Progress of Theoretical Physics, Vol. 49, No. 2, February 1973

Critical Exponents and Scaling Relations in $1/n$ Expansion

Ryuzo ABE and Shinobu HIKAMI

Department of Pure and Applied Sciences
University of Tokyo, Komaba, Meguro-Ku, Tokyo

(Received September 9, 1972)

The critical exponents ν, η and α are derived in powers of $1/n$ to first order for the n-vector model (or n-component system) with short-range interactions. Scaling relations $\chi(q)\sim\varepsilon^{-\gamma}f(q/\varepsilon^{\nu})$, $d\nu=2-\alpha$ and $\gamma=\nu(2-\eta)$ are confirmed up to the order of $1/n$.

§ 1. Introduction

In a previous paper[1] (to be referred to as A), one of the present authors evaluated the critical exponent γ exactly up to the order of $1/n$. In this paper, generalizing the previous treatment, we will derive expressions for other critical exponents ν, η and α. (We will use the conventional notation[2] for critical exponents.)

For studying the critical exponents ν and η, it is necessary to discuss the wave-number dependent susceptibility $\chi(\boldsymbol{q})$. In § 2 we derive the expression for $\chi(\boldsymbol{q})$ exactly up to $1/n$. On the basis of this expression, in § 3 we evaluate ν up to $1/n$ and discuss the scaling form $\chi(\boldsymbol{q})\sim\varepsilon^{-\gamma}f(\boldsymbol{q}/\varepsilon^{\nu})$. In § 4 we consider $\chi(\boldsymbol{q})$ at the critical point and obtain the expression for η. In this paper, we restrict ourselves to the case above the transition temperature and without external magnetic field. The critical exponent of specific heat α is calculated in the Appendix. Finally, § 5 is devoted to discussion.

§ 2. Wave-number dependent susceptibility $\chi(q)$ up to $1/n$

The wave-number dependent susceptibility $\chi(\boldsymbol{q})$ except for a trivial factor μ^2/kT (μ: magnetic moment) is given by

$$\chi(\boldsymbol{q})=\frac{1}{nN}\langle\sum_{i,j}\boldsymbol{S}_i\cdot\boldsymbol{S}_j e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\rangle\,. \tag{2·1}$$

If we introduce Z_λ defined by

$$Z_\lambda=\int_{-\infty}^{\infty}\exp\left[\frac{1}{2}\sum_{i,j}K_{ij}\boldsymbol{S}_i\cdot\boldsymbol{S}_j+\frac{\lambda}{N}\sum_{i,j}\boldsymbol{S}_i\cdot\boldsymbol{S}_j e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\right]\prod_j\delta[n-\sum_m\sigma_j^2(m)]\prod_{k,m}d\sigma_k(m), \tag{2·2}$$

$\chi(\boldsymbol{q})$ is expressed as

$$\chi(\boldsymbol{q})=n^{-1}(\partial\ln Z_\lambda/\partial\lambda)_{\lambda=0}\,. \tag{2·3}$$

Using an integral representation for δ function in Eq. (2·2), we have

$$Z_\lambda=\frac{1}{(2\pi i)^N}\int_{\alpha-i\infty}^{\alpha+i\infty}\prod_j dt_j\int_{-\infty}^{\infty}\prod_{k,m}d\sigma_k(m)\exp\Big\{\frac{1}{2}\sum_{ijm}K_{ij}\sigma_i(m)\sigma_j(m)$$

$$+\frac{\lambda}{N}\sum_{ijm}\sigma_i(m)\sigma_j(m)e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}+\sum_j t_j[n-\sum_m\sigma_j^2(m)]\Big\} \qquad (2\cdot4)$$

with an appropriate constant α. If we put

$$f(\lambda;t_1,\cdots,t_N)=\int_{-\infty}^{\infty}\exp\Big[\frac{1}{2}\sum_{ij}K_{ij}\sigma_i\sigma_j-\sum_j t_j\sigma_j^2+\frac{\lambda}{N}\sum_{ij}\sigma_i\sigma_je^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\Big]\prod_k d\sigma_k\,, \qquad (2\cdot5)$$

Z_λ is written as

$$Z_\lambda=\frac{1}{(2\pi i)^N}\int_{\alpha-i\infty}^{\alpha+i\infty}\prod_j dt_j\exp\{n[\sum_j t_j+\ln f(\lambda;t_1,\cdots,t_N)]\}. \qquad (2\cdot6)$$

We use the same notation for t_λ as in A, namely t_λ makes the exponent of Eq. (2·6) extremum. We introduce "λ-average" defined by

$$\langle\cdots\rangle_\lambda=\int_{-\infty}^{\infty}\cdots\Big[\frac{1}{2}\sum_{ij}K_{ij}\sigma_i\sigma_j-t_\lambda\sum_j\sigma_j^2+\frac{\lambda}{N}\sum_{ij}\sigma_i\sigma_je^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\Big]\prod_k d\sigma_k/f_0(\lambda,t_\lambda) \qquad (2\cdot7)$$

with

$$f_0(\lambda,t_\lambda)=\int_{-\infty}^{\infty}\exp\Big[\frac{1}{2}\sum_{ij}K_{ij}\sigma_i\sigma_j-t_\lambda\sum_j\sigma_j^2+\frac{\lambda}{N}\sum_{ij}\sigma_i\sigma_je^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}\Big]\prod_k d\sigma_k\,. \qquad (2\cdot8)$$

Introducing the Fourier transforms defined by

$$\sigma_j=N^{-1/2}\sum_{\boldsymbol{q}}\sigma(\boldsymbol{q})e^{i\boldsymbol{q}\cdot\boldsymbol{r}_j}, \qquad (2\cdot9)$$

$$K_{ij}=N^{-1}\sum_{\boldsymbol{q}}K(\boldsymbol{q})e^{-i\boldsymbol{q}\cdot(\boldsymbol{r}_i-\boldsymbol{r}_j)}, \qquad (2\cdot10)$$

we have

$$\langle\cdots\rangle_\lambda=\int\cdots\exp\Big\{-\Big[t_\lambda-\frac{1}{2}K(0)\Big]x_0^2-\sum_{\boldsymbol{k}\neq0,\boldsymbol{q}}{}'\Big[t_\lambda-\frac{1}{2}K(\boldsymbol{k})\Big](x_k^2+y_k^2)$$

$$-\Big[t_\lambda-\frac{1}{2}K(\boldsymbol{q})-\frac{\lambda}{2}\Big](x_q^2+y_q^2)\Big\}dx_0\prod_{k\neq0}{}'dx_kdy_k\Big/f_0(\lambda,t_\lambda), \qquad (2\cdot11)$$

where we have used the following change of variables as in A:

$$\left.\begin{array}{lll}\sigma(\boldsymbol{k})=(x_k+iy_k)/\sqrt{2}\,, & (\boldsymbol{k}\neq0) & \sigma(0)=x_0\,,\\ x_{-k}=x_k\,, & y_{-k}=-y_k\,, & \prod_k d\sigma_k=dx_0\prod_{k\neq0}{}'dx_kdy_k\,.\end{array}\right\} \qquad (2\cdot12)$$

The $\ln f_0(\lambda,t_\lambda)$ is calculated to be

$$\ln f_0(\lambda, t_\lambda) = \frac{N}{2}\ln\pi - \frac{1}{2}\ln\Big[t_\lambda - \frac{1}{2}K(0)\Big] - \frac{1}{2}\sum_{k\neq 0,q}{}' \ln\Big[t_\lambda - \frac{1}{2}K(\boldsymbol{k})\Big]$$
$$-\frac{1}{2}\ln\Big[t_\lambda - \frac{1}{2}K(\boldsymbol{q}) - \frac{\lambda}{2}\Big]. \qquad (2\cdot 13)$$

Choosing a path of integration with $\alpha = t_\lambda$ in Eq. (2·6), making a change of variables $t_j = t_\lambda + ix_j$ $(j=1, 2, \cdots, N)$, and putting

$$f(\lambda; t_1, \cdots, t_N)/f_0(\lambda, t_\lambda) = \langle \exp(-i\sum_j x_j\sigma_j^2)\rangle_\lambda \equiv G, \qquad (2\cdot 14)$$

we have

$$Z_\lambda = \exp\{n[Nt_\lambda + \ln f_0(\lambda, t_\lambda)]\}\frac{1}{(2\pi)^N}\int_{-\infty}^{\infty}\prod_j dx_j \exp\{n[i\sum x_j + \ln G]\}. \qquad (2\cdot 15)$$

By means of linked cluster expansion for $\ln G$, the Z_λ is expressed with $y_j = n^{1/2}x_j$ as

$$Z_\lambda = \exp\{n[Nt_\lambda + \ln f_0(\lambda, t_\lambda)]\}Z_\lambda'/(2\pi)^N n^{N/2},$$
$$Z_\lambda' = \int_{-\infty}^{\infty}\prod dy_j \exp\Big[\sum_{m=2}^{\infty}\frac{(-i)^m 2^{m-1}}{mn^{(m-2)/2}}\sum_{j_1,\cdots,j_m} y_{j_1}\cdots y_{j_m} g_\lambda(j_1, j_2) g_\lambda(j_2, j_3)\cdots g_\lambda(j_m, j_1)\Big]. \qquad (2\cdot 16)$$

Here $g_\lambda(j, l)$ is defined by

$$g_\lambda(j, l) = N^{-1}\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_l)} g_\lambda(\boldsymbol{k}), \qquad (2\cdot 17)$$

$$g_\lambda(\boldsymbol{k}) \equiv \langle\sigma(\boldsymbol{k})\sigma(-\boldsymbol{k})\rangle_\lambda$$
$$= \begin{cases} 1/[2t_\lambda - K(\boldsymbol{k})], & (\boldsymbol{k} \neq \boldsymbol{q}, -\boldsymbol{q}) \\ 1/[2t_\lambda - K(\boldsymbol{q}) - \lambda]. & (\boldsymbol{k} = \boldsymbol{q}, -\boldsymbol{q}) \end{cases} \qquad (2\cdot 18)$$

We consider $1/n$ expansion up to the order of $1/n$. Z_λ' under this approximation is denoted by $Z_{\lambda 0}'$:

$$Z_{\lambda 0}' = \int_{-\infty}^{\infty}\prod dy_j \exp[-\sum_{j,l} y_j y_l g_\lambda^2(j, l)]. \qquad (2\cdot 19)$$

If we introduce the Fourier transform

$$g_\lambda^2(j, l) = N^{-1}\sum_{\boldsymbol{k}} \nu_\lambda(\boldsymbol{k}) e^{i\boldsymbol{k}\cdot(\boldsymbol{r}_j - \boldsymbol{r}_l)}, \qquad (2\cdot 20)$$

$Z_{\lambda 0}'$ is calculated to be

$$Z_{\lambda 0}' = \pi^{N/2}[\nu_\lambda(0)]^{-1/2}\prod_{\boldsymbol{k}\neq 0}{}'[\nu_\lambda(\boldsymbol{k})]^{-1}. \qquad (2\cdot 21)$$

Taking logarithm of Eq. (2·16) and differentiating by λ, we have from Eq. (2·21)

$$\frac{\partial \ln Z_\lambda}{\partial\lambda} = n\Big[N\frac{\partial t_\lambda}{\partial\lambda} + \frac{\partial \ln f_0(\lambda, t_\lambda)}{\partial\lambda}\Big] - \frac{1}{2}\sum_{\boldsymbol{q}}\frac{\partial \ln \nu_\lambda(\boldsymbol{q})}{\partial\lambda} + O\Big(\frac{1}{n^2}\Big). \qquad (2\cdot 22)$$

Substituting Eq. (2·22) into Eq. (2·3), using extremum condition

$$1-N^{-1}\sum_{k} g_\lambda(\boldsymbol{k})=0 \tag{2·23}$$

and putting $\lambda=0$, we obtain the following expression for $\chi(\boldsymbol{q})$:

$$\chi(\boldsymbol{q})=\chi_0(\boldsymbol{q})-\frac{1}{2n}\sum_{k}\left[\frac{\partial \ln \nu_\lambda(\boldsymbol{k})}{\partial\lambda}\right]_{\lambda=0}+O(1/n^2), \tag{2·24}$$

where $\chi_0(\boldsymbol{q})$ is the wave-number dependent susceptibility of the spherical model:

$$\chi_0(\boldsymbol{q})=\frac{1}{2t-K(\boldsymbol{q})}. \tag{2·25}$$

The $\nu_\lambda(\boldsymbol{k})$ is written as

$$\nu_\lambda(\boldsymbol{k})=N^{-1}\sum_{k'} g_\lambda(\boldsymbol{k}')g_\lambda(\boldsymbol{k}-\boldsymbol{k}'). \tag{2·26}$$

Differentiating Eq. (2·26) by λ and using Eq. (2·18), we have

$$\sum_{k}\left[\frac{\partial \ln \nu_\lambda(\boldsymbol{k})}{\partial\lambda}\right]_{\lambda=0}=4N^{-1}\sum_{k}\left[\left\{g^2(\boldsymbol{q})g(\boldsymbol{k}-\boldsymbol{q})-\left(\frac{\partial t_\lambda}{\partial\lambda}\right)_{\lambda=0}\sum_{k'} g^2(\boldsymbol{k}')g(\boldsymbol{k}-\boldsymbol{k}')\right\}\Big/\nu_\lambda(\boldsymbol{k})\right]_{\lambda=0}. \tag{2·27}$$

Here, we denote t_λ with $\lambda=0$ by t. $g(\boldsymbol{k})$ and $\nu(\boldsymbol{k})$ are given by

$$g(\boldsymbol{k})=1/[2t-K(\boldsymbol{k})], \qquad \nu(\boldsymbol{k})=N^{-1}\sum_{k'} g(\boldsymbol{k}')g(\boldsymbol{k}-\boldsymbol{k}'). \tag{2·28}$$

The quantity $\partial t_\lambda/\partial\lambda$ is obtained by differentiating Eq. (2·23) with respect to λ,

$$(\partial t_\lambda/\partial\lambda)_{\lambda=0}=g^2(\boldsymbol{q})/\sum_{k} g^2(\boldsymbol{k}). \tag{2·29}$$

Therefore $\chi(\boldsymbol{q})$ is expressed as

$$\chi(\boldsymbol{q})=\chi_0(\boldsymbol{q})-\frac{2A}{nN^2}\sum_{k,k'}\frac{g^2(\boldsymbol{k}')[g(\boldsymbol{k}-\boldsymbol{q})-g(\boldsymbol{k}-\boldsymbol{k}')]}{\nu(\boldsymbol{k})}+O(1/n^2), \tag{2·30}$$

where A is defined by

$$A=g^2(\boldsymbol{q})/N^{-1}\sum_{k'} g^2(\boldsymbol{k}'). \tag{2·31}$$

In this way, the expression for $\chi(\boldsymbol{q})$ up to $1/n$ is derived.

§ 3. Critical exponent ν

We now proceed to derive the expression for the critical exponent ν. We consider the d-dimensional simple hypercubical system with nearest-neighbor interactions. For this system $K(\boldsymbol{q})$ is given by

$$K(\boldsymbol{q})=2K(\cos q_1+\cos q_2+\cdots+\cos q_d). \tag{3·1}$$

Introducing a parameter s defined by

$$s=2(z-d), \qquad t=Kz, \tag{3·2}$$

the $g(\boldsymbol{k})$ is approximated as

$$g(\boldsymbol{k})\approx 1/K(s+k^2). \tag{3·3}$$

As shown in A, we have

$$s=\{2^d\pi^{d/2}[(d/2)-1]/\Gamma[2-d/2]\}^{2/(d-2)}\varepsilon_0^{2/(d-2)}, \tag{3·4}$$

where

$$\varepsilon_0=K_{c0}-K \tag{3·5}$$

and K_{c0} is the critical point of the spherical model.

In the spherical model, $\chi_0(\boldsymbol{q})$ is given by

$$\chi_0(\boldsymbol{q})\approx 1/K(s+q^2). \tag{3·6}$$

Therefore $\chi_0(\boldsymbol{q})$ has the following scaling form:

$$\chi_0(\boldsymbol{q})\sim\varepsilon_0^{-\gamma_0}f(q/\varepsilon_0^{\nu_0}) \tag{3·7}$$

with

$$\gamma_0=2\nu_0=2/(d-2). \tag{3·8}$$

It is generally considered that the $\chi(\boldsymbol{q})$ has the scaling form[3)~7)]

$$\chi(\boldsymbol{q})\sim\varepsilon^{-\gamma}f(q/\varepsilon^{\nu}) \tag{3·9}$$

with ε given by K_c-K (K_c: exact critical point).

If we assume this scaling form for $\chi(\boldsymbol{q})$ up to the order of $1/n$, we can expand $\chi(\boldsymbol{q})$ as

$$\chi(\boldsymbol{q})=C_0\varepsilon^{-\gamma_0-\gamma_1/n}+C_2\varepsilon^{-\gamma_0-\gamma_1/n-2\nu_0-2\nu_1/n}q^2+C_4\varepsilon^{-\gamma_0-\gamma_1/n-4\nu_0-4\nu_1/n}q^4+\cdots. \tag{3·10}$$

In actual, the amplitude C_0, C_2, $\cdots$ and ε can depend on $1/n$. However, since this dependence is not important in discussing the critical behavior of $\chi(\boldsymbol{q})$, we will neglect them. Omitting these terms, we can write Eq. (3·10) as

$$\chi(\boldsymbol{q})=\chi_0(\boldsymbol{q})+\frac{1}{n}\{-\gamma_1\chi_0(0)\ln\varepsilon+K(\gamma_1+2\nu_1)\chi_0^2(0)\ln\varepsilon\cdot q^2 -K^2(\gamma_1+4\nu_1)\chi_0^3(0)\ln\varepsilon\cdot q^4+\cdots\}+O\left(\frac{1}{n^2}\right). \tag{3·11}$$

Therefore calculating the coefficients of the term of the order $[\chi_0(0)]^{n+1}q^{2n}\ln\varepsilon$ ($n=1, 2, \cdots$), we can obtain the deviation of ν from the spherical model value. If these coefficients give the same ν_1, the scaling form of $\chi(\boldsymbol{q})$, Eq. (3·9), is confirmed up to $1/n$.

We firstly consider the coefficient of q^2 in Eq. (2·30). It is convenient to divide the second term on the right-hand side of Eq. (2·30) into two parts:

$$-\frac{2A}{nN^2}\sum_{k,k'}\frac{g^2(\boldsymbol{k'})\,[g(\boldsymbol{k}-\boldsymbol{q})-g(\boldsymbol{k}-\boldsymbol{k'})]}{\nu(\boldsymbol{k})}$$

$$=-\frac{2}{nN^2}\frac{g^2(\boldsymbol{q})}{g^2(0)}\cdot\frac{g^2(0)}{\nu(0)}\sum_{k,k'}\frac{g^2(\boldsymbol{k'})\,\{g(\boldsymbol{k})-g(\boldsymbol{k}-\boldsymbol{k'})\}}{\nu(\boldsymbol{k})}$$

$$-\frac{2g^2(\boldsymbol{q})}{nN}\sum_{k}\frac{\{g(\boldsymbol{k}-\boldsymbol{q})-g(\boldsymbol{k})\}}{\nu(\boldsymbol{k})}. \qquad (3\cdot12)$$

The first term of Eq. (3·12) except for the factor $g^2(\boldsymbol{q})/g^2(0)$ is the $1/n$-order term of the susceptibility $\chi(0)$. (See Eq. (5·19) of A.) Therefore the first term of Eq. (3·12) has the logarithmic term

$$-(g^2(\boldsymbol{q})/ng^2(0))\,\gamma_1\chi_0(0)\ln\varepsilon\,. \qquad (3\cdot13)$$

Expanding $g^2(\boldsymbol{q})/g^2(0)$, we have the coefficient of q^2 in Eq. (3·13)

$$(2\gamma_1K\chi_0{}^2(0)\ln\varepsilon)/n\,, \qquad (3\cdot14)$$

where $\chi_0(0)=1/Ks$, and

$$\gamma_1=-\frac{6\sin(d\pi/2)\Gamma(d-1)}{(2-d)\pi[\Gamma(d/2)]^2}\,. \qquad (3\cdot15)$$

For the calculation of the second term of Eq. (3·12), we expand

$$g(\boldsymbol{k}-\boldsymbol{q})-g(\boldsymbol{k})=K^{-1}\left\{\frac{2q\cos\theta}{k^3}+\frac{(4\cos^2\theta-1)}{k^4}q^2+O\left(\frac{1}{k^5}\right)\right\}. \qquad (3\cdot16)$$

In A, $\nu(\boldsymbol{k})$ was calculated to be

$$\nu(\boldsymbol{k})=K^{-2}2^{-d}\pi^{-d/2}\Gamma(2-d/2)F(2-d/2,1/2,3/2,k^2/(4s+k^2))/(s+k^2/4)^{2-d/2}. \qquad (3\cdot17)$$

From Eq. (3·17) we have

$$\frac{1}{\nu(\boldsymbol{k})}=\frac{K^2 2^d\pi^{d/2}\Gamma[(d-1)/2]2^{d-4}}{\Gamma(2-d/2)\Gamma(3/2)\Gamma(d/2-1)k^{d-4}}+\cdots. \qquad (3\cdot18)$$

The summation over $\boldsymbol{k}$ in Eq. (3·12) is changed to the integral by means of the formula[8)]

$$\frac{1}{N}\sum_{k}\rightarrow\frac{1}{(2\pi)^d}\int d^dk=(2\pi)^{-1}K_{d-1}\int_0^\infty dk\int_0^\pi d\theta k^{d-1}(\sin\theta)^{d-2},$$
$$K_d=2^{-(d-1)}\pi^{-d/2}[\Gamma(d/2)]^{-1}. \qquad (3\cdot19)$$

Since our expansions, Eqs. (3·16) and (3·17), are valid for $s\ll k^2$, we set the lower limit of the integration of k to be $s^{1/2}$.

The logarithmic term is derived only from the second term of Eq. (3·16). The first term of Eq. (3·16) vanishes after the integration over θ. Thus, we calculate the following term:

$$-\frac{2g^2(\boldsymbol{q})}{nN}\sum_{k}\frac{K2^d\pi^{d/2}\Gamma[(d-1)/2]}{\Gamma[2-(d/2)]\Gamma(3/2)\Gamma[(d/2)-1]}\cdot\frac{2^{d-4}(4\cos^2\theta-1)q^2}{k^d}. \qquad (3\cdot20)$$

From Eqs. (3·19) and (3·4), Eq. (3·20) is calculated to be

$$-\frac{2Kg^2(\boldsymbol{q})}{n}\left(\frac{4}{d}-1\right)\frac{\Gamma(d-1)\sin(d\pi/2)\ln\varepsilon\cdot q^2}{[\Gamma(d/2)]^2\pi(d-2)}. \tag{3·21}$$

With the use of the expression of γ_1 (Eq. (3·15)), Eq. (3·21) is written as

$$-\frac{Kg^2(\boldsymbol{q})}{3n}\left(\frac{4}{d}-1\right)\gamma_1 q^2\ln\varepsilon. \tag{3·22}$$

Expanding $g^2(\boldsymbol{q})$ as

$$g^2(\boldsymbol{q})=\frac{1}{K^2(s+q^2)^2}=\frac{1}{K^2}\left(\frac{1}{s^2}-\frac{2q^2}{s^3}+\frac{3q^3}{s^4}-\frac{4q^6}{s^5}+\cdots\right), \tag{3·23}$$

we obtain the desired term from Eq. (3·22)

$$-\frac{K}{3n}\left(\frac{4}{d}-1\right)\chi_0^2(0)\gamma_1\ln\varepsilon\cdot q^2. \tag{3·24}$$

From Eqs. (3·14) and (3·24), we have the q^2 term for Eq. (3·12):

$$\frac{1}{n}\left[2-\frac{1}{3}\left(\frac{4}{d}-1\right)\right]K\gamma_1\chi_0^2(0)\ln\varepsilon\cdot q^2. \tag{3·25}$$

This term should be equal to the following term of Eq. (3·11)

$$n^{-1}(\gamma_1+2\nu_1)K\chi_0^2(0)\ln\varepsilon\cdot q^2. \tag{3·26}$$

Therefore we obtain the expression for critical exponent ν_1

$$\nu_1=\frac{2}{3}\left(1-\frac{1}{d}\right)\gamma_1. \tag{3·27}$$

Since the logarithmically divergent term is derived only from the second term of Eq. (3·16) multiplied by the first term of Eq. (3·18), it is sufficient to consider Eq. (3·22) as the contribution of the second term of Eq. (3·12) for our purpose. Therefore, by the use of Eq. (3·13), Eq. (3·12) is reduced to

$$-\frac{1}{n}\frac{g^2(\boldsymbol{q})}{g^2(0)}\gamma_1\chi_0(0)\ln\varepsilon-\frac{Kg^2(\boldsymbol{q})}{3n}\left(\frac{4}{d}-1\right)\gamma_1 q^2\ln\varepsilon. \tag{3·28}$$

Using Eq. (3·23), we find that the above equation is expanded as

$$-\frac{1}{n}\gamma_1\chi_0(0)\ln\varepsilon+\frac{1}{n}\sum_{m=1}^{\infty}(-1)^{m+1}\left[(m+1)\gamma_1-\frac{\gamma_1}{3}\left(\frac{4}{d}-1\right)m\right]K^m\chi_0^{m+1}(0)\ln\varepsilon\cdot q^{2m}. \tag{3·29}$$

From Eq. (3·11) the coefficient of q^{2m} term should be equal to

$$n^{-1}(-1)^{m+1}(\gamma_1+2m\nu_1)K^m\chi_0^{m+1}(0)\ln\varepsilon. \tag{3·30}$$

Therefore we have

$$\gamma_1+2m\nu_1=(m+1)\gamma_1-\frac{\gamma_1}{3}\left(\frac{4}{d}-1\right)m\,,\qquad m=1,2,\cdots. \tag{3·31}$$

From Eq. (3·31) we can derive the same ν_1 for any m as

$$\nu_1=\frac{2}{3}\gamma_1\left(1-\frac{1}{d}\right). \tag{3·32}$$

Thus the scaling form for $\chi(\boldsymbol{q})$ of Eq. (3·9) is confirmed up to $1/n$.

§ 4. Critical exponent η

At the exact critical point, $\chi(\boldsymbol{q})$ of Eq. (2·30) is considered to have the form

$$\chi(\boldsymbol{q})\sim C/q^{2-\eta}. \tag{4·1}$$

Since s is related to ε_0 as in Eq. (3·4), s is not zero similarly to ε_0 at the exact critical point. We denote ε which vanishes at the critical point by

$$\varepsilon=K_c-K\,. \tag{4·2}$$

We expand ε as

$$\varepsilon=K_{c0}+\frac{K_c'}{n}-K+O\left(\frac{1}{n^2}\right), \tag{4·3}$$

where K_{c0} denotes the critical value of the spherical model. From Eqs. (4·3) and (3·4), we can also expand s in powers of $1/n$,

$$s=s_{\rm tr}+\frac{s_1}{n}+O\left(\frac{1}{n^2}\right). \tag{4·4}$$

The quantity $s_{\rm tr}$ becomes zero at the exact critical point.

Putting $s=s_1/n$, we consider $\chi(\boldsymbol{q})$ of Eq. (2·30) up to $1/n$ order:

$$\chi(\boldsymbol{q})=[\chi_0(\boldsymbol{q})]_{s=s_1/n}-\left[\frac{2A}{nN^2}\sum_{\boldsymbol{k},\boldsymbol{k}'}\frac{g^2(\boldsymbol{k}')\{g(\boldsymbol{k}-\boldsymbol{q})-g(\boldsymbol{k}-\boldsymbol{k}')\}}{\nu(\boldsymbol{k})}\right]_{s=s_1/n}+O\left(\frac{1}{n^2}\right). \tag{4·5}$$

The critical exponent η is expanded in powers of $1/n$

$$\eta=\eta_0+\eta_1/n+O(1/n^2). \tag{4·6}$$

Using this expansion, we have from Eq. (4·1)

$$\begin{aligned}\chi(\boldsymbol{q})&=Cq^{-2}+C\eta_1 n^{-1}q^{-2}\ln q+O(1/n^2),\\ C&=K^{-1}.\end{aligned} \tag{4·7}$$

For obtaining η_1, we need to get $\ln q$ term from Eq. (4·5). The first term of Eq. (4·5) includes no $\ln q$ term, and the second term of Eq. (4·5) is of the order $1/n$. Therefore we can neglect the difference of s and take $s=0$ up to $1/n$ order.

From Eq. (4·5) we can easily see the term including $\ln q$ to be

$$-\frac{2g^2(\boldsymbol{q})}{nN}\sum_{\boldsymbol{k}}\frac{g(\boldsymbol{k}-\boldsymbol{q})}{\nu(\boldsymbol{k})}. \tag{4·8}$$

Putting $s=0$, $g(\boldsymbol{k}-\boldsymbol{q})$ is expanded as

$$[g(\boldsymbol{k}-\boldsymbol{q})]_{s=0}=\frac{1}{K}\left\{\frac{1}{k^2}+\frac{2\cos\theta}{k^3}q+\frac{(4\cos^2\theta-1)}{k^4}\cdot q^2+O\left(\frac{1}{k^5}\right)\right\}. \tag{4·9}$$

From Eq. (3·17), we have

$$\left[\frac{1}{\nu(\boldsymbol{k})}\right]_{s=0}=\frac{K^2 2^d\pi^{d/2}\Gamma[(d-1)/2]}{\Gamma(3/2)\Gamma(d/2-1)\Gamma(2-d/2)}\cdot\frac{4^{d/2-2}}{k^{d-4}}. \tag{4·10}$$

We can obtain the desired term from Eq. (4·8),

$$-\frac{\ln q}{nKq^2}\left\{\frac{2\sin(d\pi/2)\Gamma(d-1)}{\pi[\Gamma(d/2)]^2}\left(\frac{4}{d}-1\right)\right\}. \tag{4·11}$$

Using the expression of γ_1, Eq. (3·15), and comparing with Eq. (4·7), we have

$$\eta_1=\frac{2-d}{3}\left(\frac{4}{d}-1\right)\gamma_1. \tag{4·12}$$

From Eqs. (3·32) and (4·12), we can verify the following scaling relation up to $1/n$:

$$\begin{aligned}\nu(2-\eta)&=(\nu_0+n^{-1}\nu_1)(2-n^{-1}\eta_1)+O(1/n^2)\\&=\gamma_0+n^{-1}\gamma_1+O(1/n^2).\end{aligned} \tag{4·13}$$

§ 5. Discussion

The critical exponent of specific heat α is evaluated in the Appendix. The expression for α is given by

$$\alpha=\frac{d-4}{d-2}-\frac{2}{3n}(d-1)\gamma_1+O\left(\frac{1}{n^2}\right). \tag{5·1}$$

Therefore the following scaling relation

$$\alpha=2-d\nu \tag{5·2}$$

can be also verified up to the order of $1/n$.

The specific heat of the spherical model does not diverge at the critical point. Namely, the first term of Eq. (5·1) has the negative value for $2<d<4$. Putting $n=3$, this corresponds to classical Heisenberg model, α yields the positive value for three dimension:

$$\alpha=0.08076+O(1/n^2). \tag{5·3}$$

For other critical exponents, putting $d=3$ and $n=3$, we have $\gamma=1.18943$, $\nu=0.63975$, $\eta=0.09006$. The numerical estimates from high-temperature expansions for classical Heisenberg model yield: $\gamma=1.375$, $\nu=0.703$, $\eta=0.043$.[7] Our lowest-order terms in $1/n$ expansion thus lead to fair agreements with the numerical values. We hope that the agreement would be improved if the higher-order terms are taken into account; this is a future problem.

Wilson and Fisher,[9] firstly, obtained critical exponents in powers of $\varepsilon(=4-d)$, by the renormalization group method which was introduced by Wilson.[10] Afterwards, Wilson developed expansion using Feynman graph calculation.[8] He obtained the critical exponents γ, η and φ up to the order of ε^2.

Recently Ma[11] calculated the exponents of the Bose system by Wilson's expansion method. He obtained the critical exponents γ and η, assuming the scaling law. His results coincide with ours for γ and η.

In this paper, we have restricted ourselves to the case above transition temperature and without external magnetic field. We hope that we are able to extend the present treatment to the case with external magnetic field or below transition temperature.

Appendix

——*Calculation of critical exponent* α——

Putting $\lambda=0$ in Eq. (2·16), $(nN)^{-1}\ln Z$ is given up to $1/n$ by

$$\ln Z/nN=t+N^{-1}\ln f_0+(2n)^{-1}\ln\pi-(2nN)^{-1}\sum_q \ln\nu(\boldsymbol{q})-n^{-1}\ln(2\pi n^{1/2}). \tag{A·1}$$

Here we define the energy E by

$$E=kT^2\frac{\partial}{\partial T}\left(\frac{1}{nN}\ln Z\right). \tag{A·2}$$

For the spherical model $(n\to\infty)$, we have

$$\ln Z/nN=t+N^{-1}\ln f_0 \tag{A·3}$$

and

$$\begin{aligned} E&=kT^2\frac{\partial}{\partial T}\left\{t-\frac{1}{2N}\sum_q \ln\left[t-\frac{K(\boldsymbol{q})}{2}\right]\right\} \\ &=-J\{z-1/2K\}. \end{aligned} \tag{A·4}$$

We note that E can be expanded as

$$E=A_1+A_2\varepsilon^{1-\alpha}+\cdots. \tag{A·5}$$

Omitting the shifts of critical temperature and amplitude, Eq. (A·5) is given in the expansion of $1/n$ by

$$E=A_1+A_2\varepsilon^{1-\alpha_0}-A_2\varepsilon^{1-\alpha_0}\left(\frac{\alpha_1}{n}\right)\ln\varepsilon+O\left(\frac{1}{n^2}\right). \tag{A·6}$$

The third term of Eq. (A·6) can be derived from

$$-\frac{1}{2nN}\sum_{q}\ln\nu(\boldsymbol{q}). \tag{A·7}$$

From Eq. (A·2), we have

$$-\frac{kT^2}{2N}\frac{\partial}{\partial T}\sum_{q}\ln\nu(\boldsymbol{q})=-\frac{kT^2}{2N}\sum_{q}\frac{1}{\nu(\boldsymbol{q})}\left\{\left(\frac{\partial s}{\partial T}\right)\left(-\frac{2K}{N}\sum_{k}g^2(\boldsymbol{k})g(\boldsymbol{q}-\boldsymbol{k})\right)\right.$$
$$\left.-\frac{2}{K}\left(\frac{\partial K}{\partial T}\right)\nu(\boldsymbol{q})\right\}. \tag{A·8}$$

The logarithmic term is given from the first term of Eq. (A·8),

$$JTN^{-1}\sum_{q}\frac{\eta(\boldsymbol{q})}{\nu(\boldsymbol{q})}\left(\frac{\partial s}{\partial T}\right),$$
$$\eta(\boldsymbol{q})=N^{-1}\sum_{k}g^2(\boldsymbol{k})g(\boldsymbol{q}-\boldsymbol{k}). \tag{A·9}$$

This term can be calculated to be

$$JT\left(\frac{\partial s}{\partial T}\right)\left(\frac{\pi^{-(d+1)/2}}{2\Gamma[(d-1)/2]}\right)\left(-\frac{1}{2}\ln s\right)s^{d/2-1}\frac{(1-d)\Gamma[(d-1)/2]}{2\Gamma(3/2)\Gamma(d/2-1)}. \tag{A·10}$$

Noting that

$$\frac{\partial s}{\partial T}=s\varepsilon^{-1}T^{-1}2/(d-2), \tag{A·11}$$

we can write Eq. (A·10), using the expression of γ_1, as

$$\left(-\frac{Js}{2}\right)\left\{-\frac{2}{3}(1-d)\gamma_1\ln\varepsilon\right\}. \tag{A·12}$$

From Eq. (A·4) we have

$$A_2\varepsilon^{1-\alpha_0}=-\frac{J}{2}s. \tag{A·13}$$

Therefore, comparing Eq. (A·12) with Eq. (A·6), we have the expression for critical exponent α_1:

$$\alpha_1=-\tfrac{2}{3}(d-1)\gamma_1. \tag{A·14}$$

References

1) R. Abe, Prog. Theor. Phys. **49** (1973), 113.
2) M. E. Fisher, Repts. Prog. Phys. **30** (1967), 615.
3) C. Domb and D. H. Hunter, Proc. Phys. Soc. **86** (1965), 1147.
4) R. Abe, Prog. Theor. Phys. **38** (1967), 568.
5) M. Suzuki, J. Phys. Soc. Japan **22** (1967), 757.
6) M. E. Fisher and R. J. Burford, Phys. Rev. **156** (1967), 583.
7) D. S. Ritchie and M. E. Fisher, Phys. Rev. **B5** (1972), 2668.
8) K. G. Wilson, Phys. Rev. Letters **28** (1972), 548.
9) K. G. Wilson and M. E. Fisher, Phys. Rev. Letters **28** (1972), 240.
10) K. G. Wilson, Phys. Rev. **B4** (1971), 3174, 3184.
11) S. K. Ma, Phys. Rev. Letters **29** (1972), 1311 and to be published.