^{-\frac{1}{2}} (-\alpha_1^{(1)})$ | $-\Omega^{\frac{1}{2}} [-a_1^{(3)} / \frac{1}{2} a_2^{(0)}]^{\frac{1}{2}} \varphi_{\pm}^2$ |

completeness of the eigenspectrum of the original master equation. Nonetheless, the set of irreducible eigensolutions is complete, independent of detailed balance. Therefore it is formally possible to proceed with the expansion even in such cases (see footnote in [1], section 5), but one can not be sure in general that this procedure converges to the true solutions.

APPENDIX: MEAN FIELD ISING MODEL

In the mean field Ising model short range interactions between constituent spins are smeared out effectively over the entire system. Physically the model's pump parameter (i.e. $\alpha_1^{(1)}$, or κ) is a temperature*. To be more precise, it is the difference between the actual temperature and its critical value. Above the critical temperature there is no spontaneous macroscopic magnetization (monostable case). Below the critical temperature the nonzero spontaneous macroscopic magnetization may be either plus or minus (spin up or down; bistable case). The model has been studied previously in the critical region only numerically [29]. Presently we briefly discuss the correct lowest order of the stochastic description including the critical point.

A1. The master equation

Let us consider a system of N spins and denote the number of down-spins by n_1 , the number of up-spins by n_2 . Trivially, $n_1+n_2=N$. The system will be described by the bivariate master equation

$$\begin{aligned} \dot{P}(n_1, n_2; t) = & W(n_1, n_2 | n_1+1, n_2-1) P(n_1+1, n_2-1; t) \\ & + W(n_1, n_2 | n_1-1, n_2+1) P(n_1-1, n_2+1; t) \\ & - W(n_1-1, n_2+1 | n_1, n_2) P(n_1, n_2; t) \\ & - W(n_1+1, n_2-1 | n_1, n_2) P(n_1, n_2; t) \quad , \end{aligned} \quad (A1.1)$$

which is of the one step type. The spin flip transition rates are taken to be [29]

$$\begin{aligned} W(n_1-1, n_2+1 | n_1, n_2) &= n_1 \exp[-\beta(n_1-n_2)/N] \quad , \\ W(n_1+1, n_2-1 | n_1, n_2) &= n_2 \exp[-\beta(n_2-n_1)/N] \quad . \end{aligned} \quad (A1.2)$$

The parameter β defines a dimensionless temperature $T=1/\beta$. As N is kept fixed it is convenient to introduce the nett magnetization $m=n_2-n_1$. That is, $m=N$ means all spins up, $m=-N$ all down. In terms of m the equation describes a two step process,

$$\begin{aligned} \dot{P}(m, t) = & W(m | m-2) P(m-2, t) + W(m | m+2) P(m+2, t) \\ & - [W(m+2 | m) + W(m-2 | m)] P(m, t) \quad , \end{aligned} \quad (A1.3)$$

where

$$\begin{aligned} W(m+2 | m) &= \frac{1}{2}(N-m) e^{+\beta m/N} \quad , \\ W(m-2 | m) &= \frac{1}{2}(N+m) e^{-\beta m/N} \quad . \end{aligned} \quad (A1.4)$$

*The model has been used occasionally also to describe nonphysical systems, in particular in considering social phenomena [35-37].

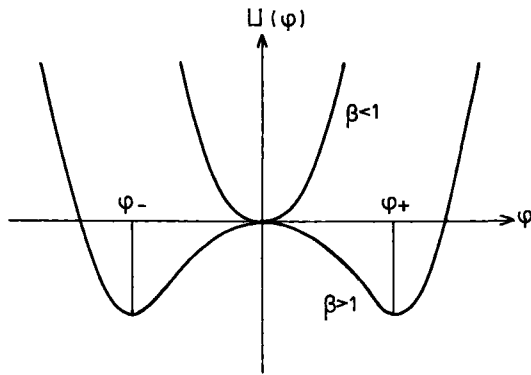


Figure A1. The macroscopic potential $U(\varphi)$ of the mean field Ising model according to (A1.8).

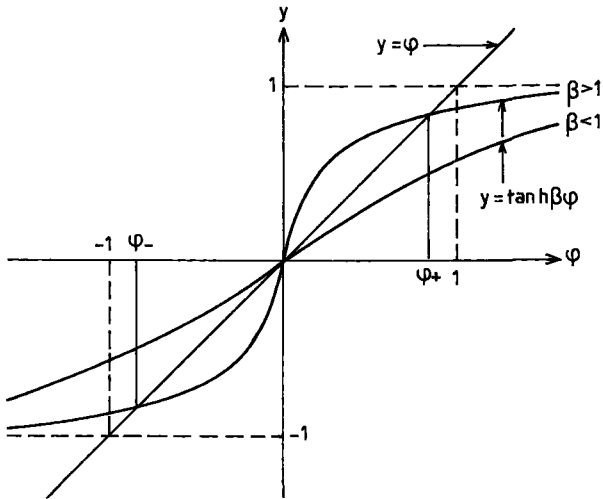


Figure A2. Solving (A1.7) for the stationary macroscopic state.

The first and second extensive jump moment are obtained from

$$a_k(m) = \sum_{m'} (m' - m)^k W(m' | m) \quad , \quad (A1.5)$$

with $k=1$ resp. 2. See also (2.5). For the corresponding intensive jump moments $\alpha_k(\varphi) = a_k(N\varphi)/N$ one thus finds

$$\begin{aligned} \alpha_1(\varphi) &= 2(\sinh\beta\varphi - \varphi \cosh\beta\varphi) \quad , \\ \alpha_2(\varphi) &= 4(\cosh\beta\varphi - \varphi \sinh\beta\varphi) \quad . \end{aligned} \quad (A1.6)$$

For the pure one step process higher moments are simply multiples of (A1.6). In view of the macroscopic evolution equation (2.7), the stationary solutions follow from

$$\tanh\beta\varphi = \varphi \quad . \quad (A1.7)$$

The stable solutions correspond to the minima of the macroscopic potential (see (2.8))

$$U(\varphi) = \frac{2}{\beta} \left(1 + \frac{1}{\beta}\right) (1 - \cosh\beta\varphi) + \frac{2}{\beta} \varphi \sinh\beta\varphi \quad , \quad (A1.8)$$

which is shown in figure A1. The transcendental equation (A1.7) is illustrated in figure A2. Obviously, if $\beta \leq 1$ there is just one solution $\varphi_0 = 0$. However, if $\beta > 1$ there exist two nonzero solutions φ_{\pm} , tending from zero for $\beta=1$ to ± 1 for $\beta \rightarrow \infty$. See figure A3.

From (A1.6) and the definition (2.9) one obtains the relevant coefficients

$$\alpha_1^{(1)} = 2(\beta - 1); \quad \alpha_1^{(3)} = \frac{1}{3}(\beta - 3)\beta^2; \quad \alpha_2^{(0)} = 4 \quad . \quad (A1.9)$$

Note that $\alpha_1^{(3)} < 0$ at the critical point $\beta=1$, which guarantees stability. The mean field Ising model clearly falls within the framework of the general theory presented in the main text. The irreducible, zeroth order description including the critical point is given by the standard quartic potential Fokker-Planck equation (3.1).

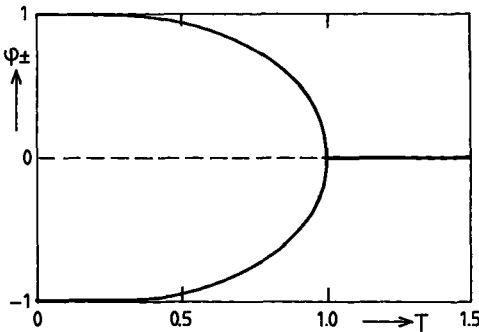


Figure A3.

The stationary macroscopic magnetic moment φ_{\pm} of the mean field Ising model as a function of temperature $T (= 1/\beta)$.

A1.1 Above and at the critical point

Here $T > 1$, i.e. $\beta \leq 1$. From (A1.9) $\alpha_1^{(1)} \leq 0$ and $\alpha_1^{(3)} < 0$. Denoting by λ the eigenvalues belonging to the original time scale t (so that $\lambda t = \mu s$), defining the ratio $\lambda/\mu = \kappa$, and using (A1.9) into (2.10) and (2.13), one obtains explicitly

$$\begin{aligned} \kappa &= (T-1)[2N/(1-\frac{1}{3}\beta)]^{\frac{1}{2}}, \\ \kappa &= 2\beta[2N/(1-\frac{1}{3}\beta)]^{-\frac{1}{2}}. \end{aligned} \quad (A1.10)$$

A1.2 Below and at the critical point

In this case $T \leq 1$, i.e. $\beta \geq 1$. Using the pertinent renormalized coefficients belonging to (A1.6) into (2.18) and (2.20) one finds

$$\begin{aligned} \kappa &= -\varphi_{\pm} [(N/2)(\cosh^2 \beta \varphi_{\pm} - \beta)] , \\ \kappa &= \frac{1}{\varphi_{\pm} \cosh \beta \varphi_{\pm}} [(2/N)(\cosh^2 \beta \varphi_{\pm} - \beta)] . \end{aligned} \quad (A1.11)$$

A2. Discussion

A2.1 Macroscopic magnetization

The solutions of the macroscopic equation (A1.7) are shown in figure A3. Near the critical point one has

$$\varphi_{\pm} \cong \pm [3(1-T)]^{\frac{1}{2}}, \text{ as } T \rightarrow 1(\beta \rightarrow 1) . \quad (A2.1)$$

Well below the critical point one finds

$$\varphi_{\pm} \cong \pm (1-2e^{-2\beta}), \text{ as } T \rightarrow 0(\beta \rightarrow \infty) . \quad (A2.2)$$

A2.2 Pump parameter and time scaling

Well above the critical point one infers from (A1.10) that

$$\frac{\kappa}{\sqrt{2N}} \cong T, \quad \kappa \sqrt{\frac{N}{2}} \cong \beta, \text{ as } T \rightarrow \infty(\beta \rightarrow 0) . \quad (A2.3)$$

Approaching the critical point from above one gets

$$\frac{\kappa}{\sqrt{3N}} \cong T-1, \quad \kappa \cong \frac{2}{\sqrt{3N}}, \text{ if } T \cong 1(\beta \cong 1) . \quad (A2.4)$$

The same result is obtained slightly below the critical point from (A1.11), appropriately expanding the cosh and using (A2.1). Well below the critical point the cosh in (A1.11) becomes progressively dominant and one obtains

$$\frac{\kappa}{\sqrt{2N}} \cong -\frac{1}{2}e^{\beta}, \quad \kappa \sqrt{\frac{N}{2}} \cong 1-4\beta^{-2\beta}, \text{ as } T \rightarrow 0(\beta \rightarrow \infty) . \quad (A2.5)$$

A2.3 Eigenvalues and susceptibility

One now easily obtains the curves $\lambda(T)$, which are distorted versions of the standard $\mu(\kappa)$. See [1] figure 3, and [21]. Well above the critical point $\mu \approx n\kappa$ ($n=0, 1, \dots$), so that $\lambda = \mu \approx n\kappa$. By means of (A2.3) one finds

$$\lambda \approx 2n, \text{ as } T \rightarrow \infty (\beta \rightarrow 0) \quad (A2.6)$$

Near the critical point $\mu \approx \mu(0)$ is of order one. Using (A2.4) one gets

$$\lambda \approx 2\mu(0)/\sqrt{3N}, \text{ if } T \approx \beta \approx 1 \quad (A2.7)$$

Well below the critical point $\mu \approx -n\kappa$ ($n=0, 1, \dots$). By (A2.5) one obtains

$$\lambda \approx \frac{1}{2}ne^{\beta}, \text{ as } T \rightarrow 0 (\beta \rightarrow \infty) \quad (A2.8)$$

As a typical illustration of the above analysis, the third eigenvalue of the irreducible problem (corresponding to $n=3$ in (A2.6) and $n=2$ in (B2.8)) is sketched in figure A4 for the unrealistic total spin number $N=50$. Note from figure 3 of [1] that the true critical region where substantial deviations from the asymptotic values $\mu = n|\kappa|$ occur, is given roughly by $|\kappa| \lesssim 10$. By (A2.4) this implies a critical temperature range $\Delta T \approx 10N^{-\frac{1}{2}}T_C$. Hence, with a critical temperature $T_C \approx 300^\circ\text{K}$, one requires at least 10^{13} spins for the critical range to be of the order of a millikelvin [29, 38].

Let us examine the eigenvalues as $N \rightarrow \infty$. Above the critical point the standard asymptotes are $\mu = n\kappa$, and (A1.10) leads to $\lambda = 2n(1-\beta)$. The below-critical standard asymptotes are $\mu = -n\kappa$. Using (A1.11) and a simple hyperbolic relation, one finds

$$\lambda = n[(1-\varphi_{\pm}^2)^{-\frac{1}{2}} - \beta(1-\varphi_{\pm}^2)^{\frac{1}{2}}] \quad (A2.9)$$

With $\varphi_{\pm} = 0$ this includes the above-critical result. With $n=2$, (A2.9) agrees with the result (4.11) of [29]. This case corresponds to the first excited state of the outer solutions of the irreducible problem (3.1).

Using the above discussed irreducible description, one can calculate the lowest order approximations of (dynamical) equilibrium quantities such as the correlation function, spectral density and susceptibility. For example, the static susceptibility (see (3.5)) above the critical point becomes

$$\chi(0) = \kappa \varphi_{00}^1 D_{-\frac{3}{2}}\left(\frac{\kappa}{\sqrt{2}}\right) / \sqrt{2} D_{-\frac{1}{2}}\left(\frac{\kappa}{\sqrt{2}}\right) \quad (A2.10)$$

where $\varphi_{00}^1 = d\varphi_0/dh$ at $h=0$, h being the external magnetic field; $\varphi_{00}^1 = 1/(1-\beta)$, see [29]. Well above the critical point (A2.10) reduces to the normal Curie-Weiss law. By means of (A2.4) at the critical point (A2.10) leads to

$$\chi_c(0) = \sqrt{3N} \Gamma\left(\frac{3}{2}\right) / 2\Gamma\left(\frac{1}{2}\right) \approx 1.17\sqrt{N} \quad (A2.11)$$

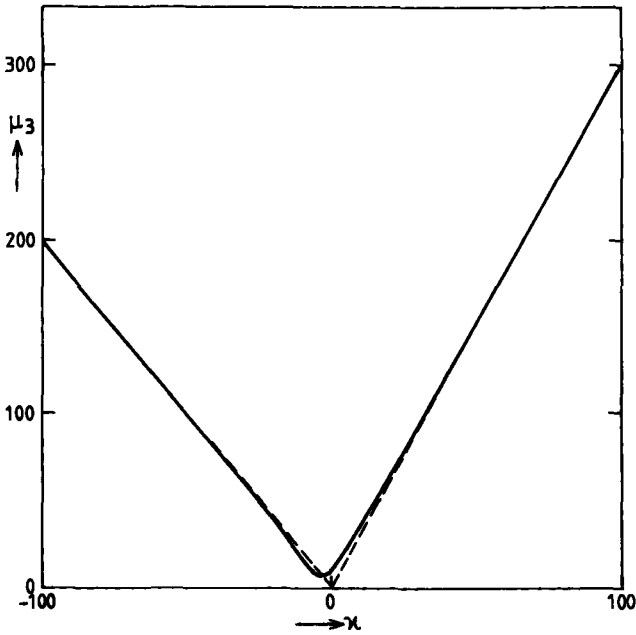


Figure A4a. General eigenvalue μ_3 as function of pump parameter κ .

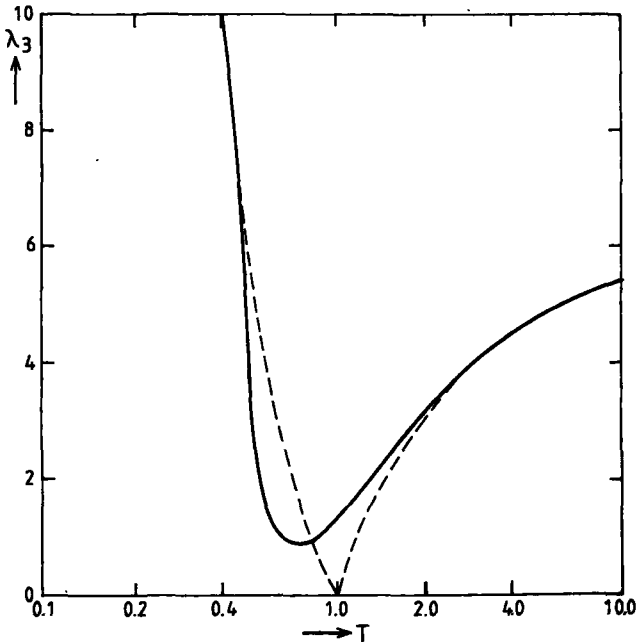


Figure A4b. Eigenvalue λ_3 for Ising model with $N=50$ spins, as function of temperature T . The value $\kappa=100$ in fig. 4a corresponds to $T \approx 10$, while $\kappa \approx -100$ corresponds to $T \approx 0.27$. Dotted lines represent asymptotic values, $N = \infty$.

Further, using the standard results of the irreducible eigenspectrum [1, 21], it is not difficult to compute the dynamical susceptibility (3.5). Defining $\chi = \chi' - i\chi''$, for the mean field Ising model one obtains at the critical point

$$\chi'_c(\omega) = \chi_c(0) \left[\frac{0.9844}{1+0.401\tilde{\omega}^2} + \frac{0.0152}{1+0.011\tilde{\omega}^2} + \frac{0.0004}{1+0.002\tilde{\omega}^2} + \dots \right], \quad (\text{A2.12})$$

$$\chi''_c(\omega) = \chi_c(0) \left[\frac{0.6230\tilde{\omega}}{1+0.401\tilde{\omega}^2} + \frac{0.0016\tilde{\omega}}{1+0.011\tilde{\omega}^2} + \dots \right], \quad (\text{A2.13})$$

where $\tilde{\omega} = \omega\sqrt{N}$. See figure A5. Contributions from higher eigensolutions than shown are zero at the displayed numerical accuracy. Higher order corrections in terms of $N^{-\frac{1}{2}}$ can be calculated systematically as discussed in section 4.

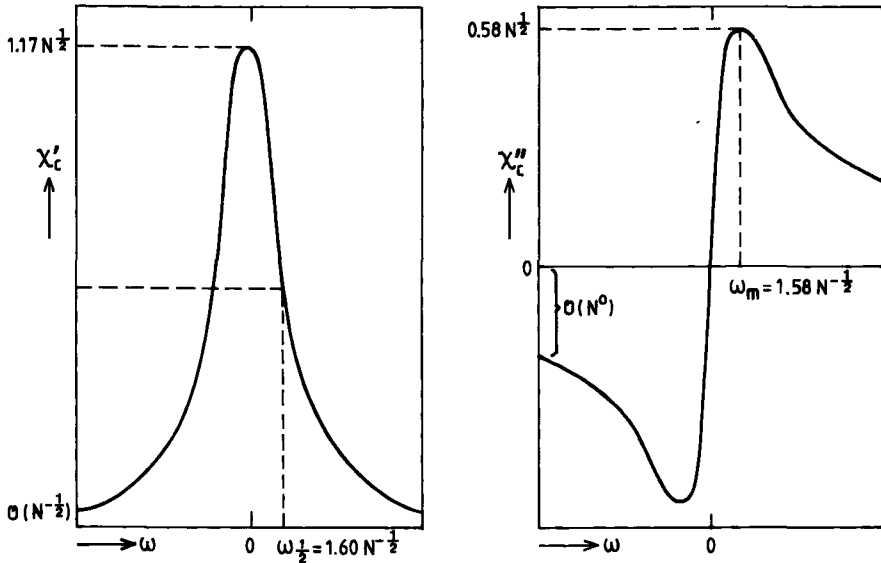


Figure A5. Real and imaginary part of dynamical susceptibility $\chi = \chi' - i\chi''$ as functions of the frequency ω at the critical point $T=1$ for the mean field Ising model. See (A2.12 & 13). Curves have been drawn using $N=50$ spins.

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CURRICULUM VITAE

ACKNOWLEDGEMENTS

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It is a great pleasure to thank professor N.G. van Kampen for his stimulating interest and cooperation during all stages of this work. The many discussions we had at the Institute for Theoretical Physics of the University of Utrecht considerably deepened my understanding of stochastic processes.

Looking somewhat further back in time, I wish to acknowledge professors R. Kronig and D. Polder for the manner in which they taught me the beauty of theoretical physics. It has become a way of living.

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I credit Beethoven, especially for his string quartets which often energized my thinking. I also thank Joke for her interest and affection. I realize that it is not the easiest way of living, together with a theoretical physicist.

The debts to my parents, and in particular to my mother, are thousandfold. I deeply admire the way in which she guided my life, especially after the decease of my father. To both I dedicate this thesis.

The contents of this treatise will be published in *Physica A* (Leiden). The permission from the Physics Laboratory TNO is highly appreciated.

Non scholae, sed vitae discimus
Seneca, Epistulae 106,11

CURRICULUM VITAE

Hans Dekker was born in 1947 in Amsterdam, where he received his elementary and high school education. He studied physics at the Technological University of Delft from 1964 to 1971, where he got involved in particular in acoustics and electromagnetic theory (candidate work), superconductivity and electron microscopy (doctoral work), and journalism. During 1970 he worked several months with the Industrial and Radiological Hygiene Branch of Tennessee Valley Authority at Muscle Shoals, Alabama, on luminescent dosimetry.

Having completed his college education receiving the Physics Study Award, he joined the laser research group of the Physics Laboratory TNO in Den Haag as a theoretical physicist. In the summers of 1972 and 1974 he visited the 'International School of Quantum Electronics' in Erice, Sicily. In 1977 he participated in the 'Nato Advanced Study Institute on Path Integrals' in Antwerp. He lectured in Den Haag (1972-1979), and at the Universities of Stuttgart (1975, on synergetics), Leuven (1979, on path integrals and stochastic processes), München (1979, on dissipative systems) and Louvain-la-Neuve (1979, on functional integration). He is the author of several scientific papers.

He first met professor N.G. van Kampen in 1975, originally on synergetics. The essential idea that led to the present thesis, emerged at the end of 1978 from a discussion at the Institute for Theoretical Physics of the University of Utrecht.

Critical Dynamics

The Expansion of the Master Equation including a Critical Point

H. Dekker, Utrecht, 1980 april 21

STELLINGEN

1 De wijd verbreide idee dat het niet mogelijk zou zijn een systematische ontwikkeling van de master vergelijking te geven welke geldig blijft in een kritisch punt, is met dit proefschrift definitief weerlegd.

2 In tegenstelling tot hetgeen Larson en Kostin beweren, is Van Kampen's correctie op Kramers' diffusie formule niet van de orde $\exp(-\Delta U)$, maar wel degelijk van de orde $1/\Delta U$.

*N.G. van Kampen, J.Stat.Phys.*17(1977)71.

*R.S. Larson & M.D. Kostin, J.Chem.Phys.*69(1978)4821.

3 In bepaalde gevallen dient Van Kampen's formule voor de kritische schaling te worden gegeneraliseerd.

*N.G. van Kampen, Adv.Chem.Phys.*34(1976)245.

*H. Dekker, J.Chem.Phys.*72(1980)189.

4 Een uniform geldige ontwikkeling van de master vergelijking voor de relaxatie van een normale instabiele toestand is nog niet bekend.

*M. Suzuki, J.Stat.Phys.*16(1977)11.

*F. Haake, Phys.Rev.Lett.*41(1978)1685.

5 Het beschrijven van kritische fluctuaties in diffusie processen met behulp van de 'meest waarschijnlijke paden' is onjuist.

*D. Dürr & A. Bach, Z.Phys.*B32(1979)413.

6 Berekening van padintegralen met behulp van alleen het 'meest waarschijnlijk pad over korte tijdintervallen' behelst een ad-hoc voorschrift, dat slechts in vlakke ruimtes achteraf kunnen worden gerechtvaardigd.

H. Dekker, in Proc.Int.Workshop on Funct.Int.(Plenum, New York, 1980).

7 Een eenvoudig model van een niet single-mode laser, geldig in de buurt van het kritische punt, is ten onrechte vrijwel onopgemerkt gebleven.

H. Dekker, Optics Comm. 10(1974)114.

R.L. Stratonovich, Sov.J.Qu.Electr. 7(1977)1225.

8 De beschrijving van de quantum mechanica van dissiperende systemen met behulp van tijdafhankelijke Hamiltonianen en afzonderlijke toegevoegde ruisbronnen (ten einde de onzekerheids relaties te waarborgen) leidt tot resultaten welke dusdanig afhangen van speciale begincondities, dat zij als onjuist moet worden beschouwd.

I.R. Svin'in, Teor.Mat.Fiz. 27(1976)270.

J. Messer, Acta Phys.Austr. 50(1977)75.

9 De bewering dat in het waterstof atoom 'banen' met gelijk hoofd-quantum getal door de letters s, p, d, f, g, ... in volgorde van afnemende excentriciteit zijn aangegeven, is gebaseerd op de oude quantum theorie en om die reden onjuist voor de s-'baan' ($\ell=0$).

J.A. Prins, Grondbeginselen van de hedendaagse natuurkunde (Wolters, Groningen, 1963).

D. ter Haar, The Old Quantum Theory (Pergamon, Oxford, 1967).

10 De relatie tussen de quantum-electrodynamische interactie Hamiltoniaan in de dipool benadering in termen van de vector potentiaal en in termen van de elektrische veldsterkte is, hoewel eenvoudig, aanleiding geweest tot verwarring. Een correcte behandeling is te vinden in *C.W. Lamberts & H. Dekker, Phys.Lab.TNO Rept. PHL 1975-49.*

11 Een foton $\hbar\omega$ is het kleinste energie quantum dat één Fourier component (met frequentie ω) van het electromagnetisch veld kan bevatten. Dynamische verschijnselen behoeven noodzakelijkerwijs meer dan één Fourier component. De gebruikelijke aanduiding 'emissie van een foton' is dan ook semantisch gezien onjuist, evenals de in de quantum electronica ingeburgerde zegswijze 'fotonentellen'. Een correct alternatief voor dit laatste is 'foto-electronen tellen'.

12□ Verder onderzoek naar de eigenschappen, zoals fading en interne excitatie, van thermoluminescente dosimeters blijft gewenst.

Z. Spurný & J. Kvasnička, *Int.Conf.Lum.Dos.* (Kraków, Poland, 1974)255.

H. Dekker, *Health Physics* 30(1976)399.

13□ Supergeleidende objectief lenzen voor electronen microscopie, en de Faraday methode ter bepaling van hun kwaliteiten, zijn ten onrechte ondergewaardeerd.

H. Dekker, *J.Phys.E* 5(1972)368.

P.W. Hawkes & U. Valdrè, *J.Phys.E* 10(1977)309.

14□ Bij het ontwerpen van conferentie- en concertzalen wordt het effectief dissipatieve randeffect gewoonlijk ten onrechte niet in beschouwing genomen.

H. Dekker, *J.Sound & Vibr.*32(1974)199.

15□ Het invoeren van veel niet goed bekende parameters in wiskundige modellen, zoals gebruik voor bijvoorbeeld economische voorspellingen, geeft hun uitkomsten een ruisachtig karakter. Deze stochastiek zou - ten eerste - op de juiste wijze behandeld, en - ten tweede - niet in het eindresultaat verbloemd moeten worden.

16□ De negatieve invloed van de Nederlandse vertaling van een boek van de Franse katholieke Venette aan het eind van de zeventiende eeuw op de maatschappelijke waardering van de vrouw in de daaropvolgende eeuwen dient nader te worden onderzocht.

N. Venette, *Venus Minsieke Gasthuis* (Amsterdam, 1687).

N.F. Noordam, *Spiegel Historiae* 12(1979)650.

17□ Het gezegde 'als de nood het hoogst is, is redding nabij' moet op logische gronden als ofwel onjuist ofwel triviaal worden beschouwd.