CRITICAL SPEED FOR QUENCHING

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A one-dimensional heat equation with a nonlinear, concentrated quenching source that moves with constant speed through a diffusive medium is examined. Bounds are established for a critical speed above which quenching will not occur. When quenching does occur, bounds are given for the quenching time. For the special case of a power law nonlinearity, the growth rate near quenching is derived. The analysis is conducted in the context of a nonlinear Volterra integral equation that is equivalent to the initial-boundary value problem.

1. Introduction

In quenching problems the solution remains bounded while the first-order time derivative becomes unbounded in finite time. We examine a quenching problem for the heat equation in an infinite one-dimensional strip. The nonlinear, concentrated quenching source term moves with constant speed through the diffusive medium. In the case of a stationary source, it is known that, under appropriate conditions, quenching will occur [2, 3, 4, 5, 7, 9]. The main issue of interest here is to establish that there is a critical speed above which quenching will not take place. A related problem that deals with the critical speed to avoid blowup in a reactive-diffusive medium has recently been examined in [6].

A stationary quenching source, which is concentrated at a fixed position, will continuously supply heat in an attempt to raise the local temperature to a prescribed level. Under appropriate conditions, this energy supply will be adequate to attain the prescribed temperature level in a finite time, thereby achieving the quenched state. By allowing the source to move, it is perpetually exposed to new surroundings that are relatively cool. Thus, if the source moves at a sufficiently high speed, it may not be able to supply enough energy at any fixed site to achieve quenching.

To gain some insight into this phenomenon, we introduce a one-dimensional model for the temperature of a diffusive medium with a concentrated quenching source that moves at a constant speed. We consider the temperature $v(x,t)$ in an infinite slab that satisfies

$$\frac{\partial v}{\partial t}(x,t) - \frac{\partial^2 v}{\partial x^2}(x,t) = \delta(x - x_0) f\left[1 - v(x_0, t_0)\right], \quad -\infty < x < \infty, \ t > 0, \tag{1.1}$$

subject to the boundary conditions

$$v(x,t) \longrightarrow 0 \quad \text{as } x \longrightarrow \pm\infty, \ t > 0, \tag{1.2}$$

and the initial condition

$$v(x,0) = v_0(x), \quad -\infty < x < \infty. \tag{1.3}$$

The initial data is assumed to be sufficiently smooth with $v_0(x) \to 0$ as $x \to \pm\infty$ and

$$0 \leq v_0(x) \leq 1 - \delta, \quad 0 < \delta < 1. \tag{1.4}$$

The quenching source in (1.1) is concentrated by the delta function at $x = x_0$, where

$$x_0 = ct, \tag{1.5}$$

with $c > 0$ being the constant speed of translation.

Consistent with the notion of quenching, the nonlinearity $f(1-v)$ is assumed to have the properties

$$f(1-v) > 0, \quad f_v(1-v) > 0, \quad f_{vv}(1-v) > 0, \quad \text{for } 0 < v < 1,$$
$$f(1-v) \longrightarrow +\infty \quad \text{as } v \longrightarrow 1^-. \tag{1.6}$$

In view of (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6), it is clear from the maximum principle that the quenching temperature $v = 1$ can only be attained at $x = x_0$. Thus, we say that a quenching state is achieved if

$$v(x_0, t) \longrightarrow 1^-, \quad \frac{\partial v}{\partial t}(x_0, t) \longrightarrow +\infty, \quad \text{as } t \longrightarrow \hat{t} < \infty, \tag{1.7}$$

where $\hat{t}$ is called the quenching time.

To analyze whether or not (1.7) occurs, as well as the influence that the speed c has on that occurrence, we will convert (1.1), (1.2), and (1.3) into an equivalent Volterra integral equation for $v(x_0, t)$. Using the Green's function associated with the one-dimensional heat operator for $-\infty < x < \infty$, it follows that (1.1), (1.2), and (1.3) is satisfied by

$$v(x,t) = V(x,t) + \int_0^t \frac{e^{-(x-x_0)^2/4(t-s)}}{2\left[\pi(t-s)\right]^{1/2}} f\left[1 - v(x_0, s)\right] ds, \quad t \geq 0, \tag{1.8}$$

where

$$V(x,t) = \frac{1}{2(\pi t)^{1/2}} \int_{-\infty}^{\infty} e^{-(x-\xi)^2/4t} \, v_0(\xi) \, d\xi. \tag{1.9}$$

Since our interest is in the behavior of the temperature at $x = x_0 = ct$, we define

$$u(t) \equiv v(x_0, t) = v(ct, t). \tag{1.10}$$

Upon evaluating (1.8) and (1.9) at $x = x_0$, it follows that $u(t)$ is determined by the nonlinear Volterra equation,

$$u(t) = Tu(t) \equiv h(t) + \int_0^t \frac{e^{(-c^2/4)(t-s)}}{2\big[\pi(t-s)\big]^{1/2}} f\big[1 - u(s)\big] ds, \tag{1.11}$$

where

$$h(t) = V(x_0, t) = V(ct, t). \tag{1.12}$$

In the analysis to follow, we will show that for sufficiently small values of c, the solution of (1.11) quenches in the sense that

$$u(t) \longrightarrow 1^-, \quad u'(t) \longrightarrow +\infty, \quad \text{as } t \longrightarrow \hat{t} < \infty. \tag{1.13}$$

We will also be able to show that for sufficiently large values of c, the solution of (1.11) exists with $u(t) < 1$ for all $t \geq 0$. From this, we can deduce an upper bound on the critical speed $\hat{c}$ above which quenching does not occur.

2. Analysis of the integral equation

We consider the existence and nonexistence of a continuously differentiable solution of the nonlinear Volterra equation (1.11). Nonexistence is associated with the occurrence of quenching as defined by (1.13). Our approach to (1.11) will employ techniques similar to those for blowup problems developed in [8]. Our analysis employs contraction mapping arguments to establish the existence of a unique solution $0 \leq u < 1$ to (1.11) for $0 \leq t < t^*$. This provides a lower bound $t^* \leq \hat{t}$ on the quenching time. It will be shown that if c is sufficiently large, then $t^* = \infty$ and hence quenching does not occur. Using certain differential inequalities derived from (1.11), we will be able to establish the nonexistence of a solution for all $t \geq t^{**}$ provided that c is sufficiently small.

For the existence of a solution to (1.11) we have the following theorem.

THEOREM 2.1. *There exists a unique solution of (1.11) that is continuously differentiable and satisfies $0 \leq u(t) \leq M < 1$, $0 \leq t < t^*$, where*

$$\frac{1}{c} \operatorname{erf}\left(\frac{ct^{*1/2}}{2}\right) = \min\left\{\frac{\big[M - (1-\delta)\big]}{f[1-M]}, \frac{1}{c}\right\} \tag{2.1}$$

with M determined by

$$\frac{M-(1-\delta)}{f(1-M)} = \frac{1}{f_u(1-M)}. \tag{2.2}$$

Proof. From the properties of $h(t)$, $f(1-u)$ and the kernel in (1.11), it is clear that whenever there is a solution bounded above by unity, then it must be positive. Thus, we begin by examining (1.11) for the space of continuous functions $u(t)$ that satisfy

$$0 \leq u(t) \leq M < 1, \quad 0 \leq t < t^*. \tag{2.3}$$

First we must show that the integral operator T, defined in (1.11), maps $u(t)$ to the same space. Clearly, $Tu(t)$ is continuous and from the properties of $f(1-u)$ it follows that

$$Tu(t) \leq \max\left[h(t)\right] + f(1-M) I(t), \quad 0 \leq t < t^*, \tag{2.4}$$

where

$$I(t) \equiv \int_0^t \frac{e^{-(c^2/4)s}}{2(\pi s)^{1/2}} \, ds = \frac{1}{c} \operatorname{erf}\left(\frac{ct^{1/2}}{2}\right). \tag{2.5}$$

Since $V(x,t)$ satisfies (1.1), (1.2), and (1.3) with $f \equiv 0$, it follows from the maximum principle that $V(x,t) \leq v_0(x)$. Then (1.4) and (1.12) provide that $h(t) \leq 1 - \delta$. Thus, from (2.4), we achieve the desired mapping with the requirement that

$$1 - \delta + f(1-M)\frac{1}{c}\operatorname{erf}\left(\frac{ct^{*1/2}}{2}\right) \leq M. \tag{2.6}$$

To establish conditions for T to be a contraction, we use the sup norm for $0 \leq t < t^*$ to obtain

$$\left\|Tu_1 - Tu_2\right\| = \left\|\int_0^t \frac{e^{-(c^2/4)(t-s)}}{2\left[\pi(t-s)\right]^{1/2}}\left[f\left(1-u_1(s)\right) - f\left(1-u_2(s)\right)\right]ds\right\|$$

$$\leq f_u(1-M)\frac{1}{c}\operatorname{erf}\left(\frac{ct^{*1/2}}{2}\right)\left\|u_1(t) - u_2(t)\right\|. \tag{2.7}$$

For the contraction, we require that

$$\frac{1}{c}\operatorname{erf}\left(\frac{ct^{*1/2}}{2}\right)f_u(1-M) < 1. \tag{2.8}$$

To satisfy (2.6) and (2.8) for the largest possible t^*, the optimal choice of M is determined by

$$\frac{M-(1-\delta)}{f(1-M)} = \frac{1}{f_u(1-M)}. \tag{2.9}$$

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Thus, if there exists a $t^* < \infty$ such that

$$\frac{1}{c} \operatorname{erf}\left(\frac{ct^{*1/2}}{2}\right) = \frac{M - (1 - \delta)}{f(1 - M)}, \tag{2.10}$$

then (1.11) has a unique continuous solution $u(t) < 1$ for $0 \leq t < t^*$. In this case, t^* represents the lower bound on any possible quenching time $\hat{t} < \infty$.

If (2.6) and (2.8) are both satisfied with $t^* = \infty$, then (1.11) has a unique continuous solution $u(t) < 1$ for all $t \geq 0$. Clearly, this can always be fulfilled for c sufficiently large, such as

$$c \geq c^* \equiv \frac{f(1 - M)}{M - (1 - \delta)} = f_u(1 - M). \tag{2.11}$$

The implication of (2.11) is that there is a critical speed $\hat{c}$ such that there is no quenching if $c \geq \hat{c}$. Moreover, (2.11) provides that

$$\hat{c} \leq c^*. \tag{2.12}$$

To demonstrate that the continuous solution of (1.11) established by contraction mapping is also continuously differentiable for $0 < t < t^*$, we consider the derivative of (1.11),

$$u'(t) = h'(t) + \frac{e^{-(c^2/4)t}}{2(\pi t)^{1/2}} f\left[1 - h(0)\right] + \int_0^t \frac{e^{-(c^2/4)(t-s)}}{2\left[\pi(t-s)\right]^{1/2}} f_u\left[1 - u(s)\right] u'(s) \, ds. \tag{2.13}$$

With the knowledge that a continuous $u(t)$ exists for $0 \leq t < t^*$, then (2.13) can be viewed as a linear Volterra equation for $u'(t)$. Then the Neumann series generated by repeated iterations of (2.13) will provide a representation of $u'(t)$ that is continuous for $0 < t < t^*$. This establishes Theorem 2.1.   □

The solution of (1.11) quenches when (1.13) is satisfied. We want to establish that, under appropriate conditions, it is sufficient to have $u(t) \to 1$ as $t \to \hat{t} < \infty$ for quenching to occur. This follows from the following theorem.

THEOREM 2.2. *Let (1.11) have a continuously differentiable solution $u(t)$ for $0 \leq t < \hat{t}$, where $u(t) \to 1$ as $t \to \hat{t}$. If $h'(t) + e^{-(c^2/4)t}/(2(\pi t)^{1/2}) f[1 - h(0)] > 0$ for $0 < t < \hat{t}$, then $u'(t) \to \infty$ as $t \to \hat{t}$.*

Proof. Under the conditions of the theorem, it is clear from (2.13) that

$$u'(t) > 0, \quad 0 < t < \hat{t}. \tag{2.14}$$

It then follows from (2.13) that

$$u'(t) \geq h'(t) + \frac{e^{-(c^2/4)t}}{2(\pi t)^{1/2}} \left\{ f\left[1 - h(0)\right] + \int_0^t f_u\left[1 - u(s)\right] u'(s) \, ds \right\}$$

$$= h'(t) + \frac{e^{-(c^2/4)t}}{2(\pi t)^{1/2}} f\left[1 - u(t)\right]. \tag{2.15}$$

Since $h'(t)$ is bounded and $f[1-u] \to \infty$ as $u \to 1^-$, then it is clear from (2.15) that $u'(t) \to \infty$ whenever $u(t) \to 1^-$, or as $t \to \hat{t}$. This establishes Theorem 2.2. $\qquad\square$

To show that quenching does occur for sufficiently small c, we examine (1.11) for conditions under which there does not exist a continuous solution $u(t) < 1$ for $t \geq t^{**}$. This result is given by the following theorem.

THEOREM 2.3. *Whenever there exists a t^{**} such that*

$$\frac{1}{c}\,\mathrm{erf}\left(\frac{ct^{**1/2}}{2}\right) = \kappa \equiv \int_{h_0}^1 \frac{dr}{f[1-r]}, \quad h_0 \equiv \inf_{0 \leq t < \infty} h(t), \qquad (2.16)$$

then (1.11) cannot have a continuous solution $u(t) < 1$ for $t \geq t^{**}$.

Proof. Under the assumption that (1.11) has a continuous solution $0 \leq u(t) < 1$ for $0 \leq t \leq t^{**}$, we utilize the property that the kernel is decreasing in t to obtain the inequality

$$u(t) = Tu(t) \geq h(t) + J(t), \quad 0 \leq t \leq t^{**}, \qquad (2.17)$$

where

$$J(t) \equiv \int_0^t \frac{e^{-(c^2/4)(t^{**}-s)}}{2\left[\pi\left(t^{**}-s\right)\right]^{1/2}} f[1-u(s)]\,ds, \quad 0 \leq t \leq t^{**}. \qquad (2.18)$$

Differentiation of (2.18) and using (2.17) yields the differential inequality

$$
\begin{aligned}
J'(t) &= \frac{e^{-(c^2/4)(t^{**}-t)}}{2\left[\pi\left(t^{**}-t\right)\right]^{1/2}} f[1-u(t)] \\
&\geq \frac{e^{-(c^2/4)(t^{**}-t)}}{2\left[\pi(t^{**}-t)\right]^{1/2}} f[1-h_0-J(t)], \quad 0 \leq t < t^{**}.
\end{aligned}
\qquad (2.19)
$$

Integration of (2.19) yields

$$\int_0^{J(t^{**})} \frac{dJ}{f[1-h_0-J]} \geq \int_0^{t^{**}} \frac{e^{-(c^2/4)s}}{2(\pi s)^{1/2}}\,ds = \frac{1}{c}\,\mathrm{erf}\left(\frac{ct^{**1/2}}{2}\right). \qquad (2.20)$$

Nonexistence occurs if $J(t^{**}) = 1 - h(t^{**})$, since this implies that $u(t^{**}) \geq 1$. Consequently, the smallest value of t^{**} that implies a contradiction of (2.20) is given by

$$\frac{1}{c}\,\mathrm{erf}\left(\frac{ct^{**1/2}}{2}\right) = \kappa \equiv \int_{h_0}^1 \frac{dr}{f[1-r]}. \qquad (2.21)$$

Whenever (2.21) is satisfied by $t^{**} < \infty$, then there is nonexistence of a solution to (1.11) for all $t \geq t^{**}$. In this case, t^{**} represents an upper bound on the quenching time $\hat{t}$. This establishes Theorem 2.3. $\qquad\square$

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A consequence of Theorem 2.3 is that quenching always occurs for a sufficiently slow speed. We can establish bounds on this critical speed for quenching. Furthermore, when quenching does occur, we can establish bounds on the quenching time. These results are given by the following theorem.

THEOREM 2.4. *There exists a critical speed $\hat{c}$, below which quenching occurs. Bounds on the critical speed $\hat{c}$ are given by*

$$\kappa^{-1} \leq \hat{c} \leq f_u(1-M), \tag{2.22}$$

where M is determined by (2.2). Moreover, when quenching occurs, the quenching time $\hat{t}$ has the bounds

$$t^* \leq \hat{t} \leq t^{**}, \tag{2.23}$$

where t^* is given by (2.10) and t^{**} by (2.21).

Proof. To show that quenching is always achieved for some sufficiently slow speed, consider the asymptotic behavior of (2.21) for c small. It follows that

$$t^{**} = \pi \kappa^2 + O(c^2), \quad \text{as } c \longrightarrow 0. \tag{2.24}$$

It is clear that (2.24) will always be fulfilled by some $t^{**} < \infty$, which implies nonexistence and hence quenching.

Furthermore, it is seen that if $c\kappa < 1$, then (2.21) can be satisfied by some $t^{**} < \infty$, which implies quenching. Consequently, the critical speed $\hat{c}$ above which there is no quenching has the lower bound $\hat{c} \geq \kappa^{-1}$. The upper bound on the critical speed follows from the existence of a solution to (1.11) for all $t \geq 0$. In Theorem 2.1, that bound is given by (2.12). Thus, (2.22) follows.

When quenching does occur, the quenching time is limited below by the existence of a solution $u(t) < 1$. From Theorem 2.1, this implies that $\hat{t} \geq t^*$ where t^* is given by (2.10). The nonexistence result of Theorem 2.3 assures that quenching has occurred. Hence, $\hat{t} \leq t^{**}$ as given by (2.21). This establishes Theorem 2.4.   □

3. Growth rates near quenching

It is of interest to determine the asymptotic behavior of the solution to (1.11) as quenching is approached. In [10], we have presented a methodology in which the asymptotic analysis of integrals is used to find the leading order growth rate of the solution to a broad class of nonlinear Volterra equations. To apply the methodology of [10] here, it is convenient to use the differentiated form of (1.11). We express (2.13) in the form

$$u'(t) = g(t) + \int_0^t k(t-s) f_u[1-u(s)]u'(s) ds, \quad t > 0, \tag{3.1}$$

where

$$k(t-s) \equiv \frac{e^{-(c^2/4)(t-s)}}{2\big[\pi(t-s)\big]^{1/2}}, \quad t > s, \tag{3.2}$$

$$g(t) \equiv h'(t) + k(t) f\big[1 - h(0)\big]. \tag{3.3}$$

The methodology of [10] is a formal scheme. For an assumed behavior of the solution as $t \to \hat{t}$, the scheme will verify that an asymptotic balance to leading order is obtained from (3.1).

The methodology of [10] is applicable to kernels that have algebraic singularities as $s \to t$. From (3.2),

$$k(t-s) \sim k_0(t-s) = \frac{1}{2\big[\pi(t-s)\big]^{1/2}}, \quad \text{as } s \longrightarrow t. \tag{3.4}$$

In [10], a variety of nonlinearities are treated. Here, we confine our attention to one case of particular interest in applications, namely the power law

$$f[1-u] = \alpha(1-u)^{-\beta}, \quad \alpha > 0, \ \beta > 0. \tag{3.5}$$

We assume that the speed c is sufficiently small to ensure that quenching does occur, so that

$$u(t) \longrightarrow 1^-, \quad u'(t) \longrightarrow \infty, \quad \text{as } t \longrightarrow \hat{t} < \infty. \tag{3.6}$$

While the methodology of [10] does determine the leading order behavior of $u'(t)$ in (3.1), it does not determine an explicit value for the quenching time $\hat{t}$.

The approach of [10] is to shift the singularity at $\hat{t}$ to a point at infinity through the change of variables

$$\eta = (\hat{t}-t)^{-1} - \eta_0, \quad \eta_0 = (\hat{t})^{-1}, \qquad w(\eta) = u(t). \tag{3.7}$$

This converts (3.1) to

$$(\eta+\eta_0)^2 w'(\eta) = g\big[\hat{t} - (\eta+\eta_0)^{-1}\big]$$
$$+ \alpha\beta \int_0^\eta k\Big\{(\eta-\sigma)\big[(\eta+\eta_0)(\sigma+\eta_0)\big]^{-1}\Big\} \Phi(\sigma)\,d\sigma, \quad \eta \geq 0, \tag{3.8}$$

where

$$\Phi(\eta) \equiv f_w\big[1 - w(\eta)\big] w'(\eta). \tag{3.9}$$

We analyze (3.8) asymptotically in the limit $\eta \to \infty$ since the quenching described by (3.6) corresponds to

$$w(\eta) \longrightarrow 1^-, \quad \eta^2 w'(\eta) \longrightarrow \infty, \quad \text{as } \eta \longrightarrow \infty. \tag{3.10}$$

For a kernel that behaves like (3.4), it is shown in [10] that it can be replaced by its asymptotic behavior in the analysis as $\eta \to \infty$. Thus, (3.8) yields the asymptotic equation

$$\eta^2 w'(\eta) \sim g(\hat{t}) + \eta Q(\eta), \quad \text{as } \eta \longrightarrow \infty, \tag{3.11}$$

where

$$Q(\eta) \equiv \frac{1}{2} \left(\frac{\eta + \eta_0}{\eta} \right)^{1/2} \int_0^1 \frac{(\eta\tau + \eta_0)^{1/2}}{[\pi(1-\tau)]^{1/2}} \Phi(\eta\tau) \, d\tau. \tag{3.12}$$

It follows from (3.12) that

$$Q(\eta) \sim \frac{1}{2} \int_0^\infty K(\tau) F(\eta\tau) \, d\tau, \quad \text{as } \eta \longrightarrow \infty, \tag{3.13}$$

where

$$K(\tau) \equiv \frac{H(1-\tau)}{[\pi(1-\tau)]^{1/2}}, \qquad F(\eta\tau) \equiv (\eta\tau + \eta_0)^{1/2} \Phi(\eta\tau). \tag{3.14}$$

To obtain the asymptotic behavior of $Q(\eta)$ as $\eta \to \infty$, the integral in (3.13) is converted to one along a vertical path in the complex plane by application of the Parseval formula for Mellin transforms. This yields

$$Q(\eta) \sim \frac{1}{4\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} M[K(\tau); 1 - z] M[F(\eta\tau); z] \, dz, \tag{3.15}$$

where the Mellin transform is defined as

$$M[q(\tau); z] \equiv \int_0^\infty \tau^{z-1} q(\tau) \, d\tau. \tag{3.16}$$

In (3.15), the vertical path of integration in the complex z-plane lies within the common strip of analyticity for the two Mellin transforms. Further simplification of (3.15) follows from noting that

$$M[F(\eta\tau); z] = \eta^{-z} M[F(\tau); z] \tag{3.17}$$

and

$$M[K(\tau); 1 - z] = \int_0^1 \frac{\tau^{z-1}}{[\pi(1-\tau)]^{1/2}} \, d\tau = \frac{\Gamma(1-z)}{\Gamma(3/2 - z)}. \tag{3.18}$$

Then (3.11) takes the form

$$\eta^2 w'(\eta) \sim g(\hat{t}) + \frac{\eta}{4\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \eta^{-z} \frac{\Gamma(1-z)}{\Gamma(3/2 - z)} M[F(\tau); z] \, dz, \quad \text{as } \eta \longrightarrow \infty. \tag{3.19}$$

In order to proceed with the asymptotic analysis of (3.19), we must examine the properties of $M[F; z]$. These properties are governed by the asymptotic behavior of $F(\eta)$ as $\eta \to \infty$, where

$$F(\eta) = (\eta + \eta_0)^{1/2} f_w [1 - w(\eta)] w'(\eta). \tag{3.20}$$

Now examine the particular case of the power law nonlinearity given by (3.5). For this case,

$$F(\eta) = \alpha \beta (\eta + \eta_0)^{1/2} [1 - w(\eta)]^{-\beta - 1} w'(\eta). \tag{3.21}$$

To obtain the leading order behavior of the solution for this case, we assume an algebraic form

$$w(\eta) \sim 1 - A \eta^{-l}, \quad A > 0, \ 0 < l < 1, \ \text{as } \eta \longrightarrow \infty. \tag{3.22}$$

We will determine unique values of A and l so that (3.19) is formally satisfied to leading order.

From (3.21) and (3.22), it follows that

$$F(\eta) \sim \alpha \beta l A^{-\beta} \eta^{\beta l - 1/2}, \quad \text{as } \eta \longrightarrow \infty. \tag{3.23}$$

Given (3.23), it follows (see [1]) that $M[F; z]$ has a simple pole at $z = 1/2 - \beta l$. Thus, as $z \to 1/2 - \beta l$,

$$M[F; z] \sim \frac{\alpha \beta l A^{-\beta}}{z - (1/2 - \beta l)}. \tag{3.24}$$

In view of (3.24), we can now determine the asymptotic behavior of the integral in (3.19) for the case of the power law nonlinearity. The asymptotic behavior is found by displacing the vertical integration path to the right. Thus, the simple pole implied by (3.24) is encountered before the one arising from $\Gamma(1 - z)$ at $z = 1$. Then (3.19) becomes

$$A l \eta^{1-l} \sim g(\hat{t}) + \frac{\alpha \beta l A^{-\beta} \Gamma(1/2 + \beta l)}{2 \Gamma(1 + \beta l)} \eta^{1/2 + \beta l}, \quad \text{as } \eta \longrightarrow \infty. \tag{3.25}$$

Since $l < 1$, an asymptotic balance to leading order in (3.25) requires matching the two algebraically growing terms. This match yields

$$l = \frac{1}{2(\beta + 1)}, \qquad A = \left[\frac{\alpha \beta \Gamma(1/2 + \beta l)}{2 \Gamma(1 + \beta l)} \right]^{1/(\beta + 1)}. \tag{3.26}$$

Thus, the asymptotic behavior assumed in (3.22) is justified with l and A provided in (3.26). Using (3.8) to recover the original variables, it follows that

$$u(t) \sim 1 - A(\hat{t} - t)^{1/(2(\beta + 1))}, \quad \text{as } t \longrightarrow \hat{t}. \tag{3.27}$$

4. Conclusions

We have analyzed the initial-boundary problem (1.1), (1.2), (1.3), (1.4), (1.5), and (1.6), which models a concentrated source quenching moving through a diffusive medium, by converting it to an equivalent Volterra equation (1.11). Our analysis has shown that if the source moves sufficiently slowly, then quenching must occur. Moreover, if the source term moves faster than a critical speed $\hat{c}$, then quenching does not occur. Bounds on the critical speed follow from Theorem 2.4 as

$$\left[\int_{h_0}^{1} \frac{dr}{f(1-r)} \right]^{-1} \leq \hat{c} \leq f_v(1-M). \tag{4.1}$$

Here, $M < 1$ is determined from (2.9) and h_0 from (2.16).

When quenching does occur, as defined by (1.7), bounds on the quenching time $\hat{t}$ follow from Theorem 2.4 with (2.10) and (2.21) as

$$\frac{4}{c^2} \left\{ \text{erf}^{-1} \left[\frac{c}{f_v(1-M)} \right] \right\}^2 \leq \hat{t} \leq \frac{4}{c^2} \left\{ \text{erf}^{-1} \left[c \int_{h_0}^{1} \frac{dr}{f(1-r)} \right] \right\}^2 \tag{4.2}$$

The asymptotic behavior of the temperature at the concentrated source point $x_0 = ct$, as quenching is approached, has been considered for the special case in which the nonlinearity has the form

$$f\left[1 - v(x_0, t)\right] = \alpha\left[1 - v(x_0, t)\right]^{-\beta}, \quad \alpha > 0, \ \beta > 0. \tag{4.3}$$

Using formal asymptotic methods, it was verified that

$$v(x_0, ct) \sim 1 - A(\hat{t} - t)^{1/2(\beta+1)}, \quad \text{as } t \longrightarrow \hat{t}, \tag{4.4}$$

where A is given in (3.26).

Acknowledgements

References

[1]   N. Bleistein and R. A. Handelsman, *Asymptotic Expansion of Integrals*, Holt, Rinehart and Winston, New York, 1975.

[2]   C. Y. Chan, *Recent Advances in Quenching Phenomena*, Proceedings of Dynamic Systems and Applications, vol. 2, Dynamic, Atlanta, GA, 1996, pp. 107–113. MR 98a:35065. Zbl 908.35056.

[3] K. Deng and C. A. Roberts, *Quenching for a diffusive equation with a concentrated singularity*, Differential Integral Equations **10** (1997), no. 2, 369–379. MR 97g:35066. Zbl 891.35061.

[4] H. A. Levine, *The phenomenon of quenching: a survey*, Trends in the Theory and Practice of Nonlinear Analysis (Proceedings of the sixth international conference, Arlington, Tex., 1984), North-Holland, Amsterdam, 1985, pp. 275–286. MR 87b:35082. Zbl 581.35037.

[5] _____ , *Advances in quenching*, Nonlinear Diffusion Equations and their Equilibrium States (Proceedings of the conference held in Gregynog, 1989), Progress in Nonlinear Differential Equations and their Applications, vol. 7, Birkhäuser Boston, Boston, MA, 1992, pp. 319–346. CMP 1 167 848. Zbl 792.35017.

[6] W. E. Olmstead, *Critical speed for the avoidance of blow-up in a reactive-diffusive medium*, Z. Angew. Math. Phys. **48** (1997), no. 5, 701–710. MR 98j:35095. Zbl 884.35067.

[7] W. E. Olmstead and C. A. Roberts, *Quenching for the Heat Equation with a Nonlocal Nonlinearity*, Nonlinear Problems in Applied Mathematics, SIAM, Philadelphia, 1995, pp. 199–205. Zbl 886.35158.

[8] C. A. Roberts, D. G. Lasseigne, and W. E. Olmstead, *Volterra equations which model explosion in a diffusive medium*, J. Integral Equations Appl. **5** (1993), no. 4, 531–546. MR 94k:45005. Zbl 804.45002.

[9] C. A. Roberts and W. E. Olmstead, *Local and nonlocal boundary quenching*, submitted.

[10] _____ , *Growth rates for blow-up solutions of nonlinear Volterra equations*, Quart. Appl. Math. **54** (1996), no. 1, 153–159. MR 97a:45003. Zbl 916.45007.

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