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CRITICAL PHENOMENA FOR SPITZER'S REVERSIBLE NEAREST PARTICLE SYSTEMS

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Motivated by several results and open problems concerning Harris' basic contact process, we consider the relationship between the critical behavior of the finite and infinite versions of Spitzer's reversible nearest particle systems. We show that the critical values for the finite and infinite systems agree, but that the behavior of the two systems at the common critical value can differ. The Nash-Williams recurrence criterion for reversible Markov chains is an important tool used in the proofs of the main results, and we give a new treatment of that theory. Finally, we compute several critical exponents for the nearest particle systems.

1. Introduction. A physical system indexed by a parameter is said to exhibit a phase transition (= critical phenomenon) if the behavior of the system changes discontinuously at some "critical" parameter value. The modeling of such phase transitions is a prominent theme in contemporary mathematics. Recently a number of stochastic processes known as interacting particle systems have been studied in some detail (see [7] for a recent review and bibliography); the presence of critical phenomena in many of these systems is a primary motivation.

This paper deals mainly with one such probabilistic model of phase transition: Spitzer's nearest particle system [29]. Let us briefly summarize the dynamics and basic properties of Spitzer's model. The processes ω_t to be studied are continuous time Markov with state space $\Omega = \{\omega \in \{0, 1\}^Z : \sum_{x<0} \omega(x) = \sum_{x>0} \omega(x) = \infty\}$, where Z is the integers. P_ω will denote the law of ω_t started in configuration $\omega = (\omega(x);\ x \in Z) \in \Omega$. The evolution of ω_t is determined by flip rates:

$$\beta_x(\omega) = \text{the exponential birth rate at } x \text{ when } \omega(x) = 0,$$

$$\delta_x(\omega) = \text{the exponential death rate at } x \text{ when } \omega(x) = 1.$$

The birth rates β_x are assumed to be of the form

$$\beta_x(\omega) = \beta(\ell_x(\omega), r_x(\omega)) = \beta(\ell, r), \tag{1.1}$$

where $\ell_x(\omega)$ is the distance to the nearest occupied site in ω to the left of x and $r_x(\omega)$ is the distance to the nearest particle to the right of x. Moreover, the $\beta(\ell, r)$ satisfy

$$\beta(\ell, r) = \lambda f_\ell f_r / f_{\ell+r}, \qquad \ell, r \geq 1, \tag{1.2}$$

where λ is a positive parameter and f is a strictly positive function on the positive integers such that

$$z = \sum_{k=1}^{\infty} f_k < \infty \tag{1.3}$$

and

$$f_k / f_{k+1} \downarrow 1 \quad \text{as} \quad k \to \infty. \tag{1.4}$$

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Finally, the death rates δ_x are identically one:

$$\delta_x(\omega) \equiv 1. \tag{1.5}$$

For fixed f satisfying (1.3)–(1.4), the flip rates (1.1)–(1.2) and (1.5) give rise to a one parameter family of Markov particle systems $\{P_\omega^\lambda;\ \omega \in \Omega\}_{\lambda>0}$, which we call Spitzer's nearest particle systems with density f. For more details on the precise formulation of the model, see [29] and [12]. For other papers which treat this model and its generalizations, see [1], [4], [5], and [16]. Assumption (1.4) ensures that $\{P_\omega^\lambda\}$ is attractive for each λ. Letting $\mathbf{1}$ = the configuration "all ones" and ν_t^λ = the law of ω_t under $P_{\mathbf{1}}^\lambda$, it follows that there is an invariant measure ν^λ for $\{P_\omega^\lambda\}$ such that

$$\nu_t^\lambda \Rightarrow \nu^\lambda \quad \text{as} \quad t \to \infty \tag{1.6}$$

($\Rightarrow$ means weak convergence), where of course ν^λ may be the pointmass on $\mathbf{0}$ (= "all zeros"). For background on attractiveness, see e.g. [7]. Assumption (1.2) ensures that each $\{P_\omega^\lambda\}$ is formally reversible (see [29]), and that they are monotone increasing in λ for each ω. Since $\mathbf{0}$ is a trap for each $\{P_\omega^\lambda\}$ (by convention) it follows from the monotonicity that there is a critical value λ_c such that:

$$\begin{aligned} &\text{for} \quad \lambda < \lambda_c \quad \text{(subcritical case)}, \qquad \nu^\lambda = \delta_{\mathbf{0}} \\ &\text{for} \quad \lambda > \lambda_c \quad \text{(supercritical case)}, \quad \nu^\lambda \neq \delta_{\mathbf{0}}. \end{aligned} \tag{1.7}$$

($\delta_{\mathbf{0}}$ = the delta measure at $\mathbf{0}$). These "soft" results do not rule out the possibilities $\lambda_c = 0$ or $\lambda_c = \infty$, nor do they determine whether or not $\nu^{\lambda_c} = \delta_{\mathbf{0}}$. But, in fact, Spitzer and Holley have identified the ν^λ, computed the critical value, and determined the behavior in the critical case. (The proof of Holley's part of the following theorem was only suggested, but not written out, on page 886 of [29]. A full proof will therefore be included in a forthcoming paper "Attractive Nearest Particle Systems" by T. M. Liggett.)

THEOREM 1.8. ([29]). $\lambda_c = z^{-1}$. *For* $\lambda > \lambda_c$, ν^λ *is the renewal measure on* Z *with probability density*

$$f_k^\lambda = \lambda f_k s^k, \tag{1.9}$$

where s *is given by* $\sum_{k=1}^\infty f_k^\lambda = 1$. $\nu^{\lambda_c} = \delta_{\mathbf{0}}$ *if and only if* $\mu = \sum k f_k = \infty$. *If* $\mu < \infty$, *then* ν^{λ_c} *is the renewal measure with probability density* $f_k^{\lambda_c} = \lambda_c f_k$.

For the sake of comparison, we now introduce two additional one parameter families of particle systems which exhibit phase transitions. The first example, Harris' (linear) contact processes on Z [14], are the spin systems with birth and death rates

$$\beta_x(\omega) = \lambda[\omega(x-1) + \omega(x+1)], \qquad \delta_x(\omega) \equiv 1, \tag{1.10}$$

$\lambda > 0$ a parameter. The second example consists of some generalized smoothing processes on Z^3, introduced by Holley and Liggett [15]. Here the state space Ω is an appropriate subset of $[0, \infty)^{Z^3}$, and the transitions at each site $x \in \mathbb{Z}^3$ are of the form

$$\begin{aligned} \omega(x) &\to \left(1 + \frac{1}{\lambda}\right)\frac{1}{6}\textstyle\sum_{i=1}^3 [\omega(x+e_i) + \omega(x-e_i)] \quad \text{at rate } \lambda, \\ &\to 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{at rate } 1, \end{aligned} \tag{1.11}$$

where $e_i (1 \le i \le 3)$ are the standard unit vectors in $\mathbb{Z}^3$ and $\lambda > 0$ is a parameter. In both instances the $\{P_\omega^\lambda\}$ are attractive for each λ and have $\delta_{\mathbf{0}}$ as a trap. The first process is monotone increasing in λ, while for the second, Theorem 3.4 of [15] provides an analogue to this monotonicity. Thus the "soft" theory applies as before, (1.6) holds, and there is a critical value λ_c where the phase transition (1.7) occurs. The ergodic theory of the contact processes (1.10) is fairly well developed by now; see [13] for a recent survey. However there is no known analogue of Theorem 1.8: λ_c can be shown to lie between 1 and 2, but the first decimal place has not been rigorously determined, and it is still not known whether or not $\nu^{\lambda_c} = \delta_0$. Similarly, for the smoothing processes, the methods of [15] give upper and lower bounds on λ_c, but the precise value is unknown, and whether or not $\nu^{\lambda_c} = \delta_0$ is an open problem.

One of the keys to understanding the ergodic theory of the contact processes (1.10) is the behavior of these processes restricted to the denumerable space Ω^f of finitely many particles,

$$\Omega^f = \{\omega \in \{0, 1\}^Z : \omega(x) \neq 0 \quad \text{for finitely many} \quad x \in Z\}.$$

If $\omega \in \Omega^f$, then ω_t is an absorbing Markov chain on Ω^f under P_ω^λ, and interest centers on the probability of ultimate absorption in the trap $\mathbf{0}$. We let T_0 denote the hitting time for "all 0's" and write

$$\sigma_t^\lambda = P_*^\lambda(T_0 > t),$$

where $*$ is the configuration with a single particle at the origin $x = 0$. Clearly

$$\sigma_t^\lambda \downarrow \sigma^\lambda = P_*^\lambda(T_0 = \infty) \quad \text{as} \quad t \to \infty.$$

Also, since σ^λ is increasing in λ, there is a critical value for the finite systems:

(1.12) $$\lambda_c^f = \inf\{\lambda : \sigma^\lambda > 0\}.$$

If we let ρ_t^λ be the density of the measure ν_t^λ i.e.

$$\rho_t^\lambda = P_1^\lambda(\omega_t(0) = 1),$$

then the self-duality equation for the linear contact processes asserts that $\rho_1^\lambda = \sigma_1^\lambda$ and so

(1.13) $$\rho^\lambda = \sigma^\lambda, \qquad \lambda > 0.$$

We see immediately from (1.7), (1.12) and (1.13) that for the linear contact model

(1.14) $$\lambda_c^f = \lambda_c,$$

i.e. "the infinite and finite critical values agree." This phenomenon is undoubtedly widespread, but very little is known about conditions which imply (1.14) for other one parameter families. (As remarked in [8], (1.14) can be shown for any λ-increasing family of additive nearest neighbor systems on Z, but this is the only available result.) For example, it is not known whether (1.14) holds for the smoothing processes (1.11). There is a generalized potlatch model studied in [15] with a theory which parallels that of the smoothing model quite closely. In fact the two models are connected by certain duality equations, so that if one knew $\lambda_c^f = \lambda_c$ for the systems (1.11), then one could conclude that the infinite (or finite) critical value for the corresponding potlatch family coincided with λ_c (and λ_c^f), and thus resolve a special case of an open problem posed in [15].

The remarks of the last paragraph are intended to explain our interest in the finite nearest particle systems $(\omega_t, P_\omega^\lambda)$, $\omega \in \Omega^f$. These are the denumerable Markov chains on Ω^f with flip rates given by (1.1)–(1.5). Since the finite systems have a leftmost and rightmost particle, it is natural to specify their flip rates by continuity, setting

(1.15) $$\beta(k, \infty) = \beta(\infty, k) = \lambda f_k, \qquad \beta(\infty, \infty) = 0.$$

Then $\mathbf{0}$ is a trap, T_0, σ_t^λ and σ^λ can be defined as before, and we can introduce the finite critical value λ_c^f given by (1.12). The main result of this paper, which will be proved in Section 3, establishes (1.14) for Spitzer's nearest particle systems, and asserts that for any given function f the finite systems die out at the critical value. Our theorem is the following.

(1.16) THEOREM. *Given* λ, $f_k > 0$, *let* $(\omega_t, \{P^\lambda_\omega;\ \omega \in \Omega^f\})$ *be the finite nearest particle system determined by* (1.1)–(1.5) *and* (1.15). *Then* $T_0 < \infty\, P^\lambda_*$*-almost surely if and only if* $\lambda \le \lambda_c = z^{-1}$. *Thus the finite and infinite critical values agree, but the critical behavior may differ in the sense that*

$$(1.17) \qquad \sigma^{\lambda_c} = 0, \qquad \rho^{\lambda_c} > 0 \quad \textit{provided} \quad \Sigma\, kf_k < \infty.$$

As already mentioned, whether $\rho^{\lambda_c} = 0$ for the linear contact model is an important open problem. Because of (1.13), that problem is equivalent to whether or not $\sigma^{\lambda_c} = 0$; the situation (1.17) cannot arise here. Thus we feel that our result supports the prevalent belief that $\rho^{\lambda_c} = 0$ for the contact model, since extinction of finite critical systems seems to be the rule.

The proof of our theorem is based on a recurrence criterion for reversible Markov chains which appeared in a little known paper by Nash-Williams [26] written more than twenty years ago. When translated back into potential theory, his result is a simple application of the Dirichlet principle, and has an illustrious history in the theory of electric circuits. We feel that the main result of [26] constitutes an important "lost" chapter in the theory of reversible Markov chains, which may well have applications to a wide range of processes. For this reason, and because [26] is difficult to read, Section 2 below is a self-contained exposition of the Dirichlet principle and the Nash-Williams recurrence criterion, with historical remarks.

In Section 3 we prove Theorem (1.16). Section 4 deals with asymptotics for nearest particle systems near the critical value. Physical principles and empirical observation suggest that the discontinuities and divergences observed at critical values for phase transitions obey "universal power laws". Roughly, if some meaningful quantity χ^λ exhibits a critical phenomenon at $\lambda = \lambda_c$, then for some γ

$$\chi^\lambda \propto |\lambda - \lambda_c|^\gamma$$

as $\lambda \uparrow \lambda_c$ or $\lambda \downarrow \lambda_c$ (perhaps with different values γ in the two cases). The universality hypothesis asserts that even though λ_c will differ for different systems, the critical exponent γ should remain the same for whole "universality classes" of related systems. While these ideas are widespread in the literature of mathematical physics, and certainly intriguing from a mathematical point of view, there are very few models for which they can be made precise. Lang [23] provides an excellent account of two classes of systems (hierarchial models, iterated maps) which have been treated in detail. Our object in Section 4 is to illustrate the notion of universality by evaluating a few critical exponents for the nearest particle systems. The exponents for finite systems make use of computations from Section 3; those for the infinite system reduce to simple asymptotics for renewal measures. In neither case is the mathematics difficult. The point is to add to the short list of examples where the detailed analysis of critical phenomena can be carried out rigorously.

2. The Dirichlet principle for reversible Markov chains. The classical Dirichlet principle (see [31], for example) states that among all continuously differentiable functions h on a bounded domain D with given boundary values, the integral

$$\Phi(h) = \int_D\!\!\int |\operatorname{grad} h|^2\, dx\, dy$$

is minimized by the harmonic function with those boundary values. Of course, the potential theory for Markov chains is highly developed (cf. [20]), but the role of the analogous variational principle seems to have been largely overlooked in probability circles. This section presents such a Dirichlet principle for reversible denumerable Markov chains, gives some applications to problems of recurrence, and then concludes with a few remarks on the history of the results.

Let S be a denumerable set. A matrix $A = [\alpha_{ij}]_{i,j\in S}$ is called a flow matrix if $\alpha_{ij} = \alpha_{ji} \ge$

0 for all i, j, and

$$0 < \alpha_i = \sum_j \alpha_{ij} < \infty \quad \text{for each} \quad i.$$

The (discrete time) Markov chain X_n on S with flow matrix A is the reversible chain with invariant (σ-finite) measure $\alpha = (\alpha_i)_{i \in S}$ and transition matrix $P = [p_{ij}]_{i,j \in S}$ given by $p_{ij} = \alpha_{ij}/\alpha_i$. Any reversible chain comes from a flow matrix A, although A is not uniquely determined. We write P_i for the law of X_n starting from i; of course P_i depends on A. By convention, all chains X_n considered throughout this section will be assumed to be *irreducible* without further notice. Introduce the hitting times

$$T_\Lambda = \min\{n \geq 0 : X_n \in \Lambda\}, \qquad T_\Lambda^+ = \min\{n > 0 : X_n \in \Lambda\},$$

$\Lambda \subset S$ ($= \infty$ if no such n). Fix a reference state $0 \in S$, and write $T_0 = T_{\{0\}}$, $T_0^+ = T_{\{0\}}^+$. Our principal objects of study in this section are the hitting probabilities

$$p_A(\Lambda) = P_0(T_\Lambda < T_0^+).$$

In order to formulate our variational principle, we fix $0 \notin \Lambda \subset S$, let $\mathscr{H} = \{h : S \to [0, 1], h_0 = 0, h|_\Lambda = 1\}$, and for $h \in \mathscr{H}$ consider

$$\Phi(h) = \Phi_A(h) = \sum_{i,j} \alpha_{ij}(h_i - h_j)^2.$$

The Dirichlet principle for reversible chains is as follows.

(2.1) THEOREM. *Suppose $P_0(T_\Lambda < \infty) = 1$, and let $h_i^\Lambda = P_i(T_\Lambda < T_0)$. Then*

(2.2) $$\Phi(h^\Lambda) = 2\alpha_0 p_A(\Lambda),$$

and

(2.3) $$\Phi(h^\Lambda) = \min_{h \in \mathscr{H}} \Phi(h).$$

Thus, for fixed Λ, $\alpha_0 p_A(\Lambda)$ is increasing in A (componentwise) provided that $P_0(T_\Lambda < \infty) = 1$ for each A considered. In particular, if $\bar{A} \geq A$ and $\bar{\alpha}_{0j} = \alpha_{0j}$ for all j, then $p_{\bar{A}}(\Lambda) \geq p_A(\Lambda)$.

PROOF. We will prove (2.2) and (2.3) only, since the final statements are immediate consequences of them. Assume first that S is finite. Put $g = 1 - h^\Lambda$ and compute

$$\Phi(h^\Lambda) = \Phi(g) = \sum_{i \neq 0} \alpha_i g_i [\sum_j p_{ij}(g_i - g_j)] + \sum_{j \neq 0} \alpha_j g_j [\sum_i p_{ji}(g_j - g_i)] + 2\alpha_0 \sum_j p_{0j}(1 - g_j).$$

The first two sums on the right vanish since g is zero on Λ and g is harmonic off $\Lambda \cup \{0\}$. This proves (2.2). For (2.3), suppose that h minimizes Φ on $\mathscr{H}$. Taking partial derivatives with respect to the h_i, we see that h must be harmonic off $\Lambda \cup \{0\}$. Therefore $h = h^\Lambda$. Now let S be general and take S^n finite, so that $S^n \uparrow S$, $0 \in S^n$, $\Lambda \cap S^n \neq 0$, and the chain on S^n with flow matrix $\alpha_{ij}|_{S^n}$ is irreducible. All expressions with the superscript n will refer to this chain on S^n. In particular, $h_i^n = P_i^n(T_{\Lambda^n} < T_0)$ for $i \in S_n$, where $\Lambda^n = \Lambda \cap S^n$. By the already proved finite version of (2.2) and (2.3),

$$\Phi^n(h^n) = 2\alpha_0^n p_{A^n}(\Lambda^n) \quad \text{and} \quad \Phi^n(h^n) = \min_{h \in \mathscr{H}^n} \Phi^n(h).$$

Since $P_0(T_\Lambda < \infty) = 1$, it is easy to see that $h_i^n \to h_i^\Lambda$ for all i. Therefore

$$\lim_{n \to \infty} p_{A^n}(\Lambda^n) = \lim_{n \to \infty} \sum_j p_{0j}^n h_j^n = \sum_j p_{0j} h_j^\Lambda = p_A(\Lambda),$$

and also

$$\Phi(h^\Lambda) \leq \liminf_{n \to \infty} \Phi^n(h^n) = \liminf_{n \to \infty} \min_{h \in \mathscr{H}^n} \Phi^n(h)$$
$$= \liminf_{n \to \infty} \min_{h \in \mathscr{H}} \Phi^n(h) \leq \min_{h \in \mathscr{H}} \Phi(h)$$

which gives both (2.2) and (2.3), since equality must then hold throughout. □

Since the hitting probabilities $p_A(\Lambda)$ govern recurrence or transience, the monotonicity in A established by the Dirichlet principle yields the following useful comparison result.

(2.4) COROLLARY. *If the chain* $(\bar{X}_n, \{\bar{P}_i\})$ *with flow matrix* $\bar{A}$ *is recurrent, and if* $(X_n, \{P_i\})$ *has flow matrix* $A \leq \bar{A}$, *then* X_n *is recurrent.*

PROOF. Choose finite Λ_N, $N \geq 1$, so that $\Lambda_N \uparrow S$. Then $T_{\Lambda_N^c} < \infty$ P_0 – a.s. and $\bar{P}_0$ – a.s. for each N, so by Theorem (2.1),

$$P_0(T_0^+ = \infty) = \lim_{N\to\infty} p_A(\Lambda_N^c) \leq \frac{\bar{\alpha}_0}{\alpha_0} \lim_{N\to\infty} p_{\bar{A}}(\Lambda_N^c) = \bar{P}_0(T_0^+ = \infty).$$

The last probability is 0 since $\bar{X}_n$ is recurrent, so X_n is recurrent as well. □

Let us now give an application of (2.4) to what we shall call constrained Markov chains. If $(\bar{X}_n)$ is reversible on S with flow matrix $\bar{A}$, there are two natural ways to restrict $\bar{X}_n$ to a subset $S_0 \subset S$. Namely, we can define the new process X_n to have flow matrix A, where *either*

(2.5) $$\alpha_{ij} = \bar{\alpha}_{ij},\ i, j \in S_0; \quad \alpha_{ij} = 0 \text{ otherwise};$$

or

(2.6) $$\begin{aligned} &\alpha_{ij} = \bar{\alpha}_{ij},\ i, j \in S_0, \quad i \neq j \\ &\alpha_{ii} = \bar{\alpha}_{ii} + \textstyle\sum_{j \notin S_0} \bar{\alpha}_{ij}, \quad \alpha_{ij} = 0 \text{ otherwise.} \end{aligned}$$

(In either case we assume that $\alpha_i > 0$ for each $i \in S_0$.) Since both constrained processes have the same imbedded jump chain, their recurrence properties are the same. We call either version X_n *constrained* to S_0. If $S = \bar{G}$ = a graph, $\bar{N}_i$ = the neighbor set of i in $\bar{G}$, and $\bar{\alpha}_{ij} = 1$ for $j \in \bar{N}_i$ (= 0 otherwise), then we call X_n the *random walk* on $\bar{G}$. In this setting (2.5) gives rise to the random walk on the subgraph $G = \mathcal{S}_0$ of $\bar{G}$ with neighbor sets $N_i = \bar{N}_i \cap G$. Version (2.6) is random walk on G with reflection at

$$\partial G = \{j \in \bar{G} - G : j \in N_i \quad \text{for some} \quad i \in G\};$$

when the process attempts to jump from $i \in G$ to $j \in \partial G$ it remains at i. As a special case of (2.4) we have the following.

(2.7) COROLLARY. *If* $\bar{X}_n$ *is recurrent reversible on* S, *and if* X_n *is* $\bar{X}_n$ *constrained to* $S_0 \subset S$, *then* X_n *is recurrent.*

PROOF. In version (2.5), $A \leq \bar{A}$. □

For example, (2.7) asserts that simple symmetric random walk on Z^2 constrained to any (connected) subset of Z^2 is recurrent. This fact, mentioned in the 3rd edition of Feller Vol. I [9, page 425] but not in earlier editions, is neither intuitively obvious nor easy to prove by standard probabilistic techniques.

Another important application of Theorem (2.1) is the Nash-Williams recurrence criterion [26]. To state it, we now assume that S can be partitioned as $S = \sum_{k=0}^{\infty} \Lambda_k$, so that whenever $i \in \Lambda_k$ and $\alpha_{ij} > 0$, then $j \in \Lambda_{k-1} \cup \Lambda_k \cup \Lambda_{k+1} (\Lambda_{-1} = \emptyset)$. We need not assume that the Λ_k are finite, but only that

(2.8) $$\textstyle\sum_{i \in \Lambda_k} \alpha_i < \infty \quad \text{for all} \quad k.$$

Finally, for convenience, assume that $\Lambda_0 = \{0\}$. If $S = G$ = a graph, and if X_n is a reversible chain on G which can only jump from a site to neighboring sites, then it is natural to take

$$\Lambda_k = \{k\text{-neighbors of } 0\} = \left\{\begin{matrix} j \in G\text{: the minimal number of edges in} \\ \text{a chain connecting 0 to } j \text{ is } k \end{matrix}\right\}$$

provided that (2.8) is satisfied. For $k \geq 1$, set

(2.9) $$\alpha(k) = \sum_{i\in\Lambda_{k-1}} \sum_{j\in\Lambda_k} \alpha_{ij}.$$

(2.10) THEOREM. *If* $P_0(T_{\Lambda_m} < \infty) = 1$ *for all* $m \geq 1$, *and* (2.8) *holds, then*

(2.11) $$P_0(T_{\Lambda_m} < T_0^+) \leq (\alpha_0\Sigma_m)^{-1},$$

where

$$\Sigma_m = \sum_{k=1}^m \{\alpha(k)\}^{-1}.$$

Thus X_n *is recurrent provided that*

(2.12) $$\sum_{k=1}^\infty \{\alpha(k)\}^{-1} = \infty.$$

REMARKS. Nash-Williams [26] proved that (2.12) implies recurrence, and also a partial converse which is too involved to state here. We do not know whether (2.11) is implicit in [26]. For another recurrence criterion for reversible chains, which was obtained after the present paper was written, see the forthcoming paper "A Simple Criterion for Transience of a Reversible Markov Chain" by Terry Lyons.

PROOF OF THEOREM (2.10). It suffices to verify (2.11). Choose finite $\Lambda_k^\ell \uparrow \Lambda_k$, and introduce the approximating flow matrices $A^{(\ell)}$, $\ell \geq 1$, given by

$$\begin{aligned} \alpha_{ij}^{(\ell)} &= \alpha_{ij} + \ell \quad i, j \in \Lambda_k^\ell \text{ for some } k,\\ &= \alpha_{ij} \qquad \text{otherwise.}\end{aligned}$$

Then $\alpha_0^{(\ell)} \equiv \alpha_0$ and $A \leq A^{(\ell)}$, so by Theorem (2.1), for each fixed m,

$$p_A(\Lambda_m) \leq \lim_{\ell\to\infty} p_{A^{(\ell)}}(\Lambda_m).$$

We show that the right side is equal to $(\alpha_0\Sigma_m)^{-1}$. Let $\{P_i^{(\ell)}\}_{i\in S}$ be the Markov measures induced by $A^{(\ell)}$,

$$\Delta_m = \sum_{k=m}^\infty \Lambda_k, \quad \text{and} \quad h_i^{(\ell)} = P_i^{(\ell)}(T_{\Delta_m} < T_0).$$

Another appeal to Theorem (2.1) and $h|_{\Delta_m} \equiv 1$ yield

$$\begin{aligned} p_{A^{(\ell)}}(\Lambda_m) &= p_{A^{(\ell)}}(\Delta_m) = (2\alpha_0)^{-1}\Phi_{A^{(\ell)}}(h^{(\ell)})\\ &= (2\alpha_0)^{-1} \sum_{k=0}^m \sum_{i\in\Lambda_k} \sum_{j\in\Lambda_{k-1}\cup\Lambda_k\cup\Lambda_{k+1}} (h_i^{(\ell)} - h_j^{(\ell)})^2 \alpha_{ij}^{(\ell)}.\end{aligned}$$

To compute the limit on m, we first establish two claims:

(2.13) $$\lim_{\ell\to\infty} \sum_{i,j\in\Lambda_k} \alpha_{ij}^{(\ell)}(h_i^{(\ell)} - h_j^{(\ell)})^2 = 0 \quad \text{for all} \quad k,$$

and

(2.14) $$\lim_{\ell\to\infty}(h_i^{(\ell)} - h_j^{(\ell)}) = (\Sigma_m\alpha(k))^{-1}, \quad i \in \Lambda_k, \quad j \in \Lambda_{k-1}, \quad 1 \leq k \leq m.$$

To get (2.13), we first show that for each $i, j \in \Lambda_k$,

(2.15) $$|h_i^{(\ell)} - h_j^{(\ell)}| = O\{(\ell|\Lambda_k^\ell|)^{-1}\} \quad \text{as} \quad \ell \to \infty.$$

In fact, if $i, j \in \Lambda_k^\ell$, then since the h_i are harmonic probabilities,

$$\begin{aligned} |h_i^{(\ell)} - h_j^{(\ell)}| &\leq \sum_r |p_{ir}^{(\ell)} - p_{jr}^{(\ell)}|\\ &= \sum_{r\in\Lambda_k^\ell} \left|\frac{\alpha_{ir} + \ell}{\alpha_i + \ell|\Lambda_k^\ell|} - \frac{\alpha_{jr} + \ell}{\alpha_j + \ell|\Lambda_k^\ell|}\right| + \sum_{r\notin\Lambda_k^\ell} \left|\frac{\alpha_{ir}}{\alpha_i + \ell|\Lambda_k^\ell|} - \frac{\alpha_{jr}}{\alpha_j + \ell|\Lambda_k^\ell|}\right|\\ &\leq \frac{2(\alpha_i + \alpha_j)}{\ell|\Lambda_k^\ell|} + \frac{2\alpha_i\alpha_j}{\ell^2|\Lambda_k^\ell|^2} \leq \frac{4K + 2K^2}{\ell|\Lambda_k^\ell|}, \qquad \text{where } K = \sup_{i\in\Lambda_k}\alpha_i < \infty \textit{ by } (2.8).\end{aligned}$$

Now, the sum on the left in (2.13) is majorized by

$$\sum_{i,j\in\Lambda_k'} (\alpha_{ij} + \ell)\frac{K'}{\ell^2|\Lambda_k'|^2} + \sum_{i,j\in\Lambda_k, i,j \text{ not both in } \Lambda_k'} \alpha_{ij}$$

$$\leq K' \{\sup_{i,j\in\Lambda_k'} \alpha_{ij}\ell^{-2} + \ell^{-1}\} + \sum_{i,j\in\Lambda_k, i,j \text{ not both in } \Lambda_k'} \alpha_{ij}$$

which goes to zero as $\ell \to \infty$. The argument for (2.14) goes as follows. By harmonicity, for $i \in \Lambda_k$, $1 \leq k \leq m-1$, we have

$$\alpha_i^{(\ell)} h_i^{(\ell)} - \sum_{j\in\Lambda_k} \alpha_{ij}^{(\ell)} h_j^{(\ell)} = \sum_{j\in\Lambda_{k-1}} \alpha_{ij} h_j^{(\ell)} + \sum_{j\in\Lambda_{k+1}} \alpha_{ij} h_j^{(\ell)}.$$

Sum over $i \in \Lambda_k$ to get

$$\text{(2.16)} \quad \sum_{i\in\Lambda_k} \alpha_i^{(\ell)} h_i^{(\ell)} - \sum_{i,j\in\Lambda_k} \alpha_{ij}^{(\ell)} h_j^{(\ell)} = \sum_{i\in\Lambda_k, j\in\Lambda_{k-1}} \alpha_{ij} h_j^{(\ell)} + \sum_{i\in\Lambda_k, j\in\Lambda_{k+1}} \alpha_{ij} h_j^{(\ell)}.$$

(all sums are finite by (2.8).) Now from (2.13), we can find a subsequence (ℓ') such that

$$\lim_{\ell'\to\infty} h_i^{(\ell')} = h(k) \quad \text{for all} \quad i \in \Lambda_k.$$

Letting $\ell' \to \infty$ in (2.16), by dominated convergence we get

$$(\alpha(k) + \alpha(k+1))h(k) = \alpha(k)h(k-1) + \alpha(k+1)h(k+1),$$

or

$$h(k+1) - h(k) = \frac{\alpha(k)}{\alpha(k+1)}(h(k) - h(k-1)).$$

Iterate to obtain

$$h(k+1) - h(k) = \frac{\alpha(1)}{\alpha(k+1)} h(1), \quad 1 \leq k \leq m-1.$$

Since $\sum_{k=0}^{m-1} (h(k+1) - h(k)) = 1$, and since the last equation also holds for $k = 0$, it follows that $\alpha(1)h(1) = \sum_m^{-1}$. Claim (2.14) follows since the limit is independent of the subsequence ℓ'. Finally, using (2.13), (2.14) and dominated convergence, we compute:

$$\lim_{\ell\to\infty} p_{A^{(\ell)}}(\Delta_m) = \alpha_0^{-1} \sum_{k=1}^{m} \sum_{i\in\Lambda_k, j\in\Lambda_{k-1}} \lim_{\ell\to\infty} (h_i^{(\ell)} - h_j^{(\ell)})^2 \alpha_{ij}$$

$$= \alpha_0^{-1} \sum_{k=1}^{m} \{(\textstyle\sum_m \alpha(k))^{-1}\}^2 \alpha(k) = (\alpha_0 \textstyle\sum_m)^{-1},$$

as desired. □

EXAMPLE. $S = Z^2$, $\Lambda_k = \{k\text{-neighbors of } 0\}$, X_n = simple symmetric 2-dimensional random walk. Then $\alpha_0 = 4$, $\alpha(k) \leq 8k$ for $k \geq 1$, and so

$$P_0(T_{\Lambda_m} < T_0^+) \leq 2\,(\textstyle\sum_{k=1}^{m} k^{-1})^{-1} \leq 2/\log m.$$

HISTORICAL REMARKS. The connection between reversible Markov chains and electric circuits is well-known (see [20, pages 303–310] or the recent monograph [19], for example). As a consequence of Ohm's law and Kirchhoff's law, if resistance α_{ij}^{-1} is placed along each edge (= wire) of a graph, if 0 is kept at unit voltage by an outside source, and if Λ is grounded, then the charge at 0 equals $\alpha_0 p_A(\Lambda)$. In the electrical setting the results of this section are classical, but the only probabilistic treatment we know is the Nash-Williams paper [26]. The monotonicity of $\alpha_0 p_A(\Lambda)$ simply states that increasing the resistance anywhere in the network increases the effective resistance between 0 and Λ. Peter Doyle [6] has written a colorful article on the history of this principle, citing papers from the last century by Rayleigh [28], Kirchhoff [22] and Maxwell [25]. As explained by Jeans [18, pages 321–324], the functional Φ represents the heat generated in the network; harmonic voltages minimize heat. Jeans remarks parenthetically that "heat = random motion." Monotonicity of effective resistance can be used to estimate the charge at a site by the corresponding charges in simpler networks. The extreme operations are to short out edges (no resistance) and to cut edges (infinite resistance). According to Doyle, this comparison technique is called "Rayleigh's short-cut method". The material we have presented is simply a probabilistic version of the short-cut theory. For instance, constraining a chain to $S_0 \subset S$ amounts to cutting all the connections between S_0 and S_0^c. The Nash-Williams result is also easy to explain: we short out all edges within a given Λ_k in order to reduce the network to a one-dimensional one. Linear circuits have no loops, only parallel and series connections, and so we can do the computations explicitly. A limiting argument is necessary in the probabilistic treatment, since shorting out edge (i, j) amounts to taking $\alpha_{ij} = \infty$. The computations for linear circuits are entirely equivalent to those for birth-death chains, which explains the familiar form of the difference equations in the proof of Theorem (2.10). In closing this section, we note that Rayleigh's short-cut method provides an alternate proof of Polya's theorem. Recurrence of simple random walk on Z^2 (or any connected subset of Z^2) follows from the Nash-Williams criterion. To show transience on Z^3 it suffices to find a transient random walk on a tree imbedded in Z^3. We leave this as an exercise for the reader; consult [6] for help.

3. The Proof of Theorem (1.16). Our object here is to prove that the finite reversible nearest particle systems die out if and only if $\lambda \le \lambda_c$. We think of ω_t as a reversible chain on a complicated graph of configurations. The idea is to sandwich ω_t between two tractable systems by appealing to the short-cut method of the previous section.

It will be convenient to identify $\omega \in \Omega^f$ with the set $B \subset Z$ of occupied sites: $\omega = 1_B$. We will use the generic notation $B = \{x_1, \dots, x_n\}$, $x_1 < \cdots < x_n (n \ge 1)$. When we write $B - \{x\}$ it is understood that $x = x_i$ for some $1 \le i \le n$. When we write $B \cup \{x\}$, then $x_i < x < x_{i+1}$ for some $0 \le i \le n$, with the convention that $x_0 = -\infty$, $x_{n+1} = \infty$. With this notation, the Q-matrix $[q_{BC}]_{B,C\in\Omega^f}$ of jump rates for ω_t has the form

$$q_{BB\cup\{x\}} = \beta(x - x_i, x_{i+1} - x), \quad q_{BB-\{x\}} = 1,$$

with all other rates 0.

Introduce an equivalence relation on Ω^f,

$$B \sim C \quad \text{if} \quad C = B + x \quad \text{for some} \quad x \in \mathbb{Z},$$

i.e. two configurations are equivalent if they are translates of one another. Write $\tilde{B}$ for the equivalence class of B, and let $\tilde{\Omega} = \{\tilde{B} : B \in \Omega^f\}$ be the space of all such "shapes". Consider the continuous time Markov chain W_t on $\tilde{\Omega}$ given by

$$W_t = \tilde{\omega}_t.$$

Next, let X_t be W_t altered to have spontaneous birth from $\varnothing$ to $\tilde{*}$ (a singleton) at rate λ. We claim that X_t is reversible on $\tilde{\Omega}$. To see this, temporarily write $f(k) = f_k$, and define

$$\alpha_B = \lambda^n \prod_{i=1}^{n-1} f(x_{i+1} - x_i), \quad (\alpha_\varnothing = 1). \tag{3.1}$$

Let $\tilde{Q} = [\tilde{q}_{\tilde{B}\tilde{C}}]$ be the Q-matrix of X_t, and abbreviate $\tilde{q}_{\tilde{B}\tilde{C}} = \tilde{q}_{BC}$. If $1 \le i \le n - 1$, then

$$\begin{aligned} \frac{\tilde{q}_{BB\cup\{x\}}}{\tilde{q}_{B\cup\{x\}B}} &= \beta(x - x_i, x_{i+1} - x) = \lambda f(x - x_i) f(x_{i+1} - x)/f(x_{i+1} - x_i) \\ &= \frac{\lambda^{n+1}[\prod_{r=1}^{i-1} f(x_{r+1} - x_r)] f(x - x_i) f(x_{i+1} - x)[\prod_{r=i+1}^{n-1} f(x_{r+1} - x_r)]}{\lambda^n \prod_{r=1}^{n-1} f(x_{r+1} - x_r)} \\ &= \alpha_{B\cup\{x\}}/\alpha_B, \end{aligned} \tag{3.2}$$

with the same formula holding if $i = 0$ or n. Also, $\tilde{q}_{\varnothing\{0\}}/\tilde{q}_{\{0\}\varnothing} = \lambda = \alpha_{\{0\}}/\alpha_\varnothing$. If we now define $\tilde{A} = [\tilde{\alpha}_{\tilde{B}\tilde{C}}]$ *by*

$$\tilde{\alpha}_{\tilde{B}\tilde{C}} = \tilde{\alpha}_{BC} = \alpha_B \tilde{q}_{BC},$$

then it follows that $\tilde{A}$ is symmetric and the discrete time imbedded chain X_n corresponding to X_t has flow matrix $\tilde{A}$. Let $\tilde{P}^\lambda_{\tilde{B}}$ govern X_n starting from $\tilde{B}$. Since

$$P^\lambda_*(T_0 < \infty) = \tilde{P}^\lambda_{\tilde{\varnothing}}(T^+_{\tilde{\varnothing}} < \infty).$$

To show that $T_0 < \infty$ P^λ_*-a.s., it suffices to prove that X_n is recurrent. To this end, we apply Theorem (2.10). Let

$$\Lambda_k = \{\tilde{B} \in \Omega : |B| = k\}.$$

Clearly $\tilde{\Omega} = \sum_{k=0}^\infty \Lambda_k$, and whenever $X_n \in \Lambda_k$ then $X_{n+1} \in \Lambda_{k-1} \cup \Lambda_{k+1}$ since ω_t is a spin system. Also, for $k \geq 1$,

$$\inf_{\tilde{B}\in\Lambda_k} \sum_{\tilde{C}\in\Lambda_{k+1}} \tilde{q}_{\tilde{B}\tilde{C}} \geq 2 \sum_{\ell=1}^\infty \lambda f(\ell) = 2\left(\frac{\lambda}{\lambda_c}\right) > 0$$

and

$$\sup_{B\in\Lambda_k} \sum_{\tilde{C}\in\Lambda_{k-1}} \tilde{q}_{\tilde{B}\tilde{C}} \leq k,$$

so an easy argument based on the Markov property gives

$$\tilde{P}^\lambda_{\tilde{\varnothing}}(T_{\Lambda_m} < \infty) = 1 \quad \text{for all} \quad m \geq 1.$$

Next, we compute the $\alpha(k)$ defined by (2.9):

$$\begin{aligned}
\alpha(k) &= \sum_{\tilde{B}\in\Lambda_k, \tilde{C}\in\Lambda_{k-1}} \tilde{\alpha}_{\tilde{B}\tilde{C}} = \sum_{\tilde{B}\in\Lambda_k} \alpha_B \sum_{\tilde{C}\in\Lambda_{k-1}} \tilde{q}_{BC} \\
&= \sum_{\tilde{B}\in\Lambda_k} k\alpha_B = k\lambda^k \sum_{d_1\geq 1,\dots d_{k-1}\geq 1} \prod_{r=1}^{k-1} f(d_r) \qquad (3.3)\\
&= k\lambda^k \prod_{r=1}^{k-1} \left(\sum_{d_r \geq 1} f(d_r)\right) = k\lambda\left(\frac{\lambda}{\lambda_c}\right)^{k-1}.
\end{aligned}$$

Hence

$$\sum_{B\in\Lambda_k} \tilde{\alpha}_{\tilde{B}} = \sum_{\tilde{B}\in\Lambda_k} \sum_{\tilde{C}} \tilde{\alpha}_{\tilde{B}\tilde{C}} = \alpha(k) + \alpha(k+1) < \infty,$$

which establishes (2.8). The hypotheses for (2.11) hold, so we get

$$P^\lambda_*(|\omega_t| = m \quad \text{for some} \quad t) = \tilde{P}^\lambda_{\tilde{\varnothing}}(T_{\Lambda_m} < T^+_{\tilde{\varnothing}}) \leq \left[\sum_{k=1}^m \frac{1}{k}\left(\frac{\lambda_c}{\lambda}\right)^{k-1}\right]^{-1}. \qquad (3.4)$$

In particular, $T_0 < \infty$ P^λ_*-almost surely for $\lambda \leq \lambda_c$.

The "shape chain" X_n introduced above is reversible on the graph $\{(\tilde{B}, N_{\tilde{B}}); \tilde{B} \in \tilde{\Omega}\}$ with $\tilde{C} \in N_{\tilde{B}}$ if shape $\tilde{C}$ is obtained by adding or subtracting a particle from $\tilde{B}$. Applying (2.10) amounts to shorting out the connection between each pair of shapes with the same cardinality. To prove that $T_0 = \infty$ with positive P^λ_*-probability whenever $\lambda > \lambda_c$, we need only show that X_n is transient in this case. As might be expected, we establish transience by *cutting* edges. Namely, consider the graph $\{(\tilde{B}, N^0_{\tilde{B}}); \tilde{B} \in \tilde{\Omega}\}$, where

$$C \in N^0_{\tilde{B}}$$

if

$$C = B \cup \{x\} \quad \text{for some} \quad x < x_1 \quad \text{or} \quad x > x_n$$

or

$$C = B - \{x_1\} \quad \text{or} \quad B - \{x_n\}.$$

Define X_t^0 on this new graph to have Q-matrix

$$\tilde{Q}^0 = [\tilde{q}^0_{\tilde{B}\tilde{C}}], \quad \text{where}$$

$$q^0_{\tilde{B}\tilde{C}} = \tilde{q}_{BC} \quad \text{if} \quad \tilde{C} \in N^0_{\tilde{B}},$$

$$= 0 \quad \text{otherwise.}$$

Let $\tilde{A}^0$ be the flow matrix for the imbedded chain X_n^0 corresponding to X_t^0. Then $\tilde{A}^0 \le \tilde{A}$, so (2.4) applies. But $|X_t^0|$ is simply a random walk (with reflection at 0): for $n \ge 1$,

$$n \to n+1 \quad \text{at rate} \quad 2\lambda \textstyle\sum_{k=1}^{\infty} f(k) = 2\left(\dfrac{\lambda}{\lambda_c}\right),$$

$$n \to n-1 \quad \text{at rate } 2.$$

Clearly X_t^0 and X_n^0 are transient whenever $\lambda > \lambda_c$. By Corollary (2.4), so is X_n. □

Remarks. We have seen that certain hitting probabilities for the P^λ_*-evolution of ω_t are sandwiched between corresponding probabilities for a pair of birth-death processes. The monotonicity of p_A therefore gives more detailed information: e.g., from (3.4) we get for $\lambda < \lambda_c$,

$$P^\lambda_*(|\omega_t| = n \quad \text{for some} \quad t < T_0) \le e^{-C_0(n-1)}.$$

From (3.4) and the random walk comparison, we get for $\lambda = \lambda_c$,

$$\frac{C_1}{n} \le P^\lambda_*(|\omega_t| = n \quad \text{for some} \quad t < T_0) \le \frac{C_2}{\log n}. \tag{3.5}$$

It would be interesting to improve the bounds in (3.5).

4. Critical exponents. In this section we compute some critical exponents for Spitzer's nearest particle system. General background on phase transitions, critical exponents, universality and renormalization may be found in [23], [24], [27], and [30]. Our object is simply to provide some elementary examples.

As we have already seen, the finite and infinite nearest particle systems exhibit critical behavior at the same value $\lambda_c = z^{-1}$. Correspondingly, there are critical exponents associated with both the finite and infinite systems. One must also treat separately the subcritical ($\lambda \uparrow \lambda_c$) and supercritical ($\lambda \downarrow \lambda_c$) asymptotics. Thus there are four cases.

I. Finite subcritical systems. Two quantities of interest here are:

$$\chi_1^\lambda = E^\lambda_*[T_0],$$

and

$$\chi_2^\lambda = E^\lambda_*\left[\textstyle\sum_{x\in Z} \displaystyle\int_0^\infty 1_{\{\omega_t(x)=1\}}\, dt\right].$$

χ_1^λ is the expected absorption time from a singleton. χ_2^λ represents the expected space-time cluster size of the total particle production originating from a singleton. Clearly both quantities diverge as $\lambda \uparrow \lambda_c$. Analogues of χ_2^λ are familiar from percolation (e.g. [3]) and other physical models. In [2] a power series for the "generalized" susceptibility" χ_2^λ is developed to 20 decimal places in order to estimate the critical value of a "reggeon quantum spin model." Various critical exponents are also estimated. It turns out [11] that the reggeon model is isomorphic to the contact system, so they obtain an approximation for the contact critical value: $\lambda_c \approx 1.649$. This sort of numerical analysis abounds in the physics literature; while it displays an intriguing internal consistency, the methodology is *not* mathematically rigorous. As we mentioned earlier, the critical contact value is not rigorously known to even one decimal place. Understandably, then, precise evaluation of critical exponents for the contact process must be viewed as exceedingly difficult. Kesten's recent work [21] on a similar percolation problem gives some indication of the magnitude of the problem. For Spitzer's model, on the other hand, we can easily compute χ_1^λ and χ_2^λ *exactly*. The universality mentioned in the Introduction is particularly striking here: the critical exponents are entirely independent of the density function f.

(4.1) THEOREM. *For any given f,*

$$\chi_1^\lambda = \lambda_c(\lambda_c - \lambda)^{-1}, \quad \chi_2^\lambda = \lambda_c^2(\lambda_c - \lambda)^{-2}, \quad (\lambda < \lambda_c).$$

PROOF. If X_t is a continuous time recurrent Markov chain on S with invariant measure α, then for any $h: S \to R$,

$$E_0\left[\int_0^{T_0^+} h_{X_t}\,dt\right] = (q_0\alpha_0)^{-1} \textstyle\sum_\iota \alpha_\iota h_\iota. \tag{4.2}$$

(here q_0 is the jump rate for leaving state 0.) Equation (4.2) follows from the familiar representation of the invariant measures, $\alpha_j = cE_0$ [time in j before T_0^+] for some constant c. We apply (4.2) to the shape chain X_t introduced in Section 3; by (3.2) X_t has the invariant measure $\alpha_{\tilde B} = \alpha_B$ defined in (3.1). Take $h \equiv 1$, $0 = \tilde\varnothing$ in (4.2). Now compute as in (3.3) but without the factor of k which occurs there to get

$$\chi_1^\lambda = E_*^\lambda[T_\varnothing] = \tilde E_{\tilde\varnothing}^\lambda[T_{\tilde\varnothing}] - \tilde E_{\tilde\varnothing}^\lambda[T_*] = \lambda^{-1}(\textstyle\sum_{\tilde B} \alpha_{\tilde B} - 1) = \sum_{k=1}^\infty \left(\dfrac{\lambda}{\lambda_c}\right)^{k-1} = \lambda_c(\lambda_c - \lambda)^{-1}.$$

Similarly, taking $h_{\tilde B} = |B|$, we calculate

$$\chi_2^\lambda = E_*^\lambda\left[\int_0^{T_0} |\omega_t|\,dt\right] = \lambda^{-1} \textstyle\sum_{\tilde B} |B|\, \alpha_{\tilde B} = \sum_{k=1}^\infty k\left(\dfrac{\lambda}{\lambda_c}\right)^{k-1} = \lambda_c^2(\lambda_c - \lambda)^{-2}. \qquad \square$$

II. Finite supercritical systems. Here the simplest quantity of interest is the survival probability

$$\sigma^\lambda = P_*^\lambda(T_0 = \infty).$$

By using the random walk comparison from Section 3 and estimate (3.4), one can show that for any density f,

$$\lambda^{-1}(\lambda - \lambda_c) \le \sigma^\lambda \le |\log \lambda^{-1}(\lambda - \lambda_c)|^{-1}$$

for λ above λ_c. It would be interesting to know the true asymptotics for σ^λ as $\lambda \downarrow \lambda_c$, and whether the rate of decay is independent of f.

III. Infinite subcritical systems. Virtually nothing is known in this case. For $\lambda < \lambda_c$ presumably $\rho_t^\lambda = P_1^\lambda(\omega_t(0) = 1) \downarrow 0$ exponentially in t. If this were proved, then perhaps the simplest critical asymptotics would be for

$$\chi_3^\lambda = \int_0^\infty \rho_t^\lambda\,dt \quad \text{as} \quad \lambda \uparrow \lambda_c.$$

IV. Infinite supercritical systems. When a physical system has an equilibrium state

ν^{λ}, two of the most basic quantities of interest are the density and susceptibility:

$$\rho^{\lambda} = \nu^{\lambda}\{\omega(0) = 1\} \quad \text{and} \quad \chi_4^{\lambda} = \textstyle\sum_{x\in Z} \operatorname{cov}_{\nu^{\lambda}}(\omega(0), \omega(x))$$

respectively. Understanding these quantities near the critical value is one of the main objects of the study of phase transitions. For Spitzer's nearest particle model ν^{λ} is a renewal measure, and so ρ^{λ} and χ_4^{λ} can be computed explicitly in terms of the moments of the underlying renewal probability density $f_k^{\lambda} = \lambda f_k s^k$. Let μ_{λ}, υ_{λ} be the mean and variance of f^{λ}. By the Renewal Theorem,

$$\rho^{\lambda} = \mu_{\lambda}^{-1}. \tag{4.3}$$

To compute the susceptibility we use a formula from Feller [9, page 340]: if u_n^{λ} is the renewal sequence generated by f^{λ}, then

$$\textstyle\sum_{n=0}^{\infty} (u_n^{\lambda} - \mu_{\lambda}^{-1}) = (\upsilon_{\lambda} - \mu_{\lambda} + \mu_{\lambda}^2)/(2\mu_{\lambda}^2).$$

Thus

$$\begin{aligned} \chi_4^{\lambda} &= 2\mu_{\lambda}^{-1}[\textstyle\sum_{n=1}^{\infty} (u_n^{\lambda} - \mu_{\lambda}^{-1})] + (\mu_{\lambda}^{-1} - \mu_{\lambda}^{-2}) \\ &= 2\mu_{\lambda}^{-1}\left[\frac{\upsilon_{\lambda} - \mu_{\lambda} + \mu_{\lambda}^2}{2\mu_{\lambda}^2} - (1 - \mu_{\lambda}^{-1})\right] + (\mu_{\lambda}^{-1} - \mu_{\lambda}^{-2}) \\ &= \upsilon_{\lambda}/\mu_{\lambda}^3. \end{aligned} \tag{4.4}$$

Thus the computation of critical exponents for ρ^{λ} and χ_4^{λ} as $\lambda \downarrow \lambda_c$ is reduced to routine analysis. Set $\Delta = \lambda - \lambda_c$. Write $\psi(\Delta) \propto \Delta^{\gamma}$ if ψ is regularly varying of order γ as $\Delta \downarrow 0$; slowly varying terms are typically ignored in the study of phase transitions. For simplicity, suppose that

$$f_k = k^{-\theta} \quad (\theta > 1).$$

In the following table we have computed the critical exponents γ_1 and γ_2 given by

$$\rho \propto \Delta^{\gamma_1}, \quad \chi_4 \propto \Delta^{-\gamma_2}$$

as functions of the decay rate θ.

θ	γ_1	γ_2
$\in (1, 2]$	$(2-\theta)/(\theta-1)$	$(2\theta-3)/(\theta-1)$
$\in (2, 3]$	0	$3-\theta$
$\in (3, \infty)$	0	0

For $\theta > 2$, $\gamma_1 = 0$ because $\rho^{\lambda} \downarrow \rho^{\lambda_c} > 0$, i.e. the critical system is nonergodic. When $\theta \in (1, 2]$, γ_1 interpolates between ∞ and 0. For $\theta > 3$, $\gamma_2 = 0$ because the correlations of ν^{λ_c} are weak. Note that when $\theta < 3/2$, $\gamma_2 \leq 0$ also; this curious phenomenon occurs because $\rho^{\lambda} \to 0$ very rapidly. If an equilibrium ν^{λ_c} is nondegenerate, one is also interested in the critical scaling exponent. Letting ω be ν^{λ_c}-distributed, this is the value γ_3 such that

$$n^{-\gamma_3} \textstyle\sum_{x=1}^{n} [\omega(x) - \rho^{\lambda_c}]$$

converges weakly to a nondegenerate limit. In our examples γ_3 is simply the normalization exponent in the central limit theorem for the number of renewals up to time n. The relevant results are in Feller [10, page 373–374]. If $f_k = k^{-\theta}$, then

$$\begin{aligned} \gamma_3 &= (\theta - 1)^{-1}, && \theta \in (2, 3), \\ &= 1/2 && \theta \geq 3. \end{aligned}$$

For $\theta \geq 3$ the limit is Gaussian; see [17] for more detailed central limit phenomena in this case. When $\theta \in (2, 3)$ the limit is stable with index $\theta - 1$. The latter situation arises because the susceptibility diverges at λ_c and ν^{λ_c} has strong correlations. Physical systems with non-Gaussian limits and what Feller calls "extremely violent" fluctuations about the expectation are the subject of extensive study in contemporary mathematical physics. Of course, in the systems of most physical interest ν_{λ_c} is immensely more complicated than a renewal measure. Finally, a word about universality. For the infinite supercritical systems the universality classes consist of all those f which are regularly varying of order θ. Although the critical values for different f of order θ typically will differ, the critical exponents γ_1, γ_2 and γ_3 will agree with those for $k^{-\theta}$ computed above.

ADDED IN PROOF. The correct asymptotics in (3.5) and II of Section 4 are obtained under a moment assumption in a forthcoming paper by T. M. Liggett.

REFERENCES

[1] BRAMSON, M. and GRAY, L. (1981). A note on the survival of the long-range contact process. *Ann. Probab.* **9** 885–890.
[2] BROWER, R. C., FURMAN, M. A. and MOSHE, M. (1978). Critical exponents for the reggeon quantum spin model. *Physics Letters* **76B** 213–219.
[3] CARDY, J. and SUGAR, R. L. (1980). Directed percolation and Reggeon field theory. *J. Physics* **A 13** L423–L427.
[4] COCOZZA, C. and KIPNIS, C. (1977). Processus de vie et de mort sur *R* ou *Z* avec interaction selon les particules les plus proches. *C.R. Acad. Sci. Paris A* **284** 1291.
[5] COCOZZA, C. and ROUSSIGNOL, M. (1979). Unicité d'un processus de naissance et mort sur la droite réelle. *Ann. Inst. Henri Poincaré* **XV** 93–105.
[6] DOYLE, P. (1980). Unpublished manuscript. Dartmouth College.
[7] DURRETT, R. (1981). An introduction to infinite particle systems. *Stochastic Process. Appl.* **11** 109–150.
[8] DURRETT, R. and GRIFFEATH, D. (1983) Supercritical contact processes on *Z*. *Ann. Probability* **11** (To appear).
[9] FELLER, W. (1968). *An Introduction to Probability Theory and its Applications*, Vol. I. 3rd ed. Wiley, New York.
[10] FELLER, W. (1971). *An Introduction to Probability Theory and its Applications*, Vol. II, 2nd ed. Wiley, New York.
[11] GRASSBERGER, P. and DE LA TORRE, A. (1979). Reggon field theory (Schlögle's first model) on a lattice: Monte Carlo calculations of critical behaviour. *Ann. Physics* **122,** 373–396.
[12] GRAY, L. (1978). Controlled spin-flip systems. *Ann. Probability* **6** 953–974.
[13] GRIFFEATH, D. (1981). The basic contact process. *Stochastic Process. Appl.* **11** 151–186.
[14] HARRIS, T. E. (1974). Contact interactions on a lattice. *Ann. Probability* **4** 175–194.
[15] HOLLEY R. and LIGGETT, T. M. (1981). Generalized potlatch and smoothing processes. *Z. Wahrsch. verw. Gebiete* **55** 165–195.
[16] HOLLEY, R. A. and STROOCK, D. W. (1978). Nearest neighbor birth and death processes on the real line. *Acta Mathematica* **140** 103–154.
[17] HOLLEY, R. and STROOCK, D. W. (1979). Central limit phenomena of various interacting systems. *Ann. Math.* **110** 333–393.
[18] JEANS, SIR J. (1948). *The Mathematical Theory of Electricity and Magnetism*, 5th ed. Cambridge University Press. See pages 321–324, "generation of heat".
[19] KELLY, F. P. (1979). *Reversibility and Stochastic Networks.* Wiley, New York.
[20] KEMENY, J. G., SNELL, J. L. and KNAPP, A. W. (1976). *Denumerable Markov Chains.* Springer-Verlag, New York.
[21] KESTEN, H. (1980). Analyticity properties and power law estimates of functions in percolation theory. Unpublished manuscript.
[22] KIRCHOFF, G. (1891). *Vorlesungen über Electricität und Magnetismus.* Leipzig.
[23] LANG, R. (1981). An elementary introduction to the mathematical problems of the renormalization group. Unpublished manuscript.
[24] MA, S. (1976). *Modern Theory of Critical Phenomena.* Benjamin, London.
[25] MAXWELL, J. C. (1891). *A Treatise on Electricity and Magnetism*, 3rd ed. Reprinted by Dover, New York, 1954.
[26] NASH-WILLIAMS, C. ST. J. A. (1959). Random walk and electric current in networks. *Proceedings of the Cambridge Philosophical Society* **55** 181–194.
[27] PFEUTY, P. and TOULOUSE, G. (1977). *Introduction to the Renormalization Group and to Critical Phenomena.* Wiley, London.

[28] RAYLEIGH, J. W. S. (1870). On the theory of resonance. *Philosophical Transactions*, CLXI. Reprinted in *Collected Scientific Papers* I 33–75. Cambridge University Press, Mass.
[29] SPITZER, F. (1977). Stochastic time evolution of one dimensional infinite particle systems. *Bull. Amer. Math. Soc.* **83** 880–890.
[30] STANLEY, H. E. (1971). *Introduction to Phase Transitions and Critical Phenomena.* Clarendon, Oxford.
[31] WEINBERGER, H. F. (1965). *A First Course in Partial Differential Equations.* Blaisdell, Waltham, Mass.

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