Critical Examination of Incoherent Operations and a Physically Consistent Resource Theory of Quantum Coherence

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Are Incoherent Operations Physically Consistent? – A Critical Examination of Incoherent Operations

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Considerable work has recently been directed toward developing resource theories of quantum coherence. In this letter we establish a criterion of physical consistency for any resource theory. This criterion requires that all free operations in a given resource theory be implementable by a unitary evolution and projective measurement that are both free operations in an extended resource theory. We show that all currently proposed basis-dependent theories of coherence fail to satisfy this criterion. We further characterize the physically consistent resource theory of coherence and find its operational power to be quite limited. After relaxing the condition of physical consistency, we introduce the class of dephasing-covariant incoherent operations as a natural generalization of the physically consistent operations. Necessary and sufficient conditions are derived for the convertibility of qubit states using dephasing-covariant operations, and we show that these these conditions also hold for other well-known classes of incoherent operations.

Resource theories offer a powerful framework for understanding how certain physical properties naturally change within a physical system. A general resource theory for a quantum system is characterized by a pair $(\mathcal{F}, \mathcal{O})$, where $\mathcal{F}$ is a set of "free" states and $\mathcal{O}$ is a set of "free" quantum operations. Any state that does not belong to $\mathcal{F}$ is then deemed a resource state. Entanglement theory provides a prototypical example of a resource theory in which the free states are the separable or unentangled states, and the free operations are local operations and classical communication (LOCC) [1, 2]. Other examples includes the resource theories of athermality [3, 4], asymetry [5–7], and non-stabilizer states for quantum computation [8].

Any pair $(\mathcal{F}, \mathcal{O})$ defines a resource theory, provided the operations of $\mathcal{O}$ act invariantly on $\mathcal{F}$; i.e. $\mathcal{E}(\rho) \in \mathcal{F}$ for all $\rho \in \mathcal{F}$ and all $\mathcal{E} \in \mathcal{O}$. However, this is just a mathematical restriction placed on the maps belonging to $\mathcal{O}$. It does not imply that $\mathcal{E} \in \mathcal{O}$ can actually be physically implemented without generating or consuming additional resource. The issue is a bit subtle here since in quantum mechanics, physical operations on one system ultimately arise from unitary dynamics and projective measurements on a larger system, a process mathematically described by a Stinespring dilation [9]. A resource theory $(\mathcal{F}, \mathcal{O})$ defined on system A is said to be *physically consistent* if every free operation $\mathcal{E} \in \mathcal{O}$ can be obtained by an auxiliary state $\hat{\rho}_B$, a joint unitary U_{AB}, and a projective measurement $\{P_k\}_k$ that are all free in an extended resource theory $(\mathcal{F}', \mathcal{O}')$ defined a larger system AB, for which $\mathcal{F} = \mathrm{Tr}_B\mathcal{F}' := \{\mathrm{Tr}_B(\rho_{AB}) : \rho_{AB} \in \mathcal{F}'\}$.

Arguably a physically consistent resource theory is more satisfying than an inconsistent one. Indeed, without physical consistency, the notions of "free" and "resource" have very little physical meaning since resources must ultimately be consumed to implement certain operations that are supposed to be " free." As an analogy, if a car wash offers to wash your car for free, but only after you go across the street and purchase an oil change from their business partner, is the "car washing operation" really free?

Resource \ Operations	Physically Consistent	Physically Inconsistent
Entanglement	LOCC	SEP, NE
Coherence	PIO	SIO, DIO, IO, MIO

TABLE I: The class of Physically Incoherent Operations (PIO) introduced in this article represents the coherence analog to LOCC in terms of being a physically consistent resource theory. The previously studied Strictly Incoherent Operations (SIO), Incoherent Operations (IO) and Maximally Incoherent Operations (MIO) represent relaxations of PIO in the same way that Separable (SEP) and Non-Entangling (NE) operations are relaxations of LOCC. We further introduce the new class of Dephasing-covariant Incoherent Operations (DIO).

At the same time, physically inconsistent resource theories can still be of interest. In open quantum systems, for instance, one may not care about whether the interacting environment consumes resources; and even when working with closed systems, it is still valuable to consider relaxations of physical consistency. Consider again entanglement. LOCC renders a physically consistent resource theory of entanglement since any LOCC operation can be implemented using only local unitaries and projections. However, often one considers more general operational classes such as separable operations (SEP) or the full class of non-entangling operations (NE) [31]. The motivation for using SEP is that it possesses a much nicer mathematical structure than LOCC without being too much stronger. In contrast, one may turn to NE when seeking maximal strength among all operations that cannot generate entanglement. Nevertheless, despite being appealing objects of study, both SEP and NE represent physically inconsistent resource theories of entanglement.

In this letter, we analyze some of the recently proposed resource theories of quantum coherence [10–19]. We observe that none of these offer a physically consistent resource theory, and the true analog to LOCC in coherence theory has been lacking. We identify this hitherto missing piece as the class of *physically incoherent operations* (PIO), and we provide its characterization. The operations previously used to study coherence are much closer akin to SEP and NE in entanglement theory, and we clarify what sort of physical interpretations can be given to these operations.

While we find that PIO allows for optimal distillation of maximal coherence from partially coherent pure states in the asymptotic limit of many copies, the process is strongly irreversible. That is, maximally coherent states cannot be diluted into weakly coherent states at a nonzero rate, and they are thus curiously found to be the *least* powerful among all coherent states in terms of asymptotic convertibility. Given this limitation of PIO and its similar weakness on the finite-copy level, it is therefore desirable from a theoretical perspective to consider more general operations. Consequently, we shift our focus to the development of coherence resource theories under different relaxations of PIO. To this end, we introduce the class of dephasing-covariant incoherent operations (DIO), which to our knowledge has never discussed before in literature. We provide physical motivation for DIO and show that these operations are just as powerful as Maximal Incoherent Operations (MIO) when acting on qubits. Detailed proofs of our results as well as a more detailed comparison between different incoherent opartional classes can be found in an accompanying paper [20].

Quantum coherence has traditionally referred to the presence of off-diagonal terms in the density matrix. For a given (finite-dimensional) system, a complete basis $\{|i\rangle\}_{i=1}^d$ for the system is specified, accounting for all degrees of freedom, and a state is said to lack coherence (or be "incoherent") with respect to this basis if and only if its density matrix is diagonal in this basis [21, 22]. We will refer to this as a *basis-dependent definition of coherence*, and accordingly, a basis-dependent resource theory of coherence identifies the free (or "incoherent") states $\mathcal{I}$ as precisely the set of diagonal density matrices in the fixed incoherent basis [32]. We frequently use the "hat" notation $\hat{\rho}$ to indicate that the state is incoherent.

When it comes to identifying the free (or "incoherent") operations, different proposals have been made. We focus on the following three operational classes. A completely positive trace-preserving (CPTP) map $\mathcal{E}$ is said to be: a Maximal Incoherent Operation (MIO) if $\mathcal{E}(\rho) \in \mathcal{I}$ for every $\rho \in \mathcal{I}$ [10, 23]; an Incoherent Operation (IO) if $\mathcal{E}$ has a Kraus operator representation $\{K_n\}_n$ such that $K_n\rho K_n^\dagger/\mathrm{Tr}[K_n\rho K_n^\dagger] \in \mathcal{I}$ for all n and

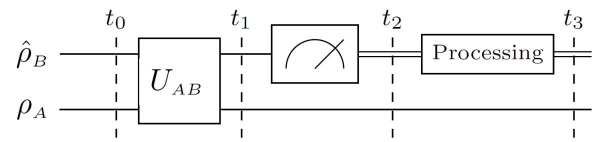

FIG. 1: This figure depicts the general process of implementing an incoherent operation on the joint system AB whose reduced action on A is the incoherent CPTP map $\rho_A \mapsto \mathcal{E}(\rho_A)$. A second system B is introduced in an incoherent state $\hat{\rho}^B$. Both the unitary U_{AB} and projective measurement are coherence non-generating. All measurement outcomes are stored in a classical register of system B so that the joint system is in a QC state at time t_2. Only maps $\mathcal{E}$ implemented in this way are physically consistent within a resource-theoretic picture.

$\rho \in \mathcal{I}$ [12]; a Strictly Incoherent Operation (SIO) if $\mathcal{E}$ has a Kraus operator representation $\{K_n\}_n$ such that $K_n\Delta(\rho)K_n^\dagger = \Delta(K_n\rho K_n^\dagger)$ for all n [16, 18], where Δ is the completely dephasing map $\Delta : \rho \mapsto \sum_{i=1}^d |i\rangle\langle i|\rho|i\rangle\langle i|$.

In each of these approaches, the allowed unitary operations and projective measurements are the same. The set of all incoherent unitary matrices forms a group which we denote by G. For a d-dimensional system, the group G consists of all $d \times d$ unitaries of the form πu, where π is a permutation matrix and u is a diagonal unitary matrix (with phases on the diagonal). We denote by $N \cong U(1)^d$ the group of diagonal unitary matrices and by Π the group of permutation matrices. Note that N is a normal subgroup of G, and $G = N \rtimes \Pi$ is the semi-direct product of N and Π. Likewise, an incoherent projective measurement consists of any complete set of orthogonal projectors $\{P_j\}$ with each P_j being diagonal in the incoherent basis.

It is crucial that a physical resource theory possess a well-defined extension to multiple systems if one allows for generalized measurements, simply because the latter describes a process that is carried out on more than one system. A natural requirement for any physical resource theory of coherence is that it satisfies the *no superactivation postulate*; that is, if ρ and σ lack quantum coherence, then so must the joint state $\rho \otimes \sigma$. Combining the basis-dependent definition of coherence with the no superactivation postulate immediately fixes the structure of multipartite incoherent states. If $\{|i\rangle_A\}_{i=1}^{d_A}$ and $\{|j\rangle_B\}_{j=1}^{d_B}$ are defined to be the incoherent bases for systems A and B respectively, then the superactivation postulate forces $\{|i\rangle_A|j\rangle_B\}_{i,j=1}^{d_A,d_B}$ to be the incoherent basis for the joint system AB.

The fact that the incoherent basis takes tensor product form when considering multiple systems has strong consequences for the physical consistency of incoherent operations. Every physical operation on some system, say A, can be decomposed into a three-step process as depicted in Fig. 1. If this operation is free within a physically consistent framework, then (i) a joint incoherent unitary U_{AB} is applied immediately prior to time t_1 on the input state ρ_A and some fixed incoherent state $\hat{\rho}_B$, (ii) an incoherent projective measurement is applied immediately prior to time t_2 with system B encoding the measurement outcome as a classical index, and (iii) a classical processing channel is applied to the measurement outcomes immediately prior to t_3. It can be assumed without loss of generality that the projective measurement in step (ii) consists of rank-one projectors P_j since the action of more general projections can be recovered by coarse-graining in step (iii). Also, note that at time t_2, the joint state is a quantum-classical (QC) state $\omega_{AB} = \sum_{j=1}^{t} \rho_{A,j} \otimes |j\rangle\langle j|_B$, where

$$\rho_{A,j} = \mathrm{Tr}_B[(\mathbb{I}_A \otimes P_j) U_{AB}(\rho_A \otimes \hat{\rho}_B) U_{AB}^\dagger].$$

With the classical processing, the final state of system A at time t_3 is given by $\mathcal{E}(\rho_A) := \sum_{k=1}^{t'} \rho'_{A,k} \otimes |k\rangle\langle k|$, where $\rho'_{A,k} = \sum_{j=1}^{t} p_{k|j} \rho_{A,j}$ for some channel $p_{k|j}$. We define the class of physical incoherent operations (PIO) to be the set of all CPTP maps $\mathcal{E}$ that can be obtained in this way.

Proposition 1. *A CPTP map $\mathcal{E}$ is a physically incoherent operation if and only if it can be expressed as a convex combination of maps each having Kraus operators $\{K_j\}_{j=1}^r$ of the form*

$$K_j = U_j P_j = \sum_x e^{i\theta_x} |\pi_j(x)\rangle\langle x| P_j, \tag{1}$$

where the P_j form an orthogonal and complete set of incoherent projectors on system A and π_j are permutations.

Proof. First consider when $\hat{\rho}_B$ is a pure state $\hat{\rho}_B = |y'\rangle\langle y'|$. A joint incoherent unitary on AB will take the form $U_{AB} = \sum_{xy} e^{i\theta_{xy}} |\pi_1(xy)\pi_2(xy)\rangle\langle xy|$, where $(\pi_1(xy), \pi_2(xy))$ is the output of a permutation π applied to (x, y). To obtain a Kraus operator representation of the map, we decompose the incoherent projective measurement into a rank-one projection in the incoherent basis $\{|y\rangle\}$. Upon projecting onto $|y\rangle$, the (unnormalized) state of system A is

$$\langle y|U_{AB}(\rho_A \otimes |y'\rangle\langle y'|)U_{AB}^\dagger|y\rangle = U_y^{(y')} P_y^{(y')} \rho_A P_y^{(y')} (U_y^{(y')})^\dagger$$

where $U_y^{(y')} = \sum_{x \in S_y^{(y')}} e^{i\theta_{xy'}} |\pi_1(xy')\rangle\langle x| + W_y^{(y')}$. Here we take $x \in S_y^{(y')}$ iff $\pi_2(xy') = y$, the operator $W_y^{(y')}$ is suitably chosen such that $U_y^{(y')}$ is unitary, and $P_y^{(y')} = \sum_{x \in S_y^{(y')}} |x\rangle\langle x|$. It is obvious that the set $\{K_y^{(y')} = U_y^{(y')} P_y^{(y')}\}_y$ forms a complete set of Kraus operators which characterizes the measurement. If $\hat{\rho}^B$ were originally a mixed state $\hat{\rho}^B = \sum_{y'} p_{y'} |y'\rangle\langle y'|$, then a complete set of Kraus operators is given by $\{\sqrt{p_{y'}} K_y^{(y')}\}_{y,y'}$ where again each Kraus operator has the form $K_y^{(y')} = U_y^{(y')} P_y^{(y')}$. □

From the proposition above it is easy to see that PIO $\subset$ SIO $\subset$ IO $\subset$ MIO, with PIO being a strict subset of the other three. To understand the physical differences between these operations let us return to Fig. 1 and for the sake of the following discussion, assume that the measurement between times t_1 and t_2 is a rank-one projection into the incoherent basis $\{|j\rangle\}_{j=1}^{d_B}$. Then the joint state at time t_2 takes the form $\sum_{j=1}^{d_B} K_j \rho_A K_j^\dagger \otimes |j\rangle\langle j|_B$ for Kraus operators $\{K_j\}_{j=1}^{d_B}$. Suppose now that the input $\hat{\rho}_A$ is incoherent so that initial joint state $\hat{\rho}_A \otimes \hat{\rho}_B$ is also incoherent. If the final state at time t_3 is always incoherent, regardless of the coherence generated during the intermediate times, then the operation is a maximally incoherent operation (MIO). If the QC joint state at time t_2 is always incoherent, then the operation is an incoherent operation (IO). If the joint state at time t_1 is always incoherent, then the operation is a physically incoherent operation (PIO), provided the subsequent projective measurement is incoherent. Conversely, every IO/MIO operation can be implemented using the scheme of Fig. 1 by taking the size of system B to be sufficiently large. Where do SIO operations fit in this picture? They are like IO in that the joint state is always incoherent at time t_1, with the added constraint that U_{AB} has the form $U_{AB} = \sum_{i,k} c_{ki} |\pi_k(i)\rangle\langle i| \otimes |k\rangle\langle 0|$, for different permutations π_k [18, 20].

The class PIO is a rather restricted class of operations. For instance, suppose that $|\psi\rangle$ and $|\phi\rangle$ are any two pure states with $\mathrm{rank}[\Delta(\psi)] = \mathrm{rank}[\Delta(\phi)]$. Then $|\psi\rangle$ can be converted to another $|\phi\rangle$ using PIO if and only if $\Delta(\psi)$ and $\Delta(\phi)$ are unitarily equivalent. The power of PIO is improved somewhat on the many-copy level. One can easily show that a state $|\psi\rangle$ can be asymptotically converted via PIO into the maximally coherent qubit state $|+\rangle = \sqrt{1/2}(|0\rangle + |1\rangle)$ at a rate equaling the von Neumann entropy of the state $\Delta(|\psi\rangle\langle\psi|)$, which is optimal [20]. On the other hand, the asymptotic conversion rate of $|+\rangle$ into *any* weakly coherent state $|\psi\rangle$ is strictly zero. The proof of this fact reveals an interesting relationship between quantum coherence and communication complexity in LOCC. Observe that for any PIO transformation $|\psi\rangle \to |\varphi\rangle$, there exists a *zero communication* LOCC protocol that transforms $|\psi^{(mc)}\rangle \to |\varphi^{(mc)}\rangle$, where $|\psi^{(mc)}\rangle$ and $|\varphi^{(mc)}\rangle$ are maximally correlated extensions of $|\psi\rangle$ and $|\varphi\rangle$; i.e. $|\psi^{(mc)}\rangle = \sum_i \sqrt{p_i}|ii\rangle_{AB}$ when $|\psi\rangle = \sum_i \sqrt{p_i}|i\rangle_A$. However the asymptotic transformation $|+^{(mc)}\rangle \to |\varphi^{(mc)}\rangle$ requires nonzero communication whenever $|\varphi^{(mc)}\rangle$ is not a product state or maximally entangled [24, 25]. Hence, rather bizarrely, in PIO theory the maximally coherent state is the weakest as it cannot be transformed into any other state that is not related by an incoherent unitary.

This result demonstrates once again that care is needed when speaking of a "maximal" resource. While the state $|+\rangle$ has maximum value according to all previously proposed coherence measures [12], its operational status as a maximal resource depends crucially on the allowed operations. This is similar to multipartite entanglement theory where the state $|GHZ\rangle = \sqrt{1/2}(|000\rangle + |111\rangle)$ maximizes certain entanglement measures (such as the tangle [26]), yet in certain operational settings it behaves weakest (such as being resistent to entanglement loss [27]).

The weakness of PIO means that the constraint of physical consistency is too strong if one wishes to have a less degenerate resource theory of coherence. This provides motivation to relax the constraint of physical consistency and to consider more general resource theories such as SIO/IO/MIO. We now turn to one such theory that has not been previously discussed, but in some sense it is the most natural one to consider.

Dephasing-Covariant Incoherent Operations. The family of Dephasing-Covariant Incoherent Operations (DIO) consists of all maps that commute with Δ. Recall that in general, for a collection of operations T, a CPTP map $\mathcal{E}$ is said to be T-covariant if $[\mathcal{E}, \tau] = 0$ for all $\tau \in T$. DIO can be seen as a natural extension of PIO in light of the following theorem, whose proof is given in Theorem 27 of [20].

Theorem 2. (a) *Let G be the group of incoherent unitaries. Then, $[\mathcal{U}, \Delta] = 0$ iff $U \in G$, where $\mathcal{U}(\rho) := U\rho U^\dagger$.* (b) *A CPTP map $\mathcal{E}$ is G-covariant iff*

$$\mathcal{E}(\rho) = q_1\rho + \frac{q_2}{d-1}\left(I - \Delta(\rho)\right) + \frac{q_3}{d-1}\left(d\Delta(\rho) - \rho\right) \quad (2)$$

for some $q_i \geq 0$ with $\sum_{i=1}^{3} q_i = 1$. (c) *A CPTP map $\mathcal{E}$ is* PIO*-covariant iff it has the form of Eq. (2) with $q_2 = 0$.*

From part (c) of Theorem 2, the commutant of PIO consists of the family of channels $\Delta_\lambda(\rho) := (1-\lambda)\rho + \lambda\Delta(\rho)$ for $\lambda \in [0, 1]$. The class DIO therefore generalizes PIO in that it is largest operational class sharing the same commutant as PIO.

Operational covariance is an important physical property as it describes an order invariance in performing a two-step process. DIO are of particular interest when observing how the probabilities $p_i = \langle i|\rho|i\rangle$ transform under a map $\mathcal{E}$. If $\mathcal{E}$ is DIO, then an experimenter can put ρ through any channel Δ_λ before applying $\mathcal{E}$ without changing the probabilities p_i. Note that DIO can also be seen as an extension of SIO to general channels.

What is the operational power of DIO? While we leave a thorough investigation of the this question for future work, here we just consider the task of transforming one qubit state ρ into another σ. It turns out that all classes of incoherent operations behave equivalently for this task, and in fact, state convertibability depends on just two incoherent monotones. The first is the *Robustness of Coherence* [28], and is defined as

$$C_R(\rho) = \min_{t \geq 0}\left\{ t \;\Big|\; \frac{\rho + t\sigma}{1+t} \in \mathcal{I},\; \sigma \geq 0 \right\}.$$

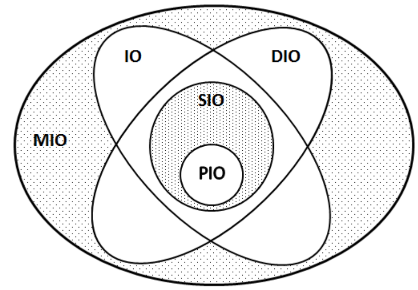

FIG. 2: Heuristic comparison between the 5 classes of incoherent operations MIO/DIO/IO/SIO/PIO.

Here we introduce a new type of robustness measure that we call the Δ-Robustness of Coherence:

$$C_{\Delta,R}(\rho) = \min_{t \geq 0}\left\{ t \;\Big|\; \frac{\rho + t\sigma}{1+t} \in \mathcal{I},\; \sigma \geq 0,\; \Delta(\sigma - \rho) = 0 \right\}.$$

While C_R is a monotone under MIO in general, for qubits $C_{\Delta,R}$ is also a MIO monotone. These two measures completely characterize qubit state transformations, as proven in Theorem 26 of [20].

Theorem 3. *For qubit state ρ and σ, the transformation $\rho \to \sigma$ is possible by either SIO, DIO, IO, or MIO if and only if both $C_R(\rho) \geq C_R(\sigma)$ and $C_{\Delta,R}(\rho) \geq C_{\Delta,R}(\sigma)$.*

Already in qutrit systems, state transformations exist that are possible by MIO but not either IO or DIO [20]. Recently, Bu and Xiong have demonstrated a state transformation this can be performed by DIO but not IO [29]. While it is easy to construct IO maps that are not DIO, it remains an open question whether or not there exists state transformations that can be implemented by IO but not by SIO or even DIO.

In conclusion, we have introduced a criterion of physical consistency for a general quantum resource theory. When applied to quantum coherence, the class PIO emerges as the physically consistent resource theory of coherence. In light of PIO's sharply limited abilities, it is desirable to enlarge the free operations. This desire may even be experimentally motivated if one is not be concerned with physical implementations, but instead just wants to know what can be accomplished with a "black box" that performs SIO/IO/DIO/MIO. Because of this, one may contest that resource theories based on the latter operations are indeed physical resource theories. But such a statement should be accompanied by a precise definition of what it means for a resource theory to be "physical." We have offered one such definition in this letter and hope it stimulates further discussion on the physical meaning of coherence resource theories.

Note Added: In the preparation of this article we became aware of independent work by Marvian and Spekkens [30], where the physical meaning of incoherent operations is analyzed and the class of dephasing-covariant incoherent operations is presented.

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[30] I. Marvian and R. Spekkens (2016), 1602.08049.
[31] It is also common to consider the class of positive partial-transpose preserving operations (PPT) as a relaxation of LOCC. However, the PPT resource theory has a different set of free states than LOCC; namely all PPT entangled state are free in the former while they are not in the latter.
[32] One can also adopt a more general notion of coherence based on asymmetry [30]. In this setting, coherence is not necessarily identified with the off-diagonal elements of a density matrix, and consequently, it is possible that all the states $|0\rangle$, $|1\rangle$, $|\psi\rangle = \sqrt{1/2}(|0\rangle + |1\rangle)$ and $\rho = 1/2(|0\rangle\langle 0| + |1\rangle\langle 1|)$ can be considered incoherent. This is a departure from traditional parlance in which $|\psi\rangle$ is called a *coherent* superposition whereas ρ is an *incoherent* superposition (see [20] for more discussion).