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CROSSED ACTIONS OF MATCHED PAIRS OF GROUPS ON TENSOR CATEGORIES

SONIA NATALE

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Abstract. We introduce the notion of (G, Γ)-crossed action on a tensor category, where (G, Γ) is a matched pair of finite groups. A tensor category is called a (G, Γ)-crossed tensor category if it is endowed with a (G, Γ)-crossed action. We show that every (G, Γ)-crossed tensor category $\mathcal{C}$ gives rise to a tensor category $\mathcal{C}^{(G,\Gamma)}$ that fits into an exact sequence of tensor categories $\operatorname{Rep} G \longrightarrow \mathcal{C}^{(G,\Gamma)} \longrightarrow \mathcal{C}$. We also define the notion of a (G, Γ)-braiding in a (G, Γ)-crossed tensor category, which is connected with certain set-theoretical solutions of the QYBE. This extends the notion of G-crossed braided tensor category due to Turaev. We show that if $\mathcal{C}$ is a (G, Γ)-crossed tensor category equipped with a (G, Γ)-braiding, then the tensor category $\mathcal{C}^{(G,\Gamma)}$ is a braided tensor category in a canonical way.

1. Introduction. Besides from their inherent algebraic appeal, monoidal and tensor categories are relevant structures in many areas of mathematics and mathematical physics. The endeavour around the far-reaching problem of their classification has seen a considerable outgrowth in the last decades. Widespread examples of tensor categories are provided by Hopf algebras and its generalizations by means of its representation theory.

The main goal of this paper is to present a construction of a class of tensor categories that generalizes and puts into a unified perspective certain renowned classes of examples.

The input for this construction consists of a matched pair of finite groups (G, Γ) plus a tensor category $\mathcal{C}$ endowed with a Γ-grading and an action of G by autoequivalences (which are not necessarily tensor functors):

$$\mathcal{C} = \bigoplus_{s\in\Gamma} \mathcal{C}_s\,, \qquad \rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}(\mathcal{C})\,,$$

that are related to each other in an appropriate sense. For reasons that might well become apparent in the sequel, we call such a data a (G, Γ)-*crossed action* on $\mathcal{C}$. We say that $\mathcal{C}$ is a (G, Γ)-*crossed tensor category*, if it is endowed with a (G, Γ)-crossed action. See Definition 4.1.

Recall that a *matched pair of groups* is a collection (G, Γ), where G and Γ are groups endowed with mutual actions by permutations

$$\Gamma \xleftarrow{\triangleleft} \Gamma \times G \xrightarrow{\triangleright} G$$

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satisfying the following conditions:

(1.1) $$s \triangleright gh = (s \triangleright g)((s \triangleleft g) \triangleright h)\,, \quad st \triangleleft g = (s \triangleleft (t \triangleright g))(t \triangleleft g)\,,$$

for all $s, t \in \Gamma$, $g, h \in G$.

The requirements in our definition of a (G, Γ)-crossed tensor category are that, for all $g \in G$, $s \in \Gamma$,

$$\rho^g(\mathcal{C}_s) = \mathcal{C}_{s \triangleleft g}\,,$$

and the existence of natural isomorphisms

$$\gamma^g_{X,Y} : \rho^g(X \otimes Y) \to \rho^{s \triangleright g}(X) \otimes \rho^g(Y)\,, \qquad X \in \mathcal{C},\ Y \in \mathcal{C}_s\,,$$

subject to certain rather natural compatibility conditions.

From a (G, Γ)-crossed tensor category $\mathcal{C}$ we produce a new tensor category that we denote $\mathcal{C}^{(G,\Gamma)}$. The tensor product in $\mathcal{C}^{(G,\Gamma)}$ is built from the tensor product of $\mathcal{C}$ and the natural isomorphisms γ. This is done in Theorem 5.1.

The main tool in the proof of Theorem 5.1 is the notion of a *Hopf monad*, introduced in [2], [3]. This notion and some of its main features are recalled in Subsection 2.3. It turns out that the data underlying a (G, Γ)-crossed tensor category $\mathcal{C}$ give rise to a monad T on $\mathcal{C}$ in such a way that the category $\mathcal{C}^T$ of T-modules in $\mathcal{C}$ identifies with $\mathcal{C}^{(G,\Gamma)}$. We show that, with respect to a suitable comonoidal structure arising from the (G, Γ)-crossed action on $\mathcal{C}$, T is in fact a Hopf monad, which allows to conclude that $\mathcal{C}^{(G,\Gamma)}$ is a tensor category.

We have that $\mathcal{C}^{(G,\Gamma)}$ is a finite tensor category if and only if the neutral homogeneous component $\mathcal{D} = \mathcal{C}_e$ of the associated Γ-grading is a finite tensor category. On the other side, $\mathcal{C}^{(G,\Gamma)}$ is a fusion category if and only if $\mathcal{D}$ is a fusion category and the characteristic of k does not divide the order of G (Proposition 6.2).

We show that, like in the case of an equivariantization under a group action by tensor autoequivalences, the category $\mathcal{C}^{(G,\Gamma)}$ fits into an exact sequence

$$\operatorname{Rep} G \longrightarrow \mathcal{C}^{(G,\Gamma)} \longrightarrow \mathcal{C}\,,$$

in the sense of the definition given in [4]. See Theorem 6.1. However, this is not an equivariantization exact sequence, unless the action $\triangleright : \Gamma \times G \longrightarrow G$ is (essentially) trivial. Dually, the category $\mathcal{C}^{(G,\Gamma)}$ is not a Γ-graded tensor category, unless the action $\triangleleft : \Gamma \times G \longrightarrow \Gamma$ is (essentially) trivial. See Propositions 6.3 and 6.8.

Let G be a group. Motivated by his developements in Homotopy Quantum Field Theory, Turaev introduced the notion of G-crossed braided categories [22], which serve as a tool in the construction of invariants of 3-dimensional G-manifolds. Müger showed in [19] (see also [13]) that G-crossed braided categories arise from the so-called Galois extensions of braided tensor categories.

As it turns out, the G-crossed categories underlying G-crossed braided categories of Turaev yield examples of crossed actions of a matched pair. Indeed, the right adjoint action $\triangleright : G \times G \longrightarrow G$ and the trivial action $\triangleleft : G \times G \longrightarrow G$ make (G, G) into a matched pair of groups. The conditions in Definition 4.1 of a (G, G)-crossed action on a tensor category $\mathcal{C}$ boil down in this case to the conditions defining a G-crossed tensor category $\mathcal{C}$.

Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category. We define in this paper a (G, Γ)-*braiding* in $\mathcal{C}$ as a triple (c, φ, ψ), where $\varphi, \psi : \Gamma \to G$ are group homomorphisms and c is a collection of natural isomorphisms

$$c_{X,Y} : X \otimes Y \to \rho^{t^{-1} \triangleright \varphi(s^{-1})}(Y) \otimes \rho^{\psi(t)}(X) , \qquad X \in \mathcal{C}_s,\ Y \in \mathcal{C}_t ,$$

satisfying certain compatibility conditions. See Definition 7.1.

Recall that a set-theoretical solution of the Quantum Yang-Baxter Equation is an invertible map $r : X \times X \to X \times X$, where X is a set, satisfying the condition $r^{12}r^{13}r^{23} = r^{23}r^{13}r^{12}$, as maps $X \times X \times X \to X \times X \times X$. A theory of set-theoretical solutions of the QYBE was developed in [11], [14], [23]. Our definition of a (G, Γ)-braiding is related to the set-theoretical solutions of the QYBE equation on the set Γ studied in [14], corresponding to appropriate actions of the group Γ on itself. We discuss this relation in Subsection 7.1.

We show that a (G, Γ)-braiding in $\mathcal{C}$ gives rise to a braiding in $\mathcal{C}^{(G,\Gamma)}$, thus providing examples of braided tensor categories. See Theorem 7.5.

In the case where $\mathcal{C}$ is a G-graded tensor category, regarded as before as (G, G)-crossed tensor category, a G-braiding c in $\mathcal{C}$ is the same thing as a (G, G)-braiding (c, φ, ψ), where $\psi = \mathrm{id}_G : G \to G$ is the identity group homomorphism and φ is the trivial group homomorphism (Proposition 8.2).

Matched pairs of groups are the main ingredients in the origin of one of the first classes of examples of non-commutative and non-cocommutative Hopf algebras discovered by G. I. Kac in the late 60's [12] (see also [16], [17], [21]). These Hopf algebras are most commonly called *abelian bicrossed products* or *abelian extensions*; they are characterized by the attribute of fitting into an exact sequence of Hopf algebras

$$k \longrightarrow k^{\Gamma} \longrightarrow H \longrightarrow kG \longrightarrow k , \tag{1.2}$$

where G and Γ are finite groups which, *a fortiori*, form a matched pair (G, Γ). This class of Hopf algebras, as well as its generalizations in different contexts, has been intensively studied in the literature.

We show that the representation category of an abelian extension of finite dimensional Hopf algebras fits into our construction. More precisely, we use the cohomological data determining an abelian exact sequence as in (1.2) to provide the tensor category $\mathcal{C}(\Gamma)$ of finite dimensional Γ-graded vector spaces with a (G, Γ)-crossed action, such that the outcoming tensor category $\mathcal{C}^{(G,\Gamma)}$ is strictly equivalent to the tensor category of finite dimensional representations of H.

Along this paper k will be an algebraically closed field. Our discussion focuses on the framework of tensor categories over k. Several pertinent definitions and facts about tensor categories are recalled in Subsection 2.2. We refer the reader to [1], [6], for a detailed treatment of the subject.

The contents of the paper are organized as follows. In Section 2 we overview the distinct concepts and basic facts on the main structures entering into the picture: matched pairs of groups, tensor categories and their module categories, Hopf monads on tensor categories and their relation with the notion of exact sequences of tensor categories. In Section 3 we discuss the main ingredients in our construction, namely, group actions on k-linear abelian categories and the related equivariantization process on one side, and group gradings on tensor categories on the other side. In Section 4 we define crossed actions of matched pairs on tensor categories. In Section 5 we present the main construction of the paper, that is, we prove here that every crossed action gives rise to a tensor category. The main general properties of this tensor category are studied in Section 6. In Section 7 we introduce (G, Γ)-crossed braidings and prove that a (G, Γ)-crossed tensor category equipped with a (G, Γ)-crossed braiding gives rise to a braided tensor category. In Section 8 we give examples of the main constructions from G-crossed categories and abelian extensions of Hopf algebras.

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2. Preliminaries.

2.1. Matched pairs of groups. A matched pair of groups is characterized by the existence of a group H endowed with an *exact factorization* into subgroups isomorphic to G and Γ, respectively. That is, H is a group containing subgroups $\tilde{G} \cong G$ and $\tilde{\Gamma} \cong \Gamma$, such that

$$H = \tilde{G}\,\tilde{\Gamma}\,, \qquad \tilde{\Gamma} \cap \tilde{G} = \{e\}\,.$$

In fact, if (G, Γ) is a matched pair, then there is a group structure, denoted $G \bowtie \Gamma$ in the cartesian product $G \times \Gamma$, defined by

$$(g, s)(h, t) = (g(s \triangleright h)\,, (s \triangleleft h)t)\,,$$

for all $g, h \in G$, $s, t \in \Gamma$. Conversely, given such a group H, we may identify G and Γ with subgroups of H. In this way the actions $\triangleleft : \Gamma \times G \to \Gamma$ and $\triangleright : \Gamma \times G \to G$ are determined by the relations

$$sg = (s \triangleright g)(s \triangleleft g)\,,$$

for all $g \in G$, $s \in \Gamma$.

Let (G, Γ) be a matched pair of groups. Relations (1.1) imply that $s \triangleright e = e$ and $e \triangleleft g = e$, for all $s \in \Gamma$, $g \in G$.

Using relations (1.1) it is also not difficult to show that the following conditions are equivalent:

(i) The action $\triangleleft$: $\Gamma \times G \longrightarrow \Gamma$ is trivial.

(ii) The action $\triangleright$: $\Gamma \times G \longrightarrow G$ is by group automorphisms.

If these conditions hold, then the group $G \bowtie \Gamma$ coincides with the semidirect product $G \rtimes \Gamma$.

Similarly, the conditions

(i') The action $\triangleright$: $\Gamma \times G \longrightarrow G$ is trivial.

(ii') The action $\triangleleft$: $\Gamma \times G \longrightarrow \Gamma$ is by group automorphisms.

are equivalent and, if they hold, then the group $G \bowtie \Gamma$ coincides with the semidirect product $G \ltimes \Gamma$.

2.2. Tensor categories. Let $\mathcal{C}$ be a monoidal category. Recall that a right dual of an object $Y \in \mathcal{C}$ is an object, denoted Y^*, endowed with morphisms $e_Y : Y^* \otimes Y \to \mathbf{1}$ and $c_Y : \mathbf{1} \to Y \otimes Y^*$ such that the compositions

$$Y \xrightarrow{c_Y \otimes \mathrm{id}} Y \otimes Y^* \otimes Y \xrightarrow{\mathrm{id} \otimes e_Y} Y, \qquad Y^* \xrightarrow{\mathrm{id} \otimes c_Y} Y^* \otimes Y \otimes Y^* \xrightarrow{e_Y \otimes \mathrm{id}} Y^*$$

coincide, respectively, with id_Y and id_{Y^*}. A left dual ${}^*Y$ of Y is an object of $\mathcal{C}$ endowed with morphisms $e'_Y : Y \otimes {}^*Y \to \mathbf{1}$ and $c'_Y : \mathbf{1} \to {}^*Y \otimes Y$ subject to similar conditions. Provided it exists, a right (respectively, left) dual of an object $Y \in \mathcal{C}$ is unique up to a unique isomorphism. The category $\mathcal{C}$ is called *rigid* if every object of $\mathcal{C}$ has right and left duals. See [1, Subsection 2.1].

A *tensor category* over k is a k-linear abelian rigid monoidal category $\mathcal{C}$ such that the tensor product $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is k-bilinear and the following conditions are satisfied:

- $\mathcal{C}$ is *locally finite*, that is, every object of $\mathcal{C}$ has finite length and Hom spaces are finite dimensional.
- The unit object $\mathbb{1} \in \mathcal{C}$ is simple.

Note that since k is algebraically closed, then an object X of $\mathcal{C}$ is simple if and only if it is scalar, that is, if and only if $\mathrm{End}(X) \cong k$.

If $\mathcal{C}$ is a tensor category over k, then the functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in both variables.

A *tensor subcategory* of a tensor category $\mathcal{C}$ is a full subcategory $\mathcal{D}$ of $\mathcal{C}$ which is closed under the operations of taking tensor products, subobjects and dual objects (so in particular $\mathbb{1} \in \mathcal{D}$). A tensor subcategory is itself a tensor category with tensor product inherited from that of $\mathcal{C}$.

A *finite tensor category* over k is a tensor category $\mathcal{C}$ over k which satisfies either of the following equivalent conditions:

- $\mathcal{C}$ has enough projective objects and finitely many simple objects.
- $\mathcal{C}$ has a projective generator, that is, an object $P \in \mathcal{C}$ such that the functor $\mathrm{Hom}_{\mathcal{C}}(P, -)$ is faithful exact.
- $\mathcal{C}$ is equivalent as a k-linear category to the category of finite dimensional representations of a finite dimensional k-algebra.

A *fusion category* over k is a semisimple finite tensor category over k.

Let G be a finite group. The category of finite dimensional representations of G over k will be denoted by $\mathrm{Rep}\, G$. This is a finite tensor category over k; it is a fusion category if and only if the characteristic of k does not divide the order of G.

All tensor categories in this paper will be assumed to be strict.

Let $\mathcal{C}, \mathcal{D}$ be tensor categories over k. A k-linear exact strong monoidal functor $F : \mathcal{C} \to \mathcal{D}$ will be called a *tensor functor*. Such functor is automatically faithful.

A *braided tensor category* over k is a tensor category $\mathcal{C}$ endowed with a *braiding*, that is, a natural isomorphism $\sigma : \otimes \to \otimes^{\mathrm{op}}$ satisfying the following *hexagon conditions*:

$$\sigma_{X,Y\otimes Z} = (\mathrm{id}_Y \otimes \sigma_{X,Z})\,(\sigma_{X,Y} \otimes \mathrm{id}_Z)\,, \quad \sigma_{X\otimes Y,Z} = (\sigma_{X,Z} \otimes \mathrm{id}_Y)\,(\mathrm{id}_X \otimes \sigma_{Y,Z})\,,$$

for all $X, Y, Z \in \mathcal{C}$.

A (left) *module category* over a tensor category $\mathcal{C}$ is a locally finite k-linear abelian category $\mathcal{M}$ endowed with a bifunctor $\overline{\otimes} : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$, which is k-bilinear and exact, and satisfies natural associativity and unit conditions.

A module category $\mathcal{M}$ is called *indecomposable* if it is not equivalent to a direct sum of two nonzero module categories. It is called *exact* if for every projective object $P \in \mathcal{C}$ and for every object $M \in \mathcal{M}$, $P\overline{\otimes}M$ is a projective object of $\mathcal{M}$. See [10].

It follows from [10, Proposition 2.1] that every tensor category $\mathcal{C}$ is an exact indecomposable module category over any tensor subcategory $\mathcal{D}$ with respect to the action $\mathcal{D}\times\mathcal{C} \to \mathcal{C}$ given by the tensor product of $\mathcal{C}$.

As a consequence of this fact, we obtain that a finite tensor category $\mathcal{C}$ is a fusion category if and only if its unit object $\mathbb{1}$ is projective.

2.3. Hopf monads on tensor categories. Let $\mathcal{C}$ be a tensor category over k. Recall that a *monad* on $\mathcal{C}$ is an endofunctor T of $\mathcal{C}$ endowed with natural transformations $\mu : T^2 \to T$ and $\eta : \mathrm{id}_{\mathcal{C}} \to T$ called, respectively, the multiplication and unit of T such that

$$\mu_X T(\mu_X) = \mu_X \mu_{T(X)}\,, \quad \mu_X \eta_{T(X)} = \mathrm{id}_{T(X)} = \mu_X T(\eta_X)\,, \tag{2.1}$$

for all objects $X \in \mathcal{C}$.

The monad T is a *bimonad* if it is a comonoidal endofunctor of $\mathcal{C}$ such that the product μ and the unit η are comonoidal transformations. That is, if the comonoidal structure of T is given by natural transformations

$$T_2(X, Y) : T(X \otimes Y) \to T(X) \otimes T(Y)\,,$$

$X, Y \in \mathcal{C}$ and $T_0 : T(\mathbb{1}) \to \mathbb{1}$, then, for all objects $X, Y \in \mathcal{C}$, we have

$$T_2(X, Y)\mu_{X\otimes Y} = (\mu_X \otimes \mu_Y)T_2(T(X), T(Y))T(T_2(X, Y))\,, \tag{2.2}$$

$$T_0\mu_{\mathbb{1}} = T_0T(T_0)\,, \quad T_2(X, Y)\eta_{X\otimes Y} = \eta_X \otimes \eta_Y\,, \quad T_0\eta_{\mathbb{1}} = \mathrm{id}_{\mathbb{1}}\,. \tag{2.3}$$

A bimonad T is called a *Hopf monad* provided that the fusion operators $H^l : T(\mathrm{id}_{\mathcal{C}} \otimes T) \to T \otimes T$ and $H^r : T(T \otimes \mathrm{id}_{\mathcal{C}}) \to T \otimes T$ defined, for every $X, Y \in \mathcal{C}$, by

$$H^l_{X,Y} := (\mathrm{id}_{T(X)} \otimes \mu_Y)\, T_2(X, T(Y)) : T(X \otimes T(Y)) \to T(X) \otimes T(Y)\,,$$

$$H^r_{X,Y} := (\mu_X \otimes \mathrm{id}_{T(Y)})\, T_2(T(X), Y) : T(T(X) \otimes Y) \to T(X) \otimes T(Y)\,,$$

are isomorphisms.

Let T be a k-linear right exact Hopf monad on $\mathcal{C}$. Then the category $\mathcal{C}^T$ of T-modules in $\mathcal{C}$ is a tensor category over k.

Recall that the objects of $\mathcal{C}^T$ are pairs (X, r), where X is an object of $\mathcal{C}$ and $r : T(X) \to X$ is a morphism in $\mathcal{C}$, such that

$$rT(r) = r\mu_X , \qquad r\eta_X = \mathrm{id}_X .$$

If $(X, r), (X', r') \in \mathcal{C}^T$, a morphism $f : (X, r) \to (X', r')$ is a morphism $f : X \to X'$ in $\mathcal{C}$ such that $fr = r'T(f)$.

The tensor product of two objects $(X, r), (X', r') \in \mathcal{C}^T$ is defined by

$$(X, r) \otimes (X', r') = (X \otimes X', (r \otimes r')T_2(X, X')) , \tag{2.4}$$

and the unit object of $\mathcal{C}^T$ is $(\mathbb{1}, T_0)$. See [2], [3], [4, Proposition 2.3].

Moreover, in this situation, the forgetful functor $F : \mathcal{C}^T \to \mathcal{C}$, $F(X, r) = X$, is a strict tensor functor. The functor F is dominant if and only if the Hopf monad T is faithful.

A *quasitriangular Hopf monad* on a tensor category $\mathcal{C}$ is a Hopf monad T equipped with an *R-matrix* R, that is, R is a $*$-invertible natural transformation

$$R_{X,Y} : X \otimes Y \to T(Y) \otimes T(X) , \quad X, Y \in \mathcal{C} ,$$

satisfying the following conditions, for all objects $X, Y, Z \in \mathcal{C}$:

$$(\mu_X \otimes \mu_Y)R_{TX,TY}T_2(X, Y) = (\mu_X \otimes \mu_Y)T_2(TY, TX)T(R_{X,Y}) , \tag{2.5}$$

$$(\mathrm{id}_{TZ} \otimes T_2(X, Y))R_{X\otimes Y,Z} = (\mu_Z \otimes \mathrm{id}_{TX\otimes TY})(R_{X,TZ} \otimes \mathrm{id}_{TY})(\mathrm{id}_X \otimes R_{Y,Z}) , \tag{2.6}$$

$$(T_2(Y, Z) \otimes \mathrm{id}_{TX})R_{X,Y\otimes Z} = (\mathrm{id}_{TY\otimes TZ} \otimes \mu_X)(\mathrm{id}_{TY} \otimes R_{TX,Z})(R_{X,Y} \otimes \mathrm{id}_Z) . \tag{2.7}$$

The $*$-invertibility of R means that the natural morphisms

$$R^{\#}_{(X,r),(Y,s)} = (s \otimes r)R_{X,Y} : X \otimes Y \to Y \otimes X ,$$

are isomorphisms, for all objects $(X, r), (Y, s) \in \mathcal{C}^T$. See [2, Subsection 8.2].

In view of [2, Theorem 8.5], if T is a quasitriangular Hopf monad on $\mathcal{C}$, then $\mathcal{C}^T$ is a braided tensor category with braiding $\sigma_{(X,r),(Y,s)} : (X, r) \otimes (Y, s) \to (Y, s) \otimes (X, r)$, defined in the form $\sigma_{(X,r),(Y,s)} = (s \otimes r)R_{X,Y}$.

2.4. Exact sequences of tensor categories. Let $\mathcal{C}, \mathcal{C}''$ be tensor categories over k. A tensor functor $F : \mathcal{C} \to \mathcal{C}''$ is called *normal* if every object X of $\mathcal{C}$, there exists a subobject $X_0 \subset X$ such that $F(X_0)$ is the largest trivial subobject of $F(X)$.

If the functor F has a right adjoint R, then F is normal if and only if $R(\mathbb{1})$ is a trivial object of $\mathcal{C}$ [4, Proposition 3.5].

For a tensor functor $F : \mathcal{C} \to \mathcal{C}''$, let $\mathfrak{Ker}_F$ denote the tensor subcategory $F^{-1}(\langle \mathbb{1} \rangle) \subseteq \mathcal{C}$ of objects X of $\mathcal{C}$ such that $F(X)$ is a trivial object of $\mathcal{C}''$.

Let $\mathcal{C}', \mathcal{C}, \mathcal{C}''$ be tensor categories over k. An *exact sequence of tensor categories* is a sequence of tensor functors

$$\mathcal{C}' \xrightarrow{\ f\ } \mathcal{C} \xrightarrow{\ F\ } \mathcal{C}'' \tag{2.8}$$

such that the tensor functor F is dominant and normal and the tensor functor f is a full embedding whose essential image is $\mathfrak{Ker}_F$. See [4].

The *induced Hopf algebra* H of the exact sequence (2.8) is defined as the coend of the fiber functor $\omega_F = \mathrm{Hom}_{\mathcal{C}''}(\mathbb{1}, Ff) : \mathcal{C}' \to \mathrm{Vec}_k$. There is an equivalence of tensor categories $\mathcal{C}' \simeq$ comod-H. See [4, Subsection 3.3].

By [4, Theorem 5.8] exact sequences (2.8) with finite dimensional induced Hopf algebra H are classified by normal faithful right exact k-linear Hopf monads T on $\mathcal{C}''$, such that the Hopf monad of the restriction of T to the trivial subcategory of $\mathcal{C}''$ is isomorphic to H. Recall that a k-linear right exact Hopf monad T on a tensor category $\mathcal{C}''$ is called *normal* if $T(\mathbb{1})$ is a trivial object of $\mathcal{C}''$.

3. Group actions and group gradings on k-linear and tensor categories. In this section we discuss some facts on group actions and group gradings on k-linear and tensor categories that will be used later on.

3.1. Group actions on k-linear abelian categories. Let G be a group and let $\mathcal{C}$ be a k-linear abelian category.

Let $\underline{G}$ be the strict monoidal category whose objects are the elements of G and morphisms are identities, with tensor product defined as the multiplication in G and unit object $e \in G$. Let also $\underline{\mathrm{Aut}}\,\mathcal{C}$ be the strict monoidal category whose objects are k-linear autoequivalences of $\mathcal{C}$, morphisms are natural transformations, with tensor product defined by composition of endofunctors and natural transformations and unit object $\mathrm{id}_{\mathcal{C}}$.

Consider the strict monoidal category $\underline{G}^{\mathrm{op}}$ obtained from $\underline{G}$ by reversing the tensor product. That is, the underlying category of $\underline{G}^{\mathrm{op}}$ is $\underline{G}$, while the tensor product in $\underline{G}^{\mathrm{op}}$ is defined by $g \otimes h = hg$, $g, h \in G$.

By a *right action* of G on $\mathcal{C}$ by k-linear autoequivalences we shall understand a monoidal functor $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}\,\mathcal{C}$. That is, for every $g \in G$, we have a k-linear functor $\rho^g : \mathcal{C} \to \mathcal{C}$ and natural isomorphisms

$$\rho_2^{g,h} : \rho^g \rho^h \to \rho^{hg}, \quad g, h \in G,$$

and $\rho_0 : \mathrm{id}_{\mathcal{C}} \to \rho^e$, satisfying

$$(\rho_2^{ba,c})_X \, (\rho_2^{a,b})_{\rho^c(X)} = (\rho_2^{a,cb})_X \, \rho^a((\rho_2^{b,c})_X), \tag{3.1}$$

$$(\rho_2^{a,e})_X \rho^a(\rho_{0X}) = \mathrm{id}_{\rho^a(X)} = (\rho_2^{e,a})_X (\rho_0)_{\rho^a(X)}, \tag{3.2}$$

for all $X \in \mathcal{C}$, $a, b, c \in G$.

3.2. Equivariantization. Let $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}\,\mathcal{C}$ be a right action of G on $\mathcal{C}$ by k-linear autoequivalences. A G-equivariant object is a pair (X, r), where X is an object of $\mathcal{C}$ and $r = (r^g)_{g \in G}$ is a collection of isomorphisms $r^g : \rho^g X \to X$, $g \in G$, satisfying

$$r^g \rho^g(r^h) = r^{hg} (\rho_2^{g,h})_X, \quad \forall g, h \in G, \qquad r^e \rho_{0X} = \mathrm{id}_X. \tag{3.3}$$

A G-equivariant morphism $f : (X, r) \to (Y, r')$ is a morphism $f : X \to Y$ in $\mathcal{C}$ such that $f r^g = r'^g \rho^g(f)$, for all $g \in G$.

The category of G-equivariant objects and morphisms is a k-linear abelian category, denoted $\mathcal{C}^G$, called the *equivariantization* of $\mathcal{C}$ under the action ρ.

Suppose that G is a finite group. Let $T^\rho : \mathcal{C} \to \mathcal{C}$ be the endofunctor of $\mathcal{C}$ defined by $T^\rho = \bigoplus_{g\in G} \rho^g$. Then T^ρ is a k-linear exact monad on $\mathcal{C}$ with multiplication $\mu : T^{\rho 2} = \bigoplus_{g,h\in G} \rho^g\rho^h \to \bigoplus_{g\in G} \rho^g = T^\rho$, given componentwise by the isomorphisms $\rho_2^{g,h} : \rho^g\rho^h \to \rho^{hg}$, and unit $\eta = \rho_0 : \mathrm{id}_\mathcal{C} \to \rho^e \to T^\rho$.

Since the unit η of T^ρ is a monomorphism, then T^ρ is a faithful endofunctor of $\mathcal{C}$ [4, Lemma 2.1].

Extending the terminology of [4], we shall call T^ρ the monad of the group action ρ. (Note however, that the group actions considered *loc. cit.* are by tensor autoequivalences on tensor categories.)

The canonical isomorphisms

$$\mathrm{Hom}_\mathcal{C}(\bigoplus_{g\in G} \rho^g(X), X) \cong \prod_{g\in G} \mathrm{Hom}_\mathcal{C}(\rho^g(X), X)\,,$$

$X \in \mathcal{C}$, induce an equivalence of categories over $\mathcal{C}$ between the category $\mathcal{C}^{T^\rho}$ of T^ρ-modules in $\mathcal{C}$ and the equivariantization $\mathcal{C}^G$. See [4, Subsection 5.3].

REMARK 3.1. Suppose that $\mathcal{C}$ is a tensor category. Assume in addition that the action of G is given by tensor autoequivalences of $\mathcal{C}$, that is, the endofunctor ρ^g is a tensor functor, for all $g \in G$, and $\rho_2^{g,h} : \rho^g\rho^h \to \rho^{hg}$, $\rho_0 : \mathrm{id}_\mathcal{C} \to \rho^e$ are natural isomorphisms of monoidal functors.

Then T^ρ is Hopf monad on $\mathcal{C}$ with comonoidal structure

$$T_2(X, Y) : \bigoplus_{g\in G} \rho^g(X \otimes Y) \to \bigoplus_{g,g'\in G} \rho^g(X) \otimes \rho^g(Y)\,,$$

and $T_0 : \bigoplus_{g\in G} \rho^g(\mathbb{1}) \to \mathbb{1}$, given componentwise by the monoidal structure $\rho_2^g : \rho^g \circ \otimes \to \rho^g \otimes \rho^g$ and $\rho_0^g : \rho^g(\mathbb{1}) \to \mathbb{1}$ of the functors ρ^g, $g \in G$.

Thus the equivariantization $\mathcal{C}^G$ is a tensor category with tensor product defined by the formula

$$(X, r) \otimes (X', r') = (X \otimes X', (r \otimes r')(\rho_2)_{X,X'})\,. \tag{3.4}$$

In addition, if $\mathcal{C}$ is a finite tensor category, then so is $\mathcal{C}^G$. If $\mathcal{C}$ is a fusion category and the characteristic of k does not divide the order of G, then $\mathcal{C}^G$ is also a fusion category.

Furthermore, T^ρ is a normal cocommutative Hopf monad on $\mathcal{C}$ and the forgetful functor $F : \mathcal{C}^G \to \mathcal{C}$ gives rise to a central exact sequence of tensor categories

$$\mathrm{Rep}\, G \longrightarrow \mathcal{C}^G \longrightarrow \mathcal{C}\,. \tag{3.5}$$

See [4, Corollary 2.22], [5, Example 2.5].

3.3. Group gradings on tensor categories. Let G be a group and let $\mathcal{C}$ be a tensor category over k. Let

$$\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g \,, \tag{3.6}$$

be a G-*grading* on $\mathcal{C}$. That is, for every $g \in G$, $\mathcal{C}_g$ is a full subcategory of $\mathcal{C}$ and the following conditions hold:

- For every object X of $\mathcal{C}$ we have a decomposition $X \cong \bigoplus_{g \in G} X_g$, where $X_g \in \mathcal{C}_g$, for all $g \in G$.
- For all $X \in \mathcal{C}_g$, $Y \in \mathcal{C}_h$, $g \neq h \in G$, we have $\mathrm{Hom}_{\mathcal{C}}(X, Y) = 0$.
- $\mathcal{C}_g \otimes \mathcal{C}_h \subseteq \mathcal{C}_{gh}$, for all $g, h \in G$.

The subcategories $\mathcal{C}_g$, $g \in G$, are called the *homogeneous components* of the grading. A G-grading (3.6) is called *faithful* if $\mathcal{C}_g \neq 0$, for all $g \in G$.

Let $\mathcal{C} = \bigoplus_{g \in G} \mathcal{C}_g$ and $\mathcal{D} = \bigoplus_{g \in G} \mathcal{D}_g$ be G-graded k-linear abelian categories. A functor $F : \mathcal{C} \to \mathcal{D}$ will be called a G-*graded functor* if $F(\mathcal{C}_g) \subseteq \mathcal{D}_g$, for all $g \in G$.

LEMMA 3.2. *Let $F : \mathcal{C} \to \mathcal{D}$ be a G-graded functor between G-graded k-linear abelian categories categories $\mathcal{C}$, $\mathcal{D}$. Suppose F is dominant. Then, for all $g \in G$, F induces by restriction a dominant functor $F : \mathcal{C}_g \to \mathcal{D}_g$.*

PROOF. Let $g \in G$ and let Y be any object of $\mathcal{D}_g$. Since F is dominant, there exists $X \in \mathcal{C}$ such that Y is a subobject of $F(X)$. Let $X \cong \bigoplus_{h \in G} X_h$ be decomposition of X into a direct sum of homogeneous objects $X_h \in \mathcal{C}_h$. Since F is a G-graded functor, $F(X) \cong \bigoplus_{h \in G} F(X_h)$ is a decomposition of $F(X)$ into a direct sum of homogeneous objects $F(X_h) \in \mathcal{D}_h$. Then Y must be a subobject of $F(X_g)$, because $\mathrm{Hom}_{\mathcal{D}}(Y, F(X_h)) = 0$, for all $h \neq g$. This proves the lemma. □

REMARK 3.3. Let $\mathcal{C}$ be a G-graded tensor category. Since the unit object $\mathbf{1}$ is simple, then it is isomorphic to an object of $\mathcal{C}_e$. Without loss of generality, we shall assume that $\mathbf{1}$ belongs to $\mathcal{C}_e$.

Suppose $Y \in \mathcal{C}_g$ is a nonzero homogeneous object. Let Y^* and ${}^*Y$ be, respectively, a right and a left dual of Y (see Subsection 2.2). Then Y^* and ${}^*Y$ are isomorphic to objects of $\mathcal{C}_{g^{-1}}$. In fact, suppose that $h \neq g^{-1} \in G$ and let $X \in \mathcal{C}_h$. We have an isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(X, Y^*) = \mathrm{Hom}_{\mathcal{C}}(X, \mathbb{1} \otimes Y^*) \cong \mathrm{Hom}_{\mathcal{C}}(X \otimes Y, \mathbb{1}) \,,$$

by [1, Lemma 2.1.6]. Hence $\mathrm{Hom}_{\mathcal{C}}(X, Y^*) = 0$ because $X \otimes Y \in \mathcal{C}_{hg}$ and $hg \neq e$. Similarly one can see that $\mathrm{Hom}_{\mathcal{C}}(X, {}^* Y) = 0$. Therefore we shall also assume without loss of generality that the duals of an object of $\mathcal{C}_g$ have been chosen so that they belong to $\mathcal{C}_{g-1}$.

PROPOSITION 3.4. *Let $\mathcal{C}$ be a G-graded tensor category. Then the neutral homogeneous component $\mathcal{D} = \mathcal{C}_e$ is a tensor subcategory of $\mathcal{C}$. Besides, every homogeneous component $\mathcal{C}_g$, $g \in G$, is an indecomposable exact left* (*and right*) *module category with the action given by the tensor product of $\mathcal{C}$.*

PROOF. In view of Remark 3.3, $\mathcal{D}$ contains the unit object and is closed under the operations of taking duals. This implies that $\mathcal{D}$ is a tensor subcategory of $\mathcal{C}$.

Let now $g \in G$. By the definition of a G-grading we have $\mathcal{D}\otimes\mathcal{C}_g \subseteq \mathcal{C}_g$, and $\mathcal{C}_g\otimes\mathcal{D} \subseteq \mathcal{C}_g$. Therefore $\mathcal{C}_g$ is both a left and right module subcategory of $\mathcal{C}$ over the tensor subcategory $\mathcal{D}$. Since $\mathcal{C}$ is an exact module category over $\mathcal{D}$, then so is $\mathcal{C}_g$.

It remains to prove the indecomposability of $\mathcal{C}_g$. We may assume that $\mathcal{C}_g \neq 0$. Let X, Y be any nonzero objects of $\mathcal{C}_g$. Again in view of Remark 3.3, we get that $Z = Y^* \otimes X \in \mathcal{D}$.

Observe that the functor $- \otimes X : \mathcal{C} \to \mathcal{C}$ is faithful exact. Then, since by rigidity $\operatorname{Hom}_{\mathcal{C}}(\mathbb{1}, Y \otimes Y^*) \neq 0$, we obtain that $\operatorname{Hom}_{\mathcal{C}}(X, Y \otimes Z) = \operatorname{Hom}_{\mathcal{C}}(X, Y \otimes Y^* \otimes X) \neq 0$. This implies that $\mathcal{C}_g$ is indecomposable as a right module category over $\mathcal{D}$. Indecomposability as a left module category is shown similarly. This finishes the proof of the proposition. □

COROLLARY 3.5. *Suppose G is a finite group. Let $\mathcal{C}$ be a G-graded tensor category with neutral homogeneous component $\mathcal{D}$. Then $\mathcal{C}$ is a finite tensor category* (*respectively, a fusion category*) *if and only if so is $\mathcal{D}$.*

PROOF. Any tensor subcategory of a finite tensor category (respectively, of a fusion category) is itself a finite tensor category (respectively, a fusion category). Then we only need to show the ‘if’ direction.

Suppose first that $\mathcal{D}$ is a finite tensor category. Let $P \in \mathcal{D}$ be a projective generator, that is P is an object of $\mathcal{D}$ such that the functor $\operatorname{Hom}_{\mathcal{D}}(P, -)$ is faithful exact. Since the group G is finite, it will be enough to show that every homogeneous component is a finite k-linear abelian category.

Let $g \in G$ such that $\mathcal{C}_g \neq 0$. Note that since $\mathcal{C}$ is a tensor category, then $\mathcal{C}_g$ is locally finite (that is, it has finite dimensional hom spaces and every object has finite length). Therefore it will be enough to show that $\mathcal{C}_g$ has a projective generator. Let $X_0 \in \mathcal{C}_g$ be any nonzero object.

By exactness of the left $\mathcal{D}$-module category $\mathcal{C}_g$, $P \otimes X_0$ is a projective object of $\mathcal{C}_g$. Hence the functor $\operatorname{Hom}_{\mathcal{C}_g}(P \otimes X_0, -) = \operatorname{Hom}_{\mathcal{C}}(P \otimes X_0, X)$ is exact. In addition, using the rigidity of $\mathcal{C}$, we get for all $X \in \mathcal{C}_g$ a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(P \otimes X_0, X) \cong \operatorname{Hom}_{\mathcal{C}}(P, X \otimes X_0^*) = \operatorname{Hom}_{\mathcal{D}}(P, X \otimes X_0^*)\,.$$

Since both functors $\operatorname{Hom}_{\mathcal{D}}(P, -)$ and $-\otimes X_0^*$ are faithful, then $\operatorname{Hom}_{\mathcal{C}_g}(P\otimes X_0, -)$ is faithful. Hence $P \otimes X_0$ is a projective generator of $\mathcal{C}_g$. Thus we obtain that $\mathcal{C}$ is a finite tensor category.

Suppose next that $\mathcal{D}$ is a fusion category. In particular it is a finite tensor category and hence so is $\mathcal{C}$, by the previous part. Since $\mathcal{C}$ is an exact module category over $\mathcal{D}$, then $\mathcal{C}$ is semisimple (see [10, Example 3.3]). Hence $\mathcal{C}$ is also a fusion category, as claimed. □

REMARK 3.6. Suppose k is of characteristic zero. Let $\mathcal{C}$ be a fusion category over k. Group gradings on $\mathcal{C}$ were classified in [9].

By [8, Proposition 2.9], $\mathcal{C}$ admits a faithful G-grading if and only if its Drinfeld center $\mathcal{Z}(\mathcal{C})$ contains a Tannakian subcategory $\mathcal{E}$ such that $\mathcal{E} \cong \operatorname{Rep} G$ as symmetric categories and $\mathcal{E}$ is contained in the kernel of the forgetful functor $U : \mathcal{Z}(\mathcal{C}) \to \mathcal{C}$.

Observe that endowing $\mathcal{C}$ with a G-grading is equivalent to providing a map $\partial : \mathrm{Irr}(\mathcal{C}) \to G$ such that $\partial(Z) = \partial(X)\partial(Y)$, for all simple objects X, Y and Z of $\mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(Z, X \otimes Y) \neq 0$.

The grading corresponding to a Tannakian subcategory $\mathcal{E} \cong \mathrm{Rep}\, G$ of the center of $\mathcal{C}$ is defined as follows. Let X be an object of $\mathcal{C}$ and let $(V, \sigma) \in \mathcal{E}$. Then $U(V, \sigma) = V$ is a trivial object of $\mathcal{C}$ and thus it is equipped with a trivial half-brading $\tau_{X,V} : X \otimes V \to V \otimes X$. Composing with the braiding σ this gives an isomorphism $\tau_{X,V}\sigma_{V,X} : V \otimes X \to V \otimes X$.

Let X be a simple object of $\mathcal{C}$. Using that σ is a braiding, we obtain in this way a natural automorphism of tensor functors $U|_{\mathcal{E}} \to U|_{\mathcal{E}}$. This is the same as an element $g \in G$, since $U|_{\mathcal{E}}$ is a fiber functor on $\mathcal{E}$. This defines a map $\partial : \mathrm{Irr}(\mathcal{C}) \to G$, which is seen to be G-grading using the hexagon axiom for the braiding of the center.

It follows from [7, Proposition 8.20] that if $\mathcal{C}$ is a fusion category endowed with a faithful G-grading, then $\mathrm{FPdim}\, \mathcal{C} = |G|\, \mathrm{FPdim}\, \mathcal{D}$.

4. (G, Γ)-crossed actions on tensor categories. Let $\mathcal{C}$ be a tensor category over k and let (G, Γ) be a matched pair of groups.

DEFINITION 4.1. A (G, Γ)-*crossed action* on the tensor category $\mathcal{C}$ consists of the following data:

- A Γ-grading on $\mathcal{C}$: $\mathcal{C} = \bigoplus_{s \in \Gamma} \mathcal{C}_s$.
- A right action of G on $\mathcal{C}$ by k-linear autoequivalences $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}(\mathcal{C})$ such that

$$\rho^g(\mathcal{C}_s) = \mathcal{C}_{s \lhd g}, \quad \forall g \in G,\ s \in \Gamma. \tag{4.1}$$

- A collection of natural isomorphisms $\gamma = (\gamma^g)_{g \in G}$:

$$\gamma^g_{X,Y} : \rho^g(X \otimes Y) \to \rho^{t \rhd g}(X) \otimes \rho^g(Y), \quad X \in \mathcal{C},\ t \in \Gamma,\ Y \in \mathcal{C}_t. \tag{4.2}$$

- A collection of isomorphisms $\gamma^g_0 : \rho^g(\mathbb{1}) \to \mathbb{1}$, $g \in G$.

These data are subject to the commutativity of the following diagrams:

(a) For all $g \in G$, $X \in \mathcal{C}$, $s, t \in \Gamma$, $Y \in \mathcal{C}_s$, $Z \in \mathcal{C}_t$,

$$\begin{array}{ccc}
\rho^g(X \otimes Y \otimes Z) & \xrightarrow{\gamma^g_{X \otimes Y, Z}} & \rho^{t \rhd g}(X \otimes Y) \otimes \rho^g(Z) \\
\Big\downarrow {\scriptstyle \gamma^g_{X, Y \otimes Z}} & & \Big\downarrow {\scriptstyle \gamma^{t \rhd g}_{X,Y} \otimes \mathrm{id}_{\rho^g(Z)}} \\
\rho^{st \rhd g}(X) \otimes \rho^g(Y \otimes Z) & \xrightarrow[\mathrm{id}_{\rho^{st \rhd g}(X)} \otimes \gamma^g_{Y,Z}]{} & \rho^{s \rhd (t \rhd g)}(X) \otimes \rho^{t \rhd g}(Y) \otimes \rho^g(Z)
\end{array}$$

(b) For all $g \in G$, $X \in \mathcal{C}$,

$$\begin{array}{ccccc}
\rho^g(X) \otimes \rho^g(\mathbb{1}) & \xleftarrow{\gamma^g_{X,\mathbb{1}}} & \rho^g(X) & \xrightarrow{\gamma^g_{\mathbb{1},X}} & \rho^g(\mathbb{1}) \otimes \rho^g(X) \\
& {\scriptstyle \mathrm{id}_{\rho^g(X)} \otimes \gamma^g_0} \searrow & \Big\downarrow {\scriptstyle =} & \swarrow {\scriptstyle \gamma^g_0 \otimes \mathrm{id}_{\rho^g(X)}} & \\
& & \rho^g(X) & &
\end{array}$$

(c) For all $g, h \in G$, $X \in \mathcal{C}$, $s \in \Gamma$, $Y \in \mathcal{C}_s$,

$$\begin{array}{ccc}
\rho^g\rho^h(X \otimes Y) & \xrightarrow{\ {\rho_2}^{g,h}_{X\otimes Y}\ } & \rho^{hg}(X \otimes Y) \\
\Big\downarrow{\scriptstyle \rho^g(\gamma^h_{X,Y})} & & \Big\downarrow{\scriptstyle \gamma^{hg}_{X,Y}} \\
 & & \rho^{s\triangleright hg}(X) \otimes \rho^{hg}(Y) \\
 & & \Big\uparrow{\scriptstyle {\rho_2}_X^{(s\triangleleft h)\triangleright g, s\triangleright h} \otimes {\rho_2}_Y^{g,h}} \\
\rho^g(\rho^{s\triangleright h}(X) \otimes \rho^h(Y)) & \xrightarrow[\ \gamma^g_{\rho^{s\triangleright h}(X), \rho^h(Y)}\ ]{} & \rho^{(s\triangleleft h)\triangleright g}\rho^{s\triangleright h}(X) \otimes \rho^g\rho^h(Y)
\end{array}$$

(d) For all $g, h \in G$,

$$\begin{array}{ccc}
\rho^g\rho^h(\mathbb{1}) & \xrightarrow{\ (\rho_2^{g,h})_{\mathbb{1}}\ } & \rho^{hg}(\mathbb{1}) \\
\Big\downarrow{\scriptstyle \rho^g(\gamma_0^h)} & & \Big\downarrow{\scriptstyle \gamma_0^{hg}} \\
\rho^g(\mathbb{1}) & \xrightarrow[\ \gamma_0^g\ ]{} & \mathbb{1}
\end{array}$$

(e) For all $X \in \mathcal{C}$, $s \in \Gamma$, $Y \in \mathcal{C}_s$,

$$\begin{array}{ccc}
X \otimes Y & \xrightarrow{\ \rho_{0\,X\otimes Y}\ } & \rho^e(X \otimes Y) \\
 & \searrow{\scriptstyle \rho_{0\,X} \otimes \rho_{0\,Y}} & \Big\downarrow{\scriptstyle \gamma^e_{X,Y}} \\
 & & \rho^e(X) \otimes \rho^e(Y)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{1} & \xrightarrow{\ \rho_{0\,\mathbb{1}}\ } & \rho^e(X \otimes Y) \\
 & \searrow{\scriptstyle =} & \Big\downarrow{\scriptstyle \gamma_0^e} \\
 & & \mathbb{1}
\end{array}$$

We shall say that $\mathcal{C}$ is a (G, Γ)-*crossed tensor category* if it is endowed with a (G, Γ)-crossed action.

REMARK 4.2. Recall that $s \triangleright e = e$, for all $s \in \Gamma$. Thus conditions (a) and (b) in the definition of a (G, Γ)-crossed tensor category imply that $\rho^e : \mathcal{C} \to \mathcal{C}$ is a monoidal functor with monoidal structure $\gamma^e_{X,Y} : \rho^e(X \otimes Y) \to \rho^e(X) \otimes \rho^e(Y)$, $X, Y \in \mathcal{C}$, and $\gamma_0^e : \rho^e(\mathbb{1}) \to \mathbb{1}$.

Commutativity of the diagrams in condition (e) amounts to the requirement that the natural isomorphism $\rho_0 : \mathrm{id}_{\mathcal{C}} \to \rho^e$ is a monoidal isomorphism.

REMARK 4.3. Suppose that G and Γ are groups endowed with mutual actions by permutations $\Gamma \xleftarrow{\triangleleft} \Gamma \times G \xrightarrow{\triangleright} G$. Let $\mathcal{C} = \bigoplus_{s\in\Gamma} \mathcal{C}_s$ be a Γ-graded tensor category over k and let $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}(\mathcal{C})$ be a right action of G on $\mathcal{C}$ by k-linear autoequivalences satisfying (4.1), and such that there exists a collection of natural isomorphisms $\gamma = (\gamma^g)_{g\in G}$ as in (4.2), satisfying condition (c).

LEMMA 4.4. *Assume that the Γ-grading on $\mathcal{C}$ is faithful and that the action $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}(\mathcal{C})$ is faithful, that is, $\rho^g \cong \rho^e$ if and only if $g = e$. Then the actions $\triangleleft$, $\triangleright$ make (G, Γ) into a matched pair of groups.*

Note that the faithfulness of ρ holds for instance if the action $\triangleleft : \Gamma \times G \longrightarrow \Gamma$ is faithful: in fact, if $g \in G$ is such that $\rho^g \cong \rho^e$ then, since the Γ-grading is faithful, we get from (4.1) that $s \triangleleft g = s$, for all $s \in \Gamma$. Hence $g = e$.

PROOF. Let $s, t \in \Gamma$, $g \in G$. For all objects $X \in \mathcal{C}_t$, $Y \in \mathcal{C}_s$, we have $X \otimes Y \in \mathcal{C}_{ts}$. Then, by (4.1), $\rho^g(X \otimes Y) \in \mathcal{C}_{ts \triangleleft g}$. On the other hand, under the isomorphism γ^g,

$$\rho^g(X \otimes Y) \cong \rho^{s \triangleright g}(X) \otimes \rho^g(Y) \in \mathcal{C}_{t \triangleleft (s \triangleright g)} \otimes \mathcal{C}_{s \triangleleft g} \subseteq \mathcal{C}_{(t \triangleleft (s \triangleright g))(s \triangleleft g)} .$$

Since, by assumption, the Γ-grading on $\mathcal{C}$ is faithful, we may take X and Y to be nonzero objects. Hence we obtain

$$ts \triangleleft g = (t \triangleleft (s \triangleright g))(s \triangleleft g) , \qquad \text{for all } s, t \in \Gamma, \quad g \in G .$$

Let now $g, h \in G$, $s \in \Gamma$, and let X, Y be objects of $\mathcal{C}$ such that $Y \in \mathcal{C}_s$ and $Y \neq 0$. The right hand side of (c) defines a natural isomorphism $\rho^g \rho^h(X \otimes Y) \to \rho^{s \triangleright hg}(X) \otimes \rho^{hg}(Y)$. On the other hand, the left hand side of (c) defines a natural isomorphism

$$\rho^g \rho^h(X \otimes Y) \to \rho^{(s \triangleright h)((s \triangleleft h) \triangleright g)}(X) \otimes \rho^{hg}(Y) .$$

Since the tensor product of $\mathcal{C}$ is a faithful functor in each variable, we get a natural isomorphism $\rho^{s \triangleright hg} \cong \rho^{(s \triangleright h)((s \triangleleft h) \triangleright g)}$. Because of faithfulness of the action ρ, we obtain

$$s \triangleright hg = (s \triangleright h)((s \triangleleft h) \triangleright g) ,$$

for all $s \in \Gamma$, $g, h \in G$. Therefore (G, Γ) is matched pair of groups, as claimed. □

5. The category $\mathcal{C}^{(G,\Gamma)}$. Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category. Since the group G acts on $\mathcal{C}$ by k-linear autoequivalences, we may consider the equivariantization $\mathcal{C}^G$, which is a k-linear abelian category.

Let (X, r) be an equivariant object. That is, $r^g : \rho^g(X) \to X$ are isomorphisms, for all $g \in G$, satisfying the relations (3.3). Let $X \cong \bigoplus_{s \in \Gamma} X_s$ be a decomposition of X as a direct sum of homogeneous objects $X_s \in \mathcal{C}_s$, $s \in \Gamma$.

Condition (4.1) implies that, for all $g \in G$, $s \in \Gamma$, r^g induces by restriction an isomorphism $r_s^g : \rho^g(X_s) \to X_{s \triangleleft g}$.

THEOREM 5.1. *Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category. Then the equivariantization of $\mathcal{C}$ under the action ρ is a tensor category over k with tensor product defined as follows:*

$$(X, r) \otimes (Y, r') = (X \otimes Y, \tilde{r}), \tag{5.1}$$

and unit object $(\mathbb{1}, (\rho_{\mathbb{1}}^g)_{g \in G})$, where, for all $g \in G$, $\tilde{r}^g$ is defined as the composition

$$\bigoplus_{s \in \Gamma} \rho^g(X \otimes Y_s) \overset{\oplus_s \gamma^g_{X,Y_s}}{\longrightarrow} \bigoplus_{s \in \Gamma} \rho^{s \triangleright g}(X) \otimes \rho^g(Y_s) \overset{\oplus_s r^{s \triangleright g} \otimes r'^g_s}{\longrightarrow} \bigoplus_{s \in \Gamma} X \otimes Y_{s \triangleleft g} = X \otimes Y ,$$

for $Y = \oplus_{s \in \Gamma} Y_s$, $Y_s \in \mathcal{C}_s$.

Observe that the action of G on $\mathcal{C}$ is not necessarily by tensor autoequivalences. Therefore the equivariantization $\mathcal{C}^G$ is not a tensor category with the tensor product defined by formula (3.4). The tensor category in Theorem 5.1 will be indicated by $\mathcal{C}^{(G,\Gamma)}$ to emphasize this distinction.

PROOF. Consider the endofunctor $T = \bigoplus_{g\in G} \rho^g$ of $\mathcal{C}$ defined by the action of G. Then T is a k-linear exact faithful endofunctor of $\mathcal{C}$. Moreover, T is a monad on $\mathcal{C}$ with multiplication $\mu : T^2 \to T$ and unit $\eta : \mathrm{id}_{\mathcal{C}} \to T$ induced, respectively, by the morphisms $\rho_2^{g,h}$, $g, h \in G$, and ρ^e. See Subsection 3.2.

The natural isomorphisms

$$\gamma^g_{X,Y} : \rho^g(X \otimes Y) \to \rho^{s\triangleright g}(X) \otimes \rho^g(Y)\,, \quad g \in G,\ X \in \mathcal{C}, Y \in \mathcal{C}_s\,,$$

induce canonically a natural transformation

$$T_2(X,Y)\colon T(X\otimes Y) = \bigoplus_{g\in G} \rho^g(X\otimes Y) \to \bigoplus_{g,h\in G} \rho^g(X)\otimes\rho^h(Y) = T(X)\otimes T(Y)\,,$$

$X, Y \in \mathcal{C}$. Similarly, the morphisms $\gamma^g_0 : \rho^g(\mathbb{1}) \to \mathbb{1}$ induce a morphism

$$T_0 : T(\mathbb{1}) = \bigoplus_{g\in G} \rho^g(\mathbb{1}) \to \mathbb{1}\,.$$

Conditions (a) and (b) in Definition 4.1 imply that T is a comonoidal endofunctor of $\mathcal{C}$ with comonoidal structure given by T_2 and T_0. Conditions (c), (d) and (e) imply that $\mu : T^2 \to T$ and $\eta : \mathrm{id}_{\mathcal{C}} \to T$ are comonoidal transformations, that is, they satisfy the relations (2.2) and (2.3). Hence T is a bimonad on $\mathcal{C}$.

We claim that T is a Hopf monad on $\mathcal{C}$. This will entail that $\mathcal{C}^{(G,\Gamma)}$ is a tensor category with the prescribed structure since, by the definition of the tensor product of $\mathcal{C}^{(G,\Gamma)}$ given in (5.1), it coincides with the one given by formula (2.4) for the tensor product in the category $\mathcal{C}^T$ of T-modules in $\mathcal{C}$.

According to the results in [3, Section 2], to establish the claim it will be enough to show that the fusion operators H^l and H^r of T are isomorphisms. Recall that H^l and H^r are defined, respectively, by

$$H^l_{X,Y} = (\mathrm{id}_{T(X)} \otimes \mu_Y)\, T_2(X, T(Y)) : T(X\otimes T(Y)) \to T(X)\otimes T(Y)\,,$$

$$H^r_{X,Y} = (\mu_X \otimes \mathrm{id}_{T(Y)})\, T_2(T(X), Y) : T(T(X)\otimes Y) \to T(X)\otimes T(Y)\,.$$

Let X be any object of $\mathcal{C}$. For every homogenous object $Y \in \mathcal{C}_s$, $s \in \Gamma$, the operator

$$H^r_{X,Y} : \bigoplus_{g,h\in G} \rho^g(\rho^h(X)\otimes Y) \to \bigoplus_{g,h\in G} \rho^g(X)\otimes\rho^h(Y)\,,$$

is given componentwise by the composition of isomorphisms

$$(\rho_2^{s\triangleright g,h} \otimes \mathrm{id}_{\rho^g(Y)})\, \gamma^g_{\rho^h(X),Y} : \rho^g(\rho^h(X)\otimes Y) \to \rho^{h(s\triangleright g)}(X)\otimes\rho^g(Y)\,.$$

Since the map $G\times G \to G\times G$, $(g,h) \mapsto (h(s\triangleright g), g)$, is bijective for any $s \in \Gamma$, then $H^r_{X,Y}$ is an isomorphism. Therefore $H^r_{X,Y}$ is an isomorphism for all $Y \in \mathcal{C}$.

Similarly, if X is any object of $\mathcal{C}$ and $Y \in \mathcal{C}_s$ is a homogeneous object, $s \in \Gamma$, then

$$H^l_{X,Y} : \bigoplus_{g,h \in G} \rho^g(X \otimes \rho^h(Y)) \to \bigoplus_{g,h \in G} \rho^g(X) \otimes \rho^h(Y),$$

is given componentwise by the composition of isomorphisms

$$(\mathrm{id}_{\rho^{(s \lhd h) \rhd g}(X)} \otimes \rho_2^{g,h})\, \gamma^g_{X, \rho^h(Y)} : \rho^g(X \otimes \rho^h(Y)) \to \rho^{(s \lhd h) \rhd g}(X) \otimes \rho^{hg}(Y).$$

We conclude as before that $H^l_{X,Y}$ is an isomorphism for all $Y \in \mathcal{C}$. Indeed, to see that for each $s \in \Gamma$ the map $G \times G \to G \times G$, $(g, h) \mapsto ((s \lhd h) \rhd g, hg)$, is bijective, we argue as follows: the composition of this map with the bijection $\mathrm{id}_G \times (s \rhd -) : G \times G \to G \times G$ gives the map $(g, h) \mapsto ((s \lhd h) \rhd g, s \rhd hg)$. Using the compatibility condition in (1.1), the last map is bijective with inverse $(u, v) \mapsto ((s \lhd (s^{-1} \rhd vu^{-1}))^{-1} \rhd u, s^{-1} \rhd vu^{-1})$. Therefore T is a Hopf monad, and thus $\mathcal{C}^{(G,\Gamma)} = \mathcal{C}^T$ is a tensor category, as claimed. □

6. Main properties. In this section we study the structure of the tensor category $\mathcal{C}^{(G,\Gamma)}$ arising from a (G, Γ)-crossed tensor category for a general matched pair of finite groups (G, Γ).

THEOREM 6.1. *Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category and let $\mathcal{C}^{(G,\Gamma)}$ be the category defined by Theorem* 5.1. *Then the forgetful functor $F : \mathcal{C}^{(G,\Gamma)} \to \mathcal{C}$, $F(X, r) = X$, gives rise to a perfect exact sequence of tensor categories*

$$\mathrm{Rep}\, G \longrightarrow \mathcal{C}^{(G,\Gamma)} \xrightarrow{F} \mathcal{C}, \tag{6.1}$$

with induced Hopf algebra $H \cong k^G$.

PROOF. By construction, $\mathcal{C}^{(G,\Gamma)} = \mathcal{C}^T$, where $T = \bigoplus_{g \in G} \rho^g$ is the Hopf monad associated to the (G, Γ)-crossed tensor category structure on $\mathcal{C}$. Moreover, the functor $F : \mathcal{C}^{(G,\Gamma)} \to \mathcal{C}$ coincides with the forgetful functor $\mathcal{C}^T \to \mathcal{C}$. Since T is a faithful exact endofunctor of $\mathcal{C}$, then the functor $F : \mathcal{C}^{(G,\Gamma)} \to \mathcal{C}$ is a dominant tensor functor [4, Lemma 2.1 and Proposition 2.2]. The left exactness of T implies that the functor F has an exact left adjoint; then F is a perfect tensor functor.

Furthermore, the isomorphisms γ^g_0, $g \in G$, induce an isomorphism $T(\mathbb{1}) = \bigoplus_{g \in G} \rho^g(\mathbb{1}) \cong \mathbb{1}^G$. Hence $T(\mathbb{1})$ is a trivial object of $\mathcal{C}$, and therefore T is a normal Hopf monad on $\mathcal{C}$. In view of [4, Theorem 4.8], the functor F induces an exact sequence of tensor categories

$$\text{comod-}H \longrightarrow \mathcal{C}^{(G,\Gamma)} \xrightarrow{F} \mathcal{C},$$

where H is the induced Hopf algebra of T, that is, H is the induced Hopf algebra of the restriction of T to the trivial subcategory of $\mathcal{C}$ [4, Remark 5.5]. As in the proof of [4, Theorem 5.21], the restriction of T to the trivial subcategory $\langle \mathbb{1} \rangle$ of $\mathcal{C}$ is isomorphic to the Hopf monad of the trivial action of G on $\langle \mathbb{1} \rangle$ and therefore $H \cong k^G$. Thus we obtain the perfect exact sequence (6.1). This finishes the proof of the theorem. □

We shall denote $\operatorname{Supp}\mathcal{C} \subseteq \Gamma$ the support of $\mathcal{C}$, that is,

$$\operatorname{Supp}\mathcal{C} = \{s \in \Gamma \,|\, \mathcal{C}_s \neq 0\}\,.$$

Since the functor $\otimes$ is faithful in each variable, $\operatorname{Supp}\mathcal{C}$ is a subgroup of Γ. Moreover, relation (4.1) implies that $\operatorname{Supp}\mathcal{C}$ is stable under the action $\lhd$ of G on Γ.

PROPOSITION 6.2. *Let $\mathcal{C}$ be a (G, Γ)-crossed fusion category and let $\mathcal{D} = \mathcal{C}_e$ be the neutral component of $\mathcal{C}$ with respect to the associated Γ-grading. Then we have:*

(i) *The category $\mathcal{C}^{(G,\Gamma)}$ is a finite tensor category if and only if $\mathcal{D}$ is a finite tensor category.*
(ii) *The category $\mathcal{C}^{(G,\Gamma)}$ is a fusion category if and only if $\mathcal{D}$ is a fusion category and the characteristic of k does not divide the order of G. If this is the case, then we have*

$$\operatorname{FPdim}\mathcal{C}^{(G,\Gamma)} = |G|\,|\operatorname{Supp}\mathcal{C}|\,\operatorname{FPdim}\mathcal{D}\,.$$

PROOF. (i) Assume that $\mathcal{C}^{(G,\Gamma)}$ is a finite tensor category. Since the forgetful functor $\mathcal{C}^{(G,\Gamma)} \to \mathcal{C}$ is a dominant tensor functor, then $\mathcal{C}$ is a finite tensor category and therefore so is $\mathcal{D}$.

Assume, on the other direction, that $\mathcal{D}$ is a finite tensor category. By Corollary 3.5, $\mathcal{C}$ is a finite tensor category. By construction $\mathcal{C}^{(G,\Gamma)} \cong \mathcal{C}^T$, where T is a faithful exact k-linear Hopf monad on $\mathcal{C}$. Then it follows from [18, Lemma 3.5] that $\mathcal{C}^{(G,\Gamma)}$ is a finite tensor category as well.

(ii) Assume that $\mathcal{D}$ is a fusion category and the characteristic of k does not divide the order of G. By Corollary 3.5, $\mathcal{C}$ is also a fusion category. In addition k^G is a cosemisimple Hopf algebra and therefore $\operatorname{Rep} G = \text{comod-}k^G$ is a fusion category too. It follows from [4, Corollary 4.16] that $\mathcal{C}^{(G,\Gamma)}$ is a fusion category.

Conversely, assume that $\mathcal{C}^{(G,\Gamma)}$ is a fusion category. Then, by part (i), $\mathcal{C}$ is a finite tensor category. Consider the forgetful functor $F : \mathcal{C}^{(G,\Gamma)} \to \mathcal{C}$. Since F is a dominant tensor functor, then it maps projective objects of $\mathcal{C}^{(G,\Gamma)}$ to projective objects of $\mathcal{C}$, by [10, Theorem 2.5]. Since $F(\mathbb{1}) \cong \mathbb{1}$, and $\mathbb{1}$ is a projective object of $\mathcal{C}^{(G,\Gamma)}$, then $\mathbb{1}$ is a projective object of $\mathcal{C}$ and hence $\mathcal{C}$ is a fusion category. Therefore so is its tensor subcategory $\mathcal{D}$.

In this case, it follows from [4, Proposition 4.10] that

$$\operatorname{FPdim}\mathcal{C}^{(G,\Gamma)} = \operatorname{FPdim}(\operatorname{Rep} G)\ \operatorname{FPdim}\mathcal{C} = |G|\,|\operatorname{Supp}\mathcal{C}|\,\operatorname{FPdim}\mathcal{D}\,,$$

the last equality because $\mathcal{C}$ is faithfully graded by $\operatorname{Supp}\mathcal{C}$ with neutral component $\mathcal{D}$; see [7, Proposition 8.20]. □

PROPOSITION 6.3. *Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category. Then the following statements are equivalent:*

(i) *The exact sequence* (6.1) *is an equivariantization exact sequence.*
(ii) *The action $\rhd : \operatorname{Supp}\mathcal{C} \times G \longrightarrow G$ is trivial.*
(iii) *The action $\lhd : \operatorname{Supp}\mathcal{C} \times G \longrightarrow \operatorname{Supp}\mathcal{C}$ is by group automorphisms.*

In Subsection 8.1 we shall further discuss (G, Γ)-crossed tensor categories satisfying the equivalent conditions in this proposition.

PROOF. Since $\operatorname{Supp}\mathcal{C}$ is a G-stable subgroup of Γ, then $(\operatorname{Supp}\mathcal{C}, G)$ is a matched pair by restriction. As pointed out in Subsection 2.1, the action $\triangleright$ is trivial if and only if $\triangleleft$ is an action by group automorphisms. Then (ii) and (iii) are equivalent.

Suppose that the action $\triangleright : \Gamma \times G \longrightarrow G$ is trivial. Conditions (a) and (b) imply that, for all $g \in G$, ρ^g is a tensor functor with tensor structure determined by γ_0^g and the isomorphisms γ^g in (4.2). Moreover, condition (c) becomes in this case

$$((\rho_2^{g,h})_X \otimes (\rho_2^{g,h})_Y)\, \gamma^g_{\rho^h(X),\rho^h(Y)}\, \rho^g(\gamma^h_{X,Y}) = \gamma^{hg}_{X,Y}\, (\rho_2^{g,h})_{X\otimes Y}\,, \tag{6.2}$$

for all $g, h \in G$ and for all $Y \in \mathcal{C}$. Combining this with condition (d), we obtain that $\rho_2^{g,h} : \rho^g\rho^h \to \rho^{hg}$ are isomorphisms of tensor functors. Therefore ρ is an action by tensor autoequivalences. Furthermore, the definition of tensor product in Theorem 5.1 reduces in this case to the usual tensor product (3.4) in the equivariantization $\mathcal{C}^G$. Hence (ii) implies (i).

Suppose that the exact sequence $\operatorname{Rep} G \longrightarrow \mathcal{C}^{(G,\Gamma)} \longrightarrow \mathcal{C}$ is an equivariantization exact sequence. Then, by [4, Theorem 5.21], the normal Hopf monad $T = \bigoplus_{g\in G}\rho^g$ is cocommutative, that is, for every morphism $f : T(\mathbb{1}) \to \mathbb{1}$ and for every object $X \in \mathcal{C}$, we have

$$(\mathrm{id}_{T(X)} \otimes f)\, T_2(X, \mathbb{1}) = (f \otimes \mathrm{id}_{T(X)})\, T_2(\mathbb{1}, X) : T(X) \to T(X)\,. \tag{6.3}$$

Let $s \in \Gamma$, $g \in G$. Restricting both morphisms of (6.3) to $\rho^g(X) \subseteq T(X)$, $X \in \mathcal{C}_s$, we get the commutativity of the following diagram:

$$\begin{array}{ccc} \rho^g(X) & \xrightarrow{\gamma^g_{X,\mathbb{1}}} & \rho^g(X)\otimes\rho^g(\mathbb{1}) \\ {\scriptstyle \gamma^g_{\mathbb{1},X}}\downarrow & & \downarrow{\scriptstyle \mathrm{id}\otimes f|_{\rho^g(\mathbb{1})}} \\ \rho^{s\triangleright g}(\mathbb{1})\otimes\rho^g(X) & \xrightarrow[f|_{\rho^{s\triangleright g}(\mathbb{1})}\otimes\mathrm{id}]{} & \rho^g(X)\,, \end{array}$$

for all $g \in G$ and for all morphisms $f : T(\mathbb{1}) \to \mathbb{1}$.

We may apply this to the morphism $f = \gamma_0^g \pi_g$, where π_g is the canonical projection $\pi_g : T(\mathbb{1}) = \bigoplus_{h\in G}\rho^h(\mathbb{1}) \to \rho^g(\mathbb{1})$. If $s \triangleright g \neq g$, then $f|_{\rho^{s\triangleright g}(\mathbb{1})} = 0$.

On the other hand, $(\mathrm{id}\otimes f)\,\gamma^g_{X,\mathbb{1}} = \mathrm{id}_{\rho^g(X)} : \rho^g(X) \to \rho^g(X)$, by condition (b).

Hence, if $s \in \operatorname{Supp}\mathcal{C}$, we may choose $0 \neq X \in \mathcal{C}_s$, and thus we obtain $s \triangleright g = g$. This shows that (i) implies (ii) and finishes the proof of the proposition. □

Observe that if (G, Γ) is any matched pair, where $\Gamma = \mathbb{Z}_2$ is the cyclic group of order 2, then the action $\triangleleft : \Gamma \times G \to \Gamma$ is necessarily trivial. As a consequence of Proposition 6.3 we obtain the following:

COROLLARY 6.4. *Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category, where $\Gamma \cong \mathbb{Z}_2$. Then the exact sequence* $\operatorname{Rep} G \longrightarrow \mathcal{C}^{(G,\Gamma)} \longrightarrow \mathcal{C}$ *is an equivariantization exact sequence.* □

Suppose that $\tilde{\Gamma}$ is a subgroup of Γ. Then the subcategory $\mathcal{C}_{\tilde{\Gamma}} = \bigoplus_{s\in\tilde{\Gamma}} \mathcal{C}_s$ is a tensor subcategory of $\mathcal{C}$.

PROPOSITION 6.5. *Let $\tilde{\Gamma}$ be a subgroup of Γ stable under the action $\triangleleft$ of G. Then the actions $\triangleright$ and $\triangleleft$ induce by restriction a matched pair $(G, \tilde{\Gamma})$. The category $\mathcal{C}_{\tilde{\Gamma}}$ is a $(G, \tilde{\Gamma})$-crossed tensor category by restriction and there is a strict embedding of tensor categories $\mathcal{C}_{\tilde{\Gamma}}^{(G,\tilde{\Gamma})} \to \mathcal{C}^{(G,\Gamma)}$.*

PROOF. Since $\tilde{\Gamma}$ is stable under the action $\triangleleft$, it is clear that $(G, \tilde{\Gamma})$ is a matched pair. Condition (4.1) implies that $\mathcal{C}_{\tilde{\Gamma}}$ is stable under the action ρ. It is immediate that the natural $\tilde{\Gamma}$-grading and the restriction of ρ make $\mathcal{C}_{\tilde{\Gamma}}$ into a $(G, \tilde{\Gamma})$-crossed tensor category. Finally, the embedding $\mathcal{C}_{\tilde{\Gamma}} \to \mathcal{C}$ induces a strict embedding of tensor categories $\mathcal{C}_{\tilde{\Gamma}}^{(G,\tilde{\Gamma})} \to \mathcal{C}^{(G,\Gamma)}$. □

REMARK 6.6. It follows from Definition 4.1 that the neutral homogeneous component $\mathcal{C}_e$ of $\mathcal{C}$ is a G-stable tensor subcategory. Furthermore, the action of G on $\mathcal{C}$ restricts to an action of G on $\mathcal{C}_e$ by tensor autoequivalences. Therefore $F^{-1}(\mathcal{C}_e) \subseteq \mathcal{C}^{(G,\Gamma)}$ is a tensor subcategory containing $\operatorname{Rep} G$, and in fact $F^{-1}(\mathcal{C}_e) \cong \mathcal{C}_e^G$ is an equivariantization tensor category.

More generally, let $\overline{\Gamma} \subseteq \Gamma$ be the subgroup defined as

$$\overline{\Gamma} = \{s \in \Gamma \,|\, s \triangleright g = g,\ \forall g \in G\}.$$

Let $\mathcal{C}_{\overline{\Gamma}}$ be the tensor subcategory of $\mathcal{C}$ corresponding to the subgroup $\overline{\Gamma}$, that is, $\mathcal{C}_{\overline{\Gamma}} = \bigoplus_{s\in\overline{\Gamma}} \mathcal{C}_s$.

It follows from the relations (1.1) that $\overline{\Gamma}$ is a G-stable subgroup of Γ. Let $\mathcal{C}_{\overline{\Gamma}}^{(G,\overline{\Gamma})}$ be the tensor subcategory in Proposition 6.5. Since $\overline{\Gamma}$ acts trivially on G, Proposition 6.3 gives us:

COROLLARY 6.7. *Then the induced exact sequence* $\operatorname{Rep} G \longrightarrow \mathcal{C}_{\overline{\Gamma}}^{(G,\overline{\Gamma})} \longrightarrow \mathcal{C}_{\overline{\Gamma}}$ *is an equivariantization exact sequence.* □

The following proposition and its corollary are dual to Proposition 6.3 and Corollary 6.7.

PROPOSITION 6.8. *Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category. Then the following statements are equivalent:*

(i) *The category $\mathcal{C}^{(G,\Gamma)}$ admits a Γ-grading such that the forgetful functor $F : \mathcal{C}^{(G,\Gamma)} \to \mathcal{C}$ is a Γ-graded tensor functor.*
(ii) *The action $\triangleleft : \operatorname{Supp}\mathcal{C} \times G \longrightarrow \operatorname{Supp}\mathcal{C}$ is trivial.*
(iii) *The action $\triangleright : \operatorname{Supp}\mathcal{C} \times G \longrightarrow G$ is by group automorphisms.*

PROOF. The equivalence of (ii) and (iii) follows from relations (1.1). We shall show that (i) is equivalent to (ii).

Assume (ii). For every $s \in \Gamma$, let $\mathcal{C}_s^{(G,\Gamma)}$ denote the full subcategory of $\mathcal{C}^{(G,\Gamma)}$ of all objects $(X, r) \in \mathcal{C}^{(G,\Gamma)}$ such that $X \in \mathcal{C}_s$. If $(X, r) \in \mathcal{C}_s^{(G,\Gamma)}$ and $(X', r') \in \mathcal{C}_t^{(G,\Gamma)}$, $s, t \in \Gamma$, then $(X, r)\otimes(X', r') = (X\otimes X', r'')$ is an object of $\mathcal{C}_{st}^{(G,\Gamma)}$, because $\mathcal{C}_s\otimes\mathcal{C}_t \subseteq \mathcal{C}_{st}$. In addition, if $s \neq t$, then $\operatorname{Hom}_{\mathcal{C}}(X, X') = 0$ and therefore we obtain $\operatorname{Hom}_{\mathcal{C}^{(G,\Gamma)}}((X, r), (X', r')) = 0$.

Let now (X, r) be any object of $\mathcal{C}^{(G,\Gamma)}$. Then, for all $g \in G$, $r^g : \rho^g(X) \to X$ is an isomorphism in $\mathcal{C}$. We have a decomposition $X \cong \bigoplus_{s\in\Gamma} X_s$, where $X_s \in \mathcal{C}_s$, for all $s \in \Gamma$. In view of condition (4.1), r^g induces by restriction an isomorphism $r_s^g : \rho^g(X_s) \to X_s$, for all $g \in G$, $s \in \Gamma$, because the action $\lhd$ of G on $\operatorname{Supp}\mathcal{C}$ is trivial by assumption. Moreover, (X_s, r_s) is an object of $\mathcal{C}^{(G,\Gamma)}$, where $r_s = \{r_s^g\}_{g\in G}$ is the restriction of r to X_s, and thus $(X, r) \cong \bigoplus_{s\in\Gamma}(X_s, r_s)$ is a decomposition of (X, r) into a direct sum of objects $(X_s, r_s) \in \mathcal{C}_s^{(G,\Gamma)}$. This shows that $\mathcal{C}^{(G,\Gamma)} = \bigoplus_{s\in\Gamma} \mathcal{C}_s^{(G,\Gamma)}$ is a Γ-grading in $\mathcal{C}^{(G,\Gamma)}$. Moreover, for all $(X, r) \in \mathcal{C}_s^{(G,\Gamma)}$, $s \in \Gamma$, we have $F(X, r) = X \in \mathcal{C}_s$, that is, the functor F is a Γ-graded tensor functor. Then we get (i).

Conversely, assume that (i) holds. Let $s \in \operatorname{Supp}\mathcal{C}$ and let $0 \neq Y \in \mathcal{C}_s$. Since F is a dominant Γ-graded tensor functor, there exists $(X, r) \in \mathcal{C}_s^{(G,\Gamma)}$ such that $Y \subseteq F(X, r) = X$ and $X \in \mathcal{C}_s$ (see Lemma 3.2). In particular, $X \neq 0$ and for all $g \in G$, $r^g : \rho^g(X) \to X$ is an isomorphism in $\mathcal{C}$. It follows from condition (4.1) that $s \lhd g = s$, for all $g \in G$. Since $s \in \operatorname{Supp}\mathcal{C}$ was arbitrary, we get (ii). This shows that (i) and (ii) are equivalent and finishes the proof of the proposition. □

REMARK 6.9. The proof of (i) ⇒ (ii) in Proposition 6.8 shows that in fact, if (X, r) is an object of $\mathcal{C}^{(G,\Gamma)}$ such that X is a nonzero homogeneous object of $\mathcal{C}$, then the homogeneous degree of X is a fixed point of Γ under the action of G.

REMARK 6.10. Suppose that the action $\lhd : \operatorname{Supp}\mathcal{C} \times G \longrightarrow \operatorname{Supp}\mathcal{C}$ is trivial. Consider the Γ-grading of $\mathcal{C}^{G,\Gamma}$ given by Proposition 6.8. Observe that the neutral component $\mathcal{C}_e^{G,\Gamma}$ of this grading is the category $F^{-1}(\mathcal{C}_e)$. Therefore $\mathcal{C}_e^{G,\Gamma} \cong \mathcal{D}^G$ is an equivariantization tensor category with respect to the restriction of the action ρ to the tensor subcategory $\mathcal{D} = \mathcal{C}_e$. See Remark 6.6.

Consider the trivial Γ-grading on $\operatorname{Rep} G$. Let us also quote that, in this context, the induced exact sequence $\operatorname{Rep} G \longrightarrow \mathcal{C}^{(G,\Gamma)} \longrightarrow \mathcal{C}$ is a *Γ-graded exact sequence*, that is, both tensor functors involved are Γ-graded tensor functors.

Let $\underline{\Gamma} \subseteq \Gamma$ be the set of fixed points of Γ under the action of G. Then $\underline{\Gamma}$ is a G-stable subgroup of Γ. Let $\mathcal{C}_{\underline{\Gamma}}^{G,\underline{\Gamma}}$ be the tensor subcategory of $\mathcal{C}^{G,\Gamma}$ given by Proposition 6.5. Since G acts trivially on the subgroup $\underline{\Gamma}$, Proposition 6.8 implies the following (c.f. Remark 6.10):

COROLLARY 6.11. *The tensor subcategory $\mathcal{C}_{\underline{\Gamma}}^{G,\underline{\Gamma}}$ is a $\underline{\Gamma}$-graded tensor category with neutral component $\mathcal{D}^G$, and with respect to the trivial $\underline{\Gamma}$-grading on* $\operatorname{Rep} G$*, the induced exact sequence* $\operatorname{Rep} G \longrightarrow \mathcal{C}_{\underline{\Gamma}}^{(G,\underline{\Gamma})} \longrightarrow \mathcal{C}_{\underline{\Gamma}}$ *is a $\underline{\Gamma}$-graded exact sequence.*

REMARK 6.12. Suppose that the neutral component $\mathcal{D} = \mathcal{C}_e$ of $\mathcal{C}$ is a fusion category. Then $\mathcal{C}^{(G,\Gamma)}$ is also a fusion category, by Proposition 6.2. Corollaries 6.7 and 6.11 imply that the fusion subcategories $\mathcal{C}_{\overline{\Gamma}}^{(G,\overline{\Gamma})}$ and $\mathcal{C}_{\underline{\Gamma}}^{(G,\underline{\Gamma})}$ are, respectively, a G-equivariantization of a group extension of $\mathcal{D}$ and a group extension of a G-equivariantization of $\mathcal{D}$. In particular, it follows from [8, Proposition 4.1] that if $\mathcal{D}$ is weakly group-theoretical, then so are the fusion subcategories $\mathcal{C}_{\overline{\Gamma}}^{(G,\overline{\Gamma})}$ and $\mathcal{C}_{\underline{\Gamma}}^{(G,\underline{\Gamma})}$.

7. **$(G, \varGamma)$-crossed braidings.** Let $(G, \varGamma)$ be a matched pair of finite groups and let $\mathcal{C}$ be $(G, \varGamma)$-crossed tensor category. We keep the notation in Section 4.

DEFINITION 7.1. A $(G, \varGamma)$-*crossed braiding* in $\mathcal{C}$ is a triple (c, φ, ψ), where

- $\varphi, \psi : \varGamma \to G$ are group homomorphisms, satisfying the following conditions, for all $s, t \in \varGamma$, $g \in G$:

$$(t^{-1} \triangleleft \varphi(s^{-1}))st = s \triangleleft \psi(t)\,, \tag{7.1}$$

$$(t \triangleright \psi(s))^{-1} = \psi(s^{-1} \triangleleft \varphi(t^{-1}))\,, \tag{7.2}$$

$$(t^{-1} \triangleright \varphi(s^{-1})))^{-1} = \varphi(s \triangleleft \psi(t))\,, \tag{7.3}$$

$$\psi(t)g = (t \triangleright g)\psi(t \triangleleft g)\,, \tag{7.4}$$

$$g\varphi(s \triangleleft g)^{-1} = \varphi(s^{-1})(s \triangleright g)\,. \tag{7.5}$$

- c is a collection of natural isomorphisms

$$c_{X,Y} : X \otimes Y \to \rho^{t^{-1} \triangleright \varphi(s^{-1})}(Y) \otimes \rho^{\psi(t)}(X)\,, \qquad X \in \mathcal{C}_s,\ Y \in \mathcal{C}_t\,. \tag{7.6}$$

For every $s, t \in \varGamma$, let $s \prec t$ and $t \succ s$ be the elements of $\varGamma$ defined, respectively, by

$$s \prec t = t^{-1} \triangleright \varphi(s^{-1})\,, \qquad t \succ s = s^{-1} \triangleleft \varphi(t^{-1})\,.$$

The isomorphisms $c_{X,Y}$ are subject to the commutativity of the following diagrams (when there is no ambiguity, we omit subscripts to denote morphisms):
(1) For all $g \in G$, $s, t \in \varGamma$, $X \in \mathcal{C}_s$, $Y \in \mathcal{C}_t$,

$$\begin{array}{ccc}
\rho^g(X \otimes Y) & \xrightarrow{\gamma^g} & \rho^{t \triangleright g}(X) \otimes \rho^g(Y) \\
\Big\downarrow{\scriptstyle \rho^g(c)} & & \Big\downarrow{\scriptstyle c} \\
\rho^g(\rho^{s \prec t}(Y) \otimes \rho^{\psi(t)}(X)) & & \rho^{(s \triangleleft (t \triangleright g)) \prec (t \triangleleft g)} \rho^g(Y) \otimes \rho^{\psi(t \triangleleft g)} \rho^{t \triangleright g}(X) \\
\Big\downarrow{\scriptstyle \gamma^g} & & \Big\downarrow{\scriptstyle \rho_2 \otimes \rho_2} \\
\rho^{(s \triangleleft \psi(t)) \triangleright g} \rho^{s \prec t}(Y) \otimes \rho^g \rho^{\psi(t)}(X) & \xrightarrow[\rho_2 \otimes \rho_2]{} & \rho^{(s \prec t)((s \triangleleft \psi(t)) \triangleright g)}(Y) \otimes \rho^{\psi(t) g}(X)
\end{array}$$

(2) For all $s, t, u \in \varGamma$, $X \in \mathcal{C}_s$, $Y \in \mathcal{C}_t$, $Z \in \mathcal{C}_u$,

$$
\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{c_{X\otimes Y,Z}} & \rho^{st\prec u}(Z) \otimes \rho^{\psi(u)}(X \otimes Y) \\
\Big\downarrow {\scriptstyle \mathrm{id}\otimes c} & & \Big\downarrow {\scriptstyle \mathrm{id}\otimes \gamma^{\psi(u)}} \\
 & & \rho^{st\prec u}(Z) \otimes \rho^{t\rhd\psi(u)}(X) \otimes \rho^{\psi(u)}(Y) \\
 & & \Big\uparrow {\scriptstyle \rho_2\otimes \mathrm{id}\otimes \mathrm{id}} \\
X \otimes \rho^{t\prec u}(Z) \otimes \rho^{\psi(u)}(Y) & \xrightarrow[c\otimes \mathrm{id}]{} & \rho^{s\prec(t\succ u)^{-1}}\rho^{t\prec u}(Z) \otimes \rho^{\psi(t\succ u)^{-1}}(X) \otimes \rho^{\psi(u)}(Y)
\end{array}
$$

(3) For all $s, t, u \in \Gamma$, $X \in \mathcal{C}_s$, $Y \in \mathcal{C}_t$, $Z \in \mathcal{C}_u$,

$$
\begin{array}{ccc}
X \otimes Y \otimes Z & \xrightarrow{c_{X,Y\otimes Z}} & \rho^{s\prec tu}(Y \otimes Z) \otimes \rho^{\psi(tu)}(X) \\
\Big\downarrow {\scriptstyle c\otimes \mathrm{id}} & & \Big\downarrow {\scriptstyle \gamma^{(tu)^{-1}\rhd s}\otimes \mathrm{id}} \\
 & & \rho^{s\prec t}(Y) \otimes \rho^{s\prec tu}(Z) \otimes \rho^{\psi(tu)}(X) \\
 & & \Big\uparrow {\scriptstyle \mathrm{id}\otimes \mathrm{id}\otimes \rho_2} \\
\rho^{s\prec t}(Y) \otimes \rho^{\psi(t)}(X) \otimes Z & \xrightarrow[\mathrm{id}\otimes c]{} & \rho^{s\prec t}(Y) \otimes \rho^{(s\lhd\psi(t))\prec u}(Z) \otimes \rho^{\psi(u)}\rho^{\psi(t)}(X)
\end{array}
$$

REMARK 7.2. Let $\mathcal{C}$ be a (G, Γ)-crossed tensor category and let φ and $\psi : \Gamma \to G$ be maps. Assume in addition that the Γ-grading on $\mathcal{C}$ is faithful and the G-action is faithful. Conditions (1), (2) and (3) in Definition 7.1 on the natural isomorphism c imply that the maps φ and ψ are group homomorphisms and that they satisfy the relations (7.1)– (7.5). This can be shown with an argument similar to that in Remark 4.3.

For instance, the relations (1.1) imply that $(t^{-1} \lhd g)^{-1} = (t \lhd (t^{-1} \rhd g))$, for all $t \in \Gamma$, $g \in G$. The existence of an isomorphism like in (7.6) makes it necessary that condition (7.1) in Definition 7.1 holds, when the Γ-grading on $\mathcal{C}$ is faithful.

REMARK 7.3. Let $\mathcal{C}$ be (G, Γ)-crossed tensor category and suppose (c, φ, ψ) is a (G, Γ)-braiding in $\mathcal{C}$. Conditions (2) and (3) in Definition 7.1 imply that the neutral homogeneous component $\mathcal{D} = \mathcal{C}_e$ of $\mathcal{C}$ is a braided tensor category with braiding induced by the restriction of the natural isomorphism c.

7.1. Crossed braidings and the set-theoretical QYBE. Let (G, Γ) be a matched pair of groups and let $G \bowtie \Gamma$ be the associated group (see Subsection 2.1). We shall identify G and Γ with subgroups of $G \bowtie \Gamma = G \times \Gamma$ in the natural way. Thus $G \bowtie \Gamma$ is endowed with an exact factorization into its subgroups G and Γ.

The exact factorization in $G \bowtie \Gamma$ induces actions of G and Γ on each other, denoted ${}^s g$, s^g, ${}^g s$, g^s, $s \in \Gamma$, $g \in G$, which are uniquely determined by the relations

$$sg = {}^s g\, s^g, \qquad gs = {}^g s\, g^s, \tag{7.7}$$

in $G \bowtie \Gamma$. See [15, Section 2].

From the definition of the group $G \bowtie \Gamma$, we obtain the following relations:

$$ {}^{s}g = s \triangleright g\,, \qquad s^{g} = s \triangleleft g\,, \qquad {}^{g}s = (s^{-1} \triangleleft g^{-1})^{-1}\,, \qquad g^{s} = (s^{-1} \triangleright g^{-1})^{-1}\,, $$

for all $g \in G$, $s \in \Gamma$.

Let $\varphi, \psi : \Gamma \to G$ be group homomorphisms. The conditions (7.1)– (7.5) in Definition 7.1 are equivalent, respectively, to the following conditions:

$$ st = {}^{\psi(s)}t\, s^{\varphi(t)}\,, \tag{7.8} $$

$$ \psi(s)^{t} = \psi(s^{\varphi(t)})\,, \tag{7.9} $$

$$ {}^{s}\varphi(t) = \varphi({}^{\psi(s)}t)\,, \tag{7.10} $$

$$ \psi({}^{g}t)g^{t} = \psi(t)g\,, \tag{7.11} $$

$$ \varphi({}^{g}s)g^{s} = g\varphi(s)\,, \tag{7.12} $$

for all $s, t \in \Gamma$, $g \in G$. Compare with [15, Proposition 1].

An alternative formulation for the conditions on the data $(G, \Gamma, \varphi, \psi)$, in terms of group actions by automorphisms and 1-cocycles, is explained in [15, Theorem 2].

REMARK 7.4. Consider the map $b_{\varphi,\psi} : \Gamma \times \Gamma \to \Gamma \times \Gamma$, given by

$$ b_{\varphi,\psi}(s,t) = ((t^{-1} \triangleleft \varphi(s^{-1}))^{-1}, s \triangleleft \psi(t))\,, \quad s, t \in \Gamma\,. $$

In terms of the actions (7.7), this map has the following expression:

$$ b_{\varphi,\psi}(s,t) = ({}^{\varphi(s)}t, s^{\psi(t)})\,, \qquad s, t \in \Gamma\,. $$

It turns out that $b_{\varphi,\psi}(s,t)$ coincides with the map $\mathcal{R}^{-1}(t,s)$, where $\mathcal{R} : \Gamma \times \Gamma \to \Gamma \times \Gamma$ is the (invertible) set-theoretical solution of the QYBE on the set Γ given in [14], corresponding to the actions ${}^{\varphi(s)}t$ and $s^{\psi(t)}$ of Γ on itself. The relevant condition for the result of [14] is (7.8) or, equivalently, (7.1). In particular the map $b_{\varphi,\psi}$ is bijective.

7.2. Braiding in the category $\mathcal{C}^{(G,\Gamma)}$. We next show that a (G, Γ)-crossed braiding in $\mathcal{C}$ induces a braiding in the associated tensor category $\mathcal{C}^{(G,\Gamma)}$.

THEOREM 7.5. *Let $\mathcal{C}$ be (G, Γ)-crossed tensor category and let (c, φ, ψ) be a (G, Γ)-braiding in $\mathcal{C}$. Then $\mathcal{C}^{(G,\Gamma)}$ is a braided tensor category with brading*

$$ c_{(X,r),(Y,l)} : (X, r) \otimes (Y, l) \to (Y, l) \otimes (X, r)\,, $$

defined componentwise by the isomorphisms

$$ (l^{t^{-1} \triangleright \varphi(s^{-1})} \otimes r^{\psi(t)})\, c_{X_s, Y_t} : X_s \otimes Y_t \to Y_{t \triangleleft (t^{-1} \triangleright \varphi(s^{-1}))} \otimes X_{s \triangleleft \psi(t)}\,, \tag{7.13} $$

where $X = \bigoplus_{s \in \Gamma} X_s$ and $Y = \bigoplus_{t \in \Gamma} Y_t$.

PROOF. Recall that $\mathcal{C}^{(G,\Gamma)} = \mathcal{C}^T$, where $T = \bigoplus_{g\in G} \rho^g$ is the Hopf monad in Theorem 5.1. The natural isomorphisms $c_{X,Y} : X \otimes Y \to \rho^{t^{-1}\triangleright\varphi(s^{-1})}(Y) \otimes \rho^{\psi(t)}(X)$, $X \in \mathcal{C}_s$, $Y \in \mathcal{C}_t$, induce canonically a natural transformation

$$R_{X,Y} : X \otimes Y \to \bigoplus_{g,h\in G} \rho^g(Y) \otimes \rho^h(X) .$$

The commutativity of the diagrams (1), (2) and (3) in Definition 7.5 imply, respectively, that the natural transformation R satisfies conditions (2.5), (2.6) and (2.7).

Let $(X, r), (Y, l) \in \mathcal{C}^{(G,\Gamma)}$. Then the natural transformation

$$R^{\#}_{(X,r),(Y,l)} = (l \otimes r)R_{X,Y} : X \otimes Y \to Y \otimes X,$$

is given componentwise by isomorphisms

$$(l^{t^{-1}\triangleright\varphi(s^{-1})} \otimes r^{\psi(t)})\, c_{X_s,Y_t} : X_s \otimes Y_t \to Y_{t\triangleleft(t^{-1}\triangleright\varphi(s^{-1}))} \otimes X_{s\triangleleft\psi(t)} ,$$

where $X = \bigoplus_{s\in\Gamma} X_s$ and $Y = \bigoplus_{t\in\Gamma} Y_t$. Recall that $t \triangleleft (t^{-1} \triangleright \varphi(s^{-1})) = (t^{-1} \triangleleft \varphi(s^{-1}))^{-1}$, for all $s, t \in \Gamma$. It was observed in Remark 7.4 that the map $b_{\varphi,\psi} : \Gamma \times \Gamma \to \Gamma \times \Gamma$, defined by $b_{\varphi,\psi}(s, t) = ((t^{-1} \triangleleft \varphi(s^{-1}))^{-1}, s \triangleleft \psi(t))$, is bijective. This implies that $R^{\#}$ is an isomorphism.

We have thus shown that T is a quasitriangular Hopf monad on $\mathcal{C}$. Therefore $\mathcal{C}^{(G,\Gamma)}$ is a braided tensor category with the braiding induced by the R-matrix R, which is easily seen to coincide with (7.13). This finishes the proof of the theorem. □

8. Some families of examples.

8.1. G-crossed categories. Let G be a finite group. Then there is a matched pair (G, Γ), where $\Gamma = G$, $\triangleleft : G \times \Gamma \to G$ is the trivial action and $\triangleright : G \times \Gamma \to \Gamma$ is the adjoint action.

A (G, G)-crossed tensor category is the same as a G-graded tensor category $\mathcal{C} = \bigoplus_{g\in G} \mathcal{C}_g$, endowed with a G-action by tensor autoequivalences $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}_{\otimes}(\mathcal{C})$ such that $\rho^g(\mathcal{C}_h) = \mathcal{C}_{g^{-1}hg}$, for all $g, h \in G$.

Thus, as a monoidal category, a (G, G)-crossed tensor category is a G-crossed category as defined in [24, Section 3.2]. See also [22, Chapter VI].

In this case the exact sequence of tensor categories given by Theorem 6.1,

$$\operatorname{Rep} G \to \mathcal{C}^{(G,G)} \to \mathcal{C} ,$$

is an equivariantization exact sequence; see Proposition 6.3.

REMARK 8.1. Consider a matched pair (G, Γ) such that the action $\triangleright$ is trivial or, equivalently, such that the action $\triangleleft$ is by group automorphisms. In this context, the notion of (G, Γ)-crossed fusion category is not new. In fact, any (G, Γ)-crossed fusion category associated to such a matched pair can be recovered from the G-crossed categories of [24].

This is due to the well-known fact that any action by group automorphisms can be recovered from an adjoint action, and can be formulated as follows.

Suppose that the action $\lhd : \Gamma \times G \to \Gamma$ is by group automorphisms. Let $S = \Gamma \rtimes G$ be the semidirect product associated to this action, so that the following relations hold in S:

$$g^{-1} s g = s \lhd g \,, \tag{8.1}$$

for all $s \in \Gamma$, $g \in G$.

Consider an S-crossed (tensor) category $\mathcal{C} = \bigoplus_{s \in S} \mathcal{C}_s$. Since Γ is a normal subgroup of S then the tensor subcategory $\mathcal{C}_\Gamma = \bigoplus_{s \in \Gamma} \mathcal{C}_s$ is stable under the adjoint action of S. Hence it is also stable under the adjoint action of G. Relation (8.1), together with the conditions defining a G-crossed category in [24, Subsection 3.2], imply that $\mathcal{C}_\Gamma$ is a (G, Γ)-crossed tensor category.

Conversely, suppose that $\mathcal{C} = \bigoplus_{s \in \Gamma} \mathcal{C}_s$ is a (G, Γ)-crossed tensor category. Condition (4.1) in Definition 4.1 implies that

$$\rho^g(\mathcal{C}_s) = \mathcal{C}_{g^{-1} s g} \,,$$

for all $s \in \Gamma$, $g \in G$, in view of (8.1).

The Γ-grading on $\mathcal{C}$ induces an S-grading $\mathcal{C} = \bigoplus_{s \in S} \mathcal{C}_s$, letting $\mathcal{C}_s := 0$, for all $s \in S \backslash \Gamma$.

Similarly, the action $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}(\mathcal{C})$ which, by Proposition 6.3, is in this case an action by tensor autoequivalences, induces an action by tensor autoequivalences $\tilde{\rho} : \underline{S}^{\mathrm{op}} \to \underline{\mathrm{Aut}}(\mathcal{C})$ in the form $\tilde{\rho}^s = \rho^{\overline{s}}$, for all $s \in S$, where $\overline{s} \in G$ denotes the image of s under the canonical projection $S \to G$.

The remaining conditions in Definition 4.1 imply that $\mathcal{C}$ becomes in this way an S-crossed category.

Recall that a G-braiding in a G-crossed category $\mathcal{C}$ is a collection of natural isomorphisms $\alpha_{X,Y} : X \otimes Y \to Y \otimes \rho^t(X)$, $Y \in \mathcal{C}_t$, called a *G-braiding*, satisfying appropriate compatibility conditions. See [22], [24, Subsection 3.3].

PROPOSITION 8.2. *Let $\mathcal{C}$ be a G-crossed tensor category. Then the following data are equivalent:*

(i) *A G-braiding c in $\mathcal{C}$.*
(ii) *A (G, G)-crossed braiding (c, φ, ψ) in $\mathcal{C}$, where $\psi = \mathrm{id}_G : G \to G$ and $\varphi : G \to G$ is the trivial group homomorphism.*

Note that the trivial homomorphism φ and the identity homomorphism ψ satisfy conditions (7.1)–(7.5) in Definition 7.1. The map $b_{\varphi,\psi}$ is given in this case by

$$b_{\varphi,\psi}(s, t) = (t, t^{-1} s t), \qquad s, t \in G \,.$$

PROOF. It is enough to observe that the commutativity of the diagrams (1)–(3) in Definition 7.1 in the case where ψ is the identity homomorphism and $\varphi : G \to G$ is the trivial group homomorphism, is equivalent to the commutativity of the diagrams in [24, Subsection 3.3]. □

REMARK 8.3. Let $\mathcal{C}$ be a G-braided tensor category regarded as a (G, G)-crossed tensor category. Suppose (c, φ, ψ) is any (G, G)-braiding in $\mathcal{C}$ where $\psi = \mathrm{id}_G$. It follows from conditions (7.1)–(7.5) that φ is a group homomorphism $\varphi : G \to Z(G)$.

8.2. Abelian exact sequences of Hopf algebras. Consider a matched pair of finite groups (G, Γ). Let also $\sigma : G \times G \to (k^*)^\Gamma$ and $\tau : \Gamma \times \Gamma \to (k^*)^G$ be normalized 2-cocycles, that is, using the notation $\sigma_s(g, h) = \sigma(g, h)(s)$ and $\tau_g(s, t) = \tau(s, t)(g)$, $s, t \in \Gamma$, $g, h \in G$, the following relations hold:

$$\sigma_{s \triangleleft g}(h, l)\sigma_s(g, hl) = \sigma_s(g, h)\sigma_s(gh, l)\,, \tag{8.2}$$

$$\sigma_s(e, g) = \sigma_s(g, e) = 1\,, \tag{8.3}$$

$$\tau_g(st, u)\tau_{u \triangleright g}(s, t) = \tau_g(t, u)\tau_g(s, tu)\,, \tag{8.4}$$

$$\tau_g(e, s) = \tau_g(s, e) = 1\,, \tag{8.5}$$

for all $g, h, l \in G$, $s, t, u \in \Gamma$.

Assume in addition that σ and τ satisfy the following compatibility conditions:

$$\sigma_{st}(g, h)\tau_{gh}(s, t) = \sigma_s(t \triangleright g\,,\, (t \triangleleft g) \triangleright h)\sigma_t(g, h)\tau_g(s, t)\tau_h(s \triangleleft (t \triangleright g), t \triangleleft g)\,, \tag{8.6}$$

$$\sigma_e(g, h) = \tau_e(s, t) = 1, \tag{8.7}$$

for all $s, t \in \Gamma$, $g, h \in G$.

Then the vector space $H = k^\Gamma \otimes kG$ becomes a Hopf algebra with the crossed product algebra structure and crossed coproduct coalgebra structure, denoted $H = k^\Gamma\, {}^\tau\#_\sigma kG$. The multiplication and comultiplication of H are defined, for all $g, h \in \Gamma$, $g, h \in G$, in the form

$$(e_s\#g)(e_t\#h) = \delta_{s \triangleleft g, h}\, \sigma_s(g, h) e_s\#gh\,, \tag{8.8}$$

$$\Delta(e_s\#g) = \sum_{tu=s} \tau_g(t, u)\, e_t\#(u \triangleright g) \otimes e_u\#g\,. \tag{8.9}$$

It is well-known that H is a semisimple Hopf algebra if and only if the characteristic of k does not divide the order of G.

Let $i = \mathrm{id} \otimes u : k^\Gamma \to H$ and $p = \varepsilon \otimes \mathrm{id} : H \to kG$ be the canonical Hopf algebra maps. Then we have an exact sequence of Hopf algebras

$$k \longrightarrow k^\Gamma \xrightarrow{\ i\ } H \xrightarrow{\ p\ } kG \longrightarrow k\,. \tag{8.10}$$

By [4, Proposition 3.9] this exact sequence gives rise to an exact sequence of tensor categories

$$\operatorname{Rep} G \xrightarrow{\ p^*\ } \operatorname{Rep} H \xrightarrow{\ i^*\ } \mathcal{C}(\Gamma)\,, \tag{8.11}$$

where $\mathcal{C}(\Gamma) = \operatorname{Rep} k^\Gamma$ is the category of finite dimensional Γ-graded vector spaces.

The category $\mathcal{C}(\Gamma)$ is a (G, Γ)-crossed fusion category with respect to the following data:

(a) The Γ-grading $\mathcal{C}(\Gamma) = \bigoplus_{s\in\Gamma} \mathcal{C}(\Gamma)_s$, where, for all $s \in \Gamma$, $\mathcal{C}(\Gamma)_s$ is the category of finite dimensional vector spaces of degree s.

(b) The action $\rho : \underline{G}^{\mathrm{op}} \to \underline{\mathrm{Aut}}(\mathcal{C}(\Gamma))$ is given by $\rho^g(V) = V$ with G-grading $\rho^g(V)_s = V_{s\triangleleft g}$.

The monoidal structure of ρ is given by $\rho_0 = \mathrm{id} : \rho^e \to \mathrm{id}_{\mathcal{C}(\Gamma)}$, and $\rho_2^{g,h} = \sigma(h, g)^{-1} : \rho^g\rho^h(V) \to \rho^{hg}(V)$, that is,

$$\rho_2^{g,h}(v) = \sigma_{|v|}(h, g)^{-1} v\,,$$

for every homogeneous element $v \in V$ of degree $|v|$.

(c) For all $U \in \mathcal{C}(\Gamma)$, $V \in \mathcal{C}(\Gamma)_s$, the natural isomorphisms $\gamma^g_{U,V} : \rho^g(U \otimes V) \to \rho^{s\triangleright g}(U) \otimes \rho^g(V)$, are given by

$$\gamma^g_{U,V}(u \otimes v) = \tau_g(|u|, s)\, u \otimes v\,,$$

on homogeneous elements $u \in U$ of degree $|u|$.

(d) The isomorphisms $\gamma^g_0 : \rho^g(k) = k \to k$ are identities, for all $g \in G$.

The next theorem relates the tensor category associated to the (G, Γ)-crossed tensor category $\mathcal{C}(\Gamma)$ with the Hopf algebra H.

THEOREM 8.4. *There is a strict equivalence of tensor categories*

$$\mathcal{C}(\Gamma)^{(G,\Gamma)} \cong \mathrm{Rep}\, H\,.$$

PROOF. Since $H = k^\Gamma\#_\sigma kG$ is a crossed product as an algebra, it follows from [20, Proposition 3.2] that ρ is an action by k-linear autoequivalences and there is an equivalence of k-linear categories $K : \mathrm{Rep}\, H \cong \mathcal{C}(\Gamma)^G = \mathcal{C}(\Gamma)^{(G,\Gamma)}$, where for all H-module W, $K(W) = (W|_{k^\Gamma}, g^{-1}|_W)$. The inverse equivalence maps an object (V, r) of $\mathcal{C}(\Gamma)^{(G,\Gamma)}$ to the vector space V endowed with the H-action $(e_s\#g).v = (r^g)^{-1}(v_s)$, $v \in V$.

It is straightforward to verify that K is a strict equivalence of tensor categories. This proves the theorem. □

REMARK 8.5. Consider the case where the exact sequence (8.10) is a split exact sequence. This corresponds to the situation where σ and τ are the trivial 2-cocycles.

Regard the category $\mathcal{C}(\Gamma)$ as a (G, Γ)-crossed tensor category as above. Suppose (c, φ, ψ) is a (G, Γ)-braiding in $\mathcal{C}(\Gamma)$. It follows from [15, Theorem 1] that the compatibility conditions between φ and ψ given in Definition 7.1 imply that pairs (φ, ψ), satisfying the compatibility conditions in Definition 7.1, are in bijective correspondence with positive quasitriangular structures in the Hopf algebra H. In fact, the conditions in [15, Theorem 1] are equivalent to the conditions (7.1)–(7.5), in view of [15, Proposition 1]. See Subsection 7.1.

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Facultad de Matemática
Astronomía y Física
Universidad Nacional de Córdoba
CIEM – CONICET
Ciudad Universitaria
(5000) Córdoba
Argentina

E-mail address: natale@famaf.unc.edu.ar
URL: http://www.famaf.unc.edu.ar/∼natale