

CROSSED PRODUCTS BY SEMIGROUPS OF ENDOMORPHISMS AND THE TOEPLITZ ALGEBRAS OF ORDERED GROUPS

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ABSTRACT. Let Γ^+ be the positive cone in a totally ordered abelian group Γ . We construct crossed products by actions of Γ^+ as endomorphisms of C^* -algebras, and give criteria which ensure a given representation of the crossed product is faithful. We use this to prove that the C^* -algebras generated by two semigroups $V, W: \Gamma^+ \rightarrow B(H)$ of nonunitary isometries are canonically isomorphic, thus giving a new, self-contained proof of a theorem of Murphy, which includes earlier results of Coburn and Douglas.

INTRODUCTION

Let Γ be a totally ordered abelian group with positive cone Γ^+ . Murphy showed in [6] that the C^* -algebras generated by two semigroups $V, W: \Gamma^+ \rightarrow B(H)$ of nonunitary isometries are canonically isomorphic [6, Theorem 2.9], generalising well-known theorems of Coburn [3] for the case $\Gamma = \mathbb{Z}$, and of Douglas [5] for subgroups of \mathbb{R} . His proof involved an analysis of the ideal structure of the ordered group Γ , and used results of Olesen and Pedersen [9, §8.11] on the primitivity of certain crossed products. Here we obtain a characterization of faithful representations of crossed products by semigroups of endomorphisms, and use it to give a short, self-contained proof of Murphy's theorem.

In §1, we discuss our notion of crossed products by semigroups of endomorphisms of C^* -algebras, which is based on ideas of Stacey [11]. Our characterization of faithful representations in Theorem 1.2 extends that for crossed products by single endomorphisms in [1]; we prove a more general theorem than we need later, because we want to fill a gap in [1] — see the remarks preceding Lemma 1.3. In §2, we show that the universal C^* -algebra $C^*(\Gamma^+)$ for isometric representations of Γ^+ is such a crossed product (Proposition 2.2). To prove the main result, Theorem 2.4, we need to verify that the hypotheses of Theorem 1.2 hold when the isometric representation consists of nonunitary isometries, and we do this by adapting an argument from [4]. It follows immediately that the Toeplitz algebra of an ordered group is isomorphic to $C^*(\Gamma^+)$.

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1. CROSSED PRODUCTS

Throughout Γ will be a totally ordered discrete abelian group with positive cone Γ^+ . An isometric representation of Γ^+ is a homomorphism of the semigroup Γ^+ into the semigroup of isometries $\text{Isom}(H)$ on a Hilbert space H . We now give our basic definitions.

Definition. A *dynamical system* is a triple (A, Γ^+, α) where A is a unital C^* -algebra, and α is an action of Γ^+ on A by endomorphisms. A *covariant representation* of a dynamical system (A, Γ^+, α) is a pair (π, V) , where π is a nondegenerate representation of A on H , and V is an isometric representation of Γ^+ on H such that $\pi(\alpha_x(a)) = V_x \pi(a) V_x^*$ for $a \in A$ and $x \in \Gamma^+$. A *crossed product* for a dynamical system (A, Γ^+, α) is a unital C^* -algebra B together with a unital homomorphism $i_A: A \rightarrow B$ and a homomorphism i_{Γ^+} of Γ^+ into the semigroup of isometries in B satisfying

1. $i_A(\alpha_x(a)) = i_{\Gamma^+}(x) i_A(a) i_{\Gamma^+}(x)^*$ for $a \in A, x \in \Gamma^+$;
2. for every covariant representation (π, V) of (A, Γ^+, α) , there is a unital representation $\pi \times V$ of B with $(\pi \times V) \circ i_A = \pi$ and $(\pi \times V) \circ i_{\Gamma^+} = V$; and
3. B is generated by $\{i_A(a)\}$ and $\{i_{\Gamma^+}(x)\}$.

Remark 1.1. (i) In fact, B is spanned by $\{i_{\Gamma^+}(x)^* i_A(a) i_{\Gamma^+}(y) : a \in A, x, y \in \Gamma^+\}$; cf. [1, Lemma 1.1].

(ii) As in [10, Proposition 3] and [11, Proposition 3.2], each system has, up to isomorphism, exactly one crossed product, denoted by $(A \rtimes_{\alpha} \Gamma^+, i_A, i_{\Gamma^+})$.

(iii) If there is a covariant representation (π, V) with π faithful, condition 2 implies that i_A is injective. However, there might not be such a representation; see [11, 2.2].

Theorem 1.2. *Suppose (π, V) is a covariant representation of (A, Γ^+, α) such that*

- (i) π is faithful, and
- (ii) for all finite subsets F of Γ^+ and all choices of $a_{x,y} \in A$,

$$\left\| \sum_{x \in F} V_x^* \pi(a_{x,x}) V_x \right\| \leq \left\| \sum_{x,y \in F} V_x^* \pi(a_{x,y}) V_y \right\|.$$

Then $\pi \times V$ is a faithful representation of $A \rtimes_{\alpha} \Gamma^+$.

Proof. Our strategy is a familiar one. The uniqueness of the crossed product gives a continuous action β of the compact group $\widehat{\Gamma}$ on $A \rtimes_{\alpha} \Gamma^+$, characterised by $\beta_{\gamma}(i_A(a)) = i_A(a)$ and $\beta_{\gamma}(i_{\Gamma^+}(x)) = \gamma(x) i_{\Gamma^+}(x)$ (β is the *dual action*). The formula $\theta(b) = \int_{\widehat{\Gamma}} \beta_{\gamma}(b) d\gamma$ defines a norm-decreasing projection θ of $A \rtimes_{\alpha} \Gamma^+$ onto the fixed-point algebra $(A \rtimes_{\alpha} \Gamma^+)^{\beta}$, which is faithful in the sense that if $\theta(b^*b) = 0$ for some $b \in A \rtimes_{\alpha} \Gamma^+$, then $b = 0$ (e.g., see the proof of [1, Lemma 2.2]). Since $\int \gamma(y-x) d\gamma = 0$ for $x \neq y$, we have

$$(1.1) \quad \theta\left(\sum_{x,y} i_{\Gamma^+}(x)^* i_A(a_{x,y}) i_{\Gamma^+}(y)\right) = \sum_x i_{\Gamma^+}(x)^* i_A(a_{x,x}) i_{\Gamma^+}(x),$$

and the inequality in (ii) above extends to $\|\pi \times V(\theta(b))\| \leq \|\pi \times V(b)\|$ for all $b \in A \rtimes_{\alpha} \Gamma^+$.

The theorem will follow once we show that $\pi \times V$ is faithful on the fixed-point algebra $(A \times_{\alpha} \Gamma^+)^{\beta} = \theta(A \times_{\alpha} \Gamma^+)$. For if $\pi \times V(b^*b) = 0$, then $\pi \times V(\theta(b^*b)) = 0$; because $\pi \times V$ is faithful on the range of θ , this forces $\theta(b^*b) = 0$, and hence also $b = 0$.

So we have only to prove that $\pi \times V$ is faithful on $(A \times_{\alpha} \Gamma^+)^{\beta}$. From (1.1) and the continuity of θ , we deduce

$$(1.2) \quad (A \times_{\alpha} \Gamma^+)^{\beta} = \overline{\text{span}}\{i_{\Gamma^+}(x)^*i_A(a)i_{\Gamma^+}(x) : a \in A, x \in \Gamma^+\}.$$

If we had the extra condition

$$(1.3) \quad i_{\Gamma^+}(x)^*i_A(A)i_{\Gamma^+}(x) \subset i_A(A),$$

it would immediately follow that $(A \times_{\alpha} \Gamma^+)^{\beta} = i_A(A)$, and $\pi \times V$ would be faithful there because $\pi = (\pi \times V) \circ i_A$ is faithful on A by assumption (i). Since (1.3) holds in the situation of §2, the weaker version of Theorem 1.2 we have just proved would suffice there.

One reason we want to prove Theorem 1.2 in the stated generality is to complete the proof of Proposition 2.1 of [1], which is the special case $\Gamma^+ = \mathbb{N}$, and where the extra generality was crucial for the proposed improvement to [8]; it was not checked in [1] that $\pi \times V$ is faithful on the fixed-point algebra. Lemma 1.5 will complete the proof of Theorem 1.2 by verifying this. First, we need a basic lemma about ideals of direct limits which is implicit in [2, Lemma 3.1]. \square

Lemma 1.3. *Let C be a C^* -algebra, and $\{B_{\alpha}\}$ a family of C^* -subalgebras such that $C = \overline{\bigcup_{\alpha} B_{\alpha}}$. If J is a closed ideal in C , then $J = \overline{\bigcup_{\alpha} (J \cap B_{\alpha})}$.*

Proof. Let $J_{\alpha} = J \cap B_{\alpha}$. We want to show that $J = \overline{\bigcup_{\alpha} J_{\alpha}}$. Obviously $\overline{\bigcup_{\alpha} J_{\alpha}} \subset J$, so suppose $x \notin \overline{\bigcup_{\alpha} J_{\alpha}}$. Then $\inf_{y \in \bigcup_{\alpha} J_{\alpha}} \|x - y\| = \epsilon > 0$. Since $C = \overline{\bigcup_{\alpha} B_{\alpha}}$, we can choose λ and $x_{\lambda} \in B_{\lambda}$ such that $\|x_{\lambda} - x\| < \frac{\epsilon}{3}$. Thus, for any y in J_{λ} ,

$$\|x_{\lambda} - y\| \geq \|x - y\| - \|x - x_{\lambda}\| \geq \epsilon - \|x - x_{\lambda}\| \geq \epsilon - \frac{\epsilon}{3} = \frac{2\epsilon}{3}.$$

If $\rho : C \rightarrow C/J$ is the quotient map, then $\ker \rho|_{B_{\lambda}} = J \cap B_{\lambda} = J_{\lambda}$, and we have

$$\|\rho(x_{\lambda})\| = \inf_{y \in J_{\lambda}} \|x_{\lambda} - y\| \geq \frac{2\epsilon}{3},$$

because ρ induces an isomorphism of the C^* -algebra B_{λ}/J_{λ} onto $\rho(B_{\lambda})$ which is necessarily isometric. Then

$$\|\rho(x)\| = \|\rho(x_{\lambda}) + \rho(x - x_{\lambda})\| \geq \|\rho(x_{\lambda})\| - \|\rho(x - x_{\lambda})\| \geq \frac{2\epsilon}{3} - \frac{\epsilon}{3},$$

since ρ is norm-decreasing. Hence $\rho(x) \neq 0$, and $x \notin \ker \rho = J$. \square

Remark 1.4. This is the proof of [2, Lemma 3.1], slightly modified to avoid using the hypothesis that the B_{α} form an increasing sequence. The finite-dimensionality of B_n was not used in [2] except to ensure that B_n is a C^* -algebra.

Lemma 1.5. *Let (π, V) be as in Theorem 1.2. Then $\pi \times V$ is faithful on $(A \times_\alpha \Gamma^+)^\beta$.*

Proof. We claim that $\pi \times V$ is faithful on $\text{span}\{i_{\Gamma^+}(x)^* i_A(a) i_{\Gamma^+}(x) : a \in A, x \in \Gamma^+\}$, which is dense in $(A \times_\alpha \Gamma^+)^\beta$ by (1.2). Suppose $b = \sum_x i_{\Gamma^+}(x)^* i_A(a_x) i_{\Gamma^+}(x)$ satisfies $\pi \times V(b) = 0$, so that $\sum_x V_x^* \pi(a_x) V_x = 0$. Choose x_0 greater than every x for which $a_x \neq 0$. Then

$$\begin{aligned}
 0 &= V_{x_0} \left(\sum_x V_x^* \pi(a_x) V_x \right) V_{x_0}^* = \sum_x V_{x_0-x} V_x V_x^* \pi(a_x) V_x V_x^* V_{x_0-x}^* \\
 (1.4) \qquad &= \sum_x V_{x_0-x} \pi(\alpha_x(1) a_x \alpha_x(1)) V_{x_0-x}^* \\
 &= \sum_x \pi(\alpha_{x_0-x}(\alpha_x(1) a_x \alpha_x(1))),
 \end{aligned}$$

so, since π is faithful, $\sum_x \alpha_{x_0-x}(\alpha_x(1) a_x \alpha_x(1)) = 0$ in A , and

$$i_A \left(\sum_x \alpha_{x_0-x}(\alpha_x(1) a_x \alpha_x(1)) \right) = 0 \text{ in } A \times_\alpha \Gamma^+.$$

Thus, reversing calculation (1.4), we obtain

$$b = i_{\Gamma^+}(x_0)^* (i_{\Gamma^+}(x_0) b i_{\Gamma^+}(x_0)^*) i_{\Gamma^+}(x_0) = 0,$$

verifying the claim.

Next, let

$$B_z = \text{span}\{i_{\Gamma^+}(x)^* i_A(a) i_{\Gamma^+}(x) : a \in A, 0 \leq x \leq z \in \Gamma^+\},$$

so that $\bigcup_{z \in \Gamma^+} B_z$ is dense in $(A \times_\alpha \Gamma^+)^\beta$ by (1.2). We claim that each B_z is a C^* -algebra. It is certainly a $*$ -algebra, so it is enough to show B_z is closed. Suppose $\{b_n\} \subset B_z$ and $b_n \rightarrow c$ in $A \times_\alpha \Gamma^+$. This implies that $i_{\Gamma^+}(z) b_n i_{\Gamma^+}(z)^* \rightarrow i_{\Gamma^+}(z) c i_{\Gamma^+}(z)^*$. A calculation like (1.4), with z in place of x_0 , shows that each $i_{\Gamma^+}(z) b_n i_{\Gamma^+}(z)^*$ has the form $i_A(a_n)$ for some a_n in A . Since $\{i_A(a_n)\}$ is Cauchy and the range of i_A is closed, $i_{\Gamma^+}(z) b_n i_{\Gamma^+}(z)^* = i_A(a_n)$ converges to $i_A(a)$, for some a in A . But then $b_n = i_{\Gamma^+}(z)^* i_{\Gamma^+}(z) b_n i_{\Gamma^+}(z)^* i_{\Gamma^+}(z)$ converges to the element $i_{\Gamma^+}(z)^* i_A(a) i_{\Gamma^+}(z)$ of B_z , and B_z is closed. We can now apply Lemma 1.3 to deduce that $\ker(\pi \times V) = \overline{\bigcup_z \ker(\pi \times V) \cap B_z}$. Since we proved in the first paragraph that $\pi \times V$ is faithful on $\bigcup_z B_z$, we have

$$\bigcup_z \ker(\pi \times V) \cap B_z = \ker(\pi \times V) \cap \left(\bigcup_z B_z \right) = \{0\}.$$

This completes the proofs of Lemma 1.5 and Theorem 1.2. \square

2. THE C^* -ALGEBRA GENERATED BY A SEMIGROUP OF ISOMETRIES

We now consider a particular crossed product which is universal for isometric representations of Γ^+ , in a sense to be made precise in Proposition 2.2. The algebra in the dynamical system is the closed subspace B_{Γ^+} of $\ell^\infty(\Gamma)$ spanned

by $\{1_x : x \in \Gamma^+\}$, where

$$1_x(y) = \begin{cases} 1 & \text{if } y \geq x, \\ 0 & \text{otherwise;} \end{cases}$$

since we have $1_x 1_y = 1_{\max\{x,y\}}$, this space is actually a C^* -subalgebra of $\ell^\infty(\Gamma^+)$. For each $x \in \Gamma^+$, the automorphism $\tau_x \in \text{Aut } \ell^\infty(\Gamma)$ defined by $\tau_x(f)(y) = f(y - x)$ satisfies $\tau_x(1_y) = 1_{x+y}$, and hence τ restricts to an action α of Γ^+ by endomorphisms of B_{Γ^+} . (The algebra B_{Γ^+} was introduced in [5], and is the restriction to Γ^+ of the algebra $\mathcal{S}_1(\Gamma)$ used in [6].) Before proving that $B_{\Gamma^+} \rtimes_\alpha \Gamma^+$ has the required universal property, we need a lemma.

Lemma 2.1. *Let P_i be a finite family of projections such that $P_1 \geq P_2 \geq \dots \geq P_N$. Then for any $\lambda_i \in \mathbb{C}$,*

$$\left\| \sum_{i=1}^N \lambda_i P_i \right\| \leq \max_{1 \leq n \leq N} \left| \sum_{i=1}^n \lambda_i \right|;$$

we have equality if $P_i \neq P_{i+1}$ for all i .

Proof. Since the projections $P_n - P_{n+1}$ are mutually orthogonal, the result follows from the identity

$$\sum_{i=1}^N \lambda_i P_i = \sum_{n=1}^{N-1} \left(\sum_{i=1}^n \lambda_i \right) (P_n - P_{n+1}) + \left(\sum_{i=1}^N \lambda_i \right) P_N. \quad \square$$

Proposition 2.2. *Let Γ be a totally ordered abelian group and let $B_{\Gamma^+} \rtimes_\alpha \Gamma^+$ be the crossed product of the system defined above.*

(i) *If ρ is a nondegenerate representation of $B_{\Gamma^+} \rtimes_\alpha \Gamma^+$, then $\rho \circ i_{\Gamma^+}$ is an isometric representation of Γ^+ .*

(ii) *Whenever $V : \Gamma^+ \rightarrow \text{Isom}(H)$ is an isometric representation of Γ^+ , there is a representation π_V of B_{Γ^+} such that (π_V, V) is a covariant representation of $(B_{\Gamma^+}, \Gamma^+, \alpha)$. If each V_x is nonunitary, then π_V is faithful.*

(iii) *$B_{\Gamma^+} \rtimes_\alpha \Gamma^+$ is generated by $\{i_{\Gamma^+}(x) : x \in \Gamma^+\}$ — indeed,*

$$B_{\Gamma^+} \rtimes_\alpha \Gamma^+ = \overline{\text{span}}\{i_{\Gamma^+}(x)i_{\Gamma^+}(y)^* : x, y \in \Gamma^+\}.$$

(iv) *$i_{B_{\Gamma^+}} : B_{\Gamma^+} \rightarrow B_{\Gamma^+} \rtimes_\alpha \Gamma^+$ is injective.*

Proof. The first part is clear because each $i_{\Gamma^+}(x)$ is an isometry. For (ii), we note that the representation π_V must satisfy

$$\pi_V \left(\sum_{x \in F} \lambda_x 1_x \right) = \left(\sum_{x \in F} \lambda_x \pi_V(\alpha_x(1)) \right) = \sum_{x \in F} \lambda_x V_x V_x^*.$$

We show that this formula gives a well-defined linear map π_V on $\text{span}\{1_x : x \in \Gamma^+\}$, and simultaneously that π_V extends to B_{Γ^+} , by showing that

$$\left\| \sum \lambda_x V_x V_x^* \right\| \leq \left\| \sum \lambda_x 1_x \right\|.$$

Given a finite linear combination $\sum_{x \in F} \lambda_x 1_x$, we can index F so that $x_1 < x_2 < \dots < x_N$ because Γ is totally ordered. Then $\sum_x \lambda_x 1_x = \sum_{i=1}^N \lambda_{x_i} 1_{x_i}$ is a linear combination of projections such that $1_{x_1} > 1_{x_2} > \dots > 1_{x_N}$, and

Lemma 2.1 implies

$$\left\| \sum_{x \in F} \lambda_x 1_x \right\| = \max_{1 \leq n \leq N} \left| \sum_{i=1}^n \lambda_{x_i} \right|.$$

For $y > x$,

$$(2.1) \quad V_x V_x^* - V_y V_y^* = V_x V_x^* - V_x V_{y-x} V_{y-x}^* V_x^* = V_x (1 - V_{y-x} V_{y-x}^*) V_x^*,$$

so $V_x V_x^* \geq V_y V_y^*$. Thus $\sum_{x \in F} \lambda_x V_x V_x^*$ is a linear combination of projections $V_{x_i} V_{x_i}^*$ such that $V_{x_1} V_{x_1}^* \geq V_{x_2} V_{x_2}^* \geq \dots \geq V_{x_N} V_{x_N}^*$, and Lemma 2.1 gives

$$(2.2) \quad \left\| \sum_{x \in F} \lambda_x V_x V_x^* \right\| = \left\| \sum_{i=1}^n \lambda_{x_i} V_{x_i} V_{x_i}^* \right\| \leq \max_{1 \leq n \leq N} \left| \sum_{i=1}^n \lambda_{x_i} \right| = \left\| \sum_{x \in F} \lambda_x 1_x \right\|,$$

as claimed. Finally, π_V is a *-homomorphism because $1_x 1_y = 1_{\max\{x,y\}}$ and $V_x V_x^* V_y V_y^* = V_{\max\{x,y\}} V_{\max\{x,y\}}^*$. It follows easily from the formula $\alpha_x(1_y) = 1_{x+y}$ that (π_V, V) is covariant on $\text{span}\{1_x : x \in \Gamma^+\}$, and hence, by continuity, on B_{Γ^+} .

Suppose now that V is a nonunitary representation. Then for $y > x$, $1 - V_{y-x} V_{y-x}^*$ is a nonzero projection, (2.1) gives $V_x V_x^* > V_y V_y^*$, and Lemma 2.1 gives equality in (2.2). Thus in this case π_V is actually isometric, and hence faithful on B_{Γ^+} . This proves (ii).

The elements $i_{\Gamma^+}(x)^* i_{B_{\Gamma^+}}(1_y) i_{\Gamma^+}(z) = i_{\Gamma^+}(x)^* i_{\Gamma^+}(y) i_{\Gamma^+}(y)^* i_{\Gamma^+}(z)$ span a dense subspace of $B_{\Gamma^+} \times_{\alpha} \Gamma^+$. We claim that, for any $x, y, z \in \Gamma^+$,

$$(2.3) \quad i_{\Gamma^+}(x)^* i_{\Gamma^+}(y) i_{\Gamma^+}(y)^* i_{\Gamma^+}(z) = i_{\Gamma^+}(a) i_{\Gamma^+}(b)^*,$$

for some $a, b \in \Gamma^+$. There are six cases which need to be considered separately, and in each case we can write down formulas for a and b . For example, if $z \leq y \leq x$, writing i for i_{Γ^+} ,

$$\begin{aligned} i(x)^* i(y) i(y)^* i(z) &= (i(y) i(x - y))^* i(y) (i(z) i(y - z))^* i(z) \\ &= i(x - y)^* i(y - z)^* \\ &= (i(y - z) i(x - y))^* = i(x - z)^*, \end{aligned}$$

so $a = 0$ and $b = x - z$ will do. The other orderings of x, y , and z can be handled similarly, and (iii) follows. To prove (iv), it is enough by Remark 1.1(iii) to produce an isometric representation V of Γ^+ for which π_V is faithful, and hence, by (ii), enough to produce one consisting of nonunitary isometries. The semigroup T on $\ell^2(\Gamma^+)$ defined by $T_x(\delta_y) = \delta_{y+x}$ will do. \square

Remark 2.3. Parts (i) and (ii) of Proposition 2.2 say that $B_{\Gamma^+} \times_{\alpha} \Gamma^+$ has the universal property which characterises the C^* -algebra $C^*(\Gamma^+)$ of the semigroup Γ^+ , as studied in [7] (see [7, p. 324]).

Theorem 2.4. *Let Γ be a totally ordered abelian group and $V : \Gamma^+ \rightarrow \text{Isom}(H)$ an isometric representation of Γ^+ . Then the representation $\pi_V \times V$ of Proposition 2.2(ii) is an isomorphism of $C^*(\Gamma^+) = B_{\Gamma^+} \times_{\alpha} \Gamma^+$ onto $C^*(V_x : x \in \Gamma^+)$ if and only if V is nonunitary.*

Proof. Since $\pi_V \times V(i_{\Gamma^+}(x)) = V_x$, and $\{i_{\Gamma^+}(x) : x \in \Gamma^+\}$ generates $B_{\Gamma^+} \times_{\alpha} \Gamma^+$, the range of $\pi_V \times V$ is precisely $C^*(V_x : x \in \Gamma^+)$. If V_z is unitary for some

nonzero z , then $\pi_V \times V(i_{B_{\Gamma^+}}(1 - 1_z)) = \pi_V(1 - 1_z) = 1 - V_z V_z^* = 0$. Since $i_{B_{\Gamma^+}}$ is injective and $1 - 1_z \neq 0$, this shows that $\pi_V \times V$ is not faithful.

Suppose now that V is nonunitary. Then Proposition 2.2 says that π_V is faithful, so by Theorem 1.2, $\pi_V \times V$ will be faithful if

$$\left\| \sum_{x \in H} V_x^* \pi_V(f_{x,x}) V_x \right\| \leq \left\| \sum_{x,y \in H} V_x^* \pi_V(f_{x,y}) V_y \right\|,$$

for all finite subsets H of Γ^+ and all choices of $f_{x,y} \in B_{\Gamma^+}$. Since $\text{span}\{1_y : y \in \Gamma^+\}$ is dense in B_{Γ^+} , it suffices to show

$$\left\| \sum \mu_{x,y,x} V_x^* V_y V_y^* V_x \right\| \leq \left\| \sum \mu_{x,y,z} V_x^* V_y V_y^* V_z \right\|,$$

for all $\mu_{x,y,z} \in \mathbb{C}$. As in (2.3), for each choice of $x, y, z \in \Gamma^+$, there are $a, b \in \Gamma^+$ such that

$$V_x^* V_y V_y^* V_z = V_a V_b^*;$$

an inspection of the cases shows that, because each V_x is nonunitary, $a = b$ precisely when $x = z$. Thus it is enough to show $\left\| \sum_{x \in F} \lambda_{x,x} V_x V_x^* \right\| \leq \left\| \sum_{x,y \in F} \lambda_{x,y} V_x V_y^* \right\|$ for all finite subsets F of Γ^+ and $\lambda_{x,y} \in \mathbb{C}$. To do this, we borrow an idea from the proof of [4, Proposition 1.7]: given $\sum_{x,y \in F} \lambda_{x,y} V_x V_y^*$, we construct a projection Q such that

$$(2.4) \quad Q V_x V_y^* Q = 0 \quad \text{for } x, y \in F \text{ with } x \neq y,$$

and

$$(2.5) \quad \left\| Q \left(\sum_{x \in F} \lambda_{x,x} V_x V_x^* \right) Q \right\| = \left\| \sum_{x \in F} \lambda_{x,x} V_x V_x^* \right\|,$$

so that

$$\begin{aligned} \left\| \sum_{x,y} \lambda_{x,y} V_x V_y^* \right\| &\geq \left\| Q \left(\sum_{x,y} \lambda_{x,y} V_x V_y^* \right) Q \right\| = \left\| Q \left(\sum_x \lambda_{x,x} V_x V_x^* \right) Q \right\| \\ &= \left\| \sum_x \lambda_{x,x} V_x V_x^* \right\|. \end{aligned}$$

By Lemma 2.1 we can find $x_M \in F$ such that

$$(2.6) \quad \left\| \sum_{x \in F} \lambda_{x,x} V_x V_x^* \right\| = \left| \sum_{x \leq x_M} \lambda_{x,x} \right|.$$

Let $\delta = \min\{x - y : x, y \in F \text{ and } x > y\}$, and take $Q = V_{x_M}(1 - V_\delta V_\delta^*)V_{x_M}^*$. We prove condition (2.5) first. For $x \in F$, $x > x_M$ implies $x \geq x_M + \delta$, and hence

$$(2.7) \quad \begin{aligned} Q V_x V_x^* &= V_{x_M} V_{x_M}^* V_x V_x^* - V_{x_M+\delta} V_{x_M+\delta}^* V_x V_x^* \\ &= \begin{cases} V_x V_x^* - V_x V_x^* = 0 & \text{if } x > x_M, \\ V_{x_M} V_{x_M}^* - V_{x_M+\delta} V_{x_M+\delta}^* = Q & \text{if } x \leq x_M. \end{cases} \end{aligned}$$

By (2.6) it follows that

$$\left\| Q \left(\sum_{x \in F} \lambda_{x,x} V_x V_x^* \right) Q \right\| = \left\| \sum_{x \leq x_M} \lambda_{x,x} Q \right\| = \left| \sum_{x \leq x_M} \lambda_{x,x} \right| = \left\| \sum_{x \in F} \lambda_{x,x} V_x V_x^* \right\|,$$

as required. To prove condition (2.4), we note from (2.7) that $QV_yV_y^*$ is self-adjoint for every $y \in F$, so Q commutes with $V_yV_y^*$. Thus

$$QV_xV_y^*Q = \begin{cases} QV_{x-y}V_yV_y^*Q = QV_{x-y}QV_yV_y^* & \text{if } x > y, \\ QV_{y-x}^*V_yV_y^*Q = QV_{y-x}^*QV_yV_y^* & \text{if } x < y, \end{cases}$$

and it is enough to show $QV_{x-y}Q = 0$ whenever $x > y$ and $x, y \in F$. But

$$\begin{aligned} QV_{x-y}Q &= V_{x_M}(1 - V_\delta V_\delta^*)V_{x_M}^*V_{x-y}V_{x_M}(1 - V_\delta V_\delta^*)V_{x_M}^* \\ &= V_{x_M}(1 - V_\delta V_\delta^*)V_{x-y}(1 - V_\delta V_\delta^*)V_{x_M}^*, \end{aligned}$$

which equals zero because $(1 - V_\delta V_\delta^*)V_{x-y} = V_{x-y} - V_\delta V_{x-y-\delta} = 0$. This completes the proof of Theorem 2.4. \square

Corollary 2.5 [6, Theorem 2.9]. *Let Γ be a totally ordered abelian group, and V, W two representations of Γ^+ as nonunitary isometries on Hilbert space. Then the map $V_x \mapsto W_x$ extends to an isomorphism of $C^*(V_x : x \in \Gamma^+)$ onto $C^*(W_x : x \in \Gamma^+)$.*

Proof. Theorem 2.4 gives us isomorphisms $\pi_V \times V$ of $B_{\Gamma^+} \times_\alpha \Gamma^+$ onto $C^*(V_x : x \in \Gamma^+)$ such that $\pi_V \times V(i_{\Gamma^+}(x)) = V_x$, and $\pi_W \times W$ of $B_{\Gamma^+} \times_\alpha \Gamma^+$ onto $C^*(W_x : x \in \Gamma^+)$ such that $\pi_W \times W(i_{\Gamma^+}(x)) = W_x$. Then $(\pi_W \times W) \circ (\pi_V \times V)^{-1}$ is the isomorphism we want. \square

Definition. The family $\{\epsilon_x : x \in \Gamma\}$ of evaluation maps is an orthonormal basis for $L^2(\widehat{\Gamma})$, and the Hardy space $H^2(\Gamma^+)$ is by definition the closed linear span of $\{\epsilon_x : x \in \Gamma^+\}$. Let P denote the projection of L^2 onto H^2 . For each $\phi \in C(\widehat{\Gamma})$, the Toeplitz operator T_ϕ with symbol ϕ is the operator on $H^2(\Gamma^+)$ defined by $T_\phi(f) = P(\phi f)$. The Toeplitz algebra $\mathcal{T}(\Gamma)$ of the totally ordered abelian group Γ is the C^* -subalgebra of $B(H^2(\Gamma^+))$ generated by the Toeplitz operators $\{T_\phi : \phi \in C(\widehat{\Gamma})\}$.

Corollary 2.6 [6, Theorem 3.14]. *Let Γ be a totally ordered abelian group and V a representation of Γ^+ as nonunitary isometries on Hilbert space. Then there is a unique isomorphism ϕ of $\mathcal{T}(\Gamma)$ onto $C^*(V_x : x \in \Gamma^+)$ such that $\phi(T_{\epsilon_x}) = V_x$ for all $x \in \Gamma^+$.*

Proof. Since the map $\phi \mapsto T_\phi$ is $*$ -linear, the Stone-Weierstrass theorem implies that $\mathcal{T}(\Gamma)$ is generated by the semigroup $\{T_{\epsilon_x} : x \in \Gamma^+\}$. Since each T_{ϵ_x} is a nonunitary isometry, the result follows from Corollary 2.5. \square

Our next corollary says that every nonzero ideal in $C^*(\Gamma^+) = B_{\Gamma^+} \times_\alpha \Gamma^+$ has nonzero intersection with the copy of B_{Γ^+} . We believe that this will be a useful tool in understanding the ideal structure of $C^*(\Gamma^+)$.

Corollary 2.7. *Let Γ be a totally ordered abelian group. If I is a nonzero closed ideal in $C^*(\Gamma^+) = B_{\Gamma^+} \times_\alpha \Gamma^+$, then there exists a nonzero $x \in \Gamma^+$ with*

$$1 - i_{\Gamma^+}(x)i_{\Gamma^+}(x)^* \in I.$$

Proof. The quotient $(B_{\Gamma^+} \times_\alpha \Gamma^+)/I$ is generated by the semigroup of isometries $i_{\Gamma^+}(x) + I$ with $x \in \Gamma^+$. Since I is nonzero, the quotient map determined by $i_{\Gamma^+}(x) \mapsto i_{\Gamma^+}(x) + I$ is not an isomorphism. Thus, by Theorem 2.4, there exists a nonzero $x \in \Gamma^+$ for which $i_{\Gamma^+}(x) + I$ is unitary, that is, for which $1 - i_{\Gamma^+}(x)i_{\Gamma^+}(x)^* \in I$. \square

To finish, we shall discuss the relationship between these corollaries and other interesting results on $C^*(\Gamma^+)$ in [6]. First, it follows immediately from Corollary 2.7 that $C^*(\Gamma^+)$ is prime: indeed, the intersection of any two nonzero ideals meets $i_{B_{\Gamma^+}}(B_{\Gamma^+})$ (cf. [6, Theorem 2.11]). As Murphy points out, one can also prove this by showing that the Toeplitz representation is faithful and irreducible: the irreducibility can be proved directly (the result for partially ordered groups in [6, Theorem 3.13] is harder), the faithfulness follows from Corollary 2.6, and that primitive implies prime is an elementary argument. Second, the simplicity of the commutator ideal $\mathcal{E}(\Gamma)$ of $C^*(\Gamma^+)$ for $\Gamma \subset \mathbb{R}$ follows from Corollary 2.5, exactly as in [5, p. 147] and even more easily from Corollary 2.7; the same applies to Murphy's generalisation [6, Theorem 2.11]. Murphy also proved, conversely to Douglas's theorem, that $\mathcal{E}(\Gamma)$ simple implies $\Gamma \subset \mathbb{R}$; again, his result concerns partially ordered groups, and is relatively straightforward if one is only interested in totally ordered groups (cf. [6, p. 324–325]).

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