

## Critical Initial States in Collisionless Plasmas

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We show that for collisionless plasmas with initial conditions (ICs) near “single-humped” linearly stable equilibria, there exist *critical initial states* that mark the transition between the ICs from which the electric field evolves to a nonzero time-asymptotic state  $A(x, t)$ , and those from which it Landau damps to zero. We develop an equation for  $A(x, t)$  and study it as a bifurcation problem, and we obtain the asymptotic field, at leading order, as a finite superposition of waves whose frequencies obey a Vlasov dispersion relation and whose amplitudes satisfy a set of nonlinear algebraic equations. [S0031-9007(98)07827-2]

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The propagation of longitudinal “electrostatic” waves in collisionless plasmas is a fundamental phenomenon in many space and laboratory settings. For such waves, the Vlasov-Maxwell equations of kinetic theory reduce to the one-dimensional nonlinear Vlasov-Poisson (VP) equations

$$\frac{\partial f_\alpha}{\partial t} + v \frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha} E \frac{\partial f_\alpha}{\partial v} = 0, \quad (1a)$$

$$\frac{\partial E}{\partial x} = 4\pi \sum_\alpha q_\alpha \int dv f_\alpha, \quad (1b)$$

where  $f_\alpha(x, v, t)$  is the distribution function for particle species  $\alpha$ ,  $\alpha = 1, \dots, N$  and  $E$  is the self-consistent longitudinal electric field.

Much of the existing kinetic theory of plasma waves is based on the linearization of these equations about a Vlasov equilibrium  $F_\alpha(v)$ ,  $\alpha = 1, \dots, N$ . According to Landau’s classic analysis of the linearized VP system [1], if  $\sum_\alpha \frac{q_\alpha^2}{m_\alpha} F_\alpha$  decreases monotonically with  $|v|$  all spatially periodic initial perturbations lead to fields that damp exponentially in time. However, because nonlinear particle trapping occurs in many important cases, this Landau damping does not give the correct time-asymptotic behavior in general. Rather, it typically takes place only on a relatively short time scale. In fact, O’Neil [2] has shown how the damping can be dramatically altered because of the nonlinear energy exchange between a wave and the resonant particles trapped in its potential wells. As the particles begin bouncing back and forth in the wells, they drive the wave amplitude through a sequence of oscillations, whose magnitude decreases as the particles become more and more phase mixed and unable to exchange energy with the electric field in any coherent manner. Thus, in the time-asymptotic limit the wave amplitude reaches a final nonzero constant value. Such nonzero, traveling-wave final states have been observed both in experiments [3] and in numerical simulations [4–6]. However, a satisfactory general quantitative analysis is still lacking, since O’Neil’s study deals only with a limiting case, namely, that in which the trapping effects are so dominant that the

amplitude of an isolated wave is approximately constant. At the opposite extreme, the standard linear theory is valid when strong initial Landau damping occurs before the nonlinear effects come into play.

In this Letter we report the results of a systematic study of the long-time evolution of a spatially periodic perturbation of a Vlasov equilibrium and focus on the transition between the initial conditions that lead to nonzero time-asymptotic states via nonlinear particle trapping and those that Landau damp completely. In particular, we determine the *critical initial states* that mark this transition. The existence of these critical states is important for the understanding of nonlinear plasma dynamics, and especially for the analysis of plasma stability. The results include as limiting cases both the O’Neil and the Landau scenarios, and they provide a solid foundation for the replies [6,7] to a recent statement that electric fields in collisionless plasmas in general damp to zero [8].

*General development.*—We introduce the representation of the electric field as the sum of a transient part and a time-asymptotic part

$$E(x, t) = T(x, t) + A(x, t), \quad (2)$$

where  $T(x, t)$  and  $A(x, t)$  are spatially periodic and  $\lim_{t \rightarrow \infty} T(x, t) = 0$ . Then, we seek solutions for  $A(x, t)$  in the form of a superposition of traveling waves [9]; mathematically, this can be expressed by taking  $A$  as an *almost periodic* function of  $t$ . Correspondingly, the complete field  $E$  will be an *asymptotically almost periodic* function [10] of  $t$ . We substitute Eq. (2) into the Poisson equation, Eq. (1b), and separate the time-asymptotic and transient parts of the resulting equation

$$\frac{\partial A}{\partial x} = 4\pi P_a \sum_\alpha \int dv f_\alpha(A + T), \quad (3a)$$

$$\frac{\partial T}{\partial x} = 4\pi(I - P_a) \sum_\alpha \int dv f_\alpha(A + T). \quad (3b)$$

Here  $P_a$  is the operator that annihilates all but the time-asymptotic part of an asymptotically almost periodic

function; by definition  $P_a E = A$  and  $(I - P_a)E = T$ . The notation  $f_\alpha(A + T)$  indicates that the  $f_\alpha$  depend nonlinearly on  $A$  and  $T$  via the Vlasov equation, Eq. (1a). To begin, we set  $A \equiv 0$  in Eq. (3a) and study the cases in which the electric field is purely transient ( $E = T$ ). As long as  $T$  decays in time at least as  $t^{-\eta}$ ,  $\eta > 2$ , it is easy to see that the corresponding single-particle trajectories tend to straight lines as  $t \rightarrow \infty$ , and  $f_\alpha(T)$  tends to a *time-asymptotic Vlasov equilibrium*  $F_\alpha^T$  such that

$$0 = \sum_\alpha q_\alpha \int dv F_\alpha^T = P_a \sum_\alpha q_\alpha \int dv f_\alpha(T). \quad (4)$$

Hence,  $A \equiv 0$  satisfies Eq. (3a) for any choice of the initial condition, even though, of course, it does not necessarily correspond to a solution of the complete system, Eqs. (3a) and (3b).

Fundamental to our study is the concept of a *critical initial state*: given a family of initial conditions of the form  $f_\alpha(x, v, 0) = F_\alpha(v) + h_\alpha(x, v)$ , we call a function  $h_\alpha^0(x, v)$  a critical initial state if  $F_\alpha(v) + h_\alpha^0(x, v)$  leads to  $A \equiv 0$ , but arbitrarily close to  $h_\alpha^0$  there exist other  $h_\alpha$  that generate a “branch” of nonzero small amplitude time-asymptotic fields (Fig. 1). Physically, this corresponds to a continuous transition between the initial states from which the field Landau damps to zero and those that produce, via nonlinear particle trapping, final states with a nonzero field. Combining this definition and the fact that  $A \equiv 0$  is always a solution to Eq. (3a) leads to an important conclusion: if there exists a function  $h_\alpha^0$  that is a critical initial state for the VP system, then  $(A, h_\alpha) = (0, h_\alpha^0)$  is a *bifurcation point* for Eq. (3a) in isolation, where  $h_\alpha$  plays the role of an infinite dimensional parameter.

This fact suggests that we analyze Eq. (3a) using the methods of bifurcation theory. The general structure of the solution for  $A$  near any bifurcation point  $(0, h_\alpha^0)$  can be obtained by *linearizing* Eq. (3a) about the critical state. Interestingly, this is not what would be obtained by linearizing the VP system and then taking the limit  $t \rightarrow \infty$ . Whereas the standard linearization gives results that are not uniformly valid in time, Eq. (3a) allows us to linearize *after* the time-asymptotic limit has been taken via  $P_a$ . In practice, we take this limit using the *Fourier-Bohr* (FB) transform, which for an asymptotically almost periodic function  $G(x, t)$  is defined as

$$g_{k, \omega_i} = \frac{1}{\pi} \lim_{\sigma \rightarrow \infty} \frac{1}{\sigma} \int_0^\sigma dt \int_{-\pi}^{+\pi} dx e^{-ikx - i\omega_i t} G(x, t) \equiv \mathcal{B}G. \quad (5)$$

In this transformation the traditional Fourier integral in time is replaced by the Bohr average  $\lim_{\sigma \rightarrow \infty} \frac{1}{\sigma} \int_0^\sigma dt$ , which has the important property of *filtering out* the transient part of  $G$  [10]. After carrying out this operation on Eq. (3a), we substitute the expression for  $f_\alpha$  obtained by integrating the nonlinear Vlasov equation along straight line trajectories. Following an integration by parts in  $t$ , the transient contributions disappear under the action of  $\mathcal{B}$ , and Eq. (3a) becomes

$$a_{k, \omega_i} = \frac{2}{k} \sum_\alpha \frac{q_\alpha^2}{m_\alpha} \lim_{\sigma \rightarrow \infty} \frac{1}{\sigma} \int_0^\sigma dt \int_{-\pi}^{+\pi} dx e^{-i\omega_i t - ikx} \times A(x, t) P \int dv \frac{f'_\alpha(x, v, t)}{\omega_i + kv}, \quad (6)$$

which can easily be linearized about the bifurcation point  $(A, h_\alpha) = (0, h_\alpha^0)$ . A simple application of the Riemann-Lebesgue lemma shows that the linearized time-asymptotic equation becomes

$$a_{k, \omega_i} \mathcal{D}(k, \omega_i) = 0, \quad (7)$$

where  $\mathcal{D}(k, \omega_i)$  is the Vlasov dielectric function [11]

$$\mathcal{D}(k, \omega_i) = 1 - \frac{4\pi}{k} \sum_\alpha \frac{q_\alpha^2}{m_\alpha} P \int dv \frac{F_\alpha^{T_0}(v)}{\omega_i + kv}. \quad (8)$$

Here,  $F_\alpha^{T_0}$  is the time-asymptotic Vlasov equilibrium determined by the still unknown transient field  $T_0$  that corresponds to the critical state. From Eq. (7) it follows that, at leading order, the only nonzero FB coefficients of  $A$  are those such that  $k$  and  $\omega_i$  satisfy the time-asymptotic Vlasov dispersion relation  $\mathcal{D}(k, \omega_i) = 0$ . Typically, the corresponding dispersion curve is qualitatively similar to that for a Maxwellian (Fig. 2), and  $\mathcal{D}(k, \omega_i)$  has a finite set of pairs of roots  $\pm \omega_j(k)$ . Hence, in the small-amplitude limit the general solution for  $A$  is a finite superposition of traveling waves

$$A(x, t) = \sum_{j=1}^M a_{k, \omega_j} e^{ikx + i\omega_j(k)t} + o(a_{k, \omega_j}), \quad (9)$$

where each  $\omega_j(k)$  satisfies the Vlasov dispersion relation.

We next use these results from the linearized time-asymptotic problem to reduce Eq. (3a) to a finite-dimensional bifurcation problem for the  $M$  amplitudes  $a_{k, \omega_j(k)}$  in Eq. (9). At leading order, this can be done by simply evaluating the FB transform of Eq. (3a) at all the pairs  $[k, \omega_j(k)]$  and showing that the effects of the higher-order terms in Eq. (9) on the distribution functions are negligible. The problem is, however, that  $f_\alpha$  still has to be determined from the nonlinear Vlasov equation, Eq. (1a). This is very difficult, because the characteristics associated with the general electric field  $E$  cannot be obtained. However, the decomposition in

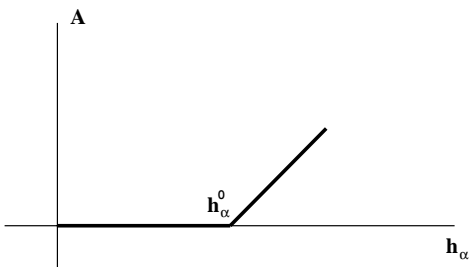


FIG. 1. A time-asymptotic bifurcating branch (schematic).

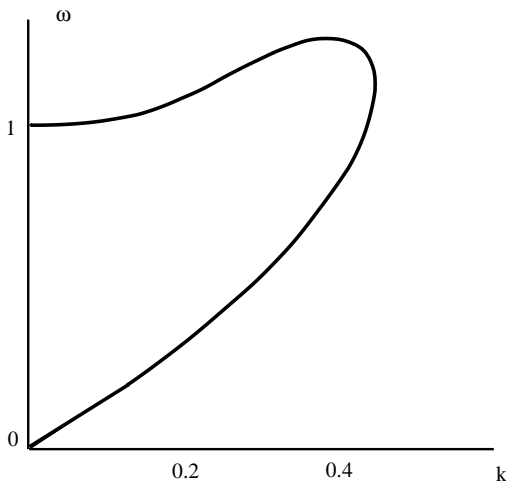


FIG. 2. The Vlasov dispersion curve for a thermal  $e$ - $p$  plasma with  $T_e = T_p$ ; the units for  $k$  and  $\omega$  are the inverse electron Debye length and the electron plasma frequency.

Eq. (2) enables us to overcome this difficulty by linearizing *only* the transient dynamics, while retaining the full nonlinear interaction between  $f_\alpha$  and  $A$ . This is done by approximating Eq. (1a) by its *transiently linearized* form

$$\frac{\partial f_\alpha}{\partial t} + v \frac{\partial f_\alpha}{\partial x} + \frac{q_\alpha}{m_\alpha} A \frac{\partial f_\alpha}{\partial v} = -\frac{q_\alpha}{m_\alpha} T \frac{\partial \mathcal{F}_\alpha}{\partial v}, \quad (10)$$

where the function  $\frac{\partial f_\alpha}{\partial v}$  that multiplied the transient field  $T$  has been replaced by the corresponding derivative of the initial condition  $\mathcal{F}_\alpha(x, v) \equiv f_\alpha(x, v, 0)$ , but the nonlinear term  $A \frac{\partial f_\alpha}{\partial v}$  remains unchanged. Thus, the approximation affects only the interaction between the plasma and the transient electric field. In general, as time grows,  $\frac{\partial \mathcal{F}_\alpha}{\partial v}$  ceases to be a good approximation to  $\frac{\partial f_\alpha}{\partial v}$ ; however, this does not generally invalidate Eq. (10), because in the same limit  $T \rightarrow 0$ . If  $\tau_T$  denotes a typical time scale for the decay of  $T$  and  $\tau_b$  is the familiar “trapping time scale” [2], the condition for the uniform validity of Eq. (10) is  $\tau_T \ll \tau_b$ , which holds in much greater generality than the criterion  $\tau_{\text{Landau}} \ll \tau_b$  for the validity of the standard linearized Vlasov equation required by O’Neil [2]. Indeed, Eq. (10) includes both the limiting cases discussed by O’Neil. The one extreme when the trapping time scale is much longer than the time scale for linear Landau damping corresponds to  $A \equiv 0$ ; then Eq. (10) reduces to a linearized Vlasov equation with  $E = T$  and  $F_\alpha$  generalized to  $\mathcal{F}_\alpha$ . Landau’s solution is then recovered at leading order. In the other extreme, O’Neil’s assumption of an almost undamped strongly trapping wave corresponds to  $T \equiv 0$ ; in this case Eq. (10) simply becomes the nonlinear Vlasov equation with  $E \equiv A$ , which O’Neil solved assuming a field comprised of a single sinusoidal wave. Of course, this second limiting case also includes all exact undamped traveling-wave Bernstein-Greene-Kruskal solutions [12–14] and their generalizations [9].

Equation (10) can be solved analytically if the characteristics associated with  $A$  can be determined. For a general multiple-wave field like the dominant term in Eq. (9), we have obtained the characteristics perturbatively using the sequence of canonical transformations introduced by Buchanan and Dornig [9]. Thus, we have solved Eq. (10) in the form

$$f_\alpha(x, v, t) = \mathcal{F}_\alpha[x_0^A(x, v, t), v_0^A(x, v, t)] - \frac{q_\alpha}{m_\alpha} \int_0^t d\tau \left\{ T \frac{\partial \mathcal{F}_\alpha}{\partial v} \right\}_{[x_\tau^A(x, v, t), v_\tau^A(x, v, t)]}, \quad (11)$$

where  $[x_\tau^A(x, v, t), v_\tau^A(x, v, t)]$  is the location in phase space at time  $\tau$  of a particle that arrives at the point  $(x, v)$  at time  $t$ . Equation (11) is then substituted into Eq. (3a) to obtain an explicit equation for  $A$  with  $T$  and  $h_\alpha$  (through  $\mathcal{F}_\alpha$ ) as parameters. However, our final goal is to obtain the self-consistent solutions  $T$  and  $A$  to the complete system, Eqs. (3a) and (3b), both of which depend on  $f_\alpha$ . Hence, we substitute Eq. (11), first, into the transient equation, Eq. (3b). Solving for  $T$  in terms of the small amplitudes  $a_{k, \omega_j}$ , then substituting  $f_\alpha$  and  $T$  into Eq. (3a) and carrying out the FB transform yields an explicit system of nonlinear algebraic equations for the FB coefficients  $a_{k, \omega_j}$  in Eq. (9). For small initial perturbations, the transient equation is amenable to a standard perturbation analysis, since the decay properties of  $T$  neutralize the secularities that arise in a direct perturbation solution of the original VP system in which the time-asymptotic part  $A$  has not been separated. The perturbative solution we obtain for  $T$  is very similar to the standard Landau solution for the linearized VP problem and decays exponentially in time (which is consistent with the requirement introduced above that  $T$  decay at least as  $t^{-\eta}$  for  $t \rightarrow \infty$ ,  $\eta > 2$ ).

*Two-wave case.*—In order to give explicit analytical results we consider the classic problem of a sinusoidal perturbation to a linearly stable (e.g., Maxwellian) Vlasov equilibrium, i.e.,  $\mathcal{F}_\alpha(x, v) \equiv F_\alpha(v) + \epsilon h_\alpha(v) \cos x$ . Near a critical state, the fundamental Langmuir modes will be dominant at long times, whereas all other modes can be neglected. This fact, and symmetry considerations [9], lead to  $A$  as a pair of counterpropagating waves

$$A(x, t) = a \sin(x - v_p t) + a \sin(x + v_p t), \quad (12)$$

where  $\pm v_p$  correspond to the dominant roots of Eq. (8). Because of the symmetry of this case, it also follows that  $F_\alpha^T \equiv F_\alpha$ . As  $t \rightarrow \infty$ , spatially periodic plateaus develop on the distribution functions  $f_\alpha$  at  $v = \pm v_p$  whenever  $a \neq 0$ ; of course, at the bifurcation point (about which the expansion is made)  $a \rightarrow 0$  and  $f_\alpha \rightarrow F_\alpha^{T_0}$  as  $t \rightarrow \infty$ . The characteristics  $[x_\tau^A, v_\tau^A]$  for the field  $A$  in Eq. (12) have been calculated via Hamiltonian perturbation theory as in [9]. Substituting  $[x_\tau^A, v_\tau^A]$  and the perturbative solution for  $T$  into Eq. (11) gives the explicit solution

for the distribution function  $f_\alpha$  in Eq. (3a). Then, the FB transform of Eq. (3a) evaluated at  $(k, \omega_i) = (1, \nu_p)$  yields a simple nonlinear scalar equation for  $a$  in terms of  $\epsilon$ . An asymptotic expansion in  $a$  of this equation gives

$$[\beta\epsilon - \Gamma(\epsilon)]a^{1/2} + \sigma(\epsilon)a^{3/2} = O(a^2), \quad (13)$$

where the coefficients  $\beta$ ,  $\Gamma(\epsilon)$ , and  $\sigma(\epsilon)$  depend on the initial distribution function [15] and  $\Gamma(0) = 0$ . Clearly, nonzero solutions of Eq. (13) for  $a$  can bifurcate from the trivial solution branch  $a = 0$  only at the *critical initial amplitudes*  $\epsilon = \epsilon^0$  that satisfy the *threshold equation*

$$\beta\epsilon^0 = \Gamma(\epsilon^0). \quad (14)$$

$$\frac{\sum_\alpha q_\alpha [P \int \frac{h_\alpha(v)}{(v \mp \nu_p)} + i\pi h_\alpha(\nu_p)]}{\sum_\alpha \frac{q_\alpha^2}{m_\alpha} [P \int \frac{F_\alpha''(v)}{(v \mp \nu_p)} + i\pi F_\alpha''(\nu_p)]} = \frac{\sum_\alpha q_\alpha [\frac{|q_\alpha|}{m_\alpha}]^{1/2} h_\alpha(\nu_p)}{\sum_\alpha q_\alpha^2 [\frac{|q_\alpha|}{m_\alpha^3}]^{1/2} F_\alpha''(\nu_p)}. \quad (17)$$

Physically, Eq. (16) implies a slow linear damping rate, since the Landau damping coefficient [1] is proportional to  $\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \frac{dF_\alpha}{dv}(\pm \nu_p)$ ; if this were not the case, all small-amplitude initial conditions would be completely damped, and one should look for critical initial amplitudes that are not small, for example, by solving Eqs. (3b) and (14) numerically. Similarly, Eq. (17) gives a condition for the *nonlinear* dynamics not to damp every small initial perturbation. When both these conditions are satisfied, the expansion of Eq. (14) gives a small critical initial amplitude at

$$\epsilon_1^0 = \frac{2[\beta - \Gamma'(0)]}{\Gamma''(0)} = \frac{\sigma \{ \sum_\alpha q_\alpha [\frac{|q_\alpha|}{m_\alpha}]^{1/2} h_\alpha(\nu_p) - S_{1,1} \sum_\alpha q_\alpha^2 [\frac{|q_\alpha|}{m_\alpha^3}]^{1/2} \frac{dF_\alpha}{dv}(\nu_p) \}}{\sum_\alpha q_\alpha [\frac{|q_\alpha|^3}{m_\alpha^3}]^{1/2} [2S_{2,2} \frac{dF_\alpha}{dv}(\nu_p) + S_{1,2} \frac{dh_\alpha}{dv}(\nu_p)]}. \quad (18)$$

Here  $\sigma = -19.58$ ;  $S_{i,j} = \int_0^\infty d\tau T_{0,i}(\tau) \sin j\omega\tau$ , where  $T_{0,i}$  is the  $i$ th spatial Fourier coefficient of the transient field at the critical point, which coincides with Landau's solution for the initial condition with amplitude  $\epsilon_1^0$ . The final electric field is zero for  $\epsilon \leq \epsilon_1^0$ , and for  $\epsilon \geq \epsilon_1^0$  it is a nonlinear superposition of traveling waves, Eq. (12), with amplitude

$$a = \frac{\sum_\alpha q_\alpha [\frac{|q_\alpha|}{m_\alpha}]^{1/2} h_\alpha(\nu_p)}{\sum_\alpha q_\alpha^2 [\frac{|q_\alpha|}{m_\alpha^3}]^{1/2} \frac{d^2 F_\alpha}{dv^2}(\nu_p)} (\epsilon - \epsilon_1^0). \quad (19)$$

These theoretical results establishing the existence of critical initial states have implications concerning plasma stability. According to our nonlinear analysis, for "single-humped" Vlasov equilibria, which traditionally have been described as "linearly stable," the electric field does damp to zero as  $t \rightarrow \infty$  if  $\epsilon \leq \epsilon_1^0$  (although the  $f_\alpha$  do not). However, our analysis also shows that the electric field does not damp to zero for initial conditions with  $\epsilon > \epsilon_1^0$ , which in fact can be very small. This result refutes a conclusion, from a recent analysis [8], which already has been called into question in recent Letters [6,7].

Expanding Eq. (13) in powers of  $\epsilon - \epsilon^0$  near  $\epsilon^0$  yields the local nonzero solution for  $a$

$$a = -\frac{\beta - \Gamma'(\epsilon^0)}{\sigma(\epsilon^0)} (\epsilon - \epsilon^0) + O[(\epsilon - \epsilon^0)^{3/2}]. \quad (15)$$

In the case of small initial perturbations about a Vlasov equilibrium, any critical initial amplitude  $\epsilon_1^0$  also must be small, and Eqs. (3b) and (14) can be expanded about  $\epsilon_1^0 = 0$  [which is a root of Eq. (14) since  $\Gamma(0) = 0$ ]. The analysis that follows shows such a small critical amplitude exists only if the initial distribution functions satisfy the following two conditions:

$$\sum_\alpha \frac{q_\alpha^2}{m_\alpha} \frac{dF_\alpha}{dv}(\pm \nu_p) = O(\epsilon_1^0), \quad (16)$$

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