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# CRITICAL VALUES FOR DUNCAN'S NEW MULTIPLE RANGE TEST 

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SUMMARY
David B. Duncan [2] has formulated a new multiple range test making use of special protection levels based upon degrees of freedom. Duncan [Tables II and III] has also tabulated the critical values (significant studentized ranges) for 5 percent and 1 percent level new multiple range tests, based upon tables by P'earson and Hartley [8] and by Beyer [1]. Unfortunately, there are sizable errors in some of the published critical values. This fact was discovered and reported by the author [4], who instigated the computation at Wright-Patterson Air Force Base of more accurate tables of the probability integrals of the range and of the studentized range than those published by l'carson and Hartley $[7,8]$. This extensive computing project, of which one of the primary objectives was the determination of more accurate eritical values for Duncan's test, has now been completed. The purpose of this paper is to report critical values (to four significant figures) which have been found by inverse interpolation in the new table of the probability integral of the studentized range. Included are corrected tables for significance levels $\alpha=0.05,0.01$ and new tables for significance levels $\alpha=0.10,0.005,0.001$-all with sample sizes $n=2(1) 20(2) 40(10) 100$ and degrees of freedom $\nu=1(1) 20,24,30,40,60,120, \infty$.

## INTRODUCTION

Multiple range tests are used for testing the significance of the range of $\boldsymbol{p}$ successive values out of an ordered arrangement of $m$ means of samples of size $N$, where $p=2, \cdots, m$. First one tests the significance of the range of all $m$ means by comparing it with the critical range for the desired level of significance. If the range of all $m$ means is found to be significant, one next tests the significance of the range of ( $m-1$ ) successive means, omitting first the largest and then the smallest (or vice versa-order is unimportant); if either of these tests on ( $m-1$ )
means shows significance, one then proceeds with tests on ( $m-2$ ) successive means, and so on until no further groups are found to have significant ranges. Whenever the range of any group is found to be non-significant, one concludes that the entire group has come from a homogeneous source, and no test is made on the range of any subgroup of that group. Multiple range tests differ from fixed range tests in that the critical range of $p$ means usually decreases as $p$ decreases, rather than remaining constant.

The new multiple range test proposed by Duncan [2] makes use of special protection levels based upon degrees of freedom. Let $\gamma_{2 . a}=$ $1-\alpha$ be the protection level for testing the significance of a difference between two means; that is, the probability that a significant difference between sample means will not be found if the population means are equal. Duncan reasons that one has $(p-1)$ degrees of freedom for testing $p$ means, and hence one may make ( $\beta-1$ ) independent tests, each with protection level $\gamma_{2 . a}$. Hence the joint protection level is

$$
\begin{equation*}
\gamma_{p, a}=\left(\gamma_{2, a}\right)^{p-1}=(1-\alpha)^{p-1} \tag{1}
\end{equation*}
$$

that is, the probability that one finds no significant differences in making ( $p-1$ ) independent tests, each at protection level $\gamma_{z, a}$, is $\gamma_{3, a}^{p-1}$, under the hypothesis that all $p$ population means are equal.

## CRITICAL VALUES FOR DUNCAN'S TEST

On the basis of protection levels $\gamma_{p, \alpha}$ given by (1) for tests on $p$ means, Duncan [2, Tables II and III] has tabulated the factor $Q(p, z, \alpha)$ by which the standard error of the mean must be multiplied in order to obtain the critical range for Duncan's new multiple range test, for $\alpha=0.05,0.01$. In the sequel, this factor $Q(p, \nu, \alpha)$ will be called the eritical value or the significant studentized range for Juncan's test.

As mentioned carlier, Duncan's tables of significant studentized ranges are based upon tables by Pearson and Hartley [8] and by Beyer [1]. The tabular values for $2 \leq p \leq 20$ and $10 \leq \nu \leq \infty$ were obtained by inverse interpolation in the Pearson-IIartley tables of the probability integral of the studentized range, while the remainder of the values were computed by Beyer, using new methods. The I'earson-Ilartley tables of the probability integral.$_{n}(Q)$ of the studentized range, with $\nu$ degrees of freedom for the independent $n$ stimate $s^{2}$ of population variance, are based upon their earlier tables of the probability integral $I_{n}(Q)$ of the range of $n$ observations from a normal population. To correct for finite degrees of freedom, they use the relation

$$
\begin{equation*}
{ }_{\cdot} P_{n}(Q) \doteq P_{n}(Q)+\nu^{-1} a_{n}(Q)+\nu^{-2} b_{n}(Q) \tag{2}
\end{equation*}
$$

The tables give vaiues (to four, two and one decimal places, respectively) of $P_{n}(Q), a_{n}(Q)$ and $b_{n}(Q)$ for $Q=0.00(0.25) 6.50$ and $n=3(1) 20$, with the observation that the results are somewhat inaccurate for small values of $\nu(<10)$ and large values of $Q(>6)$. Actually, the tables are inaccurate not only for $\nu<10$, but also for values of $\nu$ up to about 20 , and the inaccuracy for high values of $Q$ is much greater than was anticipated. The inaccuracies in the Pearson-Hartley tables, which were due to the limitations of formula (2), in turn caused errors in the published critical values for Duncan's test. Beyer was aware of the difficulty for $\nu<10$, and attempted to correct it by adding a term of the form $\nu^{-3} c_{n}(Q)$ to the right-hand side of (2). This alleviated the diffieulty to some extent, but did not remove it, and nothing at all was done to correct the inaccuracies for $\nu \geq 10$. Having first become aware of this situation during the course of an investigation of the relation between error rates and sample sizes of multiple comparisons tests based on the range (see reference [3]), the author [4] reported it in a paper, presented to the American Statistical Association, which included an outline of plans for the computation of more accurate tables.

## COMPUTATION OF THE TABLE

The computation of more accurate critical values for Duncan's lest required the computation of a more a ccurate table of the probability integral of the studentized range, and this in turn required the computation of a more accurate table of the probability integral of the range. Dr. (iertrude Blanch gave invaluable assistance in the numerical analysis. Donald $S$. Clemm programmed the computation of the probability integrals of the range and of the studentized range for the Univac Scientific (ERA 1103) computer. Fugene H. Guthrie programmed for the ERA 1103 A the inverse interpolation necessary to obtain the critical values for Dunran's test.

The methods of computation of the probability integrals of the range and of the studentized range, together with voluminous tables, have been reported by Harter and Clemm [5] and by Harter, Clemm and Guthrie [ 6 ], ard will not be repeated here. The method of inverse interpolation employed, an iterative ome suggested by Major John V. Trmitage, involves the following steps:
I. In the table of the probability integral of the studentized range for $n=p$ and the desired value of $\nu$, find the two sureessive probabilities, $y_{0}$ and $y_{1}$, between which the required protection level $P=\gamma_{\text {p, }}=$ $(1-\alpha)^{\nu-1}$ lies. Call the two corresponding arguments (studentized
ranges) $x_{0}$ and $x_{1}$, respectively. The required studentized range $Q=R\left(p, \nu, \gamma_{p, a}\right)$ will lie between $x_{0}$ and $x_{1}$.
2. : mpute the tolerance $T$ for $P$ corrcsponding to a tolerance $5 \times 10^{4-s}$ for ? by means of the equation $T=(\Delta P / \Delta Q) \times 5 \times 10^{4-8}$, where $\Delta P=y_{1}-y_{0}, \Delta Q=x_{1}-x_{0}$ and $u$ is the number of digits before the decimal point in numbers between $x_{0}$ and $x_{1}$.
3. Perform lin ar inverse interpolation to find an approximation $x$ to the required $\left.\hbar^{\prime} p, \nu, \gamma_{p, a}\right)$, using the relation

$$
x=\left[\left(x_{1}-x_{0}\right)\left(P-y_{0}\right) /\left(y_{1}-y_{0}\right)\right]+x_{0} .
$$

4. Perform direct interpolation, using Aitken's method with a tolerance of $5 \times 10^{-7}$ and with provision for up to 16 -point interpolation if the tolerance is not met foi fewer points, to find the probability $y$ corresponding to the value $x$ of the studentized range.
5. Compare the result $y$ of step (4) with the required probability $P$, using the tolerance $T$ computed in stup (2):
a. If $|y-P| \leq T$, stop and set $R\left(p, \nu, \gamma_{p, a}\right)=x$.
b. If $(y-P)>T$, replace $y_{1}$ by $y$ and $x_{1}$ by $x$, then repeat the process, starting with step (3).
c. If $(y-P)<-T$, replace $y_{0}$ by $y$ and $x_{0}$ by $x$, then repeat the process, starting with step (3).
Once $R\left(p, \nu, \gamma_{p, a}\right)$ has been found, the critical value $Q(p, y, \alpha)$ for Duncan's test is determined as follows: $Q(p, \nu, \alpha)=\Omega\left(p, \nu, \gamma_{p, \alpha}\right)$ for $p=2$ and $Q(p, \nu, \alpha)=\max \left[R\left(p, \nu, \gamma_{p, a}\right), Q(p-1, \nu, \alpha)\right]$ for $p>2$. The results are given in Table 1.

Values for $\nu=\infty$, obtained by inverse interpolation in the table of the probability integral of the range, are included for convenience in interpolation (linear harmonic $\nu$-wise interpolation is recommended).

## ACCURACY OF THE TABLE

The table of the probability integral of the studentized range, on which the table of eritical values for Duncan's test is based, is accurate to within a unit in the sixth decimal place (except for values of the probability greater than 0.999995 , which are given as 1.00000 ), and the interval is small enough to make interpolation possible. The tolerance for the direct interpolation was set at $5 \times 10^{-7}$ so that the interpolation crror would not add appreciably to the error already present, and hence the interpolated values are substantially as accurate as the values in the input table. Inverse interpolation is, of course, not is accurate as direct interpolation, the error being $\Delta Q / \Delta P$ times as great for inverse interpolation as for direct interpolation. Thus the tolerance for 1 ' was found by multiplying the tolerance for $Q\left(5 \times 10^{\text {n- }}\right)$
Table 1
Critical Valees for Duncan's Nef Multiple Range Test
Protection Level $P=(.90)$ - 1 ; Stgmificance Levela $=.10$

T．ABI．F： 1 （Sontinued）


| O |  |  | $\stackrel{-}{\square}$ |
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| 3 |  <br> $\infty \dot{\infty} \dot{\infty} \times$ ci ci ci ci |  <br>  |  |
| ： |  <br>  |  <br>  |  Nois si N © |
| ${ }_{7}$ |  <br>  |  <br>  |  |

## TABLE 1 (Continued)

| $p$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 3 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 | 17.97 |
| 2 | 6.08:5 | $6.05:$ | 6.08 .5 | 6.08. | 6.08 .5 | 6.08: | 6.085 | 6.085 | 6.085 | 6.085 | 6.085 | 6.085 | 6.085 | 6.09.: | 6.085 | 6.085 | 6.085 | 6.08 .5 |
| 3 | 4.301 | 4.:16 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 | 4.516 |
| 4 | 3.927 | 4.013 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 | 4.033 |
| ; | 3.63.5 | 3.749 | 3.797 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 | 3.814 |
| 6 | 3.461 | 3.587 | 3.648 | 3.680 | 3.694 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 | 3.697 |
| 7 | 3.344 | 3.437 | 3.548 | 3.888 | 3.611 | 3.622 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 | 3.626 |
| 8 | 3.261 | 3.399 | 3.475 | 3.521 | 3.549 | 3.566 | 3.575 | 3.579 | 3.579 | 3.579 | 3.579 | 3.579 | 3.579 | 3.579 | 3.579 | 3.579 | 3.579 | 3.579 |
| 9 | 3.199 | 3.339 | 3.420 | 3.470 | 3.502 | 3.523 | 3.536 | 3.544 | 3.547 | 3.547 | 3.547 | 3.547 | 3.547 | 3.547 | 3.547 | 3.547 | 3.547 | 3.547 |
| 10 | 3.151 | 3.203 | 3.376 | c. 430 | 3.465 | 3.489 | 3.505 | 3.516 | 3.522 | 3.525 | 3.526 | 3.526 | 3.526 | 3.526 | 3.526 | 3.526 | 3.526 | 3.526 |
| 11 | 3.113 | 3.256 | 3.342 | 3.397 | 3.435 | 3.462 | 3.480 | 3.443 | 3.501 | 3.506 | 3.509 | 3.510 | 3.510 | 3.510 | 3.510 | 3.510 | 3.510 | 3.510 |
| 12 | 3.082 | 3.225 | 3.313 | 3.370 | 3.410 | 3.439 | 3.459 | 3.474 | 3.484 | 3.491 | 3.496 | 3.498 | 3.499 | 3.499 | 3.499 | 3.499 | 3.499 | 3.499 |
| . 3 | 3.0 .55 | 3.200 | 3.289 | 3.348 | 3.389 | 3.419 | 3.442 | 3.458 | 3.470 | 5.478 | 3.484 | 3.488 | 3.490 | 3.490 | 2.450 | 3.490 | 3.490 | 3.490 |
| 14 | 3.033 | 3.178 | 3.268 | 3.329 | 3.372 | 3.403 | 3.426 | 3.444 | 3.457 | 3.467 | 3.474 | 3.479 | 3.482 | 3.484 | 3.454 | 3.485 | 3.485 | 3.485 |
| 15 | 3.014 | 3.160 | 3.2:0 | 3.312 | 3.356 | 3.389 | 3.413 | 3.432 | 3.446 | 3.457 | 3.465 | 3.471 | 3.476 | 3.478 | 3.480 | 3.481 | 3.481 | 3.481 |
| 16 | 2.988 | 3.144 | 3.235 | 3.298 | 3.343 | 3.376 | 3.402 | 3.422 | 3.437 | 3.449 | 3.458 | 3.465 | 3.470 | 3.473 | 3.477 | 3.478 | 3.478 | 3.478 |
| 17 | 2.984 | 3.130 | 3.222 | 3.285 | 3.331 | 3.366 | 3.302 | 3.412 | 3.429 | 3.441 | 3.451 | 3.458 | 3.465 | 3.469 | 3.473 | 3.475 | 3.476 | 3.476 |
| 15 | 2.971 | 3.118 | 3.210 | 3.274 | 3.321 | 3.356 | 3.383 | 3.405 | 3.421 | 3.435 | 3.445 | 3.454 | 3.460 | 3.465 | 3.470 | 3.472 | 3.474 | 3.474 |
| 19 | 2.960 | 3.107 | 3.199 | 3.264 | 3.311 | 3.347 | 3.375 | 3.397 | 3.415 | 3.429 | 3.440 | 3.449 | 3.456 | 3.462 | 3.467 | 3.470 | 3.472 | 3.473 |
| 20 | 2.980 | 3.097 | 3.190 | 3.255 | 3.303 | 3.339 | 3.368 | 3.391 | 3.409 | 3.424 | 3.436 | 3.445 | 3.453 | 3.459 | 3.464 | 3.467 | 3.470 | 3.472 |
| 24 | 2.919 | 3.066 | 3.1 c.0 | 3.226 | 3.276 | 3.315 | 3.345 | 3.370 | 3.390 | 3.406 | 3.420 | 3.432 | 3.441 | 3.449 | 3.456 | 3.461 | 3.465 | 3.469 |
| 30 | 2.885 | 3.035 | 3.131 | 3.199 | 3.2:0 | 3.290 | 3.322 | 3.349 | 3.371 | 3.389 | 3.405 | 3.418 | 3.430 | 3.439 | 3.447 | 3.454 | 3.460 | 3.466 |
| 40 | 2.888 | 3.006 | 3.102 | 3.171 | 3.224 | 3.266 | 3.300 | 3.328 | 3.352 | 3.373 | 3.390 | 3.405 | 3.418 | 3.429 | 3.439 | 3.448 | 3.456 | 3.483 |
| 60 | 2.829 | 2.976 | 3.073 | 3.143 | 3.198 | 3.241 | 3.277 | 3.307 | 3.333 | 3.355 | 3.374 | 3.391 | 3.406 | 3.419 | 3.431 | 3.442 | 3.4.51 | 3.460 |
| 1:0 | 2.800 | 2.947 | 3.04: | 3.116 | 3.172 | 3.217 | 3.254 | 3.287 | 3.314 | 3.337 | 3.359 | 3.377 | 3.394 | 3.409 | 3.423 | 3.43 .5 | 3.446 | 3.457 |
| $\infty$ | 2.772 | 2.918 | 3.017 | 3.989 | 3.146 | 3.193 | 3.232 | 3.265 | 3.294 | 3.320 | 3.343 | 3.363 | 3.382 | 3.399 | 3.414 | 3.428 | 3.442 | 3.454 |

TABLE 1 (Continued)
Critical Valces for Duncan's New Multiple Range Test Protzithon Leviz $P=(.95$, - 1 ; Sionticascr Level $\alpha=.05$


TABLE 1 (Continurl)
Critical Valieg for Duncan's New Multiple Ravie Test

|  | 20 | 22 | 24 | 26 | 23 | 30 | 32 | 31 | 36 | 3 | 40 | s0 | \% | \% | 30 | ${ }^{90}$ | 10, |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 900 03 | 90.03 | 90.03 | 90.03 | 80.03 | 90. 03 | 90.03 | 90.03 | 90.03 | 90.03 | 90.03 | 90.03 | 90.03 | 9003 | ¢0. 03 | 90.03 | 90.03 |
| \% | 14.04 | 14.04 | 14.04 | 14.04 | 14.04 | 14.04 | 14.04 | 14.04 | ${ }^{14.04}$ | ${ }^{14.04}$ | 14.04 | ${ }^{14.04}$ | ${ }^{14.04}$ | 14.04 | 14.04 | ${ }^{14.04}$ | ${ }^{14.04}$ |
| 3 | $\bigcirc 321$ | 8.321 | 8.321 | 8.321 | 8.321 | 8.321 | 8.321 | 8.321 | 8.321 | 8.321 | ${ }^{5.321}$ | 8.321 | 8.331 | , 7 |  | 8.321 |  |
| ${ }_{5}^{4}$ | ${ }_{6}^{6,766}$ | ${ }_{\substack{6.736 \\ 607 \\ \hline 0.4}}$ | ${ }_{\text {c }}^{6.756}$ | ¢..0.6 | ${ }_{\text {cke }}^{6.756}$ | 6.7.76 6.074 | $\substack{6.736 \\ 0.0 \\ 0.4}$ | 6.756 6.074 | ${ }_{\text {c }}^{6.756}$ 6.074 | 6.756 6.074 | $\underset{\substack{6.756 \\ 6.074}}{\text { col }}$ | 8.756 6.074 c. | ${ }_{\text {c }}^{6.756}$ | ${ }_{6}^{6.756}$ |  | 6.736 6.074 col | ${ }_{\substack{6.756 \\ 6.074}}$ |
| 6 | -. 803 | 5.003 | 5,763 | -i. 03 | 5.703 | ${ }_{5}^{6.703}$ | 5.703 | ${ }_{5}^{5.703}$ | 5.703 | 5.703 | 5.703 | 5.703 | 5.803 | 3.703 | -7,703 | 5.75 | ${ }_{5}^{5.703}$ |
| 7 | ${ }^{5} 548$ | 5.472 | 3.472 | ; 472 | 5.472 | 5.472 | 5.472 | 5.472 | 3.472 | 5.472 | 5.472 | 5.472 | 3.472 | 5.472 | 5.472 | 5. 472 | 3.472 |
| 8 | 5.317 | 5.317 | 5.317 | -317 | ${ }^{5.317}$ | ${ }_{5}^{5.317}$ | 5.317 | ${ }_{5}^{5.317}$ | 5.317 | 5.317 | 5.317 | 3.317 | 5.317 | 5.317 | ${ }_{5}^{5.317}$ | 5.317 | 5.317 |
| 10 | 5.206 | 5 20i | ${ }_{5}^{5.206}$ | 3.206 5.124 | 5.206 | 5. 5 | ${ }^{5} 8.200$ | ${ }^{5.208}$ | 5.208 | $\frac{5}{5} 5$ | ${ }_{5}^{5.208}$ | 5.208 | ${ }_{5}^{5.206}$ | 5.208 | 5.206 | 5.20is | 5.208 |
| 11 | 5.059 | 5.001 | 5.061 | - 060 | $\therefore .061$ | 5.061 | 5.061 | 5.061 | 5.061 | 5.061 | 5.061 | 5.061 | -. 061 | 5.061 | 5. 061 | 5.061 | 5.061 |
| 12 | 3.006 | 5.010 | 5.011 | 5011 | S. 011 | 5.011 | 3.011 | 5.011 | 3.011 | 5.011 | 5.011 | 5.011 | 5.011 | 3.011 | 3.011 | 5.011 | 5.011 |
| 13 14 18 | 4.940 | ${ }_{4}^{4.966}$ | ${ }_{4}^{4.970}$ | ${ }_{\text {4, }}^{4.972}$ | ${ }_{4.940}^{4.92}$ | ${ }_{4}^{4.972}$ | + 4.972 | 4.972 4.90 4 | 4.972 | 4.972 | 4.972 | 4.972 | ${ }_{4}^{4.972}$ | 4.972 490 | + 4.972 | 4.4 .972 | ${ }_{\text {4, }}^{4.972}$ |
| 15, | 4.587 | 4.897 | 4.904 | 4.909 | 4.912 | 4.314 | +.914 | 4.914 | 4.914 | 4.914 | 4.914 | 4.914 | 4.914 | 4.914 | 4.914 | 4.914 | 4.914 |
| 16 | 4.8 .88 | 4.869 | 4.877 | 4.893 | 4.887 | 4.890 | 4.892 | 4.892 | 4.892 | 4.592 | 4.892 | 4.992 | 4.892 | 4.892 | 4.592 | 4.892 | 4.892 |
| 17 | 4.832 | 4.84 | 4.953 | 4.530 | 4.865 | 4.869 | 4.872 | 4.873 | 4.874 | 4.874 | $4.87 \%$ | 4.974 | 4.574 | 4.874 | 4.874 | 4.874 | 4.874 |
| 18 | 4.808 | 4.821 | 4.832 | 4.839 | 4.419 | 4.850 | + 4.85 | 4.85 | ${ }^{4.8577}$ | 4.853 | 4.858 | 4.858 | ${ }^{4.838}$ | 4.858 | 4.858 | 4.838 | 4.838 |
| $1:$ | 4.788 | 4.802 | 4.812 | 4.521 | 4.528 | 4.833 | 4.838 | 4.841 | 4.843 | 4.344 | 4.54; | 4. 45 | 4.845 | 4.845 | 4.85) | 4.845 | 4.845 |
| 2 | 4.769 | 4.884 | 4.795 | 4.805 | 4.513 | 4.818 | +. 823 | 4.927 | 4.830 | +1332 | 4.533 | + 133 | 4.533 | 4.333 | 4.933 | 4.833 | 4.333 |
| 21 | 4.710 | 4.727 | 4.71 | 4.752 | 4.7\%2 | 4.770 | 4.777 | 4.783 | 4.733 | 4.791 | 4.791 | 4.803 | 4.502 | 4.902 |  | 4.502 | 4.502 |
| 30 | 4.6:0 | 4.669 | 4.655 | 4.693 | 4.711 | 4.721 | 4.730 | 4.738 | 4.744 | 4.750 | 4.735 | 4.772 | 4.787 | 4.777 | 4.777 | 4.777 | 4.77 |
| 40 | 4.91 | 4.611 | 4.630 | 4.645 | ${ }^{4.659}$ | ${ }_{4}^{4.671}$ | 4.632 | 4.692 | 4.700 | +.703 | 4.715 | 4.740 | 4.74 | 4.761 | + 7.76 | 4.764 | 4.764 |
| ${ }^{\circ}$ | 4.330 | 4.573 | 4.573 | 4.591 | 4.607 | 4.620 | 4.633 | 4.645 | 4.655 | ${ }^{4.665}$ | 4.673 | 4.707 | 4.730 | 4.74, | 4.5 | 4.761 | ${ }^{4.763}$ |
| t:0 | 4.463 | 4 | $4.51{ }^{4}$ | + +1.35 | 4.552 | 4.005 | 4.553 | +. 596 | 4.609 | 4.619 | 4.630 | 4.673 | 4.803 | 4.723 | 4.745 | + 759 | 4.770 |
|  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |  |

'TABLE 1 (Continued)
Critical Valies for ilencan's New Multiple Range Test
Procection Level $P=$ (.mos!p-i; Significance Levela $=.005$

| p | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 1.5 | 16 | 17 | 18 | 19 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1 | 180.1 | 180.1 | 180.1 | 180.1 | 180.1 | 180.1 | $1 \times 0.1$ | 1 soc 1 | 180.1 | 180.1 | 150.1 | 150.1 | 180.1 | 180.1 | 150.1 | 180.1 | 180.1 | 180.1 |
| 2 | 19.43 | 19.93 | 19.93 | 19.93 | 10.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 | 19.93 |
| 3 | 10.5 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 | 10.63 |
| 4 | 3016 | 8.126 | 8.210 | 8.239 | 8.238 | 8.238 | 8.238 | 8.238 | \$. 238 | 8.238 | 8.238 | 8.238 | 8.238 | 8.235 | 8.238 | 8.238 | 8.238 | 8.238 |
| 5 | 6.751 | 6.980 | 7.100 | 7.167 | 7.204 | 7.222 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 | 7.228 |
| i | 6.10 .5 | 6.334 | 6.466 | 6.547 | 6.600 | C 63.7 | 6.658 | 6.672 | 6.679 | 6.682 | 6.682 | 6.682 | 6.682 | 6.682 | 6.682 | 6.682 | 6.682 | 6.682 |
| 7 | 5.699 | 5.022 | 6.057 | 6.145 | 6.207 | 6.250 | 6.281 | 6.304 | 6.320 | 0.331 | 6.339 | 6.343 | 6.345 | 6.345 | 6.345 | 6.345 | 6.345 | 6.345 |
| 8 | 5.420 | 5.638 | 5.773 | 5.864 | 5.930 | 5.978 | 6.014 | 6.042 | 6.064 | 6.080 | 6.092 | 6.101 | 6.108 | 6.113 | 6.116 | 6.118 | 6.119 | 6.119 |
| 9 | 5.218 | 5.430 | 5.565 | 5.657 | 5.72. | 5.776 | 5.815 | 5.846 | 5.871 | 5.891 | 5.907 | 5.920 | 5.930 | 5.938 | 5.944 | 5.949 | 5.952 | 5.955 |
| 10 | 5.065 | 5.273 | 5.405 | 5.498 | 5.567 | 5.620 | 5.662 | 5.695 | 5.722 | 5.744 | 5.762 | 5.777 | 5.790 | 5.800 | 5.809 | 5.816 | 5.821 | 5.826 |
| 11 | 4.945 | 5.149 | 5.280 | 5.372 | 5.442 | 5.496 | 5.539 | 5.574 | 5.603 | 5.626 | 5.646 | 5.663 | 5.678 | 5.690 | 5.700 | 5.709 | 5.716 | 5.722 |
| 12 | 4.849 | 5.048 | 5.178 | 5.270 | 5.341 | 5.396 | 5.439 | 5.475 | 5.505 | 5.531 | 5.552 | 5.570 | 5.585 | 5.599 | 5.810 | 5.620 | 5.629 | 5.838 |
| 13 | 4.770 | 4.966 | 3.094 | 5.186 | 5.258 | 5.312 | 5.3:6 | 5.393 | 5.424 | 5.450 | 5.472 | 5.492 | 5.508 | 5.523 | 5.535 | 5.546 | 5.558 | 5.584 |
| 14 | 4.704 | 4.897 | 5.023 | 5.116 | 5.185 | 5.241 | 5.286 | 5.324 | 5.355 | 5.382 | 5.405 | 5.425 | 5.442 | 5.458 | 5.471 | 5.483 | 5.494 | 5.503 |
| $\therefore$ | 4.647 | 4.838 | 4.964 | 5.055 | 5.125 | 5.181 | 5.226 | 5. 204 | 5.297 | 5.324 | 5.348 | 5.368 | 5.388 | 5.402 | 5.416 | 5.429 | 5.440 | 5.450 |
| 16 | 4.599 | 4.757 | 4.912 | 5.003 | 5.073 | 5.129 | 5.17: | 5.213 | 5.245 | 5.273 | 5.298 | 5.319 | 5.338 | 5.354 | 5.368 | 5.381 | 5.393 | 5.404 |
| 17 | 4.557 | 4.744 | 4.867 | 4.958 | 5.027 | 5.081 | 5.130 | 5.168 | 5.201 | 5.229 | 5.254 | 5.275 | 5.295 | 3.311 | 5.327 | 5.340 | 5.352 | 5.363 |
| 15 | 4.521 | 4.70 .5 | 4.828 | 4.818 | 4.987 | 5.043 | 5.090 | 5.129 | 5.162 | 5.190 | 5.215 | 5.237 | 5.256 | 5.274 | 5.289 | 5.303 | 5.316 | 5.327 |
| 19 | 4.488 | 4.671 | 4.793 | 4.883 | 4.952 | 5.008 | 5.054 | 5.093 | 5.127 | 5.156 | 5.181 | 5.203 | 5.222 | 5.240 | 5.256 | 5.270 | 5.283 | 5.295 |
| 20 | 4.460 | 4.641 | 4.762 | 4.851 | 4.920 | 4.976 | 5.022 | 5.061 | 5.095 | 5.124 | 3.150 | 5.172 | 5.193 | 5.210 | 5.226 | 5.241 | 5.254 | 5.266 |
| 24 | 4.371 | 4.547 | 4.666 | 4.753 | 4.822 | 4.877 | 4.924 | 4.963 | 4.997 | 5.027 | 5.053 | 5.076 | 5.097 | 5.116 | 5.133 | 5.148 | 5.162 | 5.175 |
| 30 | 4.285 | 4.456 | 4.572 | 4.658 | 4.726 | 4.781 | 4.827 | 4.867 | 4.901 | 4.931 | 4.958 | 4.981 | 5.003 | 5.022 | 5.040 | 5.056 | 5.071 | 5.085 |
| 40 | 4.202 | 4.369 | 4.482 | 4.566 | 4.632 | 4.657 | 4.733 | 4.772 | 4.806 | 4.837 | 4.864 | 4.883 | 4.910 | 4.930 | 4.948 | 4.965 | 4.980 | 4.995 |
| to | 4.122 | 4.284 | 4.304 | 4.478 | 4.541 | 4.595 | 4.640 | 4.679 | 4.713 | 4.744 | 4.771 | 4.796 | 4.818 | 4.838 | 4.857 | 4.874 | 4.890 | 4.905 |
| 120 | 4.045 | 4.201 | 4.308 | 4.388 | 4.452 | 4.505 | 4.550 | 4.588 | 4.622 | 4.652 | 4.679 | 4.704 | 4.726 | 4.747 | 4.766 | 4.784 | 4.800 | 4.815 |
| $\therefore$ | 3.970 | 4.121 | 4.225 | 4.303 | 4.365 | 4.417 | 4.461 | 4.499 | 4.532 | 4.562 | 4.589 | 4.614 | 4.636 | 4.657 | 4.676 | 4.694 | 4.710 | 4.726 |

## TABLE 1 (Continued)

Critical Valtes for Duncan's New Multiple Range Test


TABLE 1 (Continued)
Critical Valces for Duncan's New Multiple Range Test


| 8 |  | ${ }_{\text {\% }}^{\text {\% }}$ |  |
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by $1 /(\Delta Q / \Delta P)=\Delta P / \Delta Q$. Since $u$ is defined as the number of digits before the decimal point in the studentized range interval under consideration, this would guarantee that the error in $Q$ would not exceed 5 units in the fifth significant digit if the ratio of the change in $P$ to the change in $Q$ were constant throughout the interval under consideration. This condition ( $P$ piecewise linear in $Q$ ) is obviously not satisfied in practice, but as long as the weaker condition

$$
\max \left[\Delta P_{0} / \Delta Q_{0}, \Delta P_{1} / \Delta Q_{1}\right] \leq 2 \Delta P / \Delta Q
$$

where $\Delta P_{i}=\left|y-y_{i}\right|$ and $\Delta Q_{i}=\left|x-x_{0}\right|(i=0,1)$ is satisfied, the error in $Q$ will not exceed a unit in the fourth significant digit. This weaker condition is in fact satisfied, and hence it can be stated that the error in the critical values for Duncan's test, which are given in Table 1, does not exceed a unit in the fourth and last significant digit.

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