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Critical Fluctuation of the Order Parameter in a Superconductor. I

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The fluctuation of the order parameter in a superconductor in the temperature slightly above the transition temperature is studied in the framework of the current microscopic theory. It is convenient to divide the temperature region where the fluctuation of the order parameter becomes important into two regions; the classical and the critical region. In the classical region the spatial correlation of the fluctuation is of the Ornstein-Zernike type, while in the critical region it depends on the relative distance like $r^{-3/2}$. We show that the electrical resistivity, the nuclear spin lattice relaxation time and the ultrasonic attenuation coefficient decrease like $1-a|\eta|^{-1/2}$ and $1-a'|\eta|^{-1/3}$ in the classical and in the critical region respectively, where $\eta=(T/T_c)-1$ and a and a' are constants of the order $(T_c/\mu)^{1/2}(lp_0)^{-3/2}$ and $(T_c/\mu)^{1/3}(lp_0)^{-1}$ respectively. Here μ is the chemical potential, l is the electronic mean free path and p_0 is the fermi momentum.

It is also shown that the thermal conductivity has no singular term at the transition temperature.

§ 1. Introduction

The fluctuation of the order parameter in a superconductor has been studied intensely in recent years. The fluctuation has been considered either in reference to the dynamical properties of a type II superconductor in high field region[1] or in connection with the critical behavior in various thermodynamical properties at the transition point.[2)-4)]

We are concerned with the latter problem only in the present series of papers. It is well known[2] that the random phase approximation to the propagator of the fluctuation field (of the order parameter) ends up with an inverse square root singularity in the specific heat at the transition point. However, the numerical coefficient of this singular term is so small that this singular behavior is believed to be of no practical interest. Later noticing the formal analogy between the Dyson equation describing the fluctuation field in a superconductor and the one in HeII, Baytev et al.[3] concluded that the specific heat in a superconductor has a logarithmic singularity at the transition point.

More recently in analogy with their treatment of dynamical properties in HeII, Ferrell and Schmidt[6] have suggested that in a dirty superconductor the fluctuation of the order parameter gives rise to observable effects. Making use of a semi-phenomenological assumption as to the spatial correlation of the fluctuation field, they were able to explain qualitatively the critical behavior of the electrical resistivity found experimentally by Glover.[6]

In the present paper we would like to study in the framework of the current microscopic theory, the fluctuation field (of the order parameter) and its effects on the electronic transport properties in the vicinity of the transition temperature. We limit here (i.e. in part I) our consideration in the temperature region above T_c.

Following Ferrell and Schmidt,[5] we divide the temperature region where the effect of the fluctuation is important into two different temperature regions; the classical region and the critical region. In the classical region the dispersion of the fluctuation field is obtained essentially in the random phase approximation. The spatial correlation of the fluctuation field in this region is of the classical type as in the theory of Ornstein and Zernike. In the critical region (which occupies the immediate vicinity of the transition temperature) on the other hand the nonlinear interaction between the fluctuation fields is no longer negligible. The self-consistent solution of the nonlinear equation results in the correlation function with a spatial dependence like $r^{-3/2}$ where r is the distance.

We shall discuss briefly the correlation function (or Green's function) of the fluctuation field in the next section. Section 3 is devoted to calculation of the electrical conductivity in the presence of the fluctuation field. It is shown that the electrical resistivity decreases like $(1-a|\eta|^{-1/2})$ and like $(1-a'|\eta|^{-1/3})$ in the classical and in the critical region respectively, where $|\eta|=|T/T_c-1|$, a and a' are numerical constants. It is easy to show generally that the nuclear spin lattice relaxation time and the ultrasonic attenuation coefficient decrease precisely the same way as the electrical resistivity, while the thermal conductivity does not show any singular behavior at the transition point.

We do not consider here the fluctuation of the order parameter below T_c, which will be the subject of the second part of this work.

§ 2. Spatial correlation of the fluctuation field

The fluctuation of the order parameter in the vicinity of the transition point has been considered first by Thouless,[2] who shows that the specific heat contains a term diverging like $|T-T_c|^{-1/2}$. Although his treatment is limited to a pure superconductor, an extension to an impure superconductor is essentially carried out by Gor'kov.[7] In our terminology these calculations are concerned with the classical behavior of the fluctuation. The same problem in the critical region has been formulated recently by Baytev et al.[3] In this section we present nothing new, but rather we content ourselves with rearranging known results in a convenient form for later disussions.

We start with a pairing hamiltonian given by

$$\mathscr{H}_g = -|g| \int \psi_\uparrow^\dagger(\boldsymbol{r}) \psi_\downarrow^\dagger(\boldsymbol{r}) \psi_\downarrow(\boldsymbol{r}) \psi_\uparrow(\boldsymbol{r})\, d^3r,$$
$$= -|g| \int \Psi^+(\boldsymbol{r}) \Psi(\boldsymbol{r})\, d^3r, \tag{1}$$

where

$$\Psi^+(\boldsymbol{r}) = \psi_\uparrow^\dagger(\boldsymbol{r}) \psi_\downarrow^\dagger(\boldsymbol{r}),$$
$$\Psi(\boldsymbol{r}) = \psi_\downarrow(\boldsymbol{r}) \psi_\uparrow(\boldsymbol{r}) \tag{2}$$

and $\psi_\alpha(\boldsymbol{r})$ and $\psi_\alpha^\dagger(\boldsymbol{r})$ are electron field operators.

Green's function which describes the spatio-temporal behavior of the fluctuation field is formally given by

$$\mathscr{D}(\omega.\ \boldsymbol{q}) = |g|^2 \langle\!\langle [\Psi^+,\ \Psi] \rangle\!\rangle_{\omega, q}, \tag{3}$$

where $\langle\!\langle [\quad] \rangle\!\rangle$ denotes the retarded product taken in Gibb's ensemble with the total Hamiltonian $\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_g$. Here $\mathscr{H}_0$ is the free hamiltonian describing free motion of electrons and holes in the fermi sea. We include the coupling constant $|g|$ in the definition of Green's function for later convenience.

A. Classical region

In the classical region we can make use of the random phase approximation in the calculation of Green's function[1] for the fluctuation, and (3) reduces to

$$\mathscr{D}(\omega,\ \boldsymbol{q}) \cong \frac{|g|^2 \langle [\Psi^+,\ \Psi] \rangle_{\omega, q}}{1 - |g| \langle [\Psi^+,\ \Psi] \rangle_{\omega, q}}, \tag{4}$$

where $\langle\ \rangle$ is an average over Gibb's ensemble in the absence of the fluctuation field; in a temperature above T_c the average has to be taken in the normal state, while in a temperature below T_c it has to be taken in the equilibrium BCS state.

In the following we restrict our consideration to the dirty limit unless otherwise stated explicitly, since the effects of the fluctuation field are only accessible in dirty superconductors.[8] Making use of the standard technique in the theory of metals we have[1]

$$\mathscr{D}(\omega,\ \boldsymbol{q}) = N(0)^{-1} \left\{ \ln\left(\frac{T}{T_c}\right) + \psi\left(\frac{1}{2} + \frac{-i\omega + Dq^2}{4\pi T}\right) - \psi\left(\frac{1}{2}\right) \right\}^{-1}$$
$$\cong N(0)^{-1} (|\eta| + Aq^2)^{-1},$$

with

$$A = \frac{\pi D}{8T_c}, \tag{5}$$

where $N(0)$ is the density of states at the fermi level (for electrons with a single spin direction), ψ is the di-gamma function, $D = (lv/3)$ the diffusion constant $|\eta| = |T/T_c - 1|$ and v is the fermi velocity. We put $\omega = 0$ in the final expression, since only $\mathscr{D}(\omega,\ q)$ with $\omega = 0$ is important in the critical fluctuation.

B. Critical region

In the critical region the effect of the fluctuation field becomes so important that the spectrum of the fluctuation field has to be determined self-consistently. In particular in the limit $T=T_c$ (i.e. $|\eta|=0$) we assume

$$\mathscr{D}^{-1}(\omega, \boldsymbol{q}) = \Phi(\boldsymbol{q}). \tag{6}$$

Following the scheme considered by Baytev et al.,[3] we construct the self-consistency equation for $\Phi(\boldsymbol{q})$, which runs as follows:

$$\Phi(\boldsymbol{q}) = \frac{2T^2\Gamma_0^2}{(2\pi)^6}\int d^3\boldsymbol{q}_1\int d^3\boldsymbol{q}_2\left[\frac{1}{\Phi(\boldsymbol{q}_1)\Phi(\boldsymbol{q}_2)\Phi(\boldsymbol{q}-\boldsymbol{q}_1-\boldsymbol{q}_2)} - \frac{1}{\Phi(\boldsymbol{q}_1)\Phi(\boldsymbol{q}_2)\Phi(-\boldsymbol{q}_1-\boldsymbol{q}_2)}\right], \tag{7}$$

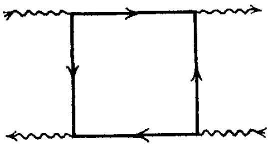

Fig. 1. The lowest order box vertex which describes the interaction between the fluctuation field is given. The wavy line describes the propagation of the fluctuation field while the bold line is that of electrons.

where we approximate the complete box diagram by the lowest order diagram. We calculate Γ_0 from the diagram given in Fig. 1 and we have*)

$$\Gamma_0 = \frac{N(0)}{8(\pi T_c)^2}\,(7\zeta(3)). \tag{8}$$

It is worth while to note that Γ_0 does not depend on the electronic mean free path or the coupling constant $|g|$. Therefore the critical region considered here exists universally in a superconductor, although its observability crucially depends on the electronic mean free path of the system. It is easy to show that (7) has a solution

$$\Phi(\boldsymbol{q}) = B'|q|^{3/2},$$

and

$$B' = \left(\sqrt{\frac{8\pi}{15}}\,\frac{T_c\Gamma_0}{2\pi^2}\right)^{1/2}. \tag{9}$$

In the immediate vicinity of the transition point it is natural to assume**)

$$\mathscr{D}_c(0, \boldsymbol{q}) = N(0)^{-1}(|\eta| + B|q|^{3/2})^{-1}, \tag{10}$$

where

*) In dirty superconductors we have to sum on other diagrams which contain corrections due to the impurity scattering as in reference 7). However, the final result (i.e. Eq.(8)) is unaffected.

**) A more complicate form of $\mathscr{D}_c$ has been considered in reference 9). We believe that the general behavior is not affected by this simplification. Perhaps the use of Eq. (7) will be more questionable, although a more complete treatment of the self-consistency equation appears almost unpractical.

$$B=B'/N(0)=\frac{1}{4\pi}\left(\frac{8\pi}{15}\right)^{1/4}\left(\frac{7\zeta(3)}{N(0)\,T_c}\right)^{1/2}\sim\left(\frac{\mu}{T_c}\right)^{1/2}p_0^{-3/2}, \tag{11}$$

$\mu=p_0^2/2m$ the chemical potential and p_0 is the fermi momentum.

In the following discussion of the electronic transport properties we refer to Eq. (10) as Green's function in the critical region.

§ 3. Transport properties

We shall consider here the modification of the electronic transport properties due to the fluctuation field. We can illustrate the general method most easily by calculating the electronic conductivity in the presence of the fluctuation field as an example. We assume here that the fluctuation field gives rise to a small correction to the response functions in the normal state, so that we can treat the effect as a perturbation. The electric conductivity is expressed in terms of the retarded product of the current operators:

$$-i\omega\sigma_{\mu\nu}(\omega)=\boldsymbol{Q}_{\mu\nu}(\omega), \tag{12}$$

where

$$\boldsymbol{Q}_{\mu\nu}(\omega)=\langle[\boldsymbol{j}_\mu, \boldsymbol{j}_\nu]\rangle(\omega),$$

and the current operator is given by

$$\boldsymbol{j}_\mu(\boldsymbol{r})=\sum_\sigma\frac{e}{2mi}(\boldsymbol{\nabla}_\mu'-\boldsymbol{\nabla}_\mu-2ie\boldsymbol{A}_\mu)\psi_\sigma^+(\boldsymbol{r}')\psi_\sigma(\boldsymbol{r})|_{\boldsymbol{r}'=\boldsymbol{r}}. \tag{13}$$

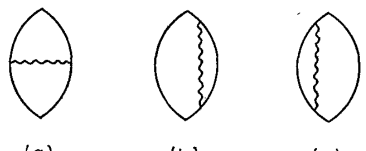

Fig. 2. The lowest order corrections in the presence of the fluctuation field are given. Only the diagram a) contributes to the critical singularity.

The lowest order corrections due to the fluctuation are given in the diagrams (Fig. 2), where the wavy line represents the propagation of the fluctuation field. Since the evaluation of each term corresponding to each diagram is similar to those in dirty type II superconductors,[9] we can immediately write down the results

$$\boldsymbol{Q}_{\mu\nu}(\omega)=\delta_{\mu\nu}\sigma\left\{-i\omega+\frac{T}{(2\pi)^3}\int d^3q\frac{\mathscr{D}(0,\boldsymbol{q})}{2\pi T}\right.$$
$$\times\left[\psi^{(1)}\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho_q\right)-\left(\frac{2\pi T}{i\omega}+\frac{2\pi T}{i\omega-Dq^2}\right)\left(\psi\left(\frac{1}{2}-\frac{i\omega}{2\pi T}+\rho_q\right)\right.\right.$$
$$\left.\left.\left.-\psi\left(\frac{1}{2}+\rho_q\right)\right)\right]\right\}, \tag{14}$$

where $\rho_q=Dq^2/4\pi T$, $\sigma=\tau e^2N/m$ the conductivity in the normal state, $\psi(z)$ and

$\psi^{(1)}(z)$ are the di-gamma and the tri-gamma function respectively.

In the low frequency limit the above expression reduces to

$$\boldsymbol{Q}_{\mu\nu}(\omega)=\delta_{\mu\nu}\,\sigma\left\{\frac{[1]}{\pi T}\psi^{(1)}\left(\frac{1}{2}\right)-i\omega\left(1+\frac{1}{2\pi T}\left(\left[\frac{1}{D\boldsymbol{q}^2}\right]\psi^{(1)}\left(\frac{1}{2}\right)\right.\right.\right.$$
$$\left.\left.\left.+3\frac{[1]}{4\pi T}\psi^{(2)}\left(\frac{1}{2}\right)\right)\right)\right\}, \tag{15}$$

where $[a(\boldsymbol{q})]=T/(2\pi)^3\int d^3q\mathscr{D}(0,\boldsymbol{q})a(\boldsymbol{q})$. We consider two separate regions here.

A. Classical region

Green's function in the classical region is given by Eq. (5) and we have

$$\sigma_c=\sigma\left(1+\frac{1}{16N(0)DA^{1/2}}\frac{1}{|\eta|^{1/2}}\right), \tag{16}$$

where we have made use of the relations

$$T\int\frac{d^3q}{(2\pi)^3}\mathscr{D}(0,\boldsymbol{q})=\frac{T}{2\pi^2}\frac{1}{N(0)}\left\{\frac{q_0}{A}-\frac{\pi}{2}\left(\frac{|\eta|}{A^3}\right)^{1/2}\right\},$$
$$T\int\frac{d^3q}{(2\pi)^3}\frac{\mathscr{D}(0,\boldsymbol{q})}{D\boldsymbol{q}^2}=\frac{T}{2\pi^2N(0)}\frac{1}{D}\frac{\pi}{2}(A|\eta|)^{-1/2}, \tag{17}$$

and q_0 is the cutoff momentum. Equation (16) is rewritten as

$$R_c=R(1-a|\eta|^{-1/2}),$$

where

$$a=\frac{1}{16N(0)DA^{1/2}}\cong 4.18(lp_0)^{-3/2}(\xi_0p_0)^{-1/2}\sim\left(\frac{T_c}{\mu}\right)^{1/2}(lp_0)^{-3/2}. \tag{18}$$

Therefore the resistivity drops like $a|\eta|^{-1/2}$ in the classical region.

B. Critical region

Here we substitute Green's function given in Eq. (10) in Eq. (15) and find

$$\sigma_c=\sigma\left(1+\frac{1}{6\sqrt{3}\,N(0)DB^{2/3}}\frac{1}{|\eta|^{1/3}}\right). \tag{19}$$

Here we have made use of the relations

$$T\int\frac{d^3q}{(2\pi)^3}\mathscr{D}(0,\boldsymbol{q})=\frac{T}{3\pi^2N(0)}\left\{\frac{1}{B}(q_0)^{3/2}-\frac{|\eta|}{B^2}\ln\left(\frac{E_0}{|\eta|}\right)\right\},$$

and

$$T\int\frac{d^3q}{(2\pi)^3}\frac{\mathscr{D}(0,\boldsymbol{q})}{D\boldsymbol{q}^2}=\frac{2}{3\sqrt{3}\,\pi}\frac{T}{N(0)D}\frac{1}{B}\left(\frac{B}{|\eta|}\right)^{1/3}. \tag{20}$$

The electrical resistivity in the critical region is given by

$$R_c = R(1 - a'|\eta|^{-1/3}), \tag{21}$$

where

$$a' = (6\sqrt{3}\, N(0) D B^{2/3})^{-1} \qquad \sim \left(\frac{T_c}{\mu}\right)^{1/3} (lp_0)^{-1}. \tag{22}$$

We note that in both cases only the diagram a) in Fig. 2 contributes to the singular term.

It is quite easy to discuss the modifications of various response functions[10] in terms of the similar diagrams. We find exactly the same behavior for the nuclear spin lattice relaxation time τ and the ultrasonic attenuation coefficient α:

$$\begin{aligned} &\tau_c = \tau_n (1 - a|\eta|^{-1/2}), \\ &\alpha_c = \alpha_n (1 - a|\eta|^{-1/2}), \qquad \text{in the classical region} \end{aligned} \tag{23}$$

and

$$\begin{aligned} &\tau_c = \tau_n (1 - a'|\eta|^{-1/3}), \\ &\alpha_c = \alpha_n (1 - \alpha'|\eta|^{-1/3}), \qquad \text{in the critical region} \end{aligned} \tag{24}$$

where the suffix n refers to the quantities in the normal state. Also we can show that the thermal conductivity K has no critical singularity (i.e. only the higher derivatives of K in the temperature shows the singularity).

§ 4. Concluding remarks

We have seen from the microscopic calculation that the electric resistivity shows a singular behavior. This partly justifies a number of premises used by Ferrell et al.[5] in their discussion of the critical phenomena.

The present calculation introduces no arbitrary phenomenological parameter so that we have a definite prediction about the extension of both the critical and the classical regions. Dropping numerical coefficients of the order of 1, we see that the classical behavior is seen in the temperature region $\Delta T_1 \sim T_c (T_c/\mu) \times (lp_0)^{-3}$. This classical behavior is taken over by the critical behavior in the immediate vicinity of T_c; $\Delta T_2 \sim 5^{-1}\, \Delta T_1$. As is easily seen from the above estimate of ΔT_1, it is essential to have a short electronic mean free path l in order to make these effects of fluctuation experimentally accessible.

Far inside the critical region the above perturbation series apparently diverges. There may exist another sub-critical region where the dynamical properties are governed by another law in the temperature region, say, $10^{-8} T_c$ or $10^{-10} T_c$ immediately above T_c. This kind of problem does not arise in HeII, where the transport properties are directly described in terms of the fluctuation field, while in a superconductor the electron plays the primary role in the transport phenomena.

Another interesting byproduct of the present study is that we expect the similar critical behavior in the nuclear spin lattice relaxation time and in the ultrasonic attenuation coefficient but not in the thermal conductivity. In this respect the experimental study of the corresponding quantities in the critical region is of particular interest.

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Note added in proof:

Since this work was completed we became aware of the fact that the diagram containing two propagators of the fluctuation field also contributes significantly to the singular term in the electric conductivity [L. G. Aslamazov and A. I. Larkin, Phys. Letters **26A**, 238 (1968)]. Taking this new term into account the expression for the electric conductivity in the classical region has to be modified (i.e. the singular term has to be multiplied by a factor 7/4). On the other hand this new term does not affect the calculation of the other transport coefficients such as the ultrasonic attenuation coefficient and nuclear spin lattice relaxation time so that the present results still hold for these quantities.

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