Critical thoughts on computing atom condensed Fukui functions

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(Received 26 March 2007; accepted 22 May 2007; published online 16 July 2007)

Different procedures to obtain atom condensed Fukui functions are described. It is shown how the resulting values may differ depending on the exact approach to atom condensed Fukui functions. The condensed Fukui function can be computed using either the fragment of molecular response approach or the response of molecular fragment approach. The two approaches are nonequivalent; only the latter approach corresponds in general with a population difference expression. The Mulliken approach does not depend on the approach taken but has some computational drawbacks. The different resulting expressions are tested for a wide set of molecules. In practice one must make seemingly arbitrary choices about how to compute condensed Fukui functions, which suggests questioning the role of these indicators in conceptual density-functional theory. © *2007 American Institute of Physics*. [DOI: 10.1063/1.2749518]

INTRODUCTION

One of the key fields of research in modern chemistry is the study of the chemical reactivity. Reactivity reflects the susceptibility of a substance towards a specific chemical reaction, and thus plays a key role in, e.g., the design of new compounds, understanding biological systems, and material science. As a consequence, there is continuing interest in rationalizing existing reactivity models and developing new reactivity models for understanding and predicting chemical reactivity. Such models range from fairly simple electrostatic models to the latest and most advanced quantum chemical approaches.

Understanding and modeling chemical reactivity is the chief aim of so-called conceptual density-functional theory (DFT).[1–6] This field has allowed putting many important well-known quantities such as electronegativity,[7–9] electronegativity equalization,[10–17] the theory of hard and soft acids and bases,[18–24] philicity,[25] and many more, on a firmer theoretical footing. The field has even led to new concepts, e.g., the maximum hardness principle.[26–32]

The Fukui function $f(\mathbf{r})$ is among the most basic and commonly used reactivity indicators. It is defined as

$$f(\mathbf{r},N)=\left(\frac{\partial(\delta E/\delta\nu_{\mathrm{ext}})_N}{\partial N}\right)_{\nu_{\mathrm{ext}}}, \tag{1}$$

where E is the energy, N the number of electrons, and ν_{ext} is the external potential.[33–35] By virtue of the Hellmann-Feynman theorem and for nondegenerate cases, in the most frequently used definition, the Fukui function is given as the change in the density function $\rho(\mathbf{r},N)$ of the molecule as a consequence of changing the number of electrons N in the molecule, under the constraint of a constant external potential.

$$f(\mathbf{r},N)=\left(\frac{\partial\rho(\mathbf{r},N)}{\partial N}\right)_{\nu_{\mathrm{ext}}}. \tag{2}$$

The Fukui function clearly reduces to the frontier molecular orbital (FMO) theory of Fukui[36] when using a frozen orbital approximation.[34,37] The Fukui function can thus be regarded as a generalization of FMO theory. The Fukui function predicts how additional electron density will be redistributed in a molecule. The dependence on the number of electrons in the molecule is included explicitly for all functions involved in Eq. (2), as this will play an important role in further derivations.

Unfortunately, the direct evaluation of the Fukui function (2) beyond FMO is not obvious. Moreover, one needs to distinguish at least two Fukui functions, depending on whether an amount of electron density is added or subtracted.[33,38–41] In practical applications, use is often made of a finite difference approximation, where one performs an *ab initio* calculation on the molecule and the corresponding singly charged molecular ions, using always the same molecular geometry. This results for the addition of an electron in

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$$f^{+}(\mathbf{r},N) \approx \rho(\mathbf{r},N+1) - \rho(\mathbf{r},N) \tag{3}$$

and

$$f^{-}(\mathbf{r},N) \approx \rho(\mathbf{r},N) - \rho(\mathbf{r},N-1) \tag{4}$$

for the loss of an electron. Obviously, a finite difference with a full electron is not always a good approximation of Eq. (2), but it avoids computations with fractional numbers of electrons although such calculations have been found instructive.[42] The finite difference approximate is exact for exact computations (full configuration interaction), but is not exact for the computationally economical methods that are commonly used in conceptual DFT.[35,43]

Another aspect of chemistry, besides reactivity, that has played a very prominent role during its development is the view that molecules consist of atoms held together by chemical bonds. Many aspects of chemical reactivity are often traced back to the atoms and the bonds that compose the molecule, as this language facilitates constructing predictive models that only require information about the composition of a reacting molecule. It is thus unsurprising that many attempts are made to decompose a molecular property in to a set of atomic contributions, one for every atom A. Such atoms in molecules will henceforth be abbreviated as AIM. This is not limited to Bader's definition[44–46] but is used more generally. Bader's AIM will henceforth be referred to as quantum chemical topology (QCT) theory. Although it should be stressed that the atom in the molecule has always been, and is likely to remain a major cornerstone in chemistry, there is much less agreement as to what extent they can be defined uniquely.[44,47–49]

The Fukui function is no exception to this procedure of introducing AIM contributions, and such atomic Fukui functions are the major topic of the present paper. AIM Fukui functions are abbreviated as $f_A^{\pm}(\mathbf{r},N)$. Often, yet another level of abstraction is introduced. This means that one does not work with the atomic Fukui functions $f_A^{\pm}(\mathbf{r},N)$ but wishes to attach one single number to every atom. Such values are then called atom condensed Fukui functions and are denoted $\mathbf{f}_A^{\pm}(N)$.[50,51] They are obtained from atomic Fukui functions by integration,

$$\mathbf{f}_A^{\pm}(N) = \int f_A^{\pm}(\mathbf{r},N)d\mathbf{r}. \tag{5}$$

Unfortunately, whereas there are good theoretical grounds for molecular Fukui functions as in Eqs. (1) and (2), atomic Fukui functions and atom condensed values are much less well established. Not only do the results of atomic Fukui functions depend on the actual approach taken to identify the AIM, one can also derive different expressions for atomic Fukui functions and hence atom condensed Fukui functions. These different approaches will be shown to yield substantially different values and it will be shown that it may also have an impact on their sign. This is not merely an academic interest, as atom condensed Fukui functions are often used in chemical reactivity studies, and many authors apparently favor certain AIM methods because they would give positive atom condensed Fukui functions for the majority of the molecules.[52,53]

ATOM CONDENSED FUKUI FUNCTIONS

The problem with atom condensed Fukui functions starts when extracting atomic Fukui functions from the derivative in Eq. (2). In what follows, we will delineate different levels of abstraction to show how these have an impact on the resulting atom condensed Fukui functions.

Prior to specific discussions on the different methods, it is worth discussing atomic weight functions. All techniques discussed here for discerning an AIM rely on distributing the electron density in every point in space to one or more atoms. An AIM density function for atom $A(\rho_A(\mathbf{r},N))$ is obtained from the molecular one $(\rho(\mathbf{r},N))$ in the following way:

$$\rho_A(\mathbf{r},N) = w_A(\mathbf{r},N)\rho(\mathbf{r},N). \tag{6}$$

Note that in the weight functions $w_A(\mathbf{r},N)$ for all atoms always sum unity,

$$\sum_A^M w_A(\mathbf{r},N) = \mathbf{1}, \tag{7}$$

and that usually a positive definite weight function is used. Note also that the weight functions in general also depend on the number of electrons contained in the molecule. In other words, in general the weight functions will be different whether the molecule has $N, N-dN$, or $N+dN$ electrons even if the geometry is the same.

Frontier molecular orbital approach

The Fukui function essentially describes a response to a perturbation, namely, the change in number of electrons under constant external potential. As mentioned above, FMO theory of Fukui and co-workers can be seen as the origin of Fukui functions in modern DFT. Within the philosophy of perturbation theory, one can opt to use only properties of the unperturbed molecule. The essence of FMO theory is that the highest occupied molecular orbital (HOMO) and lowest unoccupied molecular orbital (LUMO) suffice to describe the reactivity of a molecule. In this case one has

$$f^{-}(\mathbf{r}) = |\psi^{\mathrm{HOMO}}(\mathbf{r})|^2, \tag{8}$$

$$f^{+}(\mathbf{r}) = |\psi^{\mathrm{LUMO}}(\mathbf{r})|^2. \tag{9}$$

There are several exact expressions for the Fukui function which start with Eqs. (8) and (9), and then add on (presumably small) orbital relaxation corrections.[34,37,54–56] Equations (8) and (9) are quite simple formulas where the response is completely determined by the molecule in the neutral state and no property of the charged species appears. Only one *ab initio* calculation is needed and obviously the resulting molecular Fukui function is always positive definite. There seem to be only a few cases in the literature where orbital relaxation effects are so important that frontier orbital fails,[57–60] so basing one's analysis on FMO theory is usually justified. The neglect of electron correlation in FMO theory is believed to be less significant, though there have been no systematic studies. It is worth mentioning that there are *ab initio* analogs of Eqs. (8) and (9) in terms of the Dyson

orbitals, so the absence of electron correlation need not be regarded as an intrinsic feature of FMO theory.[61,62]

As the aim of the present paper is to investigate atomic Fukui functions and ultimately atom condensed Fukui functions, we proceed to investigate how such values can be obtained from Eqs. (8) and (9). It is clear that the results will depend on the actual AIM definition used. We therefore have chosen to investigate the (a) Mulliken,[63–66] (b) Bader's QCT,[44–46] and (c) Hirshfeld[67] and Hirshfeld-I[68,69] techniques. These techniques are among the most used projection operator based approaches, although, for example, numerical integration within Becke fuzzy atomic cells has been found robust as well.[70]

(*a*) *Mulliken*. Carbó-Dorca and Bultinck have previously shown the projection nature of the Mulliken method.[71,72] This projection operator is based on the attachment of the basis functions used in the quantum chemical calculations to atomic centers. The Mulliken weight function $w_A^M(\mathbf{r})$, which is equivalent to the projection operator, then becomes

$$w_A^M(\mathbf{r}) \equiv \prod_A = \sum_{\alpha\in A}\sum_{\lambda} S_{\lambda a}^{(-1)}|\alpha(\mathbf{r})\rangle\langle\lambda(\mathbf{r})|, \tag{10}$$

where $\mathbf{S}^{-1}=\{S_{\lambda\alpha}^{(-1)}\equiv S_{\alpha\lambda}^{(-1)}\}$ are the elements of the symmetric inverse basis set metric or overlap matrix. The first sum runs only over the basis functions centered on A, while the summation over λ runs over all basis functions. Application of the weight function $w_A^M(\mathbf{r})$ on the HOMO and LUMO is straightforward and gives for $f^-(\mathbf{r},N)$,

$$f_A^-(\mathbf{r},N) = \sum_{\alpha\in A}\sum_{\lambda} C_{\alpha,\mathrm{HOMO}}^{*} C_{\lambda,\mathrm{HOMO}}|\alpha(\mathbf{r})\rangle\langle\lambda(\mathbf{r})|. \tag{11}$$

For the atom condensed case one has

$$\mathbf{f}_A^-(N) = \sum_{\alpha\in A}\sum_{\lambda} C_{\alpha,\mathrm{HOMO}}^{*} C_{\lambda,\mathrm{HOMO}} S_{\alpha\lambda}. \tag{12}$$

This is exactly the expression published earlier by Contreras *et al.*[73] although there it is not connected to Mulliken's population analysis. Note that in the specific case of the Mulliken method the weight function $w_A^M(\mathbf{r})$ is independent of the number of electrons. This means that as long as the molecular geometry remains the same and the same basis set is used, the weight functions are the same, independent of the number of electrons.

It is worth noting that according to Roy *et al.*,[74] the Mulliken method is unsatisfactory as it would seem to depend strongly on the size of the finite difference used. However, this work makes use of debatable fractionally occupied single determinant wave functions. Moreover, his proposed Mulliken weight function is flawed. His proposal is

$$w_A^{M,\mathrm{Roy}}(N) = N_A/N. \tag{13}$$

This means that the weight function is a scalar, uniform over all space and given as the ratio of the Mulliken population versus the total number of electrons. Such a uniform scaling is certainly not chemically reasonable, since it means that every atom A will have a substantial AIM density even in the core region of a very distant atom. We therefore reject Eq. (13).

(*b*) *Bader's QCT*. Population analysis techniques that rely on projection operators can be tailored in different ways, resulting in harder or softer interatomic boundaries.[75] In Bader's QCT method, a projection operator giving very hard boundaries can be defined in the following way. First, the Cartesian space is divided in atomic basins. Within the atomic basin Ω_A for an atom A all the density in every point in that basin is attributed to atom A. So the weight function becomes a binary quantity in the sense that for every atom A, the weight function on a point in space is 1 if this point is within the basin of atom A and 0 otherwise. One has

$$w_A^{\mathrm{QCT}}(\mathbf{r},N) = \delta(r\in\Omega_A), \tag{14}$$

where the notion of a logical delta operator is used. Given Eq. (7), the basins in Bader's QCT are mutually exclusive.[44–46]

For the case of FMO, this means that $f_A^-(\mathbf{r},N)$ is simply the HOMO within the basin of atom A and that the atomic Fukui function for this atom is zero elsewhere. Bader's QCT method has attracted much attention in the calculation of atomic Fukui functions. In QCT, the properties of the total electron density suffice to determine the basins of the AIM. One could suggest using an AIM basin derived from the frontier orbital density to obtain AIM Fukui functions. However, such a procedure fails as the frontier orbital density cannot be ensured to give the correct number of AIM. As an example, in formaldehyde the LUMO produces only two critical points.[76] As a consequence, one can not derive from a frontier molecular orbital density a set of AIM densities in the QCT way. Naturally, it remains possible to use the weight functions based on the entire density. Condensed FMO Fukui functions using the FMO density and the QCT weight functions derived from the total density have been studied in detail by Bulat *et al.*[77]

As explicitly indicated, $w_A^{\mathrm{QCT}}(\mathbf{r},N)$ depends on the number of electrons in the molecule. The atomic basins are well known to change upon changing the number of electrons in the molecule.[78] As $w_A^{\mathrm{QCT}}(\mathbf{r},N)$ can be only zero or one, and density functions are positive definite, the atomic Fukui functions within the FMO method are bound to be positive definite as well.

(*c*) Hirshfeld techniques. Within this section, two different techniques are covered. First, there is the classical Hirshfeld technique which relies on weight functions given by

$$w_A^H(\mathbf{r}) = \frac{\rho_{A,Z_A}^0(\mathbf{r})}{\sum_{A=1}^M \rho_{A,Z_A}^0(\mathbf{r})}, \tag{15}$$

where $\rho_A^0(\mathbf{r},Z_A)$ is the isolated atom density for atom A.[67] The promolecular density for the molecule with M atoms $\rho^0(\mathbf{r})=\sum_{A=1}^M \rho_{A,Z_A}^0(\mathbf{r})$ is obtained by simple superposition of these isolated atom densities with the promolecular geometry the same as the actual molecular geometry. This Hirshfeld scheme has been shown to be intimately related to information entropy,[48,79–83] and is a popular method within conceptual DFT. In the classical Hirshfeld technique, always the isolated atom densities $\rho_{A,Z_A}^0(\mathbf{r})$ are obtained from the neutral atoms. The subscript Z_A is used to indicate that the atomic density used integrates to the atomic number,

$$Z_A = \int \rho^0_{A,Z_A}(\mathbf{r}) d\mathbf{r}. \tag{16}$$

This means that the promolecular density is always the same, independent of the number of electrons contained in the molecule. This introduces an unaccounted for arbitrariness, as first shown by Davidson and Chakravorty.[84] In order to solve this problem, Bultinck *et al.* have recently introduced the self-consistent Hirshfeld-I scheme.[68,69] There the atomic density functions are given by

$$w_A^{H\text{-}I}(\mathbf{r},N) = \frac{\rho^0_{A,N_A}(\mathbf{r})}{\Sigma_{A=1}^M \rho^0_{A,N_A}(\mathbf{r})}. \tag{17}$$

Here the atomic densities used in the construction of the promolecule integrate to the atomic population N_A as it appears in the molecule. As the number of electrons contained in the AIM depends on the total number of electrons in the molecule, the weight functions become dependent on the number of electrons as well. For details of the resulting iterative Hirshfeld-I procedure, the reader is referred to Bultinck *et al.*[69]

Turning now to atomic Fukui functions, according to information theory the most coherent approach would be to derive Hirshfeld and Hirshfeld-I weight factors from the FMO only. Such an approach is not tractable, as this would require a promolecular HOMO, which is not possible. Nevertheless, as was the case for QCT, it remains possible to use the regular Hirshfeld or Hirshfeld-I weight functions to distribute the frontier densities.

Fragment of molecular response approach (FMR)

A discussion of such approaches starts from the molecular Fukui function shown in Eq. (2). The question of atomic Fukui functions now becomes the question of how to divide such a response among the AIM. Ayers *et al.*[85] proposed to use the following technique:

$$f_A^{\pm} = w_A(\mathbf{r},N^0) f^{\pm}(\mathbf{r}). \tag{18}$$

The weight function is always taken to be the one from the neutral molecule, which has the number of electrons abbreviated as N^0. This means that first the molecular response of the density function is computed and then this is distributed among the AIM. This is opposed the next approach, which is based on the *response of a molecular fragment* (RMF) to the perturbation. The two approaches are generally inequivalent, as will be illustrated later. The FMR approach has the advantage that a coherent reasoning on the hardness kernel can be developed.[85]

Essentially, the FMR approach entails that any property, including a response $\vartheta(\mathbf{r})$, is always divided among the AIM in the following way:

$$\vartheta_A(\mathbf{r},N^0) = w_A(\mathbf{r},N^0)\,\vartheta(\mathbf{r},N^0). \tag{19}$$

The same goes for derivatives in general as well. An nth derivative is divided among the AIM as

$$\vartheta_A^n(\mathbf{r}) = w_A(\mathbf{r},N^0)\,\vartheta^n(\mathbf{r}). \tag{20}$$

In $w_A(\mathbf{r},N^0)$ we have explicitly denoted that the weight factor for the neutral system is used, where the number of electrons in the neutral molecule is denoted N^0.

If now, the resulting equations with different AIM schemes are investigated, the following observations can be made.

(*a*) *Mulliken*. In the case of the Mulliken approach, the weight factor $w_A^M(\mathbf{r})$ is independent of the number of electrons. The atomic Fukui function then becomes

$$f_A^{\pm}(\mathbf{r},N) = w_A^M(\mathbf{r}) f^{\pm}(\mathbf{r},N) \tag{21}$$

and in a finite difference approach, this gives, e.g.,

$$\begin{aligned} f_A^-(\mathbf{r},N) &= w_A^M(\mathbf{r})[\rho(\mathbf{r},N) - \rho(\mathbf{r},N-dN)]/dN \\ &= [w_A^M(\mathbf{r})\rho(\mathbf{r},N) - w_A^M(\mathbf{r})\rho(\mathbf{r},N-dN)]/dN \\ &= [\rho_A(\mathbf{r},N) - \rho_A(\mathbf{r},N-dN)]/dN. \end{aligned} \tag{22}$$

In other words, the atomic Fukui function is equivalent to the difference in the Mulliken AIM density functions for the molecule and the molecular ion. In the atom condensed Fukui function $\mathbf{f}_A^-(N)$ one has

$$\begin{aligned} \mathbf{f}_A^-(N) &= \int w_A^M(\mathbf{r})[\rho(\mathbf{r},N) - \rho(\mathbf{r},N-1)]d\mathbf{r} \\ &= \int [\rho_A(\mathbf{r},N) - \rho_A(\mathbf{r},N-1)]d\mathbf{r} \\ &= P_A(\mathbf{r},N) - P_A(\mathbf{r},N-1) \\ &= q_A(\mathbf{r},N-1) - q_A(\mathbf{r},N), \end{aligned} \tag{23}$$

where $P_A(\mathbf{r},N)$ is the AIM population on the atom A in the molecule with N electrons, obtained usually via

$$P_A(\mathbf{r},N) = \int \rho_A(\mathbf{r},N) d\mathbf{r} \tag{24}$$

and $q_A(\mathbf{r},N)$ is the atomic charge on the AIM A for the molecule with N electrons. This is nothing other than the atom condensed Fukui function introduced by Yang and Mortier.[50] Computing atom condensed Fukui functions as a difference in atomic charges is thus in line with the FMR approach. Note, however, that this atomic charge difference expression can be obtained solely by virtue of the fact that $w_A^M(\mathbf{r})$ is independent of the number of electrons in the molecule.

(*b*) *Bader's QCT*. In the case of QCT, an important difference with the Mulliken technique lies in the fact that $w_A^{\mathrm{QCT}}(\mathbf{r},N)$ does depend on the number of electrons in the molecule. Within the present approach, one finds, for example, for the atomic Fukui function $f_A^-(\mathbf{r},N)$,

$$\begin{aligned} f_A^-(\mathbf{r},N) &= w_A^{\mathrm{QCT}}(\mathbf{r},N)[\rho(\mathbf{r},N) - \rho(\mathbf{r},N-dN)]/dN \\ &= [w_A^{\mathrm{QCT}}(\mathbf{r},N)\rho(\mathbf{r},N) \\ &\quad - w_A^{\mathrm{QCT}}(\mathbf{r},N)\rho(\mathbf{r},N-dN)]/dN \end{aligned} \tag{25}$$

and for the atom condensed version in a finite difference,

$$\mathbf{f}_A^- = P_A^{\mathrm{QCT}}(\mathbf{r},N) - \int w_A^{\mathrm{QCT}}(\mathbf{r},N)\rho(\mathbf{r},N-1)d\mathbf{r}. \tag{26}$$

Here we clearly see that within the present approach, the atom condensed Fukui function is *not* equal to the difference of the atomic population between the neutral molecule and the molecular ion. This point was previously addressed also by Cioslowski *et al.*[78] Proper calculation of the atom condensed Fukui function in the present approach requires that one computes the atomic basins in the neutral molecule and retains the basin for splitting the density function of the molecular ion. If one opts to stick to the FMR approach, the software should be adapted as to compute Eq. (26).

(c); Hirshfeld techniques. In the Hirshfeld techniques similar observations can be made as above. In the classical Hirshfeld method,[67] one has arbitrarily chosen to use the weight factor from Eq. (15), disregarding the fact that the weight factors should change with molecular populations to keep them coherent with information entropy. As a consequence, one has for the classical Hirshfeld method that

$$\begin{aligned} f_A^-(\mathbf{r}) &= w_A^H(\mathbf{r})[\rho(\mathbf{r},N) - \rho(\mathbf{r},N-1)] \\ &= w_A^H(\mathbf{r})\rho(\mathbf{r},N) - w_A^H(\mathbf{r})\rho(\mathbf{r},N-1) \\ &= \rho_A^H(\mathbf{r},N) - \rho_A^H(\mathbf{r},N-1) \end{aligned} \tag{27}$$

and the atom condensed version becomes

$$\mathbf{f}_A^- = P_A^H(\mathbf{r},N) - P_A^H(\mathbf{r},N-1) = q_A^H(\mathbf{r},N-1) - q_A^H(\mathbf{r},N). \tag{28}$$

Despite the arbitrary choice of the promolecular density in the usual Hirshfeld method, Van Alsenoy and co-workers found that Hirshfeld based atom condensed Fukui functions do predict the same site selectivities as inferred from experiment.[86,87]

In the Hirshfeld-I scheme, very similar to the Bader scheme, the weight factors depend on the number of electrons in the molecule and one finds

$$\mathbf{f}_A^- = P_A^{H\text{-}\mathrm{I}}(\mathbf{r},N) - \int w_A^{H\text{-}\mathrm{I}}(\mathbf{r},N)\rho(\mathbf{r},N-1)d\mathbf{r}. \tag{29}$$

Computing atom condensed Fukui functions based on the self-consistent Hirshfeld-I charges means that the weight functions used for the molecular ions should be the iteratively optimized ones for the neutral molecule.

The above shows clearly that once a specific approach to atomic and atom condensed Fukui functions is chosen, one may or may not find that this is consistent with a charge difference expression. For the different AIM methods tested presently, only the Mulliken technique has some firm theoretical background and leads to an atom condensed Fukui function that can be expressed as a difference of atomic populations. For the theoretically also sound QCT and Hirshfeld-I methods, there appears no such simple formula. The Hirshfeld AIM method apparently also yields a simple population difference expression, but this is only because the weight functions have (arbitrarily) been chosen to be independent of the number of electrons.

Response of molecular fragment approach (RMF)

As stated above, the previous approach first computes the response of the molecule and then divides it into atomic terms. On the other hand, one can also opt to first divide the density functions in to atomic contributions and then perform other mathematical derivations. For the atomic Fukui function this means that one computes

$$\begin{aligned} f_A^{\pm}(\mathbf{r},N) &= \left[\frac{\partial \rho_A(\mathbf{r},N)}{\partial N}\right]_{\nu_{\mathrm{ext}}} \\ &= \left[\frac{\partial (w_A(\mathbf{r},N)\rho(\mathbf{r},N))}{\partial N}\right]_{\nu_{\mathrm{ext}}} \\ &= \left[\frac{\partial w_A(\mathbf{r},N)}{\partial N}\right]_{\nu_{\mathrm{ext}}} \rho(\mathbf{r},N) + w_A(\mathbf{r},N) \\ &\quad \times \left[\frac{\partial \rho(\mathbf{r},N)}{\partial N}\right]_{\nu_{\mathrm{ext}}} \\ &= \left[\frac{\partial w_A(\mathbf{r},N)}{\partial N}\right]_{\nu_{\mathrm{ext}}} \rho(\mathbf{r},N) + w_A(\mathbf{r},N) f^{\pm}(\mathbf{r},N). \end{aligned} \tag{30}$$

Compared to Eq. (18), an extra term has appeared. The previous approach is equivalent to the present one only if

$$\left[\frac{\partial w_A(\mathbf{r},N)}{\partial N}\right]_{\nu_{\mathrm{ext}}} = 0 \tag{31}$$

or in other words, when the weight factors do not depend on the number of electrons. Note that this difference in atomic Fukui function approach has no effect on the total molecular Fukui function since

$$\begin{aligned} \sum_A f_A^{\pm}(\mathbf{r},N) &= \sum_A \left[\frac{\partial w_A(\mathbf{r},N)}{\partial N}\right]_{\nu_{\mathrm{ext}}} \rho(\mathbf{r},N) \\ &\quad + \sum_A w_A(\mathbf{r},N) f^{\pm}(\mathbf{r},N) \\ &= \left[\frac{\partial \Sigma_A w_A(\mathbf{r},N)}{\partial N}\right]_{\nu_{\mathrm{ext}}} \rho(\mathbf{r},N) + f^{\pm}(\mathbf{r},N) \\ &= f^{\pm}(\mathbf{r},N), \end{aligned} \tag{32}$$

where we used Eq. (7) and the fact that the derivative of a scalar is zero.

Depending on the AIM technique used, a different expression may be obtained compared to the previous approach, as is shown below.

(a) Mulliken. Obviously the same expressions are found as in the FMR approach because $w_A^M(\mathbf{r})$ is independent of N. This is important: It means that when one uses Mulliken AIM one does not need to choose between the FMR and RMF approaches. The results can be interpreted in either way, based on chemical convenience.

(b) Bader's QCT. As now the dependence of $w_A^{\mathrm{QCT}}(\mathbf{r},N)$ on N appears, a different expression is found as follows:

$$f_A^{\pm}(\mathbf{r},N)=\left[\frac{\partial\rho_A(\mathbf{r},N)}{\partial N}\right]_{v_{\text{ext}}}. \tag{33}$$

In a finite different approach, this becomes for $f_A^-(\mathbf{r},N)$ as follows:

$$f_A^-(\mathbf{r},N)=\rho_A(\mathbf{r},N)-\rho_A(\mathbf{r},N-1). \tag{34}$$

In the atom condensed version, one then has

$$\mathbf{f}_A^-(N)=P_A(\mathbf{r},N)-P_A(\mathbf{r},N-1) \tag{35}$$

and so one can use the difference of AIM populations or charges as an atom condensed Fukui function.

So using such a finite difference approach immediately entails acceptance of the fact that one considers the atomic Fukui function as given by Eq. (30) rather than considering it as a division of a molecular response over the AIM. Notice also that using the finite difference approximation in the third line of Eq. (30) gives different results from using the finite difference approximation in the first line of Eq. (30) [which gives Eq. (35)]. It is also worth noting that even for full-CI calculations [where the finite difference approximation is exact for the derivative of the electron density, Eqs. (3) and (4)], the finite difference approximation is usually not exact for the atomic weight functions. This means that, even for exact calculations, Eq. (34) is generally approximate.

(c). Hirshfeld techniques. As for the original Hirshfeld method, the same expressions are found as in the previous approach due to the weight functions that are independent of the number of electrons in the molecule.

On the other hand, the Hirshfeld-I technique gives very similar expressions to that in Eq. (35) for Bader's QCT and so also a difference of atomic populations is found.

$$\mathbf{f}_A^-(N)=P_A^{H\text{-}I}(\mathbf{r},N)-P_A^{H\text{-}I}(\mathbf{r},N-1). \tag{36}$$

Again a very simple result and computationally convenient equation appears. However, this again entails that one considers the atomic Fukui function as given by Eq. (30) rather than considering it as a division of a molecular response over the AIM.

Impact of the order of response and projection

From the above, it is clear that the only AIM technique that gives atom condensed Fukui functions that can be equated in both the FMR and RMF approaches to differences in atomic populations is the Mulliken technique. There the weight factors are effectively independent of the number of electrons in the molecule and the actions of computing the response and projecting out the AIM commute. This is not the case for the other techniques. The Hirshfeld technique does apparently allow using differences of atomic populations, but this is questionable, as shown above.

Clearly in other cases besides the Mulliken method, the FMR and RMF approaches may yield substantially different results. This is quite unfortunate in the light of using such functions as reactivity descriptors as one recipe may give a totally different reactivity model as the other. It is therefore interesting to see whether some connection can be found between both approaches. Is there some argument to prefer one approach for an AIM Fukui function over another? Clearly, there is no real mathematical ground for such a preference, although there may be some chemical argumentation. When using, e.g., the QCT method of Bader *et al.*, the atomic basins may differ between a molecule with N or $N\pm dN$ electrons. As a consequence, both approaches cannot give the same result as is clear from the above discussion and formula. This was already illustrated numerically by Cioslowski *et al.*[78] who found that for formaldehyde atom condensed Fukui functions even differ in sign when including the change in weight factor.

First, it is instructive to show that the difference between the FMR and RMF approaches depends on the position where a partition of the unity is introduced when combining Eqs. (2) and (7). An obvious way to obtain atomic Fukui functions is to insert the unity (7) in Eq. (2) but depending on where this is actually done, one gets different results as

$$\left(\frac{\partial w_A(\mathbf{r},N)\rho(\mathbf{r},N)}{\partial N}\right)_{v_{\text{ext}}}\neq\left(w_A(\mathbf{r},N)\frac{\partial\rho(\mathbf{r},N)}{\partial N}\right)_{v_{\text{ext}}}. \tag{37}$$

Starting from Eq. (2) one can introduce the atomic weight factors, Eq. (7), outside the derivative operation,

$$f(\mathbf{r},N)=\mathbf{1}\left(\frac{\partial\rho(\mathbf{r},N)}{\partial N}\right)=\sum_A^M w_A(\mathbf{r},N)\left(\frac{\partial\rho(\mathbf{r},N)}{\partial N}\right)_{v_{\text{ext}}}, \tag{38}$$

from which Eq. (18) is obtained. Alternative, in the RMF approach the weight factors are introduced into the density itself,

$$f(\mathbf{r},N)=\left(\frac{\partial\mathbf{1}\rho(\mathbf{r},N)}{\partial N}\right)_{v_{\text{ext}}}=\left(\frac{\partial\Sigma_A^M w_A(\mathbf{r},N)\rho(\mathbf{r},N)}{\partial N}\right)_{v_{\text{ext}}}. \tag{39}$$

The second approach could be slightly preferred as the weight functions are intimately related to the electron density. In the FMR method, the weight functions have a less intimate connection with the response.

Another difference is based on the fact that the molecular Fukui function describes in a molecule where electron density will be added. Let us now propose that an AIM Fukui function describes how a small change in the number of electrons (N) in the molecule is spread over all AIM. Let us write the electron density for an $N+dN$ system in terms of the N electron system for an AIM,

$$\begin{aligned}\rho_A(\mathbf{r},N+dN)&=\rho_A(\mathbf{r},N)+\left[\frac{\partial\rho_A(\mathbf{r},M)}{\partial M}\right]_{M=N}dN+\cdots\\&=\rho_A(\mathbf{r},N)+f_A^+(\mathbf{r},N)dN+\cdots.\end{aligned} \tag{40}$$

Assuming that atomic derivatives of the density can be obtained as the product of a weight factor and the derivative of the density, as in the FMR method and denoting the second derivative as

$$g(\mathbf{r},N)=\left[\frac{\partial^2\rho_A(\mathbf{r},M)}{\partial M^2}\right]_{M=N}, \tag{41}$$

it is found that the density for the AIM for an $N+dN$ electron system becomes

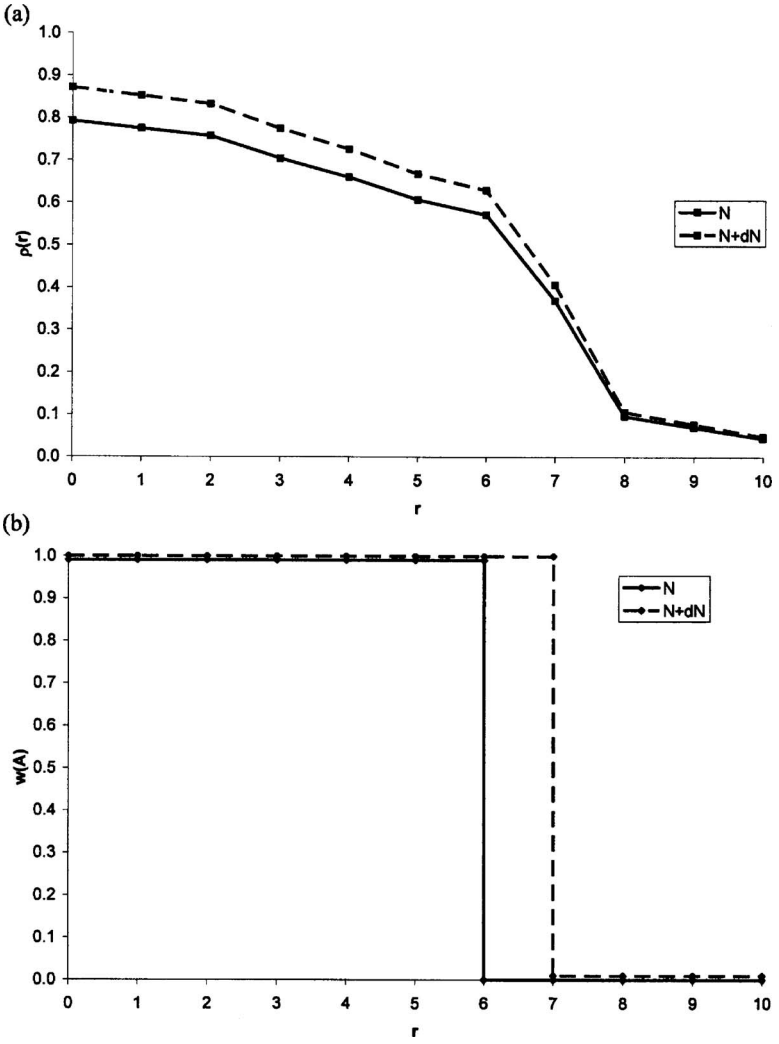

FIG. 1. (a) Molecular density functions and (b) weight functions for a QCT analysis for atom A in a molecule with N and $N+dN$ electrons.

$$\rho_A(\mathbf{r},N+dN) = w_A(\mathbf{r},N)\rho(\mathbf{r},N) + w_A(\mathbf{r},N)f^+(\mathbf{r},N)dN + w_A(\mathbf{r},N)g^+(\mathbf{r},N)dN^2 + \cdots . \quad (42)$$

Note that $g(\mathbf{r},N)$ corresponds to the Δf used as a dual reactivity descriptor by Morell *et al.*[88,89]

Let us now construct a system for which a QCT analysis is carried out. In arbitrary units, the atom in a certain direction spreads out along a certain direction where $w_A(\mathbf{r},N)=1$ for $\mathbf{r}\leqslant 6$ and $w_A(\mathbf{r},N)=0$ for $\mathbf{r}>6$ and $\mathbf{r}$ denotes the coordinate along that direction. Now let us suppose that adding dN electrons yields $w_A(\mathbf{r},N+dN)=1$ for $\mathbf{r}\leqslant 7$ and $w_A(\mathbf{r},N)=0$ for $\mathbf{r}>7$, as depicted in Fig. 1.

Such changes in weight functions can certainly occur as any QCT analysis can show. It is clear that from Eq. (42), starting from the N electron system towards the $N+dN$ system, one will never find any AIM density between $r=6$ and $r=7$ if one uses only $w_A(\mathbf{r},N)$. Nevertheless, Fig. 1 shows that in the $N+dN$ system for $\mathbf{r}$ between 6 and 7, all density corresponds to atom A as the weight factor for atom A is 1 in the $N+dN$ case. On the other hand, if one tests the AIM Fukui function according to the RMF approach (30) one finds up to second order

$$\rho_A(\mathbf{r},N+dN) = w_A(\mathbf{r},N)\rho(\mathbf{r},N) + w_A(\mathbf{r},N)f^+(\mathbf{r},M)dN + \rho(\mathbf{r},N)\frac{\partial w_A(\mathbf{r},N)}{\partial N}dN + w_A(\mathbf{r},N)g^+(\mathbf{r},N)dN^2 + \rho(\mathbf{r},N)\frac{\partial^2 w_A(\mathbf{r},N)}{\partial N^2}dN^2 + \frac{\partial w_A(\mathbf{r},N)}{\partial N}\frac{\partial \rho(\mathbf{r},N)}{\partial N}dN^2. \quad (43)$$

Assuming that $w_A(\mathbf{r},N)$ is linear in N, one finds that

$$\rho_A(\mathbf{r},N+dN) = w_A(\mathbf{r},N)\rho(\mathbf{r},N) + w_A(\mathbf{r},N)f^+(\mathbf{r},M)dN + \rho(\mathbf{r},N)\frac{\partial w_A(\mathbf{r},N)}{\partial N}dN + \frac{\partial w_A(\mathbf{r},N)}{\partial N}\frac{\partial \rho(\mathbf{r},N)}{\partial N}dN^2. \quad (44)$$

In the region between 6 and 7 in Fig. 1 [where $w_A(\mathbf{r},N)=0$ and $w_A(\mathbf{r},N+dN)=1$] and using a finite difference approach for $\partial w_A(\mathbf{r},N)/\partial N$ we find that the AIM density becomes

$$\rho_A(\mathbf{r},N+dN)|_{6<\mathbf{r}\leqslant 7}=\rho(\mathbf{r},N)+\frac{\partial\rho(\mathbf{r},N)}{\partial N}dN. \quad (45)$$

Indeed, one finds the proper density for AIM A in this region for the $N+dN$ molecule. Taking some point $\mathbf{r}$ in this region, the density attached to atom A is the molecular density at $\mathbf{r}$ in the N electron system plus the difference between the density at $\mathbf{r}$ in the N and $N+dN$ system.

Another attractive feature of the definition as in Eq. (30) is the symmetry which is obtained for an AIM density function. It is straightforward to show that for the application described above in Fig. 1, starting from $\rho_A(\mathbf{r},N+dN)$ one will find that $\rho_A(\mathbf{r},N)|_{6<\mathbf{r}\leqslant 7}=0$, as it should. One correspondingly also finds that using an AIM Fukui function according to the FMR procedure [Eq. (18)] would not yield this result.

The above derivations show that there may be different approaches to compute atom condensed Fukui functions. The FMO based approach is clearly the only approach where only one *ab initio* calculation will suffice. The FMR approach is based entirely on the response of the molecule as a whole, after which it is divided over the AIM by application of the density based weight functions for the neutral molecule. In the RMF approach, one considers also changes in the weight functions. This is the only approach where every AIM technique gives, under a finite difference approach, the atom condensed Fukui functions as a difference of atomic populations, although naturally, the finite difference approach can be severely lacking. In the case of Bader's QCT, it is not possible to predict the density function for an AIM from a known density function correctly if one does not include the N dependence of the weight functions. The important implication of this analysis is that users of atom condensed Fukui functions should decide at the very beginning which approach they want to use for atom condensed Fukui functions. This comes down to answering the question to what extent one wants to take in to account the charged system. Either nothing is considered (FMO), only the density difference (FMR) or both the density difference and difference in weight function (RMF). Once chosen a specific method, comparisons among methods should all be performed within the same method and if necessary new algorithms need to be devised or existing algorithms changed.

Given this arbitrariness, researchers should reconsider whether atomic or atom condensed Fukui functions are still useful. The interesting characteristic feature of the molecular Fukui function is precisely that it is a molecular function. Introducing AIM parts of that mainly originates an extra level of complexity, although it merits to be repeated that Yang and Mortier[50] avoided this complication by choosing a method (Mulliken population analysis) where the FMR and RMF approaches are equal. In the work of Ayers *et al.*,[85] the FMR approach is recommended. As will be shown below, simple population differences are a bad approximation to the FMR result.

COMPARISON OF PROCEDURES

It is naturally important to investigate whether one will actually find different results with the different alternative approaches to AIM Fukui functions. To that end, we computed the different AIM atom condensed Fukui functions for a wide set of organic molecules containing C, H, N, O, F, and Cl. The test set contains 168 molecules, comprising in total over 2500 atoms. This test set was previously used for electronegativity equalization calibrations[90–92] and for testing the Hirshfeld-I method.[69] First, the molecules were optimized at RHF/6-31G** level using GAUSSIAN-03.[93] UHF/6-31G** calculations were used for the molecular ions in the same geometry. Hirshfeld and Hirshfeld-I charges were obtained using a self-developed software and the STOCK program.[94] For the Hirshfeld and Hirshfeld-I charges, atomic densities for fractional atomic populations are obtained as described Ref. 69. The required integer population atomic densities are obtained from ROHF calculations obtained using the ATOMSCF program.[95] Bader's QCT charges were obtained for a smaller subset of molecules using the PROAIM program.[96] The problem with these charges is that computing them in the FMR spirit and our current programs requires quite a lot of user intervention which hampers severely their routine application. The same is true for the Hirshfeld-I FMR results.

As an example, Mulliken, QCT, Hirshfeld, and Hirshfeld-I atom condensed Fukui functions for formaldehyde, based on FMO theory, and FMR and RMF approaches are shown in Table I.

The QCT FMO results were obtained using the total density based basins for the neutral molecule. In a similar fashion, for the Hirshfeld and Hirshfeld-I FMO results the frontier molecular orbital densities were distributed over all atoms using the weight functions used normally for the partitioning of the complete density. As the example shows, even for this small molecule the results may sometimes differ quite substantially depending on the AIM method used and the approach used. In some cases, such as for Hirshfeld-I $f_A^+(\mathbf{r},N)$ values, the agreement between FMR and RMF is relatively good. Nevertheless, one cannot exclude that this could be fortuitous, and the cases of poor agreement should inspire caution using RMF values as replacement for FMR values. The RMF atom condensed Fukui functions are the most commonly reported Hirshfeld atom condensed Fukui functions in literature, although they are questionable. They are best replaced by Hirshfeld-I based atom condensed Fukui functions which are also tractable for both approaches.

We wish to focus on the effect of using different approaches for the atomic Fukui functions, so no in depth report is made of the influence of a specific choice of AIM technique. Such analysis has been performed before,[97,98] and the reader is referred to the relevant literature although one should be cautious because sometimes inappropriate comparisons are made, e.g., mixing results from different approaches.

As Table I shows, drastic changes in atom condensed Fukui functions are observed when the weight factors are allowed to differ between the molecule and its ionic counterpart. For all 168 molecules, Mulliken FMO and FMR/RMF values were computed for both $f_A^-(\mathbf{r},N)$ and $f_A^+(\mathbf{r},N)$. The same was done using the Hirshfeld-I technique using the FMR and RMF approaches. Table II reports the correlation results.

TABLE I. $f_A^{\pm}(\mathbf{r},N)$ values for the different AIM techniques within the frozen molecular orbital (FMO), fragment of molecular response (FMR), and response of molecular fragment (RMF) atom condensed Fukui function approaches.

$f_A^-(\mathbf{r},N)$	C	H	O		C	H	O
Mulliken				QCT			
FMO	0.093	0.137	0.633		0.093	0.141	0.625
FMR	0.022	0.235	0.509		0.07	0.168	0.595
RMF	0.022	0.235	0.509		−0.235	0.304	0.626
Hirshfeld				Hirshfeld-I			
FMO	0.180	0.118	0.584		0.161	0.122	0.594
FMR	0.226	0.146	0.482		0.192	0.151	0.504
RMF	0.226	0.146	0.482		−0.133	0.248	0.636
$f_A^+(\mathbf{r},N)$							
Mulliken				QCT			
FMO	0.684	0.004	0.308		0.516	0.036	0.412
FMR	0.294	0.215	0.277		0.355	0.146	0.354
RMF	0.294	0.215	0.277		0.365	0.214	0.206
Hirshfeld				Hirshfeld-I			
FMO	0.524	0.070	0.337		0.496	0.077	0.350
FMR	0.402	0.146	0.306		0.367	0.156	0.321
RMF	0.402	0.146	0.306		0.364	0.174	0.289

As Table II clearly illustrates, there is a very poor correlation between the results of different approaches to atom condensed Fukui functions. When using atom condensed Fukui functions for developing reactivity models or obtaining insight in the role of different parameters in a quantitative structure activity relationship environment, this means that in the variable elimination step, it is very likely that all atom condensed Fukui functions could be retained as sufficiently noncollinear. The model finally obtained could thus depend on the actual approach taken to compute atom condensed Fukui functions as descriptors. So depending on the choice of an atomic Fukui function expression and AIM technique, other conclusions can appear. So if one prefers the FMR approach, for example, because it gives a coherent approach to the hardness kernel,[85] the RMF atom condensed Fukui functions may be a very bad approximation. This is well illustrated, for example, in the case of the Hirshfeld-I technique where the correlation between FMR and RMF data is very poor.

The dominantly positive nature of atom condensed Fukui functions from some combination of atomic Fukui function approach and AIM technique is in some cases, such as QCT/FMO, barely any surprise. It is a direct consequence of the approach taken. Other approaches may give negative values. Of course there is no argument against the possibility of a negative atom condensed Fukui function. Their presence will depend critically on the properties of the hardness matrix.[85,99–102] Even one can sketch different requirements under which an AIM could be reduced as a consequence of a global oxidation of the molecule and vice versa.[102] It would be quite hard to unambiguously detect such a case, since this detection will also be dependent on the AIM approach. Unambiguous evidence for this phenomenon has recently been found, however, in a molecule where the redox active atoms change from low spin (d^6) to high spin (d^5).[103] Table I shows clearly how conclusions on the occurrence of negative atom condensed Fukui functions may depend on the approach taken. It is seen that, in agreement with QCT findings of Cioslowski *et al.*,[78] negative values can occur even for such a simple molecule. According to Roy *et al.*,[52,53] the Hirshfeld approach gives superior atom condensed Fukui functions because they are almost exclusively positive definite. It is found that this is valid for the Hirshfeld-I data when using the FMR approach but not for the RMF results. Considering the entire test set, approximately 2% of the values of Hirshfeld-I FMR results are negative, whereas this amounts roughly 20% for RMF data.

Considering finally the performance of the Mulliken scheme for atom condensed Fukui functions, it is clearly an advantage that one finds only one unique result from the FMR and RMF approaches. Unfortunately, the basis set dependence of the Mulliken technique hampers somewhat the routine use. Within the FMO approach, for the HOMO based $f_A^-(\mathbf{r},N)$ with the present basis set, the atom condensed Fukui function is always positive or only slightly negative (order −0.0005). For the LUMO based $f_A^+(\mathbf{r},N)$, there are many

TABLE II. Correlation coefficients (R^2) between the frozen molecular orbital (FMO), fragment of molecular response (FMR), and response of molecular fragment (RMF) atom condensed Fukui function approaches for different AIM techniques using the entire test set. Values are given for $f_A^-(\mathbf{r},N)$ and in brackets for $f_A^+(\mathbf{r},N)$.

Mulliken	FMO	FMR & RMF
FMO	1.00	0.27
	(1.00)	(0.24)
FMR & RMF	0.27	1.00
	(0.24)	(1.00)
Hirshfeld-I	FMR	RMF
FMR	1.00	0.35
	(1.00)	(0.63)
RMF	0.35	1.00
	(0.63)	(1.00)

negative values that are physically not acceptable as this means that a negative number of electrons is found for some AIM in a frontier molecular orbital.

CONCLUSIONS

Although the molecular Fukui function is deeply rooted in density functional theory, the same firm footing is not possible for its atomic counterpart. For an atom in a molecule, there are different ways to define an atomic Fukui function. Not only do values, and possibly trends, depend on the actual method used to discern an atom in a molecule, it also depends on the definition of the atomic Fukui function.

In a first approach, one can stick close to frontier molecular orbital theory. This requires only one single computation and all atomic Fukui functions can be retrieved from the atomic projection of frontier molecular orbitals.

Two other methods differ among each other in the order of computing a response and projecting out an atom. The first possibility is that the molecular response is computed and then divided among the atoms using the AIM weight functions. This gives in general different results from first projecting out the AIM density functions and computing their response to a change in the number of electrons in a molecule. Both methods have their own merits. The Mulliken method can be theoretically substantiated and gives the same results for both approaches. Unfortunately, it can give physically meaningless results. In general, computing atom condensed Fukui functions based on differences in atomic populations is only consistently coherent within the RMF approach. A molecular Fukui function is rigorously formulated in density functional theory. Given the arbitrary choices (Does one use FMO, FMR, or RMF? Which AIM scheme will be used?) that must be made to define atomic Fukui functions and their condensed values, it seems advisable to use three-dimensional molecular reactivity descriptors instead. At the very least, further study is needed, so that the choices that underlie condensed reactivity indicators can be better understood, and the implications of different choices on the data used in reactivity models can be characterized.

Note added in proof. Very recently, Otero *et al.* reported calculations that reflect the difference between the FMR and RMF procedures using Bader's atoms in molecules for computing atom condensed Fukui functions. [104] The results of that study are entirely in line with the present discussion.

ACKNOWLEDGMENTS

One of the authors (P.B.) wishes to thank Dr. Ricardo Mosquera for assistance with QCT calculations. He also thanks Ghent University and the Fund for Scientific Research-Flanders (Belgium) for their grants to the Quantum Chemistry group at Ghent University. He also thanks Pratim Chattaraj (India) for interesting discussions on conceptual DFT. One of the authors (C.V.A.) gratefully acknowledges the University of Antwerp for access to the university's CALCUA computer cluster. Another author (R.C.D.) acknowledges the Ministerio de Ciencia y Tecnología for Grant No. BQU2003-07420-C05-01, which has partially sponsored this work. One of the authors (P.W.A.) thanks NSERC and the Canada Research Chairs for partial financial support.

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