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Critical Exponents for Long-Range Interactions

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Abstract. Long-range components of the interaction in statistical mechanical systems may affect the critical behavior, raising the system's 'effective dimension'. Presented here are explicit implications to this effect of a collection of rigorous results on the critical exponents in ferromagnetic models with one-component Ising (and more generally Griffiths–Simon class) spin variables. In particular, it is established that even in dimensions $d < 4$ if a ferromagnetic Ising spin model has a reflection-positive pair interaction with a sufficiently slow decay, e.g. as $J_x = 1/|x|^{d+\sigma}$ with $0 < \sigma \leqslant d/2$, then the exponents $\hat{\beta}$, δ, γ and Δ_4 exist and take their mean-field values. This proves rigorously an early renormalization-group prediction of Fisher, Ma and Nickel. In the converse direction: when the decay is by a similar power law with $\sigma \geqslant 2$, then the long-range part of the interaction has no effect on the existent critical exponent bounds, which coincide then with those obtained for short-range models.

1. Introduction

The presence of long-range interactions may affect the critical behavior in models of statistical mechanics, raising their effective dimensionality. In this Letter, we present the specific implications of a collection of general theorems (including some recent results) which establish this phenomenon for a class of ferromagnetic models with one-component spin variables and summable interactions decaying in d dimensions as

$$J_x \approx \frac{1}{|x|^{d+\sigma}} \quad (\sigma > 0). \tag{1.1}$$

These results establish that in a class of such models, the critical exponents $\hat{\beta}$, γ, δ, and Δ_4 (defined by (2.14) below), take their mean-field values:

$$\hat{\beta} = \tfrac{1}{2}, \qquad \gamma = 1, \qquad \delta = 2, \qquad \Delta_4 = \tfrac{3}{2} \tag{1.2}$$

as soon as

$$d_{\text{bnd.}} \equiv d/\min\{1, \sigma/2\} \geqslant 4, \tag{1.3}$$

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i.e. if $\sigma \leqslant d/2$ or $d \geqslant 4$. In particular, this shows that critical exponents can be rigorously derived, even for some three dimensional models.

The results reviewed here for the above critical exponents typically come in one of two forms: either as explicit bounds in terms of d and σ, or as less explicit but sharper relations involving another critical exponent (η). All the explicit bounds depend on the dimension d and the power σ only through $d_{\text{bnd.}}$. Hence, they are unaffected by the long-range part of the interaction if $\sigma > 2$.

Both effects were predicted (on the basis of renormalization-group arguments) by Fisher–Ma–Nickel [1] who got (1.3) as the condition for 'upper-criticality', and $\sigma = 2$ as the threshold for the recovery of the short-range exponents. We note that mean-field values of critical exponents have been detected experimentally (at $d = 3$) in some highly ionic systems [2].

The major part of the analysis, whose results are reported here, was the derivation of relations in the form of partial differential inequalities (PDI), which have been obtained for the order parameter (through a collection of works) – typically by means of some geometric representation. The principal PDI are stated here in Section 4. Their analysis leads to complementary sets of critical exponent bounds: some are dimension independent, and naturally involve mean-field values, and other provide converse inequalities which depend on an 'effective dimension' $d_{\text{eff.}}$. A basic role in the extraction of an explicit dependence on d and σ is played by the 'infrared bound', of Fröhlich–Simon–Spencer [3], which, at present, has only been proven for reflection positive (RP) interactions. Let us explain here that role, and the term 'effective dimension', as used in this Letter.

For a system with a translation-invariant Hamiltonian, such as (2.1) below, it is natural to express the spins by means of Fourier transform ('spin wave') variables $\hat{\sigma}(p)$. With a suitable finite-volume normalization, the thermal average $\langle |\hat{\sigma}(p)|^2 \rangle_\beta$ coincides with the Fourier transform $G_\beta(p)$ of the correlation function $\langle \sigma_0 \sigma_x \rangle_\beta$. It turns out that for various results a key issue is the density of these 'spin wave modes' – which is $\mathrm{d}^d p$ – as a function of the 'excitation level' as measured by G. In the benchmark case of a d-dimensional finite-range Gaussian model (where $\hat{\sigma}(p)$ are independent random variables and the equipartition law holds exactly), $G(p) \approx C/p^2$ (for $p \neq 0$) and, hence, the density of levels is $\mathrm{d}G^{-d/2}$. For systems with nontrivial spin variables, the behavior of $G_\beta(p)$ is an intricate dynamical issue. Assuming that $G_{\beta_c}(p) \approx C/p^{(2-\eta)}$ with some critical exponent η, the density of levels can still be written as $\mathrm{d}G^{-d_{\text{eff}}/2}$, with an 'effective dimension' $d_{\text{eff.}} = d/(1 - \eta/2)$. As mentioned above, the general results reported here are either dimension independent or depend on it only through $d_{\text{eff.}}$. In particular, (1.2) is shown to hold whenever $d_{\text{eff.}} \geqslant 4$. For reflection-positive interactions, the 'infrared bound' of FSS [3] yields the explicit estimate

$$d_{\text{eff}} \geqslant d_{\text{bnd.}}\,. \tag{1.4}$$

In Section 3, we mention some known and some new examples of long-range reflection-positive interactions which exhibit power law with any values of d and $\sigma > 0$. These include $J_x = 1/\|x\|^{d+\sigma}$ with $\|x\|$ interpreted as either the Euclidean norm (if $d \leqslant 4$, otherwise there are some gaps in the proven values for σ), or as $\|x\| = |x_1| + \cdots + |x_d|$ (in which case RP is proven for all $\sigma \geqslant 0$, $d \geqslant 1$).

2. Critical Exponent Bounds

We consider here ferromagnetic models with pair interactions, having a Hamiltonian of the form

$$H = -\frac{1}{2} \sum_{x \neq y} J_{y-x} \sigma_x \sigma_y - \frac{h}{\beta} \sum_x \sigma_x \tag{2.1}$$

on the square lattice $\mathbb{Z}^d$. An interaction is said to be of long range if the decay of J_{x-y} is slower than exponential. Specifically, we focus on couplings which decay by a power law, satisfying (1.1) with some $\sigma > 0$. Throughout this Letter, the relation $A_x \approx B_x$ is defined to mean that

$$c_1 B_x \leqslant A_x \leqslant c_2 B_x \tag{2.2}$$

for some x-independent constants $0 < c_1, c_2 < \infty$.

While the most complete results on the critical behavior have been obtained for Ising spin models (with $\sigma_x = \pm 1$), many of them are valid more generally for one component spin variables with an a-priori measure in the Griffiths–Simon class [4]. This class consists of spin distributions which can be generated by means of distributional limits of sums of ferromagnetically coupled Ising spins. It includes the discrete variables which are equidistributed in $\{-k, -k+1, \ldots, k\}$, the continuous 'φ^4' field variables, and various other discrete and continuous spin distributions.

The Gibbs states $\langle \ \rangle_{\beta, h}$ of such systems correspond to limits of finite volume measures with weights proportional to $e^{-\beta H(\sigma)}$. Their properties are closely related with those of the first few derivatives of the free energy, which in our notation is $(-\beta)^{-1} f(\beta, h)$, with $f = \lim |\Lambda|^{-1} \ln Z_\Lambda(\beta, h)$; β being the inverse temperature and h/β the external field. In particular, the following quantities convey very significant information

$$M(\beta, h) = \partial f(\beta, h)/\partial h = \langle \sigma_0 \rangle_{\beta, h} \quad \text{(magnetization)},$$

$$\chi(\beta, h) = \partial M/\partial h \quad \text{(magnetic susceptibility)},$$

$$\mathscr{C}(\beta, h) = \partial^2 f(\beta, h)/\partial \beta^2 \quad \text{(specific heat)},$$

$$\overline{u_4} = \partial^2 f(\beta, h)/\partial h^4 .$$

Much is already known about ferromagnetic spin models with two-body interactions. By the Lee–Yang [5] theory, they can exhibit phase transitions only at $h = 0$. Along the $h = 0$ line, in the (β, h) plane, there are well-developed perturbative expansions for β either small or very large. Furthermore, nonperturbative results have been obtained about the properties of the high- and low-temperature phases, and it was shown that they extend (for $h = 0$) up to a common critical point. In addition, it is known that all the models of the above type share various basic characteristics in their critical behavior, some of which are summarized below in Proposition 1.1.

The most elementary way in which the long-range nature of the interaction is seen to affect the critical behavior is through the falloff of the two-point function, or the rate at which its Fourier transform – $G(p)$ ($= \Sigma_x \langle \sigma_0 \sigma_x \rangle e^{ixp}$) diverges as $p \to 0$ at $\beta = \beta_c$. For reflection positive interactions (with the only assumption on the spins being the integrability of $\exp(a\sigma_0^2)$ for all finite a), it is known – by the important result of Frölich–Simon–Spencer [3] – that $G(p)$ is bounded for all $\beta < \beta_c$ by

$$G(p) \leqslant \frac{1}{2\beta E(p)}, \tag{2.3}$$

with

$$E(p) = \tfrac{1}{2}\Sigma\,(1 - e^{ipx})J_x = \Sigma \sin^2\left(\frac{p\cdot x}{2}\right) J_x \geqslant C\,\Sigma_{|p\cdot x| \leqslant 1}(p\cdot x)^2 J_x. \tag{2.4}$$

(The inequality (2.3) is saturated in Gaussian models with the given couplings, in which case it is related to the equipartition law – $E(p)$ being the energy density of a 'spin wave' with momentum p. Away from $p = 0$, the inequality also holds for $\beta \geqslant \beta_c$ – for periodic b.c.)

For a long-range interaction obeying the power law (2.2),

$$E(p) \geqslant \text{Const.}\ |p|^{\min\{2,\sigma\}} = \text{Const.}\ |p|^{2-|2-\sigma|_+}. \tag{2.5}$$

[$|x|_+ = \max(x, 0)$]. Hence, if the interaction is also reflection positive, we know that

$$G(p) \leqslant C\beta^{-1}\,|p|^{-(2-|2-\sigma|_+)}, \tag{2.6}$$

for all $\beta < \beta_c(\{J\})$, with a β-independent constant C.

The bound (2.6) is not optimal in two ways: (i) its proof is restricted to RP interactions, (ii) for low d, and not too low σ, the infrared divergence of $G(P)$ is expected to be less singular (in the sense of a lower power of $|p|^{-1}$) than the one in the right side of (2.3). Since this inequality is the only way through which the range of the interaction enters much of the analysis, we state the following results under the more general assumption that there is some η with which $G(p)$ satisfies:

$$G(p) \leqslant \text{Const.}\ |p|^{-(2-\eta)} \tag{2.7}$$

with a constant which is fixed for all $\beta < \beta_c$ in a neighborhood of β.

Following is a summary of results applying to the models introduced above.

PROPOSITION 2.1. *For each distribution in the Griffiths–Simon class, and a d-dimensional Hamiltonian of the form* (2.1)*, there is a critical value β_c – which is finite if either $d \geqslant 2$ or $d = 1$ and $\sigma \leqslant 1$ – such that:*

(i) *For all $\beta < \beta_c$, the Gibbs state is unique and its correlation functions decay qualitatively as fast as the couplings:*

$$\langle \sigma_0 \sigma_x \rangle_{\beta,0} \approx \frac{C(\beta)}{|x|^{d+\sigma}} \tag{2.8}$$

(the symbol $\approx$ being defined by (2.2)).

(ii) *For $\beta > \beta_c$, there is symmetry breaking, characterized by nonvanishing spontaneous magnetization: $M^*(\beta) \equiv M(\beta, 0+) > 0$.*

(iii) *In the vicinity of the critical point $(\beta_c, 0)$, where M is singular, M satisfies the following inequalities – subject to restrictions given below – with $d_{\text{eff.}} = d/(1 - \eta/2)$ (η being any value with which* (2.7) *is satisfied).*

(a) *Along the coexistence line, $h = 0$ and $\beta \geqslant \beta_c$*

$$C(\beta - \beta_c)^{1/2} \leqslant M(\beta, 0+) \leqslant C'(\beta - \beta_c)^{1\Big/\left[2+3\left|\frac{4-d_{\text{eff}}}{2d_{\text{eff}}-7}\right|_+\right]} |\ln(\beta - \beta_c)|^{\#\,2/3}\,. \tag{2.9}$$

(b) *Along rays $\beta = \beta_c + b \cdot h$, $h \geqslant 0$:*

$$Ch^{1/3} \leqslant M(\beta_c, h) \leqslant C' h^{1\Big/\left[3+3\left|\frac{4-d_{\text{eff}}}{2d_{\text{eff}}-7}\right|_+\right]} |\ln h|^{\#}\,. \tag{2.10}$$

The constants C and C' do not depend here on b, which does, however, affect the lower-order terms

(c) *Approaching the critical point within the symmetric regime, $h = 0$ and $\beta < \beta_c$, the magnetic susceptibility $\chi(\beta, h) = \partial M(\beta, h)/\partial h$ diverges, with*

$$C(\beta_c - \beta)^{-1} \leqslant \chi(\beta, 0+) \leqslant C'(\beta_c - \beta)^{-\left[1+\left|\frac{4-d_{\text{eff}}}{d_{\text{eff.}}-2}\right|_+\right]} |\ln(\beta_c - \beta)|^{\#} \tag{2.11}$$

with the upper bound holding for any model with $d_{\text{eff.}} > 2$.

Furthermore, along this line, the specific heat and the function $|\overline{u_4}|$, introduced above satisfy:

$$C \leqslant \mathscr{C}(\beta) \leqslant C'(\beta_c - \beta)^{-\left|\frac{4-d_{\text{eff}}}{d_{\text{eff}}-2}\right|_+} |\ln(\beta_c - \beta)|^{\#}\,, \tag{2.12}$$

and

$$C\chi^4 \geqslant |\overline{u_4}| \geqslant C'(\beta_c - \beta)^{3\left|\frac{4-d_{\text{eff}}}{d_{\text{eff.}}-2}\right|_+} \chi^4\,. \tag{2.13}$$

The upper bounds for cases (a) and (b) are restricted to models with Ising spins and $d_{\text{eff.}} > 3.5$. In all the logarithmic terms $\# = 0$ unless $d_{\text{eff.}} = 4$ – in which case $\# = 1$.

Remark. The above Proposition summarizes an extensive collection of works, with the latest bounds being derived in our reference [6]. The coincidence of the boundaries of the high- and low-temperature regimes is proven in the works of Aizenman [7] and Aizenman–Barsky–Fernández [8]. The validity of (2.8) throughout the high temperature regime is based on Griffiths' inequality [9] for the lower bound, and – for the upper bound – on an extension of Simon's [10] method to long-range models found in [11]. The earliest of the mean-field critical exponent bounds is the lower one in (2.11) due to Glimm and Jaffe [12]. The complementary upper bound in (2.11) is from the work of Aizenman and Graham [13]. The lower bounds in (2.9) and (2.10) are found in [6, 7 and 14]. The upper bounds of (2.9)–(2.10) were derived in Aizenman–Fernández [6]. The upper bound on the specific heat in (2.12) is due to Sokal [15, 16]. The upper bound in (2.13) is the 'tree bound' of Aizenman [17], and Fröhlich [18], while the (unpublished) lower bound in (2.13) follows from a simple adaptation (inequality (4.12) below) of an argument we used in [6]. Naturally, the works cited above were affected by various other (rigorous and nonrigorous) developments, of which it will be impossible to give a complete account here. Let us, however, note that many of the above results involve partial differential inequalities derived for the order parameter. The main PDI are listed in Section 4.

The bounds (2.9)–(2.13) may be expressed as statements on the critical exponents, for which the standard notation is:

$$\begin{gathered} M(\beta,0) \cong C(\beta-\beta_c)^{\hat{\beta}}, \qquad M(\beta_c,h) \cong Ch^{1/\delta}, \qquad \chi(\beta,0) \cong C(\beta_c-\beta)^{-\gamma}, \\ \mathscr{C}_{\text{sing}}(\beta,h) \cong (\beta_c-\beta)^{-\alpha}, \qquad |\overline{u_4}| \cong (\beta_c-\beta)^{-(\gamma+2\Delta_4)}. \end{gathered} \tag{2.14}$$

It follows from (2.6) that for reflection positive long-range interactions $\eta \geqslant |2-\sigma|_+$, or $d_{\text{eff.}} \geqslant d_{\text{bnd.}}$. Hence, the above results carry the following explicit implications.

PROPOSITION 2.2. *In Ising spin models with reflection positive long-range interactions the critical exponents satisfy:*

(i) *For* $\sigma < 2$:
If $4d - 7\sigma > 0$

$$0 \leqslant \beta - 2, \qquad \delta - 3 \leqslant \left|\frac{2\sigma - d}{4d - 7\sigma}\right|_+ . \tag{2.15}$$

Under the weaker restriction: $d - \sigma > 0$,

$$0 \leqslant \gamma - 1, \quad \gamma - \tfrac{2}{3}\Delta_4, \qquad \alpha_{\text{sing}} \leqslant \left|\frac{2\sigma - d}{d - \sigma}\right|_+ . \tag{2.16}$$

(ii) *For* $\sigma \geqslant 2$ *the same inequalities* (2.16)–(2.20) *hold with* σ *replaced all throughout by* 2 (*short-range bounds*).

Remarks. The inequalities for critical exponents are understood as bounds on the limsup (for the upper bounds) and liminf (for the lower bounds) of the logarithmic ratios implied by the notation (2.15). As seen in Proposition 2.1, many of these bounds are valid more generally for spins in the GS class. A result of [13] (inequality (4.7) below) permits the somewhat sharper upper bound: $\Delta_4 \leqslant \gamma + \frac{1}{2}$.

3. Long Range Reflection-Positive Models

In view of the usefulness of reflection positivity, it is of interest to see a large collection of RP interactions – and, in particular, interactions exhibiting a slow falloff in dimensions $d \leqslant 4$. Such interactions were presented in Fröhlich–Israel–Lieb–Simon [19], where the focus was on dimensions $d = 1$ and 2. In this section we mention, and supplement, some of their results. Even before recalling the definition, let us list some valid examples.

Examples of long-range RP interactions:

(1) (Based on FILS [19])

$$J_x = 1/|x|^\tau \quad (|\cdot| = \text{the Euclidean norm}) \tag{3.1}$$

for $1 \leqslant d \leqslant 4$, and any $\tau \geqslant |d-2|_+$. (For $d = 1$ one may add $J_x = C/(|x| + a^2)^\tau$).

$$(2) \quad J_x = 1/(\|x\| + a^2)^\tau, \quad \text{with } \|x\| = |x_1| + \cdots + |x_d|, \tag{3.2}$$

for any $d \geqslant 1$ and $\tau \geqslant 0$.

$$(3) \quad J_x = 1 \Big/ \prod_{i=1}^{d} (|x_i| + a_i^2)^{\tau_i}, \quad \tau_i \geqslant 0. \tag{3.3}$$

It should be noted here that the value of J_x for $x = 0$ is not relevant (even if it diverges), and that reflection positivity is preserved under the addition of any nearest neighbor term – or in fact any other RP interaction.

The cases grouped above in (1) include the $d = 1, 2$ examples of FILS [19] and their natural extension to higher d. That construction yields RP interaction like (3.1) also for $d > 4$, but with a somewhat more involved restrictions on τ (as explained below). Before commenting on the other examples added here, let us recall the definition of reflection positivity.

DEFINITION. An interaction $\{J_x\}$ is called *reflection positive* (RP) (with respect to mid-plane reflections) if it satisfies:

$$\sum_{\substack{x_1, y_1 \geqslant 1 \\ x_\|, y_\| \in \mathbb{Z}^{d-1}}} \bar{z}_{(x_1, x_\|)} z_{(y_1, y_\|)} J_{(x_1 + y_1 - 1,\, x_\| - y_\|)} \geqslant 0, \tag{3.4}$$

for any finite collection of complex numbers $(z_x)_{x \in \mathbb{Z}^d}$, and if similar inequalities are satisfied for all the other orientations of the reflection hyperplanes, with the corresponding decompositions of $\mathbb{Z}^d$ into 'orthogonal' and 'parallel' components (with x_1 replaced by x_i, $i = 2, \ldots, d$).

The one-dimensional RP interactions were shown in Fröhlich–Israel–Lieb–Simon [19] to admit a spectral representation as

$$J_x = \rho_0\, \delta_{|x|,1} + \int_{-1}^{1} \lambda^{|x|-1} \rho(\mathrm{d}\lambda), \tag{3.5}$$

with $\rho(\mathrm{d}\lambda)$ a positive (and otherwise arbitrary) measure in $[-1, 1]$.

The one-dimensional examples in (3.1) were generated in [19] by choosing $\rho(\mathrm{d}s) = \mathcal{O}(s)\Gamma(\tau)^{-1}s^{\tau-1}\,\mathrm{e}^{-as}\,\mathrm{d}s$ (supported in [0, 1]). An ingenious construction is also presented in [19] for the higher-dimensional cases of (3.1). In essence, the construction takes advantage of the reflection-positivity of the well-known free field (with arbitrary mass), whose two point function may be used to show the reflection positivity of couplings with $d - 2 \leqslant \tau \leqslant d$. Higher values of τ are then generated by making use of the principle (derived in [19] by Schur's theorem on positive-definite matrices) that a pointwise product of RP interactions is also an RP interaction. For $d > 4$ (which is not of much interest for us), this construction leaves some gaps in the range of values of τ. However, examples (3.2) and (3.3) suggest that these gaps are just a spurious effect of the argument.

The interactions in (3.2)–(3.3) are obtained here by means of the following proposition, which allows us to construct d-dimensional reflection positive interactions out of one-dimensional examples.

PROPOSITION 3.1. (i) *If* $\tilde{J}_x$ *is a one-dimensional reflection positive interaction (not necessarily ferromagnetic), then the d-dimensional interaction*

$$J_x = \tilde{J}_{\|x\|}, \quad \textit{with } \|x\| = |x_1| + \cdots + |x_d|, \tag{3.6}$$

is also reflection positive (in $\mathbb{Z}^d$*).*

(ii) *If* $\tilde{J}^{(1)}, \ldots, \tilde{J}^{(d)}$ *are one-dimensional interactions with the spectral measure* ρ *– of (3.5), supported on positive* λ*, then the interaction*

$$J_x = \tilde{J}^{(1)}_{|x_1|} \cdot \ldots \cdot \tilde{J}^{(d)}_{|x_d|} \tag{3.7}$$

is RP in $\mathbb{Z}^d$.

Remarks. Unlike (3.6), (3.7) applies only to a certain subclass of RP interactions, which are automatically *ferromagnetic*. Both cases are implied by the more general statement that any interaction which may be written in the form

$$J_x = \int_{[-1,1]^d} \prod_{i=1}^{d} \lambda_i^{|x_i|-1}\,\hat{\rho}\,(\mathrm{d}\lambda_1 \ldots \mathrm{d}\lambda_d), \tag{3.8}$$

with $\hat{\rho}\,(\cdot)$ satisfying:

$$\hat{\rho}\,(\mathrm{d}\lambda)\,\mathrm{sgn}(\lambda_k) \prod_{i=1}^{d} \mathrm{sgn}(\lambda_i) \geqslant 0 \quad \text{for each } k = 1, \ldots, d, \tag{3.9}$$

is RP (in d-dimensions).

Leaving the details as an exercise, let us just comment that the reflection-positivity of interactions of the form (3.8) is implied by two different positivity statements: (i) the positivity of (3.9), and (ii) the 'positive definiteness' (as seen in the positivity of the Fourier transform) of the function $f_\lambda(x) = \lambda^{|x|}$ – on $\mathbb{R}$. (The second statement plays a role in the control of the 'parallel' direction in (3.4).) The fact that interactions of the two types mentioned in Proposition 3.1 admit the representation (3.8), satisfying (3.9), is a rather direct consequence of the FILS representation (3.5).

4. Partial Differential Inequalities

It may be interesting to note that most of the properties discussed above are consequences of a number of partial differential inequalities which over the years have been derived for the systems considered here. Without repeating the full arguments, we shall summarize here the relevant PDI, grouping them according to their most elementary applications. We denote here $|J| = \Sigma_x J_{0,x}$.

An important preliminary fact (which has a number of derivations, e.g. an argument using FKG inequalities [20]), is that these models may exhibit a first-order phase transition for a given (β, h) only if the magnetization is discontinuous there as a function of h. This can only happen at $h = 0$, by the Lee–Yang theory, or by the concavity stated below in (4.2).

(i) *Phase structure*. The magnetization is an odd function of h, which for $h > 0$ is: (a) positive (Griffiths I [9]) and monotone (Griffiths II [20, 21]) in β and h

$$\partial M(\beta, h)/\partial h\,; \quad \partial M(\beta, h)/\partial\beta \geqslant 0\,, \tag{4.1}$$

and (b) concave (GHS [22])

$$\partial^2 M(\beta, h)/\partial h^2 \leqslant 0\,. \tag{4.2}$$

While the low temperature is characterized by the nonvanishing of $M(\beta, 0+)$, it can be shown (by an application of the Simon inequality [10]) that the rapid decay of correlations which is characteristic of the high-temperature phase persists (at $h = 0$) as long as $\chi \equiv \partial M/\partial h$ is finite. The fact that the two phases extend up to a common critical point is derived (by an argument of Aizenman and Barsky [23]) from the PDI

$$M \leqslant h \cdot \partial M/\partial h + [\beta |J| M^2 + hM] \cdot \partial(\beta M)/\partial\beta \tag{4.3}$$

(obtained by Aizenman–Barsky–Fernández [8]), with the additional help of

$$\partial M/\partial\beta \leqslant |J| M\, \partial M/\partial h \tag{4.4}$$

(a consequence of GHS whose importance for the study of critical exponents was emphasized by Newman [24]).

(ii) *Mean-field bounds on the critical exponents.* (a) The use of differential inequalities for critical exponent bounds seems to have been started by Glimm and Jaffe [12], who used

$$\partial\chi/\partial\beta \leqslant |J| \chi^2 \quad (\text{at } h = 0, \beta < \beta_c) \tag{4.5}$$

(a consequence of the GHS–Lebowitz inequality [25]) for the mean-field bound on γ. The important result of McBryan and Rosen [26] that $\chi \to \infty$ as $\beta \to \beta_c$, can also be derived from a finite-volume version of (4.5) (by an argument given in [27]).

(b) The general mean-field bounds on $\hat{\beta}$ and δ (of [7, 14, 6, 8, 28]) can be derived from the PDI (4.3) and (4.4).

(c) A bound on $|\overline{u_4}|$ at $h = 0$ and $\beta < \beta_c$ – with an important implication for the 'triviality' of the scaling limits, can be expressed as

$$|\partial^3 M/\partial h^3| \leqslant 2(\beta|J|)^2 \chi^4 \tag{4.6}$$

(Aizenman [17], Fröhlich [18]). There is also a sharper relation

$$|\partial^3 M/\partial h^3| \leqslant 2|J|\chi^2 \cdot \partial(\beta\chi)/\partial\beta \tag{4.7}$$

derived in Aizenman–Graham [13].

(iii) *Bubble-corrected inequalities*
The inequalities which supplement the mean field bounds in (2.9)–(2.13) are also obtained through PDI. It turns out that for some of the pivotal quantities one may derive pairs of supplementary bounds which differ by factors involving the 'bubble diagram' $B(\beta)$, which is defined as

$$B(\beta) = \sum_x \langle \sigma_0 \sigma_x \rangle^2_{h=0} \,. \tag{4.8}$$

For large enough d, or small enough σ, the bubble diagram is finite at the critical point (a fact which for RP interactions may be ascertained by the infrared bound (2.3)), and then the supplementary bounds completely determine the critical exponents. Even when the buble diagram diverges, one obtains informative, though less definitive, results through bounds on B of the form $B \leqslant \text{Const.}\, \chi^{|(4-d_{\text{eff}})/2|_+}$ (as discussed in Sokal [16], and in Appendix A of [6]). Following are some such pairs of supplementary bounds.

(a) For the study of α

$$\text{Const.} \leqslant \mathscr{C} \leqslant |J|^2 B \,. \tag{4.9}$$

of Sokal [14] (based on [5, 25]). This is the simplest of the complementary pairs, and its discovery was a precursor of the results on the upper critical dimension.

(b) For the study of γ

$$\beta \left| \frac{\partial(\beta|J|\chi)}{\partial\beta} \right| \geqslant \text{const.}\ \frac{(\beta|J|\chi)^2}{1 + (\beta|J|)^2 B} \tag{4.10}$$

of Aizenman–Graham [13] – which supplements (4.5).

(c) For δ and $\hat{\beta}$

$$\text{Const.}\ \frac{M^4}{h^3} \geqslant \left|\frac{\partial\chi}{\partial h}\right| \geqslant \text{const.}\ \frac{|1 - hB/M|^2_+}{(1 + 2\beta|J|B)^3}\, h\chi^4 \tag{4.11}$$

of Aizenman–Fernández [6], where the lower bound (which is the more important one here) was derived only for Ising spin systems.

(d) For Δ_4

$$|\overline{u_4}| \geqslant \frac{1}{48|\beta J|^2 B + 2\chi^{-2}\beta(\partial\chi/\partial\beta)} \left(\beta\, \frac{\partial\chi}{\partial\beta}\right)^2 , \tag{4.12}$$

which complements (4.7). The (unpublished) proof of (4.12) is by a direct adaptation of the argument used to prove the lower bound in (4.10) (found on p. 435 in [6]).

Let us end this note by recalling that some other drastic effects of long-range interactions occur in the borderline case of one-dimensional $1/|x|^2$ interactions. Recent rigorous proofs and the rich history of the subject can be found in [29], and references therein.

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