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CRITICAL LENGTHS FOR SEMILINEAR SINGULAR PARABOLIC MIXED BOUNDARY-VALUE PROBLEMS

By

C. Y. CHAN AND SHANNON S. COBB

University of Southwestern Louisiana, Lafayette, Louisiana

1. Introduction. Let

$$\begin{aligned} Hu &\equiv u_{xx} - u_t, \\ \Omega &\equiv (0, a) \times (0, T), \\ \Gamma &\equiv ([0, a] \times \{0\}) \cup (\{0\} \times (0, T)), \\ S &\equiv \{a\} \times (0, T), \end{aligned}$$

where $T \leq \infty$. Also, let u be a solution of the problem:

$$Hu = -f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \qquad u = 0 \quad \text{on } S, \tag{1.1}$$

where $f(u)$ tends to infinity as u approaches c^- for some positive constant c. The length a^* is said to be the critical length for the problem (1.1) if u exists globally for $a < a^*$, and for $a > a^*$ there exists a finite time T such that

$$\max\{u(x, t) : 0 \leq x \leq a\} \to c^- \quad \text{as } t \to T^-. \tag{1.2}$$

This finite time T is called the quenching time. In the special case that $f(u) = (1-u)^{-1}$, Kawarada [9] showed that (1.2) occurred for $a > 2^{3/2}$. Acker and Walter [2] showed that under appropriate conditions on the forcing term $f(u)$, there existed a unique critical length a^* for the problem (1.1). This result was then extended to forcing terms of the type $g(u, u_x)$ by Acker and Walter [3], and to $h(x, u, u_x)$ by Chan and Kwong [7]. Results on the behavior of the solution of the problem (1.1) with $a = a^*$ were given by Levine and Montgomery [10]. Existence of the critical length a^* and its determination by computational methods were given by Chan and Chen [4] for a more general parabolic singular operator; they studied the problem:

$$Lu = -(1-u)^{-1} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \quad u = 0 \quad \text{on } S,$$

where $Lu \equiv Hu + bu_x/x$ with b a constant less than 1; in particular, $a^* = 1.5303$ (to five significant figures) for $b = 0$. Similar results were given by Chan and Kaper [6] for the problem:

$$Lu = -f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \qquad u_x = 0 \quad \text{on } S. \tag{1.3}$$

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This includes the problem (1.1) as a special case since the solution of that problem is symmetric with respect to the line $x = a/2$. We refer to the papers of Chan and Chen [4] and Chan and Kaper [6] for the significance of the expression Lu. Critical lengths for global existence of solutions for a coupled system of two semilinear parabolic equations subject to zero initial-boundary data were given by Chan and Chen [5]. Existence of the critical size for the multidimensional version of the problem (1.1) was studied by Acker and Kawohl [1].

The main purpose here is to study the critical length for the following problem:

$$Lu = -f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \Gamma, \qquad Bu = 0 \quad \text{on } S, \tag{1.4}$$

where $Bu \equiv u_x + ku$. Here, b is a constant less than 1; k is a positive constant; f is nondecreasing and continuously differentiable on $[0, c)$ for some constant c such that $f(0) > 0$; and $\lim_{u\to c^-} f(u) = \infty$. As in the papers by Chan and Chen [4] and Chan and Kaper [6], we assume existence of a solution u before its quenching time. In the problem (1.3), u attains its maxima with respect to x at $x = a$; unlike the problem (1.1), the singular term bu_x/x as well as the third boundary condition in our present problem destroys the symmetry of the solution u about the line $x = a/2$, and shifts the points where u attains its maxima with respect to x from the line $x = a/2$. Thus, they make the problem more difficult both theoretically and numerically.

In Sec. 2, we establish existence of a critical length a^*, and give a computational method to determine a^*. In Sec. 3, a method is given to determine an upper bound of the quenching time for a given a greater than a^*. An algorithm is given in Sec. 4 to compute a^*. For illustration, a numerical example is given by taking $f(u)$ to be $(1-u)^{-1}$.

2. Critical length. Let us first establish the following results.

LEMMA 1. Let u be a solution of the problem (1.4).

(a) There exists at most one solution.
(b) The solution u is positive in $\Omega \cup S$.
(c) The solution u is a strictly increasing function of t for each $x \in (0, a]$.
(d) There exists a curve $\phi(t)$ such that for each $t \in (0, T)$, u is strictly decreasing in x on $(\phi(t), a]$, and nondecreasing in x on $[0, \phi(t)]$, where $\phi(t) \in (0, a)$.

Proof. (a) Let u_1 and u_2 be two distinct solutions, and $w \equiv u_1 - u_2$. Then by the mean value theorem,

$$[L + f'(\eta)]w = 0 \quad \text{in } \Omega,$$

where η lies between u_1 and u_2. Without loss of generality, let $w > 0$ somewhere. Since $f'(\eta)$ is bounded above, it follows from the strong maximum principle (cf. Protter and Weinberger [12, pp. 168–169, 172, and 175]) that w attains its positive maximum somewhere on S. At this point, $w_x > 0$ by the parabolic version of Hopf's lemma (cf. Protter and Weinberger [12, pp. 170–172]). This contradicts $Bw = 0$ on S. Thus, there exists at most one solution.

(b) Since $f(0) > 0$, we have $Lu + f(u) < f(0)$. By the mean value theorem, $[L + f'(\eta)]u < 0$, where η lies between u and 0. The assertion then follows from the strong maximum principle and the parabolic version of Hopf's lemma.

(c) For any $h > 0$, let

$$w(x, t) = u(x, t+h) - u(x, t).$$

By the mean value theorem, $[L + f'(\eta)]w = 0$, where η lies between $u(x, t+h)$ and $u(x, t)$. Since $w(x, 0) > 0$ for $0 < x \leq a$, $w(0, t) = 0$, and $Bw = 0$ on S, it follows from the strong maximum principle and the parabolic version of Hopf's lemma that $w > 0$ on $\Omega \cup S$. The assertion is then proved.

(d) It follows from Lemma 1(b) that $u_x(a, t) = -ku(a, t) < 0$ for $0 < t < T$; by the parabolic version of Hopf's lemma, $u_x(0, t) > 0$ for $0 < t < T$. For any fixed t and any positive x_0 $(\leq a)$ such that $u_x(x_0, t) < 0$, it follows from the mean value theorem that for any positive ϵ $(\leq x_0)$,

$$0 < u(\epsilon, t) - u(0, t) = u_x(\eta, t)\epsilon \quad \text{for some } \eta \in (0, \epsilon).$$

Thus for each t (> 0), there exists a point $x \in (0, x_0)$ such that $u_x(x, t) = 0$.

Differentiating the differential equation in (1.4) with respect to x, we obtain

$$(L + f'(u) - b/x^2)u_x = 0.$$

Let G be the component containing S such that $u_x < 0$ in G. Since G does not intersect the line $x = 0$, it follows by applying the strong maximum principle that G is simply connected with $u_x = 0$ on $\partial G \cap \Omega$, where ∂G denotes the boundary of G. If $u_x(x_1, t_1) < 0$ somewhere in $\Omega \setminus G^-$, where G^- denotes the closure of G, then by the continuity of u_x, there exists a neighborhood N of (x_1, t_1) such that $u_x < 0$ in N and $u_x = 0$ on $\partial N \cap (\Omega \setminus G)$, but this contradicts the strong maximum principle. Thus, $u_x \geq 0$ in $\Omega \setminus G^-$, and $\partial G \cap \Omega = \phi(t)$.

Let

$$lU \equiv U'' + \frac{b}{x}U', \qquad \beta U \equiv U' + kU.$$

With slight modification of the proof of Theorem 3 of Chan and Kaper [6], we obtain the following result.

THEOREM 2. If $T = \infty$ and $u(x, t) \leq C < c$ for some constant C, then u converges uniformly on $[0, a]$ from below to a solution U of the singular nonlinear two-point boundary-value problem:

$$lU = -f(U), \qquad U(0) = 0 = \beta U(a). \tag{2.1}$$

Furthermore, $u < U$ in $(0, a] \times [0, \infty)$.

In order to show that beyond the critical length there exists a finite time T such that (1.2) holds, the following result is crucial.

THEOREM 3. $Bu(x, t) \geq 0$ in Ω.

Proof. For any $\epsilon \in (0, a)$, let

$$\Omega_\epsilon \equiv (\epsilon, a) \times (0, T),$$
$$\Gamma_\epsilon \equiv ([\epsilon, a] \times \{0\}) \cup (\{\epsilon\} \times (0, T)).$$

Let u_ϵ denote the solution of the (regular) problem:

$$\begin{gathered} Lu_\epsilon = -f(u_\epsilon) \quad \text{in } \Omega_\epsilon\,, \\ u_\epsilon = 0 \quad \text{on } \Gamma_\epsilon\,, \qquad Bu_\epsilon = 0 \quad \text{on } S. \end{gathered} \tag{2.2}$$

An argument as in the proofs of Lemma 1(b) and (c) shows that $u_\epsilon > 0$ in $\Omega_\epsilon \cup S$, and u_ϵ is a strictly increasing function of t for each $x \in (\epsilon, a]$. It follows from the strong maximum principle and the parabolic version of Hopf's lemma that u_ϵ strictly increases as ϵ decreases. In particular, we have $0 < u_\epsilon < u$ in Ω_ϵ. Let us differentiate (2.2) with respect to x, and denote the partial derivative of u_ϵ with respect to x by $u_{\epsilon,x}$. We obtain

$$[L + f'(u_\epsilon) - b/x^2]u_{\epsilon,x} = 0 \quad \text{in } \Omega_\epsilon\,.$$

Now,

$$u_{\epsilon,x}(x, 0) = 0 \quad \text{for } \epsilon \le x \le a.$$

For any $\tau \in (0, T)$,

$$u_{\epsilon,x}(\epsilon, t) \ge 0 \quad \text{and} \quad u_{\epsilon,x}(a, t) = -ku_\epsilon(a, t) < 0 \quad \text{for } 0 < t < \tau.$$

Let $\Omega_{\epsilon\tau} \equiv [\epsilon, a]\times[0, \tau]$. By the strong maximum principle, $u_{\epsilon,x}$ attains its negative minimum somewhere on $\Omega_{\epsilon\tau}$ at $x = a$. Since $u_\epsilon(a, t)$ increases as t increases, it follows that $u_{\epsilon,x}(x, t) \ge -ku_\epsilon(a, \tau)$ on $\Omega_{\epsilon\tau}$. An argument as in the proof of Lemma 1(d) shows that there exists a curve $\psi(t)$ such that for each $t \in (0, T)$, $\psi(t) \in (\epsilon, a)$ and u_ϵ is strictly decreasing in x on $(\psi(t), a]$ and nondecreasing in x on $[\epsilon, \psi(t)]$. Thus for $x \in (\psi(\tau), a)$, $Bu_\epsilon(x, \tau) > 0$. Because $u_\epsilon(x, \tau) > 0$ for $x \in (\epsilon, \psi(\tau)]$, $Bu_\epsilon(x, \tau) > 0$ there. Since τ is arbitrary, we have

$$Bu_\epsilon(x, t) > 0 \quad \text{in } \Omega_\epsilon\,. \tag{2.3}$$

Since u_ϵ is bounded, $\lim_{\epsilon\to 0} u_\epsilon$ exists. Let us denote this limit by Z. Then in Ω_ϵ, $0 < u_\epsilon \le Z \le u$ and $BZ \ge 0$.

To prove that $Z = u$, let $\sigma \in (\epsilon, a)$ and u_σ be the unique solution of the (regular) problem:

$$\begin{gathered} Lu_\sigma = -f(u_\sigma) \quad \text{in } \Omega_\sigma\,, \\ u_\sigma(x, 0) = 0, \qquad u_\sigma(\sigma, t) = u_\epsilon(\sigma, t), \qquad Bu_\sigma = 0 \quad \text{on } S. \end{gathered}$$

The adjoint L^* (cf. Friedman [8, p. 26]) of L in Ω_σ is given by

$$L^* v = v_{xx} - (bv/x)_x + v_t$$

with adjoint boundary conditions (cf. Polozhiy [11, p. 413]) given by

$$v(\sigma, t) = 0 = v_x(a, t) + (k - b/a)v(a, t).$$

Let $R^*(\xi, \tau; x, t)$ denote its Green's function (cf. Friedman [8, pp. 82–84 and 155]). In Green's identity (cf. Friedman [8, p. 27]),

$$vLu - uL^*v = (vu_x - uv_x + buv/x)_x - (uv)_t\,,$$

let $u = u_\epsilon$ and $v(\xi, \tau) = R^*(\xi, \tau; x, t)$. Let us integrate this over the domain $(\sigma, a) \times (0, t-\delta)$, where δ is a small positive constant less than t. By letting δ tend to zero, we obtain

$$u_\epsilon(x, t) = \int_0^t \int_\sigma^a R^*(\xi, \tau; x, t) f(u_\epsilon(\xi, \tau))\, d\xi\, d\tau + \int_0^t R_\xi^*(\sigma, \tau; x, t) u_\epsilon(\sigma, \tau)\, d\tau \quad \text{in } \Omega_\sigma .$$

Since $R^*(\xi, \tau; x, t) > 0$ for $(\xi, \tau) \in (\sigma, a) \times (0, t)$ (cf. Friedman [8, p. 84]), it follows that $R_\xi^*(\sigma, \tau; x, t) \geq 0$. As ϵ decreases, u_ϵ and $f(u_\epsilon)$ are nondecreasing. By the monotone convergence theorem (cf. Royden [13, p. 84]),

$$Z(x, t) = \int_0^t \int_\sigma^a R^*(\xi, \tau; x, t) f(Z(\xi, \tau))\, d\xi\, d\tau + \int_0^t R_\xi^*(\sigma, \tau; x, t) Z(\sigma, \tau)\, d\tau \quad \text{in } \Omega_\sigma .$$

Thus, $LZ = -f(Z)$ in Ω_σ. Since σ is arbitrary, it follows that $LZ = -f(Z)$ in Ω. Now, $Z(x, 0) = 0$ and $BZ = 0$ on S. From $0 \leq u_\epsilon \leq Z \leq u$ in Ω, we have $Z(0, t) = 0$. Since u is unique, it follows that $u = Z$. From (2.3), $Bu \geq 0$ in Ω.

Let $u(x, t; a)$ denote the solution $u(x, t)$ of the problem (1.4). Then for any positive constant α, let h be a nonnegative constant such that $h < \alpha$.

Theorem 4. If $\lim_{t \to \infty} u(\phi(t), t; a) = c$, then there exists a finite time T such that

$$\max\{u(x, t; a+\alpha) : 0 \leq x \leq a+\alpha\} \to c^- \quad \text{as } t \to T^- . \tag{2.4}$$

Proof. Let us assume that there does not exist a finite time T such that (2.4) holds. Let

$$w(x, t) = u(x+h, t; a+\alpha) - u(x, t; a).$$

By the mean value theorem,

$$[L + f'(\eta)]w = 0 \quad \text{in } \Omega,$$

where η lies between $u(x+h, t; a+\alpha)$ and $u(x, t; a)$. By Theorem 3, $Bw \geq 0$ on S. Since $w(x, 0) = 0$ and $w(0, t) \geq 0$, it follows from the strong maximum principle and the parabolic version of Hopf's lemma that $w \geq 0$ on $\Omega \cup S$. That is,

$$u(x+h, t; a+\alpha) \geq u(x, t; a) \quad \text{on } \Omega \cup S. \tag{2.5}$$

Let us choose positive numbers ϵ $(< c)$ and t_0 such that

$$f(z) \geq \frac{8\epsilon}{\alpha}\left(\frac{2}{\alpha} + \frac{2|b|}{\phi(t_0) + \alpha/4}\right) + \alpha^2$$

for $z \in [c - \epsilon, c)$ and $u(\phi(t_0), t_0; a) \geq c - \epsilon$. Also, let

$$E \equiv (\phi(t_0) + \alpha/4, \phi(t_0) + \alpha) \times (t_0, \infty).$$

By assumption, $u(x, t; a+\alpha)$ exists for all $t > 0$, and hence $u(x, t; a+\alpha) < c$ in E. From (2.5) and Lemma 1(c), $u(x, t; a+\alpha) \geq c - \epsilon$ on the parabolic boundary ∂E of E. Let

$$z(x, t) = c - \epsilon + [x - \phi(t_0) - \alpha/4][\phi(t_0) + \alpha - x](t - t_0) \quad \text{in } E.$$

On ∂E, $z = c - \epsilon$. By direct computation,

$$Lz = -2(t - t_0) + \frac{b}{x}\{2[\phi(t_0) - x] + 5\alpha/4\}(t - t_0) - [x - \phi(t_0) - \alpha/4][\phi(t_0) + \alpha - x].$$

In the domain

$$(\phi(t_0) + \alpha/4, \phi(t_0) + \alpha) \times (t_0, t_0 + 8\epsilon/\alpha^2),$$

denoted by D, we have for $z \in [c - \epsilon, c)$,

$$Lz + f(z) \geq 0 \quad \text{in } D.$$

By the strong maximum principle, $u(x, t; a + \alpha) > z$ in D. Since

$$z(\phi(t_0) + \alpha/2, t_0 + 8\epsilon/\alpha^2) = c,$$

it follows that

$$u(\phi(t_0) + \alpha/2, t_0 + 8\epsilon/\alpha^2; a + \alpha) \geq c.$$

This contradiction proves the theorem.

We remark that Theorem 2 shows that there exists a critical length a^* such that u exists globally if $a < a^*$. This critical length is determined as the supremum of all a for which a solution U of the problem (2.1) exists; if $U(a^*)$ exists, then $u(a^*, t)$ exists also. Theorem 4 shows that (1.2) holds for some finite time T when $a > a^*$.

To compute a^*, let us construct a sequence $\{U_n\}$ for $a < a^*$ by $U_0 = 0$ for $0 \leq x \leq a$, and for $n = 1, 2, 3 \ldots$,

$$lU_n + f(U_{n-1}) = 0, \qquad U_n(0) = 0 = \beta U(a). \tag{2.6}$$

In terms of Green's function $G(x; \xi)$ corresponding to l, we have

$$U_n(x) = \int_0^a \xi^b G(x; \xi) f(U_{n-1}(\xi))\, d\xi \quad \text{for } n = 1, 2, 3, \ldots, \tag{2.7}$$

where

$$G(x; \xi) = \begin{cases} (1 - q\xi^{1-b})x^{1-b}/(1 - b) & \text{for } 0 \leq x \leq \xi, \\ (1 - qx^{1-b})\xi^{1-b}/(1 - b) & \text{for } \xi \leq x \leq a, \end{cases}$$

with $q = k[(1 - b)/a^b + ka^{1-b}]^{-1}$. The sequence is well defined. From (2.7) and the positivity of Green's function, $U_n(x) > 0$ for $n \geq 1$ and $0 < x \leq a$. Since $U_n'(a) < 0$, it follows that $U_n(x)$ attains its positive maximum somewhere in $(0, a)$. With slight modification of the proof of Theorem 5 of Chan and Kaper [6], we obtain the following result.

THEOREM 5. The sequence $\{U_n\}$ converges monotonically upwards to the minimal solution U $(< c)$ of the problem (2.1); furthermore,

$$0 < U_n < U_{n+1} < U, \qquad 0 < x \leq a, \quad n = 1, 2, 3, \ldots.$$

The results established in the rest of this section are useful for computational purposes. To obtain an upper bound a_u for a^*, let us use $U_1(x)$, which is a lower bound of the solution U of the problem (2.1). From (2.7),

$$U_1(x) = f(0)\left(\left(\frac{a^{1+b}}{1+b} - \frac{qa^2}{2}\right)\frac{x^{1-b}}{1-b} + \left(\frac{1}{2} - \frac{1}{1+b}\right)\frac{x^2}{1-b}\right) \quad \text{for } b \neq -1,$$
$$U_1(x) = f(0)\left(\frac{1-qa^2}{4}x^2 + \frac{x^2}{2}\ln\frac{a}{x}\right) \quad \text{for } b = -1. \tag{2.8}$$

Differentiating (2.8) with respect to x yields

$$U_1'(x) = f(0)\left(\left(\frac{a^{1+b}}{1+b} - \frac{qa^2}{2}\right)x^{-b} + \left(1 - \frac{2}{1+b}\right)\frac{x}{1-b}\right) \quad \text{for } b \neq -1,$$
$$U_1'(x) = f(0)\left(-\frac{qa^2}{2}x + x\ln\frac{a}{x}\right) \quad \text{for } b = -1,$$

from which $U_1'(x) = 0$ occurs at

$$x_c = \{[2a^{1+b} - qa^2(1+b)]/2\}^{1/(1+b)} \quad \text{for } b \neq -1,$$
$$x_c = ae^{-qa^2/2} \quad \text{for } b = -1,$$

where $U_1'' = -f(0) < 0$. This implies that the (absolute) maximum of $U_1(x)$ occurs at the value x_c. Thus, an upper bound a_u for a^* is determined by $U_1(x_c) = c$, which yields

$$2a_u^{1+b} - q(1+b)a_u^2 = 2[2(1-b)c/f(0)]^{(1+b)/2} \quad \text{for } b \neq -1,$$
$$4c = f(0)a_u^2 e^{-ka_u/(2+ka_u)}[1 + ka_u/(2+ka_u)] \quad \text{for } b = -1. \tag{2.9}$$

To show that (2.9) determines exactly one a_u for a given b, let us differentiate (2.8) with respect to a:

$$\frac{\partial U_1}{\partial a} = \frac{q^2 f(0)x^{1-b}}{k^2}\left((1-b)a^{-b} + \frac{k^2}{2}a^{2-b} + \frac{k(2+b-b^2)}{2(1+b)}a^{1-b}\right) \quad \text{for } b \neq -1,$$
$$\frac{\partial U_1}{\partial a} = \frac{f(0)x^2(4 + 3ka + k^2a^2)}{2a(4 + 4ka + k^2a^2)} \quad \text{for } b = -1.$$

In either case, $\partial U_1/\partial a > 0$. Thus, U_1 increases as a increases. Hence for a given b, a_u is determined uniquely by (2.9). We obtain the following result.

LEMMA 6. $0 < a^* < a_u$, where a_u is determined uniquely by (2.9) for each given b.

Our next result is useful in stopping the computation of successive iterates.

LEMMA 7. For $0 < x < a$, if f' is strictly increasing and $U_{n+1} - U_n > U_n - U_{n-1}$ for some positive integer n, then $U_{m+1} - U_m > U_m - U_{m-1}$ for $m = n+1, n+2, n+3, \ldots$.

Proof. The sequences $\{U_n\}$ and $\{f(U_n)\}$ are strictly increasing. For some η between U_{n+1} and U_n, and some ζ between U_n and U_{n-1}, we have

$$\begin{aligned} U_{n+2}(x) - U_{n+1}(x) &= \int_0^a \xi^b G(x;\xi)[f(U_{n+1}(\xi)) - f(U_n(\xi))]\,d\xi \\ &= \int_0^a \xi^b G(x;\xi) f'(\eta)[U_{n+1}(\xi) - U_n(\xi)]\,d\xi \\ &> \int_0^a \xi^b G(x;\xi) f'(\zeta)[U_n(\xi) - U_{n-1}(\xi)]\,d\xi \\ &= U_{n+1}(x) - U_n(x). \end{aligned}$$

The lemma then follows by using mathematical induction.

We now show that each iterate is a unimodal function.

LEMMA 8. For $a < a^*$, and each $n \geq 1$, the function $U_n(x)$ has a unique (positive) maximum.

Proof. Let h be a critical point of $U_n(x)$ $(n \geq 1)$ in the interval $(0, a)$. From (2.6),

$$U_n''(h) = -f(U_{n-1}) < 0,$$

which shows that all critical points of $U_n(x)$ give relative maxima. Hence, there is exactly one (positive) maximum.

Since $l(U_{n+1} - U_n) \leq 0$, a proof similar to Lemma 8 gives the following result.

LEMMA 9. For $a < a^*$ and each $n \geq 0$, the difference $U_{n+1}(x) - U_n(x)$ has a unique (positive) maximum.

3. Quenching time. To obtain an upper bound for the quenching time, we may consider the singular Sturm-Liouville problem:

$$lw = -\lambda^2 w, \qquad w(0) = 0, \qquad \beta w(a) = 0.$$

Its eigenvalues λ^2 are determined by

$$\lambda J_{\nu-1}(\lambda a) + k J_\nu(\lambda a) = 0,$$

where $\nu = (1-b)/2$ and $J_\nu(x)$ is the Bessel function of the first kind of order ν. The eigenfunction corresponding to the smallest positive eigenvalue μ^2 is $x^\nu J_\nu(\mu x)$. Following the argument of Sec. 4 of Chan and Kaper [6], the upper bound t_1 for the quenching time is determined by

$$\left[\max_{0\leq x\leq a} x^\nu J_\nu(\mu x)\right] g(t_1) = c,$$

where $g(t)$ is given by the problem

$$g'(t) + \mu^2 g(t) = G(g(t)), \qquad g(0) = 0;$$

here,

$$G(g(t)) \leq \inf\left\{ \frac{f(x^\nu J_\nu(\mu x) g(t))}{x^\nu J_\nu(\mu x)} : x \in [0, a] \right\}.$$

In particular, for $f(u) = (1-u)^{-1}$,

$$t_1 = \mu^{-1}(4-\mu^2)^{-1/2}\tan^{-1}[\mu(4-\mu^2)^{-1/2}] - (2\mu^2)^{-1}\ln[(4-\mu^2)/4] + (\ln 2)(4-\mu^2)^{-1}.$$

4. Numerical algorithm. By Lemma 6, an upper bound a_u of a^* can be determined for each given b by using the subroutine DZREAL (to find, to double precision, the real zeros of a real function using Muller's method) from the IMSL MATH/LIBRARY (Version 1.1, January, 1989; MALB-USM-PERFCT-EN8901-1.1). Since 0 can be taken as a lower bound of a^*, we can use the method of bisection to approximate a^* by $a^{**} = a_u/2$. We use the representation formula (2.7) to compute $U_n(x)$ with $n \ge 1$ by using the following steps:

1. We divide the interval $[0, a^{**}]$ into 20 equal subintervals with end points x_i satisfying $0 = x_1 < x_2 < x_3 < \cdots < x_{21} = a^{**}$.

2. At the 19 interior subdivision points, we evaluate

$$y_1(x) \equiv x^{1-b}/(1-b), \qquad y_2(x) \equiv (1-qx^{1-b})/(1-b);$$

we also compute $y_2(x_{21})$. These values are stored in the memory of the computer for future use.

3. Let

$$F_{n1}(j, k) = \int_{x_j}^{x_k} \xi f(U_{n-1}(\xi))\,d\xi, \qquad F_{n2}(j, k) = \int_{x_j}^{x_k} (\xi^b - q\xi) f(U_{n-1}(\xi))\,d\xi.$$

To save computer time, we evaluate $U_n(x_{11})$ first. From (2.7),

$$U_n(x_{11}) = y_2(x_{11})F_{n1}(1, 11) + y_1(x_{11})F_{n2}(11, 21).$$

To obtain $U_n(x_{10})$, we only need to compute $F_{n1}(10, 11)$ and $F_{n2}(10, 11)$ since

$$U_n(x_{10}) = y_2(x_{10})[F_{n1}(1, 11) - F_{n1}(10, 11)] + y_1(x_{10})[F_{n2}(11, 21) + F_{n2}(10, 11)].$$

In this way, we can successively compute U_n at $x_{10}, x_9, x_8, \ldots, x_2$. Similarly,

$$U_n(x_{12}) = y_2(x_{12})[F_{n1}(1, 11) + F_{n1}(11, 12)] + y_1(x_{12})[F_{n2}(11, 21) - F_{n2}(11, 12)].$$

Proceeding in this way, we obtain successively U_n at $x_{12}, x_{13}, x_{14}, \ldots, x_{21}$.

To use a computer to calculate $U_n(x)$, we use three subroutines from the IMSL MATH/LIBRARY: DCSINT (to compute, to double precision, the cubic spline interpolant with the 'not-a-knot' condition) and DQDAG (to integrate, to double precision, a function using a globally adaptive scheme based on Gauss-Kronrod rules) with DCSVAL (to evaluate, to double precision, a cubic spline).

4. We use the subroutine DUVMGS (to find, to double precision, the minimum point of a nonsmooth (unimodal) function of a single variable) to determine $\max_{0 \le x \le a^{**}} U_n(x)$ without any initial guesswork of where its critical point is since, by Lemma 8, $U_n(x)$ is unimodal. Let us denote this maximum value by M.

5. We stop the computation of $U_n(x)$ as follows:

(a) If $M \ge c$, then $a^{**} > a^*$.

(b) If $U_n - U_{n-1} > U_{n-1} - U_{n-2}$ for some n, then, by Lemma 7, $a^{**} > a^*$, provided f' is strictly increasing.

(c) If $M < c$ and (by using Lemma 9)

$$\max_{0 \le x \le a} [U_n(x) - U_{n-1}(x)] < 5 \times 10^{-(r+1)}$$

for some arbitrarily chosen nonnegative integer r, then $a^{**} < a^*$. Here, r determines the error tolerance in computing the successive iterates.

If $a^{**} > a^*$, then we replace a_u by a^{**}; otherwise u exists globally, and we replace 0 by a^{**}. The above procedure of bisection is repeated until we reach the demanded accuracy (such as the difference between two successive approximations of a^* is less than $5 \times 10^{-(r+1)}$). Since the difference between a^* and the (ultimate) approximation a^{**} can be made as small as we like, this value a^{**} can be taken numerically to be a^*.

We apply the above algorithm to the case $f(u) = (1-u)^{-1}$ and $k = 1$. We compute critical lengths a^* for various given values of b with the use of a computer. The results with $r = 5$ are given in Table 1.

TABLE 1. Critical lengths a^* for four values of b.

b	a^*
0.40000	0.82415
0.00000	0.99514
−0.40000	1.14290
−1.00000	1.33802

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REFERENCES

[1] A. Acker and B. Kawohl, *Remarks on quenching*, Nonlinear Anal. **13**, 53–61 (1989)
[2] A. Acker and W. Walter, *The quenching problem for nonlinear parabolic differential equations*, Lecture Notes in Math., Vol. 564, Springer-Verlag, New York, 1976, pp. 1–12
[3] A. Acker and W. Walter, *On the global existence of solutions of parabolic differential equations with a singular nonlinear term*, Nonlinear Anal. **2**, 499–505 (1978)
[4] C. Y. Chan and C. S. Chen, *A numerical method for semilinear singular parabolic quenching problems*, Quart. Appl. Math. **47**, 45–57 (1989)
[5] C. Y. Chan and C. S. Chen, *Critical lengths for global existence of solutions for coupled semilinear singular parabolic problems*, Quart. Appl. Math. **47**, 661–671 (1989)
[6] C. Y. Chan and H. G. Kaper, *Quenching for semilinear singular parabolic problems*, SIAM J. Math. Anal. **20**, 558–566 (1989)
[7] C. Y. Chan and M. K. Kwong, *Existence results of steady-states of semilinear reaction-diffusion equations and their applications*, J. Differential Equations **77**, 304–321 (1989)
[8] A. Friedman, *Partial Differential Equations of Parabolic Type*, Prentice-Hall, Englewood Cliffs, NJ, 1964, pp. 26–27, 82–84 and 155
[9] H. Kawarada, *On solutions of initial-boundary problem for* $u_t = u_{xx} + (1-u)^{-1}$, Publ. Res. Inst. Math. Sci. **10**, 729–736 (1975)
[10] H. A. Levine and J. T. Montgomery, *The quenching of solutions of some nonlinear parabolic equations*, SIAM J. Math. Anal. **11**, 842–847 (1980)
[11] G. N. Polozhiy, *Equations of Mathematical Physics*, Hayden, New York, 1967, p. 413
[12] M. H. Protter and H. F. Weinberger, *Maximum Principles in Differential Equations*, Springer-Verlag, New York, 1984, pp. 168–172 and 175
[13] H. L. Royden, *Real Analysis*, 2nd ed., Macmillan, New York, 1968, p. 84