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CROSSING AND PHYSICAL PARTIAL WAVE AMPLITUDES

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ABSTRACT

We study the implications of crossing and analyticity properties on partial wave amplitudes in the physical region. We first show the physical interest of the $\pi\pi$ equations recently written by Roy ; comparison with the Chew - Mandelstam equations is particularly instructive, and we show how these equations can be used a) to construct low energy amplitudes in practice, and b) to check the consistency of experimental data (a direct method for solving the "up-down" ambiguity and calculating s wave scattering lengths is obtained). We then reconsider the Martin inequalities and the sum rules from which they are derived, we show that they can be used in a more powerful way. Our considerations and results give strong support to the model of Le Guillou, Morel and Navelet for $\pi\pi$ s waves. We finally give the general method for deriving physical region crossing equations for arbitrary processes, in particular $\pi N \to \pi N$.

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1. - INTRODUCTION

The crossing property of amplitudes plays a fundamental role in relativistic scattering. However, it has the feature of relating all partial wave amplitudes to one another. In certain cases, crossing can be used directly to infer some structures of these partial wave amplitudes - a good example is the current algebra calculation of scattering lengths. But, whenever unitarity becomes an important constraint, it is no longer simple to determine quantitatively the physical implications of crossing, although we know that "forces" are provided by crossed channel exchanges.

First implications of crossing alone were discovered [1] by Balachandran and Nuyts [2], and, later on, Roskies [3] and other authors [4] established their practical usefulness. In $\pi\pi$ scattering, these necessary and sufficient crossing conditions consist in integral relations for partial wave amplitudes in the unphysical region $0 \leq s \leq 4m_\pi^2$. Another approach, initiated by Martin [5),6)], consists in incorporating also analyticity and unitarity properties. This yields sets of inequalities for partial waves in the region $0 \leq s \leq 4m_\pi^2$. However, the main disadvantage of these methods lies in the fact that the crossing relations, or constraints, are written and used in unphysical regions.

Considering experimental data, we recall the well-known "up-down" ambiguity in the $I = 0$ $\pi\pi$ s wave, and the fact that, owing to experimental difficulties, some energy regions are not well investigated - in particular $\pi\pi$ scattering lengths are not well known. We may ask whether an appropriate use of the crossing symmetry cannot solve such ambiguities and determine the unknowns, as is strongly suggested by the work of Le Guillou, Morel and Navelet [7] (hereafter called GMN) who show that knowledge of the $I = 1$ p wave, and of some high energy parameters practically limits the possible $\pi\pi$ s waves in a very restricted domain. In order to do this without using a specific model, one must express crossing directly on measurable quantities.

First crossing constraints on physical $\pi\pi$ partial waves have been obtained by Wanders [8] and by Roskies [9]. However, their basic drawback is that s and p wave amplitudes are absent in these relations which only constrain higher waves. Recently, Roy [10] has written $\pi\pi$ equations which, in connection with the above remarks, can be seen to have the features of (a) expressing each partial wave amplitude in the physical region (including s and p waves) as an integral over physical absorptive parts, and (b) being well-defined up to energies around 1100 MeV, therefore providing direct consistency tests for experimental data. Roy uses crossing to express the t dependent subtraction functions in twice subtracted fixed t dispersion relations, and then projects on partial waves. Similar relations have been extensively studied in the πN unsubtracted cases by Steiner et al. [11] in recent years.

In this work we show that such relations are extremely useful in practice, both as constructive procedures for calculating low energy amplitudes and as consistency checks for experimental data. We shall also re-examine the question of Martin inequalities and various crossing relations which follow from them. Finally, we shall derive partial wave crossing equations for arbitrary processes.

In Section 2, we discuss the properties, practical use, and physical interest of the $\pi\pi$ equations. Our own derivation of these equations is contained in Appendix A, together with relevant formulas. We show how the Chew-Mandelstam [12] equation can be obtained as first approximations to these equations. In this respect we recall that Lovelace [13] proved that the Chew-Mandelstam equations have no solutions if the p wave absorptive part does not vanish, and therefore in order to construct s and p wave $\pi\pi$ amplitudes which satisfy unitarity and crossing, one must incorporate some information about higher waves and asymptotic contributions. In the present framework, this can be done in a very natural and systematic way. Hence, these equations constitute a very convenient tool to unitarize low energy amplitudes in a crossing symmetric way. They furthermore justify the GMN model in some sense, and we show how they can be used in practice.

In Section 3 we briefly discuss the unsubtracted case. In Section 4, we reconsider the equations used by Martin. Among other things, we show that Martin inequalities can be used in a much stronger way than has been done up to now. In definite models, we show how the high energy information can be inserted in the computation of low energy amplitudes.

Finally, in Section 5, we give the general method to write physical region crossing equations for other amplitudes than $\pi\pi$, for instance πK or πN.

2. - CROSSING EQUATIONS FOR PHYSICAL $\pi\pi$ PARTIAL WAVES

Our notations are explained in Appendix A.1. Let $F^I(s,t,u)$ be the isospin I total $\pi\pi$ amplitude, $A^I(s,t)$ its s channel and $D_u^I(u,t)$ its u channel absorptive parts, the twice-subtracted fixed t dispersion relations can be written as

$$F^I(s,t,u) = g^I(t) + (s-u)\,h^I(t) + \frac{s^2}{\pi}\int_4^\infty \frac{ds'\,A^I(s',t)}{s'^2(s'-s)} + \frac{u^2}{\pi}\int_4^\infty \frac{du'\,D_u^I(u',t)}{u'^2(u'-u)} \tag{2.1}$$

(we set $m_\pi = 1$). Roy [10] has shown that, by using the crossing symmetry, the t dependent subtraction functions $g^I(t)$ and $h^I(t)$ can be expressed in terms of physical absorptive parts and s wave scattering lengths. Our own method is given in Appendix A.2. Once the subtraction functions are explicited, one replaces absorptive parts in the right-hand side of the dispersion relation (2.1) by their partial wave expansions, and one projects on partial waves to obtain relations of the form

$$f_\ell^I(s) = S.T + \sum_{I'=0}^{2} \sum_{\ell'=0}^{\infty} (2\ell'+1) \int_4^\infty ds' \, G_{\ell,I}^{\ell',I'}(s,s') \, \mathrm{Im} f_{\ell'}^{I'}(s') \tag{2.2}$$

where S.T. represents polynomial subtraction terms, present in s and p wave amplitudes. The explicit forms of Eq. (2.2) are given in Appendix A.3.

In Eq. (2.2) the ℓ' summation will converge if the partial wave expansion of absorptive parts in (2.1) does. The convergence region can be inferred from the large Lehmann-Martin ellipses [14], and yields

$$-4 \lesssim s \lesssim 60 \tag{2.3}$$

Therefore, Eq. (2.2) has the important feature of relating only physical quantities if $4 \leq s \lesssim 60$.

We shall now explain why these equations are physically interesting.

A. - General properties

These equations are rigorous, as pointed out by Roy [10] : to derive them one only uses results of axiomatic quantum field theory - crossing, fixed t dispersion relations, number of subtractions, analyticity domain in t. It is of interest to write the equation for the $\pi^0\pi^0 \to \pi^0\pi^0$ s wave, $f_0^{00}(s) = \frac{1}{3}[f_0^0(s) + 2f_0^2(s)]$, more explicitly :

$$f_0^{00}(s) = a_0^{00} + \frac{(s-4)}{\pi}\int_4^\infty \frac{ds' \, \mathrm{Im} f_0^{00}(s')}{(s'-s)(s'-4)} + \frac{2}{\pi}\int_4^\infty \frac{ds'}{s'}\left[Q_1\left(1+\frac{2s'}{s-4}\right) - Q_0\left(1+\frac{2s'}{s-4}\right)\right] \mathrm{Im} f_0^{00}(s')$$

$$+ \sum_{\ell=2}^{\infty} (2\ell+1) \frac{1}{\pi}\int_4^\infty G_0^\ell(s,s') \, \mathrm{Im} f_\ell^{00}(s') \, ds' \tag{2.4}$$

where a_0^{00} is the s wave $\pi^0\pi^0$ scattering length, and where

$$G_0^\ell(s,s') = \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt \left\{ \frac{t(t-4)(2s'-4)}{s'(s'-4)(s'-t)(s'+t-4)} + \frac{s(s+t-4)(2s'+t-4)\, P_\ell\left(1+\frac{2t}{s'-4}\right)}{s'(s'-s)(s'+t-4)(s'+s+t-4)} \right\} \tag{2.5}$$

We notice that $G_0^\ell(s,s')$ is regular at $s' = s$ for $\ell \geq 2$. Although a unitarity cut contribution is explicit in the right-hand side of Eq. (2.4), the equation does not have the structure of a partial wave dispersion relation ; it does not provide the analytic structure of $f_0^{00}(s)$ throughout the complex s plane [see Eq. (2.3)]. However, the contribution of the left-hand cut is given <u>explicitly</u> in terms of physical absorptive parts for $-4 \leq s \leq 60$.

Since s-u symmetry (or antisymmetry) is used to obtain the equations, these are necessary crossing constraints on physical partial wave amplitudes. They are not sufficient and, in order to have full s,t,u symmetry, they must be supplemented with other conditions - for instance by imposing Bose symmetry in the s channel. This places further constraints on physical absorptive parts as shown in Appendix A.5. These supplementary conditions have been written and studied in various forms by Wanders [8] and Roskies [9], they have the important feature that they do <u>not</u> constrain s and p wave amplitudes owing to the presence of subtractions in Eq. (2.1). In the unsubtracted case, s and p waves would be constrained as well as higher waves.

Roy has argued [10] that, by making use of elastic unitarity, Eq. (2.2) provides a system of non-linear singular integral equations to determine the $f_\ell^I(s)$ for $4 \leq s \leq 16$, given the s wave scattering lengths and the absorptive parts in the inelastic region. In the past, there have been some attempts to write integral equations in the physical region [15] ; Roy's equations are different in that they are rigorously derived from field theory.

B. - <u>Convergence and various contributions</u>

In Eq. (2.2), the kernels $G(s,s')$ decrease as $1/s'^3$ as $s' \to \infty$ (see also Appendix A.3) therefore the s' integration is rapidly convergent. A consequence is that high energy contributions, i.e., Pomeron and Regge poles will be quite small at low values of s. The ℓ' summation, on the contrary, eventually diverges when s becomes too large. This will come from the summation of higher partial wave amplitudes at <u>low energies</u> since it arises when the angle $1 + 2t/(s'-4)$ is outside the Lehmann-Martin ellipses, i.e., at fixed t for s' small. At a given value of s, below 60 (or 68 if Mandelstam analyticity holds), only a finite number of ℓ' values will contribute significantly to the right-hand side of Eq. (2.2), and this number increases with s [we have a compensation between the exponential decrease of $\mathrm{Im}\, f_{\ell'}$ and exponential increase of $P_{\ell'}(1+2t/(s'-4))$ as $\ell' \to \infty$]. In practice, we remark that for $s' \geq (s-4)/2$ in Eq. (2.5) the angle $1 + 2t/(s'-4)$ is physical. Therefore, to make good use of these equations, we need to know $\mathrm{Im}\, f_\ell(s')$, $\ell \geq 2$, in the very low energy region $s' \leq (s-4)/2$. Since, experimentally, higher waves are small at low energies, we can estimate their real parts, i.e., scattering lengths, through Eq. (2.2) neglecting unitarity terms and then use the approximation $\mathrm{Im}\, f_\ell \sim (\mathrm{Re}\, f_\ell)^2$.

C. - Comparison with the Chew-Mandelstam equations

It is possible to build crossing symmetric amplitudes with only s and p wave absorptive parts ; if we set $\mathrm{Im}\, f^I_\ell = 0$ for $\ell \geq 2$ in Eq. (2.2) we obtain (see Appendix A.3)

$$
\begin{aligned}
f_0^0(s) = a_0 + (s-4)\frac{(2a_0-5a_2)}{12} &+ \frac{(s-4)}{\pi}\int_4^\infty \frac{ds'\,\rho^0(s')}{(s'-s)(s'-4)} \\
&+ \frac{2}{\pi}\int_4^\infty \frac{ds'}{s'}\left[Q_1(z')-Q_0(z')\right]\left[\frac{\rho^0(s')}{3} + 3x'\rho^1(s') + \frac{5}{3}\rho^2(s')\right] \\
&- \frac{1}{3}\frac{(s-4)}{\pi}\int_4^\infty \frac{ds'}{s'(s'-4)}\left[2\rho^0(s') - 9\rho^1(s') - 5\rho^2(s')\right]
\end{aligned}
\tag{2.6}
$$

$$
\begin{aligned}
f_0^2(s) = a_2 - (s-4)\frac{(2a_0-5a_2)}{24} &+ \frac{(s-4)}{\pi}\int_4^\infty \frac{ds'\,\rho^2(s')}{(s'-s)(s'-4)} \\
&+ \frac{2}{\pi}\int_4^\infty \frac{ds'}{s'}\left[Q_1(z')-Q_0(z')\right]\left[\frac{\rho^0(s')}{3} - \frac{3}{2}x'\rho^1(s') + \frac{1}{6}\rho^2(s')\right] \\
&+ \frac{1}{6}\frac{(s-4)}{\pi}\int_4^\infty \frac{ds'}{s'(s'-4)}\left[2\rho^0(s') - 9\rho^1(s') - 5\rho^2(s')\right]
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
f_1^1(s) = \frac{(s-4)}{72}(2a_0-5a_2) &+ \frac{(s-4)}{\pi}\int_4^\infty \frac{ds'\,\rho_1(s')}{(s'-s)(s'-4)} \\
&+ \frac{4}{\pi(s-4)}\int_4^\infty ds'\; Q_1(z')\left[\frac{\rho^0(s')}{3} + \frac{3}{2}x'\rho^1(s') - \frac{5}{6}\rho^2(s')\right] \\
&- \frac{1}{18}\frac{(s-4)}{\pi}\int_4^\infty \frac{ds'}{s'(s'-4)}\left[2\rho^0(s') + 27\rho^1(s') - 5\rho^2(s')\right]
\end{aligned}
\tag{2.8}
$$

where we have set

$$
\rho^0(s) \equiv \mathrm{Im}\, f_0^0(s)\ , \quad \rho^2(s) \equiv \mathrm{Im}\, f_0^2(s)\ , \quad \rho^1(s) \equiv \mathrm{Im}\, f_1^1(s)
\tag{2.9}
$$

and where

$$
x' = 1 + \frac{2s}{s'-4} \qquad\qquad z' = 1 + \frac{2s'}{s-4}
\tag{2.10}
$$

In this approximation, higher partial waves are real. We have, for instance

$$f_\ell^0(s) = \frac{4}{\pi(s-4)} \int_4^\infty ds'\, Q_\ell(z') \left[\frac{\rho^0(s')}{3} + 3z'\rho^1(s') + \frac{5}{3}\rho^2(s') \right] \qquad \ell \geq 2 \tag{2.11}$$

Since the supplementary conditions of Appendix A.5 place no restrictions on the ρ^I in this approximation, the amplitude so constructed satisfies crossing identically.

If unitarity can be imposed on s and p waves one gets a very reasonable low energy approximation. This is the spirit of the Chew-Mandelstam [12] and Cini-Fubini [16] approximations. By using elastic unitarity for the $\rho^I(s)$, Eqs. (2.6)-(2.8) yield a closed system of non-linear integral equations for the $\pi\pi$ s and p waves. We note that partial wave amplitudes are bounded by constants as $s \to +\infty$, owing to unitarity, therefore the linear term in s should cancel in Eqs. (2.6)-(2.8). This will happen if we have

$$2a_0 - 5a_2 = \frac{4}{\pi} \int_4^\infty \frac{ds'}{s'(s'-4)} \left[2\rho^0(s') + 27\rho^1(s') - 5\rho^2(s') \right] \tag{2.12}$$

If we insert this relation into Eqs. (2.6)-(2.8), we obtain the Chew-Mandelstam equations [12] - notice that Eqs. (2.6)-(2.8) depend on two parameters a_0 and a_2, while the Chew-Mandelstam ones depend on only one, hence the relation (2.13) between a_0 and a_2.

Lovelace [13] has shown that the Chew-Mandelstam equations do not have solutions unless the imaginary part of the p wave vanishes identically - and hence the ρ meson does not appear - with possible exceptions for imaginary parts which oscillate at infinity. This means that if, as in usual models, the phase shifts are well behaved at infinity, the problem of building unitary s and p waves which satisfy crossing exactly and where the ρ is present has <u>no solution</u> if one does not incorporate some information about higher waves and asymptotic (Regge) contributions. This brings us to a very important feature of Eq. (2.2). In fact, these equations provide a <u>direct estimation</u> of higher partial wave and asymptotic contributions. For instance, the left-hand cut contribution of the f_0 resonance can be readily computed. Also, one can cut the s' integration in two pieces : from threshold to a large energy N, and from N to infinity, where above N the Regge pole picture is a good approximation while below it the resonance picture is preferable. Low energy higher partial waves may be estimated by Eq. (2.11) (see above paragraph). These higher wave and asymptotic contributions would serve as driving terms in new Chew-Mandelstam type equations for s and p wave amplitudes.

This procedure is particularly appealing in connection with unitarization of Veneziano-type amplitudes [17]. In that case, a good approximation for the Im $f_\ell(s)$, $\ell \geq 2$ and s large, is given by the δ function approximation, which also contains the asymptotic Regge contribution - note that since we integrate over the $\mathrm{Im}\, f_\ell$, it is not really necessary to have a local description of them for $\ell \geq 2$ if one unitarizes s and p waves. Since the Veneziano amplitude is crossing symmetric, the supplementary conditions of Appendix A.5 are automatically satisfied.

The inadequacy of the original Chew-Mandelstam approximation can also be seen by writing Eq. (2.12) in terms of total amplitudes ; we have

$$2a_0 - 5a_2 = \frac{4}{\pi}\int_4^\infty \frac{ds'}{s'(s'-4)} \left[2A^0(s',0) + 9A^1(s',0) - 5A^2(s',0) \right] \qquad (2.13)$$

$$(\text{s-p approximation})$$

in contrast with the usual sum rule [18], from fixed t dispersion relations

$$2a_0 - 5a_2 = \frac{4}{\pi}\int_4^\infty \frac{ds'}{s'(s'-4)} \left[2A^0(s',0) + 3A^1(s',0) - 5A^2(s',0) \right] \qquad (2.14)$$

If this last relation is used as a low energy sum rule, i.e., by retaining only s and p wave contributions on the right-hand side, there is a clear contradiction with Eq. (2.13) unless the imaginary part of the p wave vanishes. To render these equations compatible, the extra p wave contribution in Eq. (2.13) should be equal to the f_0 and asymptotic contributions of Eq. (2.14) which is not the case in practice [7]. In including higher wave contributions in Chew-Mandelstam equations, as suggested above, Eq. (2.12) would not be true anymore, it is a consequence of the pure s-p approximation. We note also that it is not possible to build Chew-Mandelstam-type equations including a <u>finite</u> number of higher waves, as pointed out by Martin [19]. If we set $\mathrm{Im}\, f_\ell = 0$ for $\ell \geq N$, then the supplementary conditions of Appendix A.5 will impose that $\mathrm{Im}\, f_\ell = 0$ for $2 \leq \ell \leq N$.

In all these respects, the Chew-Mandelstam equations appear as "first order approximations" to the exact equations (2.2) which provide further "corrections" in a systematic way.

D. - <u>Phenomenological applications</u>

One can insert experimental quantities - phase shifts and inelasticities - diretly into Eq. (2.2) since it relates physical amplitudes. Therefore, one can check whether experimental amplitudes are consistent with crossing and analyticity requirements, by comparing both sides of the equations.

For instance, one immediate use is to solve the "up-down" ambiguity in the $I = 0$ s wave, or to determine $\pi\pi$ amplitudes in regions where they are badly known - e.g., scattering lengths - starting from the intermediate region where good results are available, by performing self-consistency calculations. The high energy contributions to the integrals may be estimated with some Regge parametrization. We have already discussed in paragraph B above, the question of low energy behaviour of higher partial waves which will be determined by s and p wave amplitudes, and are of fundamental importance in practical applications.

We note that since fixed t dispersion relations have been explicited in terms of physical quantities (see Appendix A.2), they may also be useful in phenomenological applications.

A few words are in order about the domain where these equations can be used in practice. As we saw, axiomatic analyticity yields that they are valid up to $s = 60m_\pi^2$ ($E_{cm} \sim 1100$ MeV) and Mandelstam analyticity, $s \approx 68m_\pi^2$ ($E_{cm} \sim 1160$ MeV). From a practical point of view this is satisfactory since it extends beyond the ρ and the $K\bar{K}$ threshold regions which are of interest at present. Actually, since four-pion inelasticity seems weak, one can presumably use the equations up to and above the f_o region - up to $E_{cm} \sim 1400$ MeV if the first important t channel singularity is in the ρ region.

E. - <u>Example of application : phenomenological models</u>

In recent years, the problem of building s and p wave $\pi\pi$ amplitudes consistent with crossing symmetry has been of great interest, either in the context of current algebra [20] or in various phenomenological models [7),21),22)]. We believe that, in view of the above discussion, the crossing equations (2.2) constitute the correct context for this problem.

In the phenomenological model of Le Guillou, Morel and Navelet [7)], crossing is imposed in the unphysical region $0 \leq s \leq 4$. The physical input is the position and shape of the ρ resonance, and some high energy and d wave informations ; and GMN show that this practically fixes the s waves in the low energy region. It is interesting to check in the present framework :

(a) - to what extent is crossing well satisfied by the GMN amplitudes in the <u>physical</u> region, i.e., up to what energy can their results be believed,

(b) - if the solution is really unique, up to small variations as they suggest.

We have taken Eq. (2.2) and computed the left-hand side by inserting on the right only the s and p wave absorptive parts given by the model [this amounts to using Eqs. (2.6)-(2.8) as they are written] ; we then compare the result (output with the amplitudes of the model (input). The results for the I = 0 s wave amplitude are shown on Fig. 1, and we notice the following.

- The two curves are very close up to $s \simeq 20 m_\pi^2$. Above they deviate sizeably. In fact, the "output" curve has a linear behaviour as $s \to \infty$ [see Eq. (2.6)].
- This linear behaviour of the output curve will be corrected by the inclusion of higher partial wave contributions as mentioned in paragraph B. We have here an illustration of the fact that neglecting $\ell \geq 2$ on the right-hand side of (2.2) is a good approximation up to $s \sim 16\text{-}20 m_\pi^2$.
- In this respect, the crossing properties of the GMN model seem very good up to, say, the ρ mass.

Next, keeping the <u>same</u> $I = 1$ p wave and $I = 2$ s wave amplitudes (which fit experimental data) we have computed the "input" and "output" $I = 0$ s wave starting with another form for the phase shift δ_0^0. This new phase shift B is shown on Fig. 2, compared with the GMN phase shift A. It is of the "down-up" type while the GMN one is "up-down". Both of these phase shifts correspond to the <u>same value of the scattering</u> length a_0. This is important, owing to the subtraction terms in (2.6). The resulting curves for $\mathrm{Re}\, f_0^0(s)$ are shown on Fig. 3 and show a clear disagreement even at <u>very low</u> energies. Therefore, given $f_0^2(s)$ and $f_1^1(s)$, crossing considerations eliminate the B curve as opposed to the A curve in Fig. 3. We do not claim, on this example, that we have solved the "up-down" ambiguity, in particular since scattering lengths play an important role in the discussion, however, it seems clear that this ambiguity <u>can</u>-be solved, and scattering lengths determined by such considerations. Detailed work on this subject is now underway.

3. - <u>UNSUBTRACTED CASE</u>

The combination of amplitudes which has isospin $I = 1$ in the t channel is s-u antisymmetric ; assuming, for instance, ρ Regge exchange at high energy, we may write an unsubtracted dispersion relation for this amplitude as

$$2F^0(s,t) + 3F^1(s,t) - 5F^2(s,t) = \frac{1}{\pi} \int_4^\infty ds' \left[\frac{1}{s'-s} - \frac{1}{s'+s+t-4} \right] \times$$
$$\times \left[2A^0(s',t) + 3A^1(s',t) - 5A^2(s',t) \right] \tag{3.1}$$

Projecting onto partial waves, and expanding absorptive parts, we obtain the following sum rules

$$\left.\begin{array}{ll} \ell \text{ even} & 2f_\ell^0(s) - 5f_\ell^2(s) \\ \ell \text{ odd} & 3f_\ell^1(s) \end{array}\right\} = \frac{1}{s-4} \int_{4-s}^{0} dt\, P_\ell\left(1 + \frac{2t}{s-4}\right) \frac{1}{\pi} \int_4^\infty \frac{ds'\,(2s+t-4)}{(s'-s)(s'+s+t-4)} \times$$

$$\times \left\{ \sum_{\ell' \text{even}} (2\ell'+1) \left[2\, \mathrm{Im}\, f^0_{\ell'}(s') - 5\, \mathrm{Im}\, f^2_{\ell'}(s') \right] \; P_{\ell'}\left(1 + \frac{2t}{s'-4}\right) + \right.$$

$$\left. \sum_{\ell' \text{odd}} (2\ell'+1) \; 3 \; \mathrm{Im}\, f^1_{\ell'}(s') \quad P_{\ell'}\left(1 + \frac{2t}{s'-4}\right) \right\} \tag{3.2}$$

We note the following.

- Since we cannot use Bose symmetry in the direct channel in (3.2), these relations are valid only for $0 \leq s \leq 32 m_\pi^2$.
- Asymptotic contributions are more important than in the subtracted case since the kernels, here, decrease as $(s')^{-2}$ instead of $(s')^{-3}$.
- The advantage is that we do not have subtraction terms in Eq. (3.2), of which Eq. (2.14) is a particular case ; these relations can be used in particular to check whether low energy data substantiate the assumptions leading to unsubtracted dispersion relations (not only must the Pomeranchuk theorem hold, but the real part of the $I_t = 1$ amplitude must vanish at infinity).
- Otherwise, these relations can be used in the same spirit as discussed in the previous Section. We now need a more accurate high energy parametrization.

4. - PHYSICAL-UNPHYSICAL REGION CROSSING CONSTRAINTS

Most crossing constraints used in recent years concern the unphysical region [2)-6)] $0 \leq s \leq 4$. The method initiated by Martin [5)] consists in using the Froissart-Gribov formula for $f^I_\ell(s)$, Eq. (A.8) of Appendix A, valid for $\ell \leq 2$ and $s \leq 4$, and in s and p waves a subtracted version of this formula, which is derived in Eqs. (A.15.a,b). In Eqs. (A.8) and (A.15), the absorptive parts on the right-hand sides can be expanded in partial waves if $-28 \leq s \leq 4$, and we note the following.

(a) - As noted by Martin [5)], Eq. (A.15) shows that the s and p wave amplitudes are completely determined in the region $0 \leq s \leq 4$ by the physical absorptive parts and the two s wave scattering lengths. In particular, the existence and the position of Adler zeros in the I = 0, 2 $\pi\pi$ s waves are completely determined by physical quantities. Martin's inequalities follow when one takes into account positivity of absorptive parts due to unitarity.

(b) - The crossing equations discussed previously hold for $-4 \leq s \leq 60$, the present formulae (A.8) and (A.15) relate partial wave amplitudes for $-28 \leq s \leq 4$. We can therefore compare the two representations in $0 \leq s \leq 4$, since they are not identical. This will provide an alternative way to impose full s,t,u crossing, equivalent to the relations written in Appendix A.5. We think that in theoretical models this method can be useful since Martin's relations also provide a constructive procedure for partial wave amplitudes, as we shall argue later on. Also, the two sets of definitions for $f_\ell^I(s)$ are complementary in the sense that, put together, they provide the analyticity properties of partial wave amplitudes for $-28 \leq s \leq 60$.

In usual treatments, the Martin inequalities are used as follows. In parameter-free calculations, as in the Padé approximation [23),24)], one checks the violation of crossing by testing whether the inequalities are satisfied. In phenomenological models [7),21),22)] the parameters are fixed by imposing the inequalities. Although this has proven to be very useful in practice, it is perhaps not the optimal way to use these inequalities. In fact, one would like to have more precise information about how well an amplitude satisfies crossing than the "yes or no" answer about an inequality. We will show that a) it is possible to refine Martin's inequalities, and b) they are particular cases of sum rules which may be as useful as the inequalities (what we show here is already implicit in the work of Martin [5)], we are interested in the practical application of his results). This is best exemplified by writing Eqs. (A.15.a,b) of Appendix A at $s = 0$:

$$f_0^0(0) = \frac{a_0}{3} + \frac{5a_2}{3} - \frac{1}{\pi}\int_4^\infty K^0(0,0,s')\left[\frac{A^0(s',0) + 3A^1(s',0) + 5A^2(s',0)}{3}\right] ds' \tag{4.1}$$

$$f_0^2(0) = \frac{a_0}{3} + \frac{a_2}{6} - \frac{1}{\pi}\int_4^\infty K^0(0,0,s')\left[\frac{2A^0(s',0) - 3A^1(s',0) + A^2(s',0)}{6}\right] ds' \tag{4.2}$$

$$3 f_1^1(0) = -\frac{2a_0 - 5a_2}{6} - \frac{1}{\pi}\int_4^\infty K^1(0,0,s')\left[\frac{2A^0(s',0) + 3A^1(s',0) - 5A^2(s',0)}{6}\right] ds' \tag{4.3}$$

The notations are explained in Appendix A. By the optical theorem, we see that values of s and p wave amplitudes at $s = 0$ are related to scattering lengths and integrals over total cross-sections.

For the $\pi^0\pi^0 \to \pi^0\pi^0$ s wave $f_0^{00}(s) = \frac{1}{3}[f_0^0(s) + 2f_0^2(s)]$ we get

$$f_0^{00}(4) - f_0^{00}(0) = \frac{1}{8\pi^2}\int_4^\infty ds' \left[\frac{2Q_0\left(\frac{s'}{2}-1\right) - (s'-2)\,Q_1\left(\frac{s'}{2}-1\right)}{\sqrt{s'(s'-4)}}\right]\left[\frac{\sigma_{tot}^0(s') + 2\sigma_{tot}^2(s')}{3}\right] \tag{4.4}$$

and, since the integrand is positive, the Martin inequality

$$f_0^{00}(4) - f_0^{00}(0) \geq 0 \tag{4.5}$$

However, in a definite model one can use Eq. (4.4) in a much stronger way than by its consequence (4.5) since the total cross-section must be not only positive but greater than the predicted s wave cross-section. Denoting by Σ_s the s wave contribution to the right-hand side of (4.4) we must have the stronger condition

$$f_0^{00}(4) - f_0^{00}(0) \geqslant \Sigma_s \tag{4.6}$$

One can go further if the s waves predicted by the model are believed to represent the physical situation. In fact, Σ_s is the largest contribution to Eq. (4.4) and other contributions can be estimated. For instance, we can assume that above a certain energy N, σ_{tot} has reached its asymptotic value σ_∞, while below it we retain only the contribution of s and d waves. The f_0 resonance contribution Σ_{f_0} can be computed easily, while the asymptotic contribution Σ_∞ is

$$\Sigma_\infty = \frac{\sigma_\infty}{8\pi^2}\int_4^\infty \frac{8}{3s'^2}\, ds' = \frac{\sigma_\infty}{3\pi^2 N} \tag{4.7}$$

Summing these contributions gives

$$f_0^{00}(4) - f_0^{00}(0) \simeq \Sigma_s + \Sigma_{f_0} + \Sigma_\infty \tag{4.8}$$

and since Σ_s and Σ_{f_0} are known, one deduces the value of Σ_∞ and, hence, the "asymptotic cross-section" through Eq. (4.7). We do not mean that Eq. (4.8) determines the true σ_∞ predicted by the model, but rather that one must check a) that Σ_∞ is positive and b) that it has an acceptable order of magnitude. In other words, inequalities (4.5) and (4.6) must "leave some room" for higher partial wave contributions ; they cannot be satisfied too <u>largely</u>, given our knowledge of high energy phenomenology. In fact, the exact form of Eq. (4.4) can also be written as

$$f_0^{00}(4) - f_0^{00}(0) = \sum_{\ell=2}^{\infty} (2\ell+1)\, f_\ell^{00}(0) \tag{4.9}$$

and, by the positivity of $f_\ell^{00}(0)$ $\ell \geq 2$, the inequality

$$f_0^{00}(4) - f_0^{00}(0) - \sum_{\ell=2}^{2L} (2\ell+1)\, f_\ell^{00}(0) \geqslant 0 \quad \forall L \tag{4.10}$$

which transforms into an equality as $L \to \infty$. Turning to the $\pi^+\pi^0 \to \pi^+\pi^0$ amplitude, we obtain

$$a_2 + \frac{2}{3}\left[f_0^2(0) - f_0^0(0)\right] = \frac{1}{8\pi^2}\int_4^\infty ds' \frac{\left\{2Q_0\left(\frac{s'}{2}-1\right) - (s'-2)\, Q_1\left(\frac{s'}{2}-1\right)\right\}}{\sqrt{s'(s'-4)}}\left[\sigma_{tot}^1(s') + \sigma_{tot}^2(s')\right] \tag{4.11}$$

and $a_2 + \frac{2}{3}[f_0^2(0) - f_0^0(0)]$ is positive. The s and p wave contributions to the right-hand side of (4.11), Σ'_s and Σ'_p dominate, and we may estimate the asymptotic contributions Σ'_∞, in order to get

$$a_2 + \frac{2}{3}\left[f_0^2(0) - f_0^0(0)\right] \simeq \Sigma'_s + \Sigma'_p + \Sigma'_\infty \qquad (4.12)$$

from which the value of a new "asymptotic cross-section" σ'_∞ can be deduced. We now have a further test of crossing in checking to what extent these two asymptotic cross-sections are compatible, i.e., $\sigma_\infty \sim \sigma'_\infty$.

It is interesting to see how definite models fare when tested in this spirit. We examine the GMN model [7] where Martin inequalities are imposed, and the Padé approximation to the $\lambda\varphi^4$ theory [23] where they are satisfied in their original form [4]. The results are given in the Table. In all cases, the s and p waves dominate the integrals. We note the following as relevant :

(a) - in the Padé approximation, it is at step (4.6) that the inequalities are violated ; the s waves "over-saturate" them ;

(b) - in the GMN model, the inequalities are satisfied at every stage ; also not only do the "asymptotic cross-sections" have reasonable values, but they are quite close to each other (~30%) despite the fact that they represent very small fractions of the right-hand sides of (4.8) and (4.12) (a few percent) ; in this respect, the crossing properties of the GMN model are quite satisfactory.

Equation (4.3) cannot be transformed into an inequality. However, it can be used as a <u>sum rule</u> ; again the s and p waves will dominate the integral, and asymptotic contributions are very small since Pomeron exchange is absent. We notice that it is the combination of the two relations

$$2a_0 - 5a_2 = \frac{1}{8\pi^2}\int_4^\infty \frac{ds}{\sqrt{s(s-4)}}\left[2\sigma^0(s) + 3\sigma^1(s) - 5\sigma^2(s)\right] \qquad (4.13)$$

$$f_1^1(0) = -\frac{1}{32\pi^2}\int_4^\infty ds\,\sqrt{s(s-4)}\;Q_1\!\left(\frac{s}{2}-1\right)\left[\frac{2\sigma^0(s) + 3\sigma^1(s) - 5\sigma^2(s)}{6}\right] \qquad (4.14)$$

which converge if the usual Regge pole pictures are assumed. In the GMN model, Eq. (4.14) has a central role in defining the p wave amplitude, and Eq. (4.13) is closely satisfied.

The arguments we have developed are applicable to all Martin inequalities. Our basic remark is that in definite models, the total absorptive parts are certainly <u>greater than their s or p wave contributions</u> as computed by the model, and not only positive as is assumed in the derivation of Martin. Furthermore, in many instances, and in particular near $s = 0$, the variation of s and p wave amplitudes in the unphysical region should roughly be obtained by approximating absorptive parts in Eqs. (A.15.a,b) by their s and p wave contributions, since these are dominant and since the integrals converge rapidly.

As in Section 2, we see that in principle, if we are given the absorptive parts for $\ell \geq 2$, the s and p waves should be <u>entirely determined</u> by Eqs. (A.15.a,b) through a self-consistency calculation, once the scattering lengths are known. This could be done as follows :

(a) - use a K matrix representation for the s and p waves ;

(b) - parametrize the K matrix as a function of the energy with a given functional form, e.g., a rational function ;

(c) - fix the parameters in order to obtain equality between right and left-hand sides of Eqs. (A.15.a,b).

The extension of our remarks to models where other waves than $\ell = 0$ and $\ell = 1$ are included is straightforward.

Equations (A.15.a,b) are special cases of more general equations which are written in Appendix B. In studying these equations, one obtains further constraints from crossing on physical absorptive parts, they are described in Appendix B.

5. - <u>PHYSICAL REGION CROSSING EQUATIONS FOR OTHER AMPLITUDES</u>

Crossing equations for partial wave amplitudes above thresholds can be written, quite generally, for any amplitude ; we now outline the method to obtain them. We have in mind the case of other meson-meson amplitudes, and meson-baryon amplitudes, in particular $\pi N \to \pi N$. We shall treat the very simple case of $\pi\eta \to \pi\eta$ scattering ; the inclusion of spin and isospin is only an algebraic complication.

We denote by $F(s,t,u)$ the $\pi\eta \to \pi\eta$ total amplitude where s is the square of the total c.m. energy. This amplitude is s-u symmetric

$$F(s,t,u) = F(u,t,s) \qquad (5.1)$$

The physical $\pi\pi \to \eta\eta$ amplitude is given by the value of $F(s,t,u)$ when t is the square of the $\pi\pi \to \eta\eta$ c.m. energy. Owing to s-u symmetry, the fixed t dispersion relations for $\pi\eta \to \pi\eta$ are written with two subtractions as

$$F(s,t,u) = \varphi(t) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty} \frac{ds'}{(s'-s_0)^2}\left[\frac{(s-s_0)^2}{s'-s} + \frac{(u-s_0)^2}{s'-u}\right] A(s',t,\Sigma-s'-t) \tag{5.2}$$

where μ is the π mass, m the η mass, and we have

$$\Sigma = 2m^2 + 2\mu^2 = s+t+u \tag{5.3}$$

In Eq. (5.2), s_0 is some subtraction point, $\varphi(t)$ is a subtraction function as yet undetermined, and A the absorptive part of the $\pi\eta \to \pi\eta$ amplitude which can be expanded, for t physical, as

$$A(s,t,\Sigma-s-t) = \sum_{\ell=0}^{\infty} (2\ell+1)\, \mathrm{Im}\, f_\ell^{\pi\eta\to\pi\eta}(s)\, P_\ell\left(1 + \frac{t}{2q^2}\right) \tag{5.4}$$

where

$$q^2 = [s-(m+\mu)^2][s-(m-\mu)^2]/4s \tag{5.5}$$

The partial wave expansion (5.4) can be inserted in the dispersion relation (5.2) provided it converges. This happens if we have

$$t \geq t_0 \tag{5.6}$$

where t_0 is well defined in terms of physical masses, as proven from axiomatic field theory (see for instance Martin [14]). We now calculate the $\pi\eta$ partial wave amplitudes from Eq. (5.2) as

$$f_\ell^{\pi\eta\to\pi\eta}(s) = \frac{1}{4q^2}\int_{-4q^2}^{0} dt\, P_\ell\left(1+\frac{t}{2q^2}\right)\left\{\varphi(t) + \frac{1}{\pi}\int_{(m+\mu)^2}^{\infty}\frac{ds'}{(s'-s_0)^2}\left[\frac{(s-s_0)^2}{s'-s}+\frac{(u-s_0)^2}{s'-u}\right]\times \right.$$
$$\left. \times \sum_{\ell'=0}^{\infty}(2\ell'+1)\, \mathrm{Im}\, f_{\ell'}^{\pi\eta\to\pi\eta}(s')\, P_{\ell'}\left(1+\frac{2s't}{(s'-(m+\mu)^2)(s'-(m-\mu)^2)}\right)\right\} \tag{5.7}$$

This equation is well defined for

$$4q^2 \leq -t_0 \tag{5.8}$$

and it will relate the $f_\ell^{\pi\eta\to\pi\eta}(s)$ in the physical region provided $\varphi(t)$ can be explicited.

In order to determine $\varphi(t)$ we remark that we have from (5.2)

$$\varphi(t) = F(s_0, t, \Sigma - s_0 - t) - \frac{(\Sigma - 2s_0 - t)^2}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{ds'}{(s'-s_0)^2} \frac{A(s', t, \Sigma - s' - t)}{s' + t + s_0 - \Sigma} \tag{5.9}$$

We calculate $F(s_0, t, \Sigma - s_0 - t)$ by writing a <u>fixed s</u> dispersion relation

$$F(s,t,u) = \psi_1(s) + t\,\psi_2(s) + \frac{t^2}{\pi} \int_{4\mu^2}^{\infty} \frac{dt'}{t'^2} \frac{B(t',s)}{t'-t} + \frac{(u-s_1)^2}{\pi} \int_{(m+\mu)^2}^{\infty} \frac{ds'}{(s'-s_1)^2} \frac{A(s', \Sigma - s - s', s)}{s' - u} \tag{5.10}$$

where s_1 is some subtraction point, and where $B(t,s)$ is the absorptive part of the $\pi\pi \to \eta\eta$ amplitude :

$$B(t,s) = \sum_{\ell=0}^{\infty} (2\ell+1)\ \mathrm{Im}\, g_\ell^{\pi\pi\to\eta\eta}(t)\ P_\ell(\cos\theta_t) \qquad t \geq 4\mu^2 \tag{5.11}$$

$$\cos\theta_t = \frac{s-u}{\sqrt{(t-4\mu^2)}\sqrt{(t-4m^2)}} \tag{5.12}$$

Particularizing Eq. (5.10) at $s = s_0$, we see that $\varphi(t)$ is completely determined in terms of partial wave absorptive parts up to two constants $\psi_1(s_0)$ and $\psi_2(s_0)$ which can further be expressed in terms of the s and p wave $\pi\eta \to \pi\eta$ scattering lengths a_0 and a_1.

Inserting the value of $\varphi(t)$ into Eq. (5.7) we obtain

$$f_\ell^{\pi\eta\to\pi\eta}(s) = \int_{4\mu^2}^{\infty} \sum_{\ell'} K_1^{\ell\ell'}(s,s')\ \mathrm{Im}\, g_{\ell'}^{\pi\pi\to\eta\eta}(s')\, ds' + \int_{(m+\mu)^2}^{\infty} \sum_{\ell'} K_2^{\ell\ell'}(s,s')\ \mathrm{Im}\, f_{\ell'}^{\pi\eta\to\pi\eta}(s')\ ds' \tag{5.13}$$

(with subtraction terms in s and p waves). We note that <u>the only requirement</u> that we place on s_o in Eq. (5.9) is that it must be such that the partial wave expansions of A and B in Eq. (5.10) converge for $s = s_o$.

The following remarks are in order.

(a) - Equation (5.13) does not relate only physically accessible quantities since it contains the $\pi\pi \to \eta\eta$ imaginary parts in the region $4\mu^2 \leq t \leq 4m^2$. This was expected ; unitarity only gives us the phase of $\pi\pi \to \eta\eta$ partial waves for $4\mu^2 \leq t \leq 16\mu^2$ as equal to the $\pi\pi$ phase in that region. This may turn out to be very useful in practice ; we can exploit our knowledge of $\pi\eta \to \pi\eta$ (or $\pi N \to \pi N$) amplitudes in Eq. (5.13) in order to obtain some information on $\pi\pi \to \eta\eta$ (or $\pi\pi \to N\bar{N}$) for $4\ {}^2 \leq t \leq 4m^2$. We remark that one can also write equations similar to (5.13) for $g_\ell^{\pi\pi\to\eta\eta}(t)$, by starting with Eq. (5.10) instead of (5.2).

(b) - In order to relate the constants $\Psi_1(s_o)$ and $\Psi_2(s_o)$ to a_o and a_1, the simplest way is to note that since these constants appear only in the subtraction terms of s and p wave amplitudes in Eq. (5.13), one can impose on the $\ell = 0$ and $\ell = 1$ equations written in terms of $\Psi_1(s_o)$ and $\Psi_2(s_o)$

$$\lim_{s\to(m+\mu)^2} f_0^{\pi\eta}(s) = a_0 \qquad \lim_{s\to(m+\mu)^2} \frac{f_1^{\pi\eta}(s)}{(s-(m+\mu)^2)} = a_1 \tag{5.14}$$

(c) - The generalization of Eq. (5.13) to πK and πN is obvious. The validity of the resulting equations is the following [14], in two cases of interest

$$\pi K \ : \ 4q^2 = -t_o \simeq 34.9\ m_\pi^2 \qquad E_{cm} \sim 1080 \text{ MeV}$$

$$\pi N \ : \ 4q^2 = -t_o \simeq 18\ m_\pi^2 \qquad E_{cm} \sim 1320 \text{ MeV}$$

Mandelstam analyticity would give us slightly larger domains (for instance $E_{cm} \sim 1400$ MeV for πN).

(d) - In cases where for physical reasons one can write unsubtracted dispersion relations (owing to Pomeranchuk theorem for instance) one does not have to bother with a subtraction function. As a consequence, 1) the $\pi\pi \to \eta\eta$ ($\pi\pi \to N\bar{N}$) amplitudes do not appear, one has direct relations between physical $\pi\eta \to \pi\eta$ ($\pi N \to \pi N$) ones ; 2) the kernels are more slowly convergent than in Eq. (5.13), they decrease as $(s')^{-2}$ instead of $(s')^{-3}$ and asymptotic (Regge) contributions are more important at low energies. This case has been extensively studied by Steiner et al. [11] in πN scattering ; we refer to these authors for further details.

We note finally that one can in principle write crossing equations for a general process $A + B \to C + D$. Denoting the various channels of this reaction as

$$\begin{aligned} 1 &: A + B \to C + D \\ 2 &: A + \bar{C} \to \bar{B} + D \\ 3 &: A + \bar{D} \to C + \bar{B} \end{aligned}$$

we would have

$$f_\ell^1(s) = \sum_{I=1}^{3} \int_{s_I}^{\infty} ds' \sum_{\ell'} K_{1I}^{\ell\ell'}(s,s') \, \mathrm{Im} f_{\ell'}^{I}(s') \tag{5.15}$$

There is no difficulty in obtaining this equation which expresses the crossing property of the total amplitude. However, at present we do not have enough results from axiomatic field theory - validity of fixed variable dispersion relations and number of subtractions - to determine the region where (5.15) is well defined. In practice, one can assume the Mandelstam representation to estimate this region.

6. - CONCLUSION

It is of fundamental importance that the implications of crossing can be written systematically on partial wave amplitudes in the physical region. In $\pi\pi$ physics, the interest of the physical region crossing equations is twofold. First, from a practical point of view, one may directly use these equations as a smoothing procedure in order to render experimental data compatible with crossing and analyticity requirements. Secondly, from a theoretical point of view it is clear that these equations constitute the correct framework to build low energy amplitudes. In these two respects we believe that these equations represent an important step forward in the study of $\pi\pi$ scattering.

As dynamical equations, they are more profound than, say, N/D equations, since the contribution of "left-hand singularities" is explicitly given in terms of physical absorptive parts. They provide a systematic way to implement crossing in low energy amplitudes which satisfy unitarity. However, they are not complete in the sense that in order to use these equations, one must in principle know all partial wave absorptive parts above a certain energy. In practice this is not a serious problem since a) the integrals are rapidly convergent and high energy contributions are small, and b) it is quite easy to incorporate the contributions of higher partial waves and Regge poles. In order to obtain the imaginary parts at higher energies, one could think of considering similar equations for other reactions and performing some coupled channel calculations, e.g., in order to obtain $\mathrm{Im}\, f_\ell^{\pi\pi}$ in the inelastic region, one could consider the reaction $\pi\pi \to K\bar{K}$, and hence $\pi K \to \bar{\pi}K$, etc. However, this cannot be done in a systematic way and one cannot go to high energies, as is obvious from the convergence domains of these equations : they are basically low energy equations, and, in their present version, give no information on high energy amplitudes (these have to be inserted as driving terms).

The high energy driving terms can be inferred from a phenomenological analysis. In this respect, we think that these equations give strong support to the GMN model where it was found that the $\pi\pi$ s waves are practically determined once the $I = 1$ p wave and some asymptotic contributions are given. We have explained how these equations could be used in order to unitarize s and p wave amplitudes in the Veneziano model. Phenomenological duality can also be incorporated quite naturally.

We have seen that in order to have necessary and sufficient conditions for crossing and analyticity properties to be satisfied, these equations have to be supplemented with extra conditions. The supplementary conditions can be expressed in several ways, and in theoretical analyses we believe that the Martin inequalities and sum rules are well suited for this. Furthermore, we have shown that these inequalities and sum rules can be used in a much stronger way than has been done up to now.

We have written similar equations for arbitrary processes, the main problem there lying in the derivation of the convergence domain of the relations. In πN scattering, this domain can be determined, and the resulting equations should be useful in particular in the determination of the $\pi\pi \to N\bar{N}$ amplitude below the $N\bar{N}$ threshold.

Finally, from a theoretical point of view, it seems that the last unsettled point about crossing and partial wave amplitudes is the problem of incorporating the crossing condition on complex angular momentum plane continuations of partial wave amplitudes. If this could be done in a non-trivial way - i.e., not simply equivalent to writing crossing on the sequence of $f_\ell(s)$ for $\ell = 0,1,2,\dots$ - this might give further insight on the relation between the high energy and low energy regions.

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APPENDIX A : $\pi\pi$ EQUATIONS

A.1. - Notations and conventions

We set $m_\pi = 1$ in what follows. The $\pi\pi$ total amplitudes with isospin I are denoted by $F^I(s,t,u)$ where s, t and u are the usual Mandelstam invariants. The s channel partial wave expansions are written as

$$F^I(s,t,u) = \sum_{\substack{I=0,2 \quad \ell \text{ even} \\ I=1 \quad \ell \text{ odd}}} (2\ell+1)\, f_\ell^I(s)\, P_\ell\left(1+\frac{2t}{s-4}\right) \tag{A.1}$$

Our convention for unitarity is such that in the elastic region

$$f_\ell^I(s) = \sqrt{\frac{s}{s-4}}\; e^{i\delta_\ell^I(s)} \sin\delta_\ell^I(s) \tag{A.2}$$

Therefore we have

$$F^{\binom{0}{2}}(4,0,0) = \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} \tag{A.3}$$

where a_0 and a_2 are the s wave scattering lengths. Our optical theorem is therefore

$$\operatorname{Im} F^I(s,0,4-s) = \frac{\sqrt{s(s-4)}}{16\pi}\, \sigma_{tot}^I(s) \tag{A.4}$$

and total cross-sections for definite charge configurations are

$$\sigma_{tot}^{\pi^+\pi^+} = 2\,\sigma_{tot}^2 \quad , \quad \sigma_{tot}^{\pi^+\pi^0} = \sigma_{tot}^1 + \sigma_{tot}^2$$

$$\sigma_{tot}^{\pi^+\pi^-} = \frac{2}{3}\sigma_{tot}^0 + \frac{1}{3}\sigma_{tot}^2 + \sigma_{tot}^1 \quad , \quad \sigma_{tot}^{\pi^0\pi^0} = \frac{2}{3}\sigma_{tot}^0 + \frac{4}{3}\sigma_{tot}^2$$

The absorptive part of $F^I(s,t,u)$ in the s channel is denoted by $A^I(s,t)$. The t and u channels absorptive parts are denoted by $D_t^I(t,s)$ and $D_u^I(u,t)$; they are related to the s channel absorptive parts by crossing matrices, we have

$$d_{st} = \begin{vmatrix} 1/3 & 1 & 5/3 \\ 1/3 & 1/2 & -5/6 \\ 1/3 & -1/2 & 1/6 \end{vmatrix} \tag{A.5}$$

$$\alpha_{su} = \begin{vmatrix} 1/3 & -1 & 5/3 \\ -1/3 & 1/2 & 5/6 \\ 1/3 & 1/2 & 1/6 \end{vmatrix} \tag{A.6}$$

In what follows we shall take an operator notation for integrals in the following way

$$\psi(s,t)\ A^I(t) = \frac{1}{\pi}\int_4^\infty \psi(s,t,s')\ A^I(s',t)\ ds' \tag{A.7}$$

A.2. - Fixed t dispersion relations

We want to express the t dependence of subtraction functions in twice-subtracted fixed t dispersion relations in terms of the scattering lengths and of the imaginary part of the forward amplitude. This can be done by using the crossing relation at $s = 0$, i.e., $F^I(t,0,4-t) = \alpha_{st}^{II'} F^{I'}(0,t,4-t)$ as shown by Roy [10]. We give here our own derivation which, although more complicated formally, leads directly to the result and gives as by-product the subtracted Froissart-Gribov formula for s and p waves, which we will need later on ; our method consists in subtracting s and p waves from the total amplitude. For $\ell \geq 2$, the Froissart-Gribov formula converges, we have

$$f_\ell^I(s) = \frac{4}{\pi(s-4)}\int_4^\infty Q_\ell\left(1+\frac{2s'}{s-4}\right) D_t^I(s';s)\ ds' \tag{A.8}$$

Inserting Eq. (A.8) in the expansion (A.1) and making use of the Darboux-Christoffel formula

$$\sum_{\ell=L}^{\infty} (2\ell+1)\ P_\ell(x)\ Q_\ell(z) = L\ \frac{P_L(x)\ Q_{L-1}(z) - P_{L-1}(x)\ Q_L(z)}{z-x} \tag{A.9}$$

we obtain, using our convention (A.7),

$$F^{(0)}_{(2)}(s,t) = f_0^{(0)}_{(2)}(s) + K^0(s,t)\ D_t^{0}_{2}(s) \tag{A.10.a}$$

$$F^1(s,t) = 3\left(1+\frac{2t}{s-4}\right) f_1^1(s) + K^1(s,t)\ D_t^1(s) \tag{A.10.b}$$

where the kernels K^0 and K^1 are

$$K^0(s,t,s') = \frac{4}{s-4}\left[\frac{z}{z^2-x^2} - Q_0(z)\right] \tag{A.11.a}$$

$$K^1(s,t,s') = \frac{4x}{s-4}\left[\frac{1}{z^2-x^2} - 3\,Q_1(z)\right] \tag{A.11.b}$$

with

$$x = 1 + \frac{2t}{s-4} \qquad\qquad z = 1 + \frac{2s'}{s-4} \tag{A.12}$$

The twice subtracted forward dispersion relations can be written as

$$F^I(s,0) = \begin{Bmatrix} a_0 \\ 0 \\ a_2 \end{Bmatrix} + \begin{Bmatrix} 1 \\ 1/2 \\ -1/2 \end{Bmatrix} \left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{3}\right) + U(s)\, A^I(0) + C(s)\, D_u^I(0) \tag{A.13}$$

where we have introduced the kernels

$$U(s,s') = \frac{s(s-4)}{s'(s'-4)(s'-s)} \qquad\qquad C(s,s') = \frac{s(s-4)}{s'(s'-4)(s'+s-4)} \tag{A.14}$$

If we write Eqs. (A.10.a,b) at $t = 0$ and compare with (A.13) we obtain the subtracted versions of the Froissart-Gribov formula for s and p waves :

$$f_0^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(s) = \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{3}\right) + U(s)\, A^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(0) + C(s)\, D_u^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(0) - K^0(s,0)\, D_t^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(s) \tag{A.15.a}$$

$$3 f_1^1(s) = \frac{1}{2}\left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{3}\right) + U(s)\, A^1(0) + C(s)\, D_u^1(0) - K^1(s,0)\, D_t^1(s) \tag{A.15.b}$$

We now use these expressions for the s and p wave amplitudes in Eqs. (A.10.a,b) in order to obtain the twice-subtracted fixed s dispersion relations ; taking into account the definitions of K^0 and K^1 in (A.11.a,b) we get

$$F^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(s,t) = \begin{pmatrix} a_0 \\ a_2 \end{pmatrix} + \begin{pmatrix} 1 \\ -1/2 \end{pmatrix}\left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{3}\right) + U(s)\, A^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(0) + C(s)\, D_u^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(0)$$
$$+ \frac{t(t+s-4)}{\pi}\int_4^\infty \frac{ds'\,(2s'+s-4)}{(s'-t)\,s'(s'+s-4)(s'+s+t-4)}\, D_t^{\left(\begin{smallmatrix}0\\2\end{smallmatrix}\right)}(s',s) \tag{A.16}$$

$$F^1(s,t) = \left(1+\frac{2t}{s-4}\right)\Big\{\left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{6}\right) + U(s)\,A^1(0) + C(s)\,\mathcal{D}^1_u(0)$$

$$+ \frac{(s-4)\,t\,(t+s-4)}{\pi}\int_4^\infty \frac{ds'\ \mathcal{D}^1_t\ (s',s)}{s'(s'-t)(s'+s+t-4)\,(s'+s-4)} \tag{A.17}$$

By the crossing relation

$$F^I(s,t) = \sum_{I'=0}^{2} \alpha_{st}^{II'}\,F^{I'}(t,s) \tag{A.18}$$

we then obtain the fixed t dispersion relations where the subtraction functions have been explicited.

A.3. - Partial wave equations

Owing to t-u symmetry (or antisymmetry in $I = 1$), we project out partial waves by

$$f^I_\ell(s) = \int_0^1 d\cos\theta\ F^I(s,t,u)\ P_\ell(\cos\theta) \tag{A.19}$$

and we obtain

$$f^{\binom{0}{2}}_\ell(s) = \left[\begin{pmatrix}a_0\\a_2\end{pmatrix} + \begin{pmatrix}1\\-1/2\end{pmatrix}\left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{3}\right)\right]\delta_{\ell 0} + \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\ P_\ell\left(1+\frac{2t}{s-4}\right)$$

$$\Big\{\ [U(t) + C(t)]\,\mathcal{D}_t^{\binom{0}{2}}(0) + \begin{pmatrix}1\\-1/2\end{pmatrix}\frac{2s}{t-4}\left[U(t)\ A^1(0) + C(t)\ \mathcal{D}^1_u(0)\right]$$

$$+ \frac{s(s+t-4)}{\pi}\int_4^\infty \frac{ds'}{s'(s'+t-4)(s'+s+t-4)}\left[\frac{(2s'+t-4)}{s'-s}\ A^{\binom{0}{2}}(s',t) - \begin{pmatrix}1\\-1/2\end{pmatrix} 2\,\mathcal{D}^1_t(s',t)\right]\Big\} \tag{A.20}$$

$(\ell\ \text{even})$

$$f^1_\ell(s) = \left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{18}\right)\delta_{\ell 1} + \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\ \ P_\ell\left(1+\frac{2t}{s-4}\right)\Big\{\,[U(t) - C(t)]\ \mathcal{D}^1_t(0)$$

$$+ \frac{s}{t-4}\left[\ U(t)\ A^1(0) + C(t)\ \mathcal{D}^1_u(0)\right]$$

$$+ \frac{s(s+t-4)}{\pi}\int_4^\infty \frac{ds'}{s'(s'+t-4)(s'+s+t-4)}\left[\left(\frac{2s'+t-4}{s'-s}\right) A^1(s',t) - \mathcal{D}^1_t(s',t)\right]\Big\} \tag{A.21}$$

$(\ell\ \text{odd})$

In the right-hand side of Eqs. (A.20)-(A.21), we can replace the absorptive parts by their partial wave expansions

$$A^I(s',t) = \sum_{\ell'} (2\ell'+1)\ \mathrm{Im}\, f^I_{\ell'}(s')\ P_{\ell'}\left(1+\frac{2t}{s'-4}\right) \tag{A.22}$$

This expansion converges if t lies in the Lehmann-Martin large ellipses [14] ; since s' ranges from 4 to $+\infty$, we must have

$$-28 \leq t \leq 4 \tag{A.23}$$

Since the t integration ranges from $(4-s)/2$ to zero, s must therefore be in the interval

$$-4 \leq s \leq 60 \tag{A.24}$$

Notice the importance of using $t-u$ symmetry to reduce the integration range ; using the interval $4-s \leq t \leq 0$ would give us $0 \leq s \leq 32$ instead. Assuming Mandelstam analyticity gives us $-4 \leq s \leq 68$.

We now write the partial wave equations in some detail.

A.3.1. - I = 0

$$\begin{aligned} f^0_\ell(s) = {} & \left[a_0 + \left(\frac{s-4}{4}\right)\left(\frac{2a_0 - 5a_2}{3}\right)\right] \delta_{\ell 0} \\ & + \frac{1}{\pi}\int_4^\infty ds' \left\{ \sum_{\ell' \mathrm{even}} (2\ell'+1)\left[G^{0,0}_{\ell\ell'}(s,s')\ \mathrm{Im}\, f^0_{\ell'}(s') + G^{02}_{\ell\ell'}(s,s')\ \mathrm{Im}\, f^2_{\ell'}(s')\right]\right. \\ & \left. + \sum_{\ell' \mathrm{odd}} (2\ell'+1)\ G^{0,1}_{\ell\ell'}(s,s')\ \mathrm{Im}\, f^1_{\ell'}(s')\right\} \end{aligned} \tag{A.25}$$

The kernels $G^{I,I'}_{\ell,\ell'}$ are obtained easily from (A.20)

$$\begin{aligned} G^{00}_{\ell\ell'}(s,s') = {} & \frac{2}{s-4}\int_{\frac{4-s}{2}}^0 dt\ P_\ell\left(1+\frac{2t}{s-4}\right)\left\{\frac{1}{3}\left[V(t,s') + C(t,s')\right] - \frac{2s\, C(t,s')}{3(t-4)}\right. \\ & \left. + \frac{s(s+t-4)}{s'(s'+t-4)(s'+s+t-4)}\ P_{\ell'}\left(1+\frac{2t}{s'-4}\right)\left[\frac{2s'+t-4}{s'-s} - \frac{2}{3}\right]\right\} \end{aligned} \tag{A.26}$$

$$G^{0,2}_{\ell\ell'}(s,s') = \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\; P_\ell\left(1+\frac{2t}{s-4}\right)\left\{\frac{5}{3}\left[U(t,s') + C(t,s')\right] + \frac{5}{3}\,\frac{s\,C(t,s')}{(t-4)}\right.$$
$$\left. + \frac{5}{3}\,\frac{s(s+t-4)}{s'(s'+t-4)(s'+s+t-4)}\; P_{\ell'}\left(1+\frac{2t}{s'-4}\right)\right\}$$

(A.27)

$$G^{0,1}_{\ell\ell'}(s,s') = \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\; P_\ell\left(1+\frac{2t}{s-4}\right)\left\{\left[U(t,s') + C(t,s')\right] + \frac{2s}{t-4}\left[U(t,s') + \frac{1}{2}\,C(t,s')\right]\right.$$
$$\left. - \frac{s(s+t-4)}{s'(s'+t-4)(s'+s+t-4)}\; P_{\ell'}\left(1+\frac{2t}{s'-4}\right)\right\}$$

(A.28)

where $U(t,s')$ and $C(t,s')$ are defined by Eq. (A.14).

A.3.2. - <u>I = 2</u>

$$f_\ell^2(s) = \left[a_2 - \left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{6}\right)\right]\delta_{\ell 0}$$
$$+\frac{1}{\pi}\int_4^\infty ds'\left\{\sum_{\ell'\,\mathrm{even}}(2\ell'+1)\left[G^{2,0}_{\ell\ell'}(s,s')\,\mathrm{Im}f^0_{\ell'}(s') + G^{2,2}_{\ell,\ell'}(s,s')\,\mathrm{Im}f^2_{\ell'}(s')\right]\right.$$
$$\left. + \sum_{\ell'\,\mathrm{odd}}(2\ell'+1)\quad G^{2,1}_{\ell\ell'}(s,s')\;\mathrm{Im}\,f^1_{\ell}(s')\right\} \qquad \text{(A.29)}$$

with

$$G^{2,0}_{\ell,\ell'}(s,s') = \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\; P_\ell\left(1+\frac{2t}{s-4}\right)\left\{\frac{1}{3}\left[U(t,s') + C(t,s')\right] + \frac{1}{3}\,\frac{s}{t-4}\,C(t,s')\right.$$
$$\left. + \frac{1}{3}\,\frac{s(s+t-4)}{s'(s'+t-4)(s'+s+t-4)}\; P_{\ell'}\left(1+\frac{2t}{s'-4}\right)\right\} \qquad \text{(A.30)}$$

$$G^{2,2}_{\ell,\ell'} = \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\; P_\ell\left(1+\frac{2t}{s-4}\right)\Big\{ \frac{1}{6}\left[U(t,s')+C(t,s')\right] - \frac{5}{6}\,\frac{s\;C(t,s')}{(t-4)}$$
$$+\frac{s(s+t-4)}{s'(s'+t-4)(s'+s+t-4)}\, P_{\ell'}\left(1+\frac{2t}{s'-4}\right)\left[\frac{2s'+t-4}{s'-s} - \frac{5}{6}\right]\Big\}$$

(A.31)

$$G^{2,1}_{\ell,\ell'}(s,s') = -\frac{1}{2}\, G^{0,1}_{\ell,\ell'}(s,s')$$

(A.32)

A.3.3. - I = 1

$$f^1_\ell(s) = \left(\frac{s-4}{4}\right)\left(\frac{2a_0-5a_2}{18}\right)\delta_{\ell 1}$$
$$+\frac{1}{\pi}\int_4^\infty ds' \Big\{ \sum_{\ell'\,\text{even}} (2\ell'+1)\; G^{1,0}_{\ell,\ell'}(s,s')\left[2\,\mathrm{Im} f^0_{\ell'}(s') - 5\,\mathrm{Im} f^2_{\ell'}(s')\right]$$
$$+\sum_{\ell'\,\text{odd}} (2\ell'+1)\; G^{1,1}_{\ell,\ell'}(s,s')\;\; \mathrm{Im} f^1_{\ell'}(s') \Big\}$$

(A.33)

$$G^{1,0}_{\ell,\ell'} = \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\; P_\ell\left(1+\frac{2t}{s-4}\right)\Big\{ \frac{1}{6}\left[U(t,s')-C(t,s')\right] - \frac{1}{6}\,\frac{s\,C(t,s')}{(t-4)}$$
$$-\frac{1}{6}\,\frac{s(s+t-4)}{s'(s'+t-4)(s'+s+t-4)}\, P_{\ell'}\left(1+\frac{2t}{s'-4}\right)\Big\}$$

(A.34)

$$G^{1,1}_{\ell,\ell'} = \frac{2}{s-4}\int_{\frac{4-s}{2}}^{0} dt\; P_\ell\left(1+\frac{2t}{s-4}\right)\Big\{ \frac{1}{2}\left[U(t,s')-C(t,s')\right] + \frac{s}{(t-4)}\left[U(t,s')+\frac{1}{2}C(t,s')\right]$$
$$+\frac{s(s+t-4)}{s'(s'+t-4)(s'+s+t-4)}\, P_{\ell'}\left(1+\frac{2t}{s'-4}\right)\left[\frac{2s'+t-4}{s'-s} - \frac{1}{2}\right]\Big\}$$

(A.35)

A.4. - Analytic expressions for the kernels

The various functions which appear above can be expressed in terms of two basic integrals ; we notice that the angle $z' = 1 + 2t/(s'-4)$ can be expressed in terms of $z = 1 + 2t/(s-4)$ as

$$z' = 1 + \alpha \, (z-1) \tag{A.36}$$

with

$$\alpha = \frac{s-4}{s'-4} \tag{A.37}$$

The two basic integrals of interest are

$$I_1 \, (\ell, \ell', \alpha) = \frac{1}{2} \int_{-1}^{+1} P_\ell(z) \, P_{\ell'}(z') \, dz \tag{A.38}$$

and

$$I_2 \, (\ell, \ell', \alpha, x) = \frac{1}{2} \int_{-1}^{+1} \frac{P_\ell(z) \, P_{\ell'}(z')}{x - z} \, dz \tag{A.39}$$

where we see immediately that for $\ell' < \ell$ $I_1(\ell, \ell', \alpha)$ vanishes, while for $\ell' \leq \ell$ we have

$$I_2 \, (\ell, \ell', \alpha, x) = Q_\ell(x) \, P_{\ell'}(1 + \alpha \, (x-1)) \qquad \ell' \leq \ell \tag{A.40}$$

This provides us with explicit expressions for $\ell' \leq \ell$ terms in the kernels. In practical calculations, it is presumably preferable to compute all other kernels directly on the computer, however, for theoretical purposes it is interesting to have explicit forms for the integrals I_1 and I_2. We start with the expression of Legendre polynomials [25]

$$P_\ell(z) = \sum_{\lambda=0}^{\ell} (-1)^\lambda \frac{(\ell+\lambda)!}{(\lambda!)^2(\ell-\lambda)!} \left(\frac{1-z}{2}\right)^\lambda \tag{A.41}$$

and with the formula

$$\frac{1}{2} \int_{-1}^{+1} P_\ell(z) \left(\frac{1-z}{2}\right)^n dz = \frac{(-1)^\ell \, (n!)^2}{(n-\ell)! \, (n+\ell+1)!} \tag{A.42}$$

From Eqs. (A.41) and (A.36) we get

$$P_{\ell'}(z') = \sum_{\lambda=0}^{\ell'} \alpha^\lambda (-1)^\lambda \frac{(\ell'+\lambda)!}{(\lambda!)^2(\ell'-\lambda)!} \left(\frac{1-z}{2}\right)^\lambda \tag{A.43}$$

Hence, from Eq. (A.42)

$$P_{\ell'}(z') = \sum_{m=0}^{\ell'} (2m+1)\, P_m(z)\, \alpha^m\, \Pi_m^{\ell'-m}(\alpha) \tag{A.44}$$

where the polynomial $\Pi_m^{\ell'-m}(\alpha)$ is defined as

$$\Pi_m^{\ell'-m}(\alpha) = \sum_{\lambda=m}^{\ell'} (-\alpha)^{\lambda-m} \frac{(\ell'+\lambda)!}{(\ell'-\lambda)!(\lambda-m)!(\lambda+m+1)!} \tag{A.45}$$

From the usual definition of Jacobi polynomials

$$P_n^{\alpha,\beta}(x) = \frac{(n+\alpha)!}{n!\,\alpha!}\, F\left(-n, n+\alpha+\beta+1; \alpha+1; \frac{1-x}{2}\right) \tag{A.46}$$

we see that we have

$$\Pi_m^{\ell'-m}(\alpha) = \frac{1}{\ell'+m+1}\, P_{\ell'-m}^{2m+1,\,1}(1-2\alpha) \tag{A.47}$$

Hence, the integrals I_1 and I_2 of Eqs. (A.38) and (A.39) can be expressed in terms of Jacobi polynomials and Legendre functions of second kind, since we have

$$\frac{1}{2}\int_{-1}^{+1} \frac{P_\ell(z)\, P_m(z)}{x-z}\, dz = Q_\ell(x)\, P_m(x) \qquad \ell \geq m \tag{A.48}$$

We note that owing to the presence of the factors α^m on the right-hand side of Eq. (A.44), the diagonal term in Eq. (A.38) is

$$I_1(\ell,\ell,\alpha) = \alpha^\ell = \left(\frac{s-4}{s'-4}\right)^\ell \tag{A.49}$$

hence providing us with the correct threshold factor. It is easy to see that this will happen in all kernels and therefore the crossing equations (A.25), (A.29), (A.33) can directly be written for amplitudes regularized at threshold, i.e., for $f_\ell(s)/(s-4)^\ell$.

A.5. - Supplementary conditions

These are obtained by imposing t-u symmetry (or antisymmetry) on total $\pi\pi$ amplitudes. They can be written in several ways. The simplest, formally, is to impose that

$$\sum_{I'=0}^{2} \alpha_{st}^{II'} F^{I'}(t,s) = (-1)^I \sum_{I'=0}^{2} \alpha_{su}^{II'} F(4-s-t,s) \tag{A.50}$$

where $F^I(s,t)$ is understood to be defined by Eqs. (A.16) and (A.17) ; this will therefore be an integral constraint on absorptive parts, in regions where the integrals are defined. For instance, in the $\pi^0\pi^0$ case $F^{00}(s,t) = \frac{1}{3}(F^0(s,t) + 2F^2(s,t))$ one gets

$$\int_4^\infty \frac{ds'(2s'-4)}{s'(s'-4)} \left[\frac{t(t-4)}{(s'-t)(s'+t-4)} - \frac{u(u-4)}{(s'-u)(s'+u-4)}\right] A^{00}(s',0)$$

$$- \int_4^\infty \frac{ds'(4-u-t)}{s'(s'+t+u-4)} \left[\frac{u(2s'+t-4)\,A^{00}(s',t)}{(s'+t-4)(s'-u)} - \frac{t(2s'+u-4)\,A^{00}(s',u)}{(s'+u-4)(s'-t)}\right] = 0 \qquad \text{(A.51)}$$

Another way is to impose that the odd (even) partial waves of $F^0(t,s)$ and $F^2(t,s)$ ($F^1(t,s)$) vanish identically. Again in the $\pi^0\pi^0$ case we would have

$$\int_{4-s}^0 dt\, P_{2\ell+1}\left(1+\frac{2t}{s-4}\right) \sum_{\substack{\ell'=0 \\ \ell' \text{ even}}}^{\infty} (2\ell'+1) \int_4^\infty ds'\, \mathrm{Im}\, f_{\ell'}^{00}(s') \left\{ \frac{t(t-4)(2s'-4)}{s'(s'-4)(s'-t)(s'+t-4)} \right.$$

$$\left. + \frac{s(s+t-4)(2s'+t-4)}{s'(s'-s)(s'+t-4)(s'+s+t-4)} P_{\ell'}\left(1+\frac{2t}{s'-4}\right) \right\} = 0 . \qquad \text{(A.52)}$$

which must hold for all values of ℓ and s for which the partial wave expansion converges (note that here we must use the full interval $[4-s,\, 0]$ hence we have $0 \le s \le 32$). Crossing conditions such as (A.50)-(A.52) have already been studied by Wanders [8] and Roskies [9]. The important point is that s and p wave contributions cancel out <u>explicitly</u> from these relations which therefore place no restrictions on them.

APPENDIX B : RELATIONS BETWEEN THE PHYSICAL AND THE UNPHYSICAL REGIONS

This method was introduced by Martin. We shall briefly show some further aspects, first concentrating on $\pi^0\pi^0$ for simplicity. We follow the notations of Appendix A.

B.1. - $\pi^0\pi^0$ scattering

Using the Froissart-Gribov formula for higher partial wave amplitudes, and making use of the Darboux-Christoffel identity [see Eqs. (A.10)] we obtain

$$F(s,t) = f_0(s) + K^0(s,t)\,A(s) \tag{B.1}$$

[We use the operator notation of Eq. (A.7)]. Using the crossing symmetry $F(s,t) = F(t,s)$ we can write

$$f_0(s) - f_0(t) = K^0(t,s)\,A(t) - K^0(s,t)\,A(s) \tag{B.2}$$

of which Eq. (A.15) is a particular example. Equation (B.2) is the starting point for deriving Martin inequalities for the $\pi^0\pi^0$ s wave, as is obvious from its structure ; in the region $0 \leq s \leq 4$, $0 \leq t \leq 4$, the absorptive parts on the right-hand side can be expanded in partial waves and unitarity can be used to show that these absorptive parts are positive. This equation also has the structure

$$f_0(s) - f_0(t) = \varphi(s,t) \tag{B.3}$$

By subtracting at an intermediate point t_0 we also obtain

$$f_0(s) - f_0(t) = \varphi(s,t_0) - \varphi(t,t_0) \tag{B.4}$$

and equating (B.3) and (B.4), we obtain, after inserting the expression for $\varphi(s,t)$ of Eq. (B.2)

$$\begin{aligned} K^0(s,t)\,A(s) - K^0(t,s)\,A(t) = {} & K^0(s,t_0)\,A(s) - K^0(t_0,s)\,A(t_0) \\ & - K^0(t,t_0)\,A(t) + K^0(t_0,t)\,A(t_0) \end{aligned} \tag{B.5}$$

which can also be written as

$$[K^0(s,t) - K^0(s,t_0)]\,A(s) + [K_0(t,t_0) - K_0(t,s)]\,A(t) + [K^0(t_0,s) - K^0(t_0,t)]\,A(t_0) = 0 \tag{B.6}$$

This equation is a three-point crossing constraint on absorptive parts ; if s, t and t_0 are all between 0 and 4, this will be a constraint on physical amplitudes since Legendre expansions will converge. Equation (B.6) is essentially equivalent to the supplementary conditions (A.50)-(A.52) previously discussed.

Particularizing Eq. (B.4) at $t = 4$, $t_0 = 0$ we obtain the following expression for the $\pi^0\pi^0$ s wave

$$f_0(s) = a_0 + \frac{s(s-4)}{\pi} \int_4^\infty \frac{ds'\,(2s'-4)}{s'(s'-4)(s'+s-4)(s'-s)}\, A(s',0)$$

$$- \frac{1}{\pi} \int_4^\infty ds' \frac{\left\{(s-4)\, Q_0\left(1+\frac{2s'}{s-4}\right) - (2s'+s-4)\, Q_1\left(1+\frac{2s'}{s-4}\right)\right\} A(s',s)}{s'(s'+s-4)} \tag{B.7}$$

This form is interesting in that the last term is a "slope" term, responsible for the Martin inequality (4.5) while the second term on the right-hand side, which vanishes at $s = 0$ and at $s = 4$, is a "curvature" term (which by itself would give a positive curvature to the s wave amplitude).

B.2. - Charged pions

The main difference lies in the fact that we now have a crossing matrix. We first write the total isospin amplitudes in the s channel as in Eqs. (A.10.a,b). We now consider combinations of amplitudes which are eigenvectors of the s-t crossing matrix, i.e., which have definite s-t symmetry properties, these can be chosen as

$$F^0(s,t) + 2\,F^2(s,t) \qquad s\text{-}t \text{ symmetric} \tag{B.8.a}$$

$$2F^0(s,t) + 9\,F^1(s,t) - 5\,F^2(s,t) \qquad s\text{-}t \text{ symmetric} \tag{B.8.b}$$

$$2F^0(s,t) - 3\,F^1(s,t) - 5\,F^2(s,t) \qquad s\text{-}t \text{ antisymmetric} \tag{B.8.c}$$

using the same methods as above, we obtain

$$\left[f_0^0(t) + 2 f_0^2(t)\right] - \left[s \leftrightarrow t\right] = K^0(s,t)\left[D_t^0(s) + 2D_t^2(s)\right] - \left[s \leftrightarrow t\right] \tag{B.9.a}$$

$$\left[2f_0^0(t)+27\left(1+\frac{2s}{t-4}\right)f_1^1(t)-5f_0^2(t)\right]-[s\leftrightarrow t]=\left\{K^0(s,t)\left[2D_t^0(s)-5D_t^2(s)\right]+9K^1(s,t)\,D_t^1(t,s)\right\}-\{s\leftrightarrow t\} \qquad (B.9.b)$$

$$\left[2f_0^0(t)-9\left(1+\frac{2s}{t-4}\right)f_1^1(t)-5f_0^2(t)\right]+[s\leftrightarrow t]=\left\{-K^0(s,t)\left[2D_t^0(s)-5D_t^2(s)\right]+3K^1(s,t)\,D_t^1(t,s)\right\}+\{s\leftrightarrow t\} \qquad (B.9.c)$$

where the symbol $[s\leftrightarrow t]$ means that the same expression should be taken after interchanging s and t. We note that in Eq. (B.9.c) we can immediately set $s = t$ in order to obtain

$$2f_0^0(s)-9\left(\frac{3s-4}{s-4}\right)f_1^1(s)-5f_0^2(s)=3K^1(s,s)\,D_t^1(s)-K^0(s,s)\left[2D_t^0(s)-5D_t^2(s)\right] \qquad (B.10)$$

As in Eqs. (B.5), (B.6), we can write crossing constraints on imaginary parts, equivalent to our previous equations (A.50)-(A.52), we shall not repeat this. From these equations we can get a new and amusing relation : if we add Eq. (B.10) to Eq. (B.9.b) and subtract it after changing s into t in it, we directly obtain an expression for the p wave amplitude :

$$18\left[3s+3t-8\right]\left\{\frac{f_1^1(s)}{s-4}-\frac{f_1^1(t)}{t-4}\right\}=$$
$$\left[K^0(t,s)-K^0(t,t)\right]\left[2D_t^0(t)-5D_t^2(t)\right]+3\left[K^1(t,t)+3K^1(t,s)\right]D_t^1(t)-[s\leftrightarrow t] \qquad (B.11)$$

this expression is formally similar to Eq. (B.2), but it is for the p wave regularized at threshold. More interesting is that for $s + t = \frac{8}{3}$, i.e., $u = \frac{4}{3}$ the left-hand side of Eq. (B.11) vanishes identically. Therefore, on the line $u = \frac{4}{3}$, one must have

$$\left[K^0(t,s)-K^0(t,t)\right]\left[2D_t^0(t)-5D_t^2(t)\right]+3\left[K^1(t,t)+3K^1(t,s)\right]D_t^1(t)-[s\leftrightarrow t]=0 \qquad (B.12)$$
$$\text{for } t=8/3-s$$

and for all s where these integrals are defined. Once again, we note that the s and p wave absorptive parts satisfy identically Eq. (B.12), but not the higher waves, which are therefore constrained by this equation.

Finally, let us mention another set of crossing constraints on the absorptive part in the forward direction, that has been written by Martin in the $\pi^0\pi^0$ case and can easily be generalized. In Eq. (B.9.a) for instance, one can eliminate the left-hand side by taking the derivative once with respect to t, and once with respect to s, if one then derives a third time with respect to either of these variables, upon setting $s = t = 0$ the right-hand side gives an integral crossing relation involving the first and second derivatives with respect to angle of the forward amplitude. This can be done in Eq. (B.9.c) by taking $\partial^4/\partial s^2 \partial t^2$ and setting $s = t = 0$. On Eq. (B.9.b) as it stands one would have to take a fifth order derivative, however, if one considers a different combination of amplitudes, for instance $F^0(s,t) - F^2(s,t)$ which crosses into $F^1(t,s) + F^2(t,s)$, the p wave amplitude will only appear on one side and second order derivatives in each variable will be sufficient.

A

	$f_o^{00}(4) - f_o^{00}(0)$	Σ_s	Σ_{f_o}
$\lambda\varphi 4$	$3.6177\ 10^{-2}$	$3.6177\ 10^{-2}$	$0.003\ 10^{-2}$
GMN	$1.551\ 10^{-2}$	$1.461\ 10^{-2}$	$0.017\ 10^{-2}$

B

	$a_2 + \frac{2}{3}[f_o^2(0) - f_o^0(0)]$	Σ'_s	Σ'_p
$\lambda\varphi 4$	$1.74\ 10^{-2}$	$1.82\ 10^{-2}$	$0.05\ 10^{-2}$
GMN	$0.962\ 10^{-2}$	$0.313\ 10^{-2}$	$0.437\ 10^{-2}$

C

	σ_∞	σ'_∞	$\sigma_\infty / \sigma'_\infty$
$\lambda\varphi 4$	< 0	< 0	-
GMN	50 mb	70 mb	0.72

A - Values of various contributions to sum rule (4.4), in the two cases of the Padé approximation to the $\lambda\phi^4$ theory ($\lambda\varphi^4$) and of the central solution of Ref. 7) (GMN). The f_o contribution Σ_{f_o} is calculated as given by the model in $\lambda\varphi^4$, and in terms of physical mass and width for GMN.

B - Same as above for the sum rule (4.11) ; here Σ'_p is calculated as given by these models.

C - Values of "asymptotic cross-sections" deduced from these sum rules, and their ratio.

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FIGURE CAPTIONS

Figure 1 Real part of the $I = 0$ s wave $\pi\pi$ amplitude in the GMN model. Solid curve : as given by the model ; dashed curve : as computed by Eq. (2.6) with imaginary parts given by the model. The corresponding phase shift is curve A of Fig. 2.

Figure 2 Two possibilities for the $I = 0$ s wave $\pi\pi$ phase shift.
Curve A : central solution of the GMN model ;
Curve B fits the "up" solution above the ρ and corresponds to the same scattering length as curve A.
Experimental data are from : E. Malamud and P. Schlein ("up-down" solution), Proceedings of the Argonne Conference on $\pi\pi$ and πK Interactions, Argonne (1969) ; and J.P. Baton, G. Laurens and J. Reignier - Phys.Letters 33B, 528 (1970).

Figure 3 Real part of the $I = 0$ s wave $\pi\pi$ amplitude as computed starting from curve B of Fig. 2. Solid curve : direct calculation $\left(\sqrt{\frac{s}{s-4}}\cos\delta\sin\delta\right)$. Dashed curve as computed through Eq. (2.6).

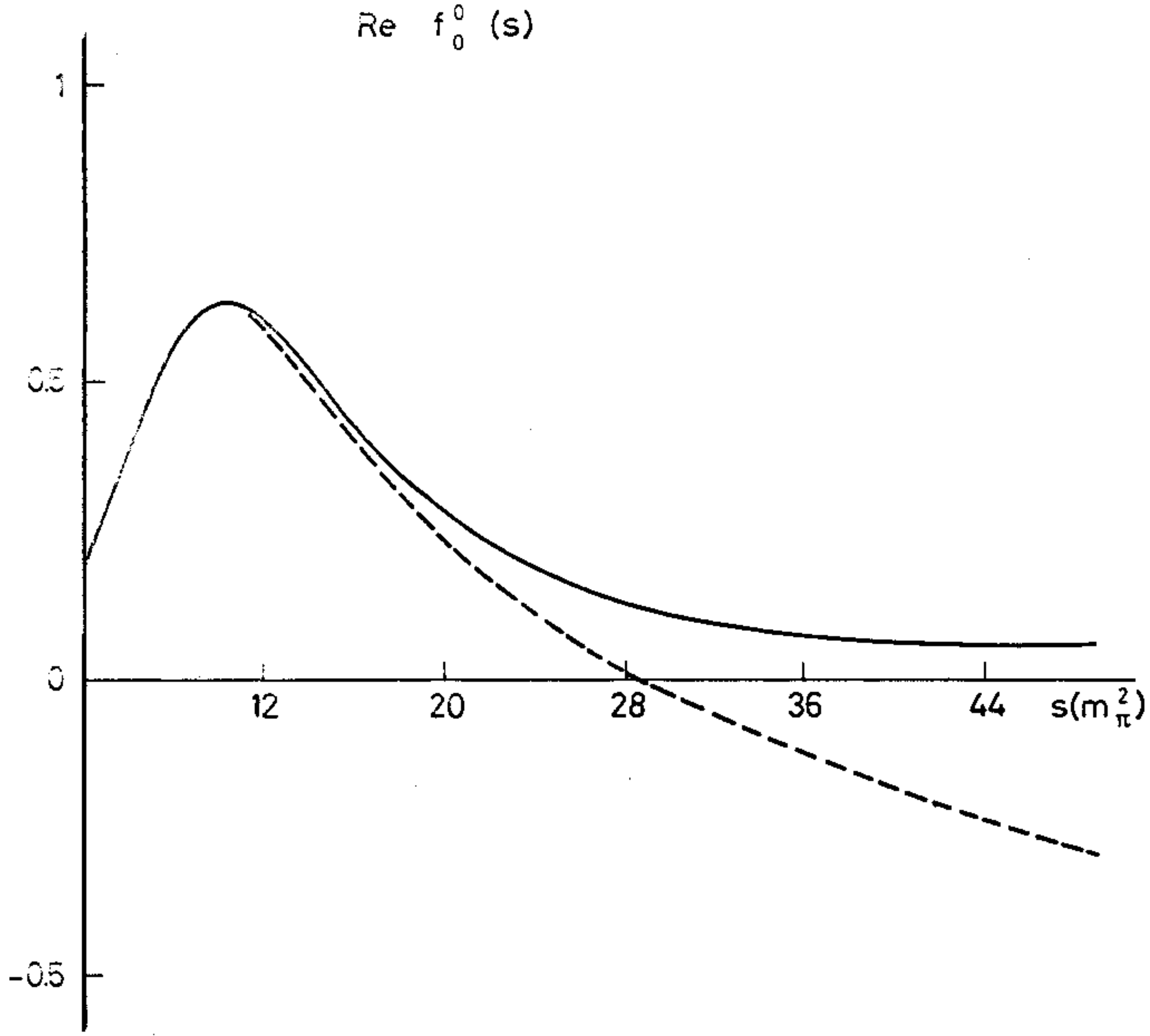
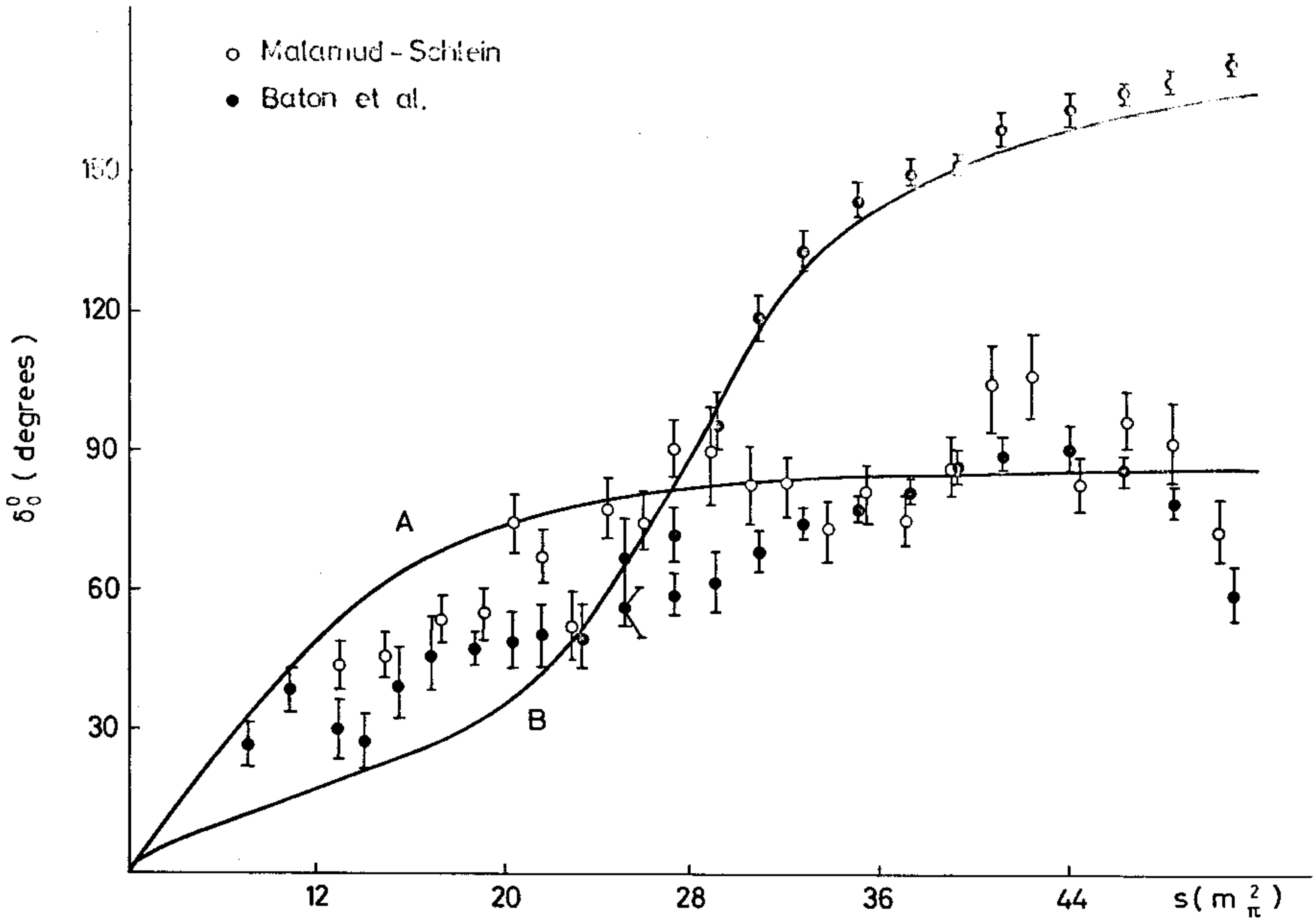


FIG. 1



Malamud - Schlein
Baton et al.
150
120
90
60
30
δ_0^0 (degrees)
A
B
12
20
28
36
44
s(m_π^2)

FIG. 2

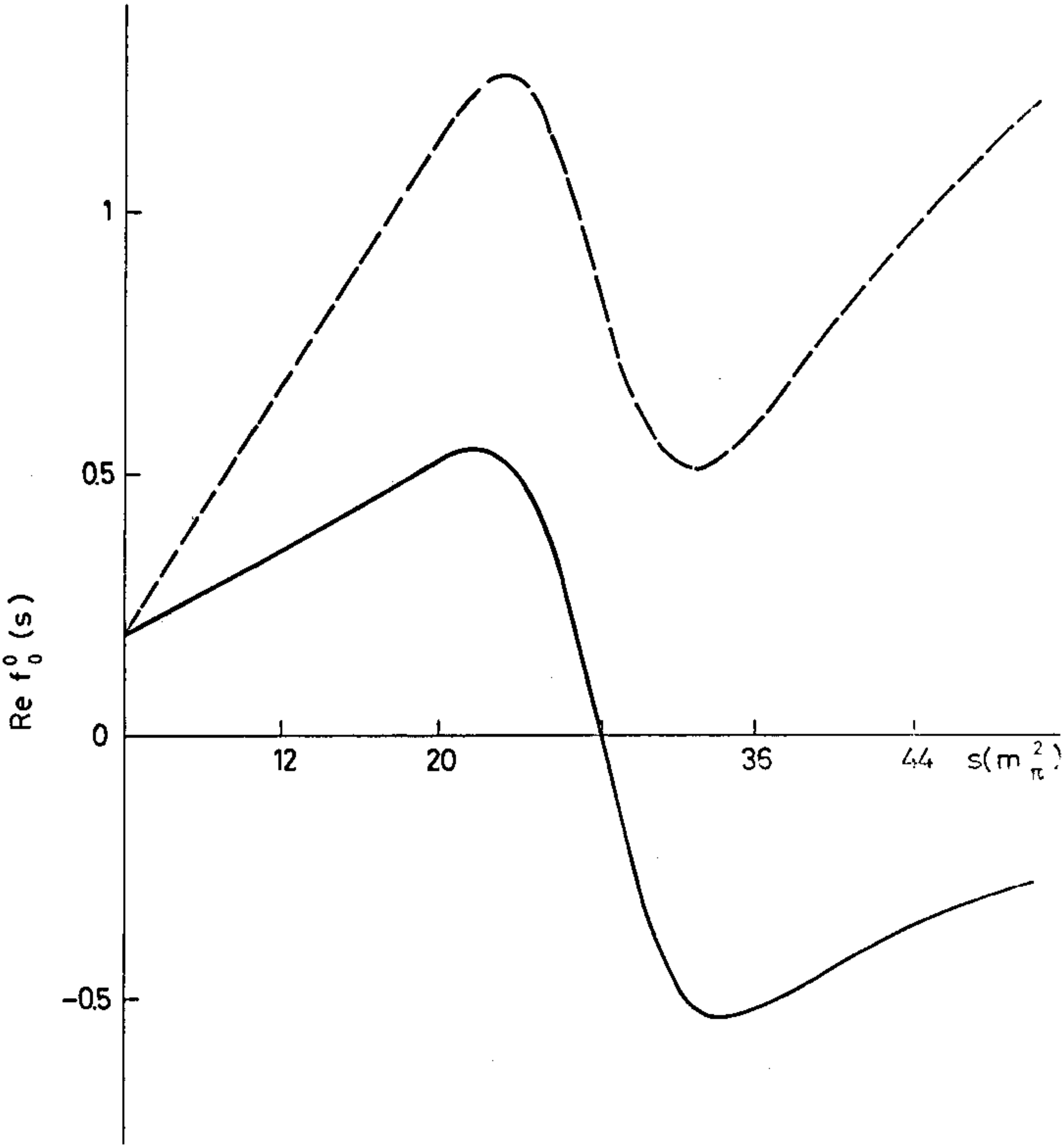


FIG. 3