Ref.TH.1008-CERN

CROSSING CONDITIONS FOR THE PION-PION AMPLITUDE

André Martin

CERN - Geneva

A B S T R A C T

We obtain a denumerable set of rigorous inequalities on the $\pi_0 \pi_0 \to \pi_0 \pi_0$ partial wave amplitudes which guarantee that the amplitude is crossing symmetric. The numerical inequalities presented for the case of S and D waves indicate that one has very little freedom on the choice of the D waves once the S waves are given.

1. INTRODUCTION

One of the problems encountered in the recent years is to construct scattering amplitudes which satisfy, among other things, the requirement of crossing symmetry and unitarity. One of the traditional approaches, both theoretical and experimental is to expand the scattering amplitude in partial wave amplitudes for which unitarity is easy to express. This is the case, for instance, when one wants to use the N/D formalism [1]. Then one has the problem of satisfying the requirement of crossing symmetry: naturally one would like to make sure that one gets the same result if one expresses the scattering amplitude by a sum over partial wave amplitudes in the s channel or in the t channel, in a region where both expansions converge. The difficulty is that if one approximates the amplitude with a finite number of partial waves the requirement of exact crossing symmetry on the approximate amplitude cannot be satisfied. More specifically, let us take the neutral pion-pion case. We have

$$ F(s, t, u) = \sum_{\ell \text{ even}} (2\ell+1) f_\ell(s) P_\ell(\cos\theta_s) \tag{1} $$

with $\cos\theta_s = 1 + \dfrac{2t}{s-4}$ (the pion mass is unity) and,

$$ F(s, t, u) = \sum_{\ell \text{ even}} (2\ell+1) f_\ell(t) P_\ell(\cos\theta_t) \tag{2} $$

with $\cos\theta_t = 1 + \dfrac{2s}{t-4}$. Expansion (1) is crossing symmetric in the exchange of t and u. From the analyticity properties of the pion-pion amplitude, we know that a common domain of convergence of both expansions (1) and (2) is the triangle $s > 0$, $t > 0$, $u > 0$. If we approximate F by a finite number of partial waves:

$$F \simeq F_L^S = \sum_{\ell=0}^{\ell=L-2} (2\ell+1) f_\ell(s) P_\ell(\cos\theta_s) \qquad (3)$$

Then the equation

$$F_L^S \equiv F_L^t \qquad (4)$$

would imply that F_L^S is a polynomial in s and t, and, due to the reality of $f_\ell(s)$ for $0 < s < 4$, a _real_ polynomial. Therefore, $\text{Im}\, F(s, t = 0, 4-s)$ is identically zero and the total cross-section, given by the optical theorem, is zero.

What is generally done to avoid this catastrophy is to impose sufficiently few crossing conditions, for instance to impose the equality of F_L^S and F_L^t and a _finite_ number of their derivatives at the symmetry point $s = t = u = 4/3$. However, there is no guarantee that it is a consistent procedure and one might very well be led into error by what might be called "unphysical intuition".

What we shall do in this paper is to use positivity to present substitutes for the unacceptable equation (4). More specifically we shall

1) show that there are subdomains inside $s > 0$, $t > 0$, $u > 0$ in which one can establish rigorous inequalities between F_L^S and F_L^t;

2) show that for $L > 4$ one can find a bound on $F - F_L^S$ in terms of the last non-neglected partial wave, $f_{L-2}(s)$, and find curves in the $s\,t$ plane along which the uncertainty in $F - F_L^S$ is minimized. This again will lead to inequalities following from crossing symmetry and involving F_L^S, $f_{L-2}(s)$, F_L^t, $f_{L-2}(t)$;

3) show that the set of inequalities thus obtained is complete
    (which does not mean optimum) in the sense that an amplitude
    satisfying all our conditions for increasing $L \to \infty$ is
    exactly crossing symmetric.

As an illustration of the strength of these restrictions
we present optimized inequalities for $L = 4$, i.e., involving only
S and D waves. It is seen that once the S wave is given the
arbitrariness in the D wave is very small.

One might question the usefulness of this work in view of
the existence of a new approach which avoids completely partial
wave expansions and nevertheless generates, by iteration procedure,
a crossing symmetric and unitary amplitude, namely that of
Atkinson [2]. For our defence we want to point out that Atkinson's
approach meets serious difficulties when one wants to incorporate
resonances and introduce subtractions. In fact, integrals over
single spectral functions can be replaced by integrals over the
imaginary part of partial wave amplitudes with small angular momentum
and it is therefore important to study the crossing conditions on
low partial waves. The other motivation is that by looking at the
S wave alone, Wanders and his collaborators [3] find families of
solutions which, in practice, depend only on a few parameters. The
question is then to understand why the Atkinson solution depends on
an arbitrary function of two real variables. One would like to see
what happens in the Wanders approach when higher waves are taken
into account.

2. COMPARISON OF THE EXACT AMPLITUDE WITH THE INCOMPLETE PARTIAL WAVE SUM

For this comparison we use two basic information which can be now considered as consequences of local field theory:

i) for $0 \leqslant s \leqslant 4$ we can write fixed s dispersion relations with two subtractions [4]:

$$F(s,t,u) = C(s) + \frac{t^2}{\pi} \int_4^\infty \frac{A_t(s,t')\,dt'}{t'^2(t'-t)} + \frac{u^2}{\pi} \int_4^\infty \frac{A_u(s,u')\,du'}{u'^2(u'-u)} \tag{5}$$

Here, of course, $A_t(s,x) = A_u(s,x)$ due to $t \rightleftharpoons u$ symmetry;

ii) from unitarity in the t channel $A_t(s,t,u)$, for $t > 4$, $0 \leqslant s \leqslant 4$, is _positive_, because it can be written as the _convergent_ series

$$A_t(s,t,u) = \sum (2\ell+1)\,\mathrm{Im}\,f_\ell(t)\,P_\ell\left(1+\frac{2s}{t-4}\right) \tag{6}$$

where $\mathrm{Im}\,f_\ell(t)$ and $P_\ell\left(1+\frac{2s}{t-4}\right)$ are positive.

We shall prefer to use the variable $\cos\Theta_s$ and write (5) as

$$F(s, \cos\theta_s) = C'(s) + \frac{2(\cos\theta_s)^2}{\pi} \int_{z_0}^{\infty} \frac{A(s,z)\,dz}{z(z^2 - \cos^2\theta_s)}$$

(7)

where $z_0 = \frac{4+s}{4-s}$.

From (7) follows the Froissart-Gribov formula for $\ell \geqslant 2$
$4 \geqslant s > 0$

$$f_\ell(s) = \frac{2}{\pi} \int_{z_0}^{\infty} A(s,z)\, Q_\ell(z)\,dz$$

(8)

One can then use (7) and (8) together with the second Darboux-Christoffel formula

$$\frac{1}{z-x} = \sum_{0}^{L-1} (2\ell+1)\, Q_\ell(z)\, P_\ell(x)$$
$$+ L\, \frac{P_L(x)\, Q_{L-1}(z) - Q_L(z)\, P_{L-1}(x)}{z-x}$$

(9)

to write F as F_L^s plus a rest:

$$F(s, \cos\theta_s) = \sum_{\ell=0}^{\ell=L-2} (2\ell+1)\, f_\ell(s)\, P_\ell(\cos\theta_s)$$

$$+ \frac{2L}{\pi} \int_{z_0}^{\infty} \frac{z\, Q_{L-1}(z)\, P_L(\cos\theta_s) - \cos\theta_s\, P_{L-1}(\cos\theta_s)\, Q_L(z)}{z^2 - \cos\theta_s^2} \cdot$$

$$\cdot A(s,z)\, dz$$

$$(10)$$

where, from Eq. (6), $A(s,z)$ is positive. Equation (10) is the basis for all what follows. First we notice that the sign of $F - F_L^S$ is known whenever the sign of the integrand in (10) is known, i.e., whenever

$$z\, Q_{L-1}(z)\, P_L(\cos\theta) - \cos\theta\, P_{L-1}(\cos\theta)\, Q_L(z) \tag{11}$$

has a constant sign when z varies from $z_0 = (4+s)/(4-s)$ to infinity. For $\cos\theta$ real, the sign of (11) for $z \to \infty$ is given by $P_L(\cos\theta)$ since $z Q_{L-1}(z) \sim \text{const.}(z)^{-L+1}$ while $Q_L \sim \text{const. } z^{-L-1}$. Therefore, (11) has a constant sign, equal to the sign of $P_L(\cos\theta)$ if for $z \geqslant z_0$

$$\frac{z\, Q_{L-1}(z)}{Q_L(z)} > \frac{\cos\theta\, P_{L-1}(\cos\theta)}{P_L(\cos\theta)}$$

but since Q_{L-1}/Q_L is an increasing function of z for $z > 1$ it is sufficient to require

$$\frac{z_0 \, Q_{L-1}(z_0)}{Q_L(z_0)} > \frac{\cos\theta \, P_{L-1}(\cos\theta)}{P_L(\cos\theta)} \tag{12}$$

We therefore get for $-1 < \cos\theta < +1$ a set of intervals in which $F-F_L^S$ has a known sign. For $0 < \cos\theta$ the right extremities of the intervals are given by

$$P_L(\cos\theta) = 0 \tag{13}$$

the left extremities by

$$P_L(\cos\theta) \frac{z_0 \, Q_{L-1}(z_0)}{Q_L(z_0)} - \cos\theta \, P_{L-1}(\cos\theta) = 0 \tag{14}$$

The gaps, in which the sign of the integrand in (10) changes with z, are extremely narrow for s close to 4, which corresponds to z_0 very large and hence $z_0 Q_{L-1}(z_0)/Q_L(z_0)$ very large. For $-1 < \cos\theta < +1$ the number of intervals with definite, alternate signs of $F-F_L^S$ is $L+1$. Since F_L^S is a polynomial of degree $L-2$ in $\cos\theta$

$$\left(\frac{d}{d\cos\theta}\right)^L (F - F_L^S) \equiv \left(\frac{d}{d\cos\theta}\right)^L F$$

which is easily shown from (7) to be positive. Therefore, $F-F_L^S$ vanishes at most L times and since there are L gaps in which it vanishes at least once, it vanishes once and only once in each gap.

To illustrate the situation we have shown on Fig. 1 the case $L = 4$, i.e., $F_L^S = f_0(s) + 5P_2(\cos \Theta_s)f_2(s)$. The shaded regions correspond to definite signs of $F-F_L^S$. Since the picture may be difficult to read let us indicate as an example that for $s = 2$ we have

$$F - F_4^S > 0 \quad \text{for} \quad 0.8640 < |\cos \Theta| < 1$$

$$F - F_4^S < 0 \quad \text{for} \quad 0.3439 < |\cos \Theta| < 0.8611$$

$$F - F_4^S > 0 \quad \text{for} \quad |\cos \Theta| < 0.3400$$

$$(15)$$

The fact that the zeros of $F-F_L^S$ are sharply fixed is not astonishing. In fact, this whole thing is the development of an idea of Burkhardt [5] who noticed that near the line $P_L(\cos \Theta) = 0$ the error committed in neglecting waves with $\ell \gg L$ is of the order of f_{L+2} and not f_ℓ as it would be for $P_L(\cos \Theta) \neq 0$.
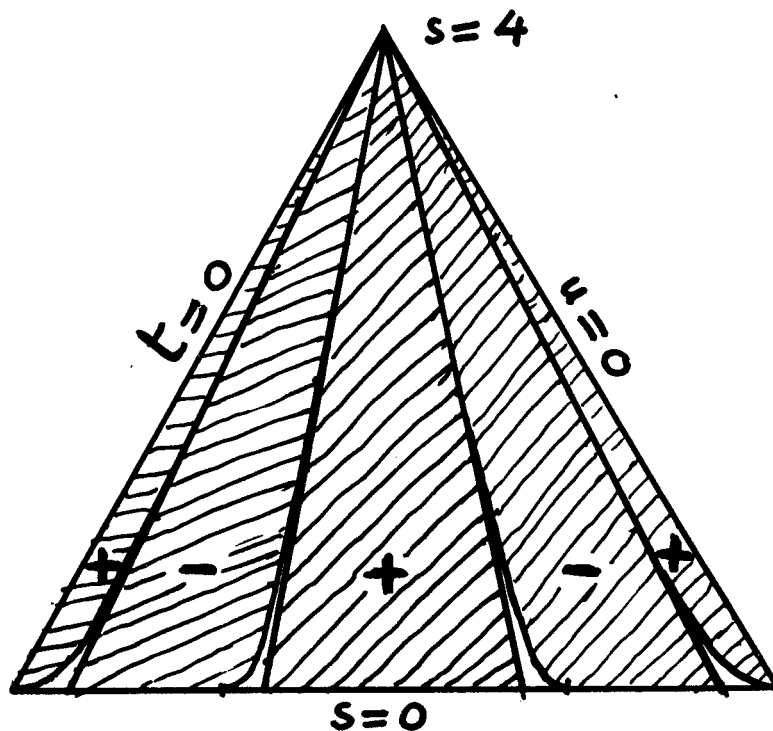
- Figure 1 -

So far we have only discussed the sign of $F-F_L^S$. Now we would like to put upper and lower bounds on $F-F_L^S$ in terms of the last partial wave amplitude appearing in F_L^S, namely $f_{L-2}(s)$. This is possible only for $L \geqslant 4$ since the Froissart-Gribov representation holds only for $\ell \geqslant 2$. After some manipulations using recursion formulas one can rewrite $F-F_L^S$ as

$$F - F_L^S = \frac{2(L-1)}{\pi} \int_{z_0}^{\infty} \frac{\cos\theta \, P_{L-1}(\cos\theta) Q_{L-2}(z) - z Q_{L-1}(z) P_{L-2}(\cos\theta)}{z^2 - \cos^2\theta} \times A(s,z) \, dz$$

Hence, comparing with (8) we get

$$(L-1) m_L(s,\cos\theta_s) f_{L-2}(s) < F - F_L^S < (L-1) M_L(s,\cos\theta_s) f_{L-2}(s) \tag{16}$$

with

$$M_L(s,\cos\theta_s) = \sup_{z_0 < z < \infty} \frac{\cos\theta_s P_{L-1}(\cos\theta_s) - \dfrac{z Q_{L-1}(z)}{Q_{L-2}(z)} P_{L-2}(\cos\theta_s)}{z^2 - \cos^2\theta_s}$$

$$m_L(s,\cos\theta_s) = \inf_{z_0 < z < \infty} \frac{\cos\theta_s P_{L-1}(\cos\theta_s) - \dfrac{z Q_{L-1}(z)}{Q_{L-2}(z)} P_{L-2}(\cos\theta_s)}{z^2 - \cos^2\theta_s}$$

Clearly, M_L and m_L are finite for any point $(s, \cos\theta_s)$, such that $s > 0$, $t > 0$, $u > 0$. However, we would like to select points where the uncertainty $M_L - m_L$ is minimum. From the previous analysis we suspect that the uncertainty is minimum in the neighbourhood

of the lines $P_L(\cos \Theta_s) = 0$ or of the lines (14) which are very close, at least for s not too close to zero.

We shall prove that

$$\frac{M_L(s, \cos\Theta_s) - m_L(s, \cos\Theta_s)}{P_{L-2}(\cos\Theta_s)}$$

is minimum along the lines (14).

Let

$$\phi(z) = \frac{z \, Q_{L-1}(z)}{Q_{L-2}(z)}$$

$$\lambda = \frac{\cos\Theta \, P_{L-1}(\cos\Theta)}{P_{L-2}(\cos\Theta)}$$

What we have to study is

$$\sup_{z_0 < z < \infty} y(\lambda, z) - \inf y(\lambda, z)$$

with

$$y(\lambda, z) = \frac{\lambda - \phi(z)}{z^2 - \cos^2\Theta}$$

Of course $\cos\Theta$ is a function of λ, but since z is most of the time large as compared to $\cos\Theta$ we neglect the variation of $\cos\Theta$. $\phi(z)$ is easily shown to be a <u>decreasing</u> function of z, as

$$(2L-1)\,\phi(z) = L-1 + L \, Q_L(z)\big/ Q_{L-2}(z)$$

and it has been shown [6] that Q_L/Q_{L-2} is decreasing.

We must now distinguish three cases

a) $\lambda > \phi(z_o)$ then $\inf y = 0$ and

$$\sup(y) = \frac{\lambda - \phi(z_M(\lambda))}{z_M^2 - \cos^2\theta}$$

where $z_M \geqslant z_o$.

If $z_M > z_o$

$$\frac{\partial}{\partial z} y(\lambda, z_M) = 0$$

and

$$\frac{d}{d\lambda}\left[y(\lambda, z_M(\lambda))\right] = \frac{1}{z_M^2 - \cos^2\theta} > 0$$

If $z_M = z_o$

$$\frac{d}{d\lambda}\left[y(\lambda, z_o)\right] = \frac{1}{z_o^2 - \cos^2\theta} > 0$$

so we make y minimum by reducing λ. Hence, we must take $\lambda \leqslant \phi(z_o)$.

b) $\phi(\infty) \lesssim \lambda \lesssim \phi(z_0)$

$$\sup(y) - \inf(y)$$
$$= \frac{\lambda - \phi(z_M(\lambda))}{z_M^2 - \cos^2\theta} - \frac{\lambda - \phi(z_0)}{z_0^2 - \cos^2\theta}$$

With $\phi(z_M) < \lambda$ and $z_M > z_0$

$$\frac{d}{d\lambda}\left[\sup(y) - \inf(y)\right]$$
$$= \frac{1}{z_M^2 - \cos^2\theta} - \frac{1}{z_0^2 - \cos^2\theta} < 0$$

Hence, we must take λ as large as possible.

c) $\lambda < \phi(\infty)$

$$\sup(y) - \inf(y) = -\frac{\lambda - \phi(z_0)}{z_0^2 - \cos^2\theta}$$

Here again we must take λ as large as possible.

The conclusion is that the optimum choice is $\lambda = \phi(z_0)$ which corresponds precisely to Eq. (14). Notice that there is a break in the derivative of $\sup(y) - \inf(y)$ at $\lambda = \phi(z_0)$ which makes the minimum sharper and which justifies a posteriori the neglect of the variation of $\cos\theta$ in the denominator of y.

Therefore the uncertainty on $F-F_L^S$, when f_{L-2} is given, is minimum along the lines

$$P_L(\cos\theta_s)\,\frac{z_0\,Q_{L-1}(z_0)}{Q_L(z_0)} - \cos\theta_s\,P_{L-1}(\cos\theta_s) = 0$$

or equivalently

$$P_{L-2}(\cos\theta_s)\,\frac{z_0\,Q_{L-1}(z_0)}{Q_{L-2}(z_0)} - \cos\theta_s\,P_{L-1}(\cos\theta_s) = 0 \tag{17}$$

Then, the uncertainty is given by

$$0 < \frac{F - F_L^S}{f_{L-2}^{(s)}\,P_{L-2}(\cos\theta_s)} < (L-1)\,\sup\left(\frac{\phi(z_0) - \phi(z)}{z^2 - \cos^2\theta}\right) \tag{18}$$
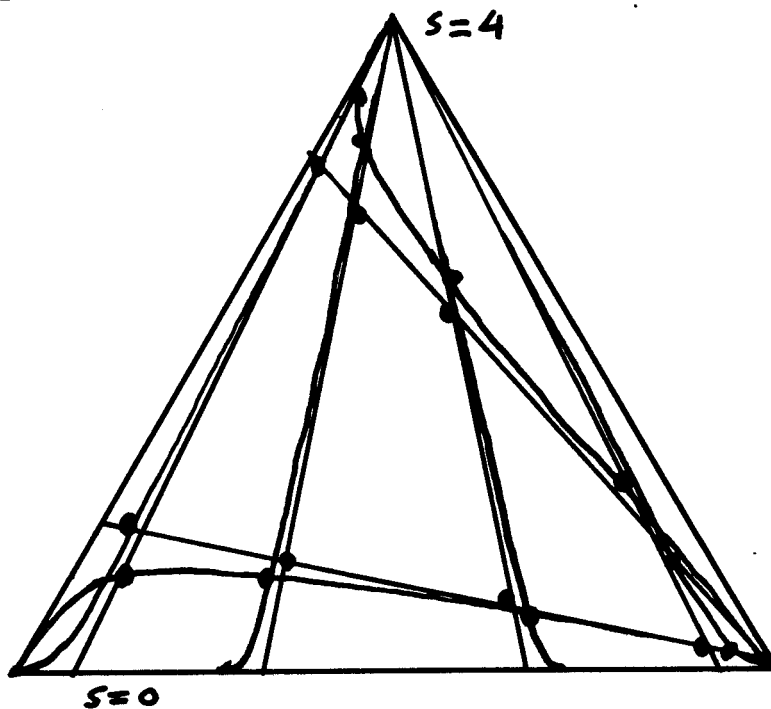
with

$$\phi(z) = \frac{z\,Q_{L-1}(z)}{Q_{L-2}(z)}$$

3. INEQUALITIES FOLLOWING FROM CROSSING SYMMETRY

The first and simplest way to get interesting inequalities following crossing symmetry consists in taking the intersection of a region where $F-F_L^s$ is, say, positive and that of a region where $F-F_{L'}^t$ is negative. Then we have $F_{L'}^t > F_L^s$. Clearly, the sharpest inequalities will be obtained on the borders, or more precisely the corners, of such regions. A particularly simple case of such inequalities is the case $L = L' = 0$ and has already been reported [7]; the most stringent inequality obtained was

$$f_0(3.205) > f_0(0.2134) > f_0(2.9863)$$

(19)

The same idea can be applied to a comparison of F_{L+2}^s and F_L^t. This has the advantage of producing inequalities for the Lth partial wave amplitude at one given energy in terms of lower partial wave amplitudes. For instance, this has been done for F_4^s and F_2^t as is shown on Fig. 2

- Figure 2 -

The results are given on Table I. The inequalities are given in pairs going in opposite direction.

The advantage of Table I is its simplicity: given any model or suggestion for the S wave one can find the D wave in the interval $0 < s < 4$ with an uncertainty of the order of $\pm 15\%$, if one interpolates smoothly between the points where one has inequalities. This is already rather remarkable if we remember that Atkinson has shown that the pion-pion amplitude depends on an arbitrary function of two real variables.

However, the most stringent results are obtained by the second method, in which one compares F_L^s and F_L^t at the intersections of the lines of minimum uncertainty given by Eq. (14) and the analogue obtained by exchanging s and t. In this way one obtains $\dfrac{L(L-1)}{2}$ pairs of inequalities. In a given pair of inequalities only the coefficients change, not the values of s involved, so that one can very easily see whether the inequalities are constraining or not. Table II exhibits the "complete" set of optimized inequalities for $L = 4$, obtained by taking the intersections of the curves

$$P_4(\cos\theta_s)\frac{z_0(s)\,Q_3\big(z_0(s)\big)}{Q_4\big(z_0(s)\big)} - \cos\theta_s\,P_3(\cos\theta_s) = 0$$

and

$$P_4(\cos\theta_t)\frac{z_0(t)\,Q_3\big(z_0(t)\big)}{Q_4\big(z_0(t)\big)} - \cos\theta_t\,P_3(\cos\theta_t) = 0$$

At these intersections represented on Fig. 3, one applied inequality (18).
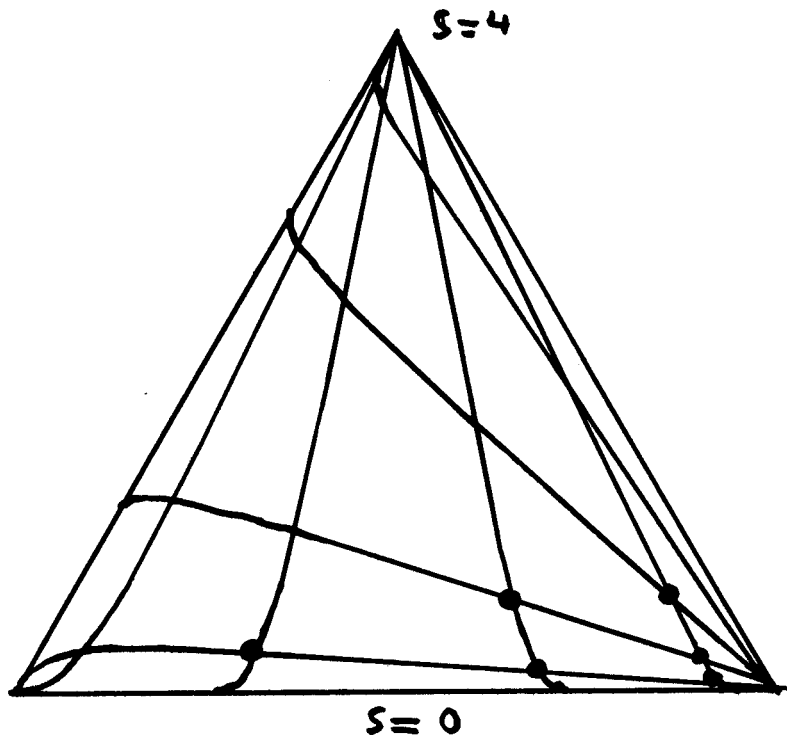
- Figure 3 -

We see that the double inequalities listed in Table II
can be considered as approximate equations with small uncertainties
on the coefficients of the D wave amplitudes. If we exclude the
last double inequality which is rather poor, we see that the accuracy
of the D wave coefficients varies from $\pm 0.8\%$ to $\pm 6\%$. Let us
repeat that this is a rather striking fact. It would even be more
striking if we considered higher waves, but then it would be
necessary to increase the number of figures used to indicate the
various critical energies.

4. COMPLETENESS OF THE SET OF INEQUALITIES

The procedure that we described in the previous Section, which consists in comparing F_L^s and F_L^t at selected points inside the triangle $s > 0$, $t > 0$, $u > 0$, yields for each L $\frac{L(L-1)}{2}$ pairs of inequalities. The question is then this: suppose we have a scattering amplitude given by the sequence

$$f_0(s) , f_2(s) , f_4(s) , f_6(s) \ldots$$

suppose that the partial sums satisfy the inequalities previously described; is the scattering amplitude thus constructed crossing symmetric?

In fact, we have to add additional requirements which follow from positivity and the Froissart-Gribov representation:

i) $f_\ell(s) > 0$ for $0 < s < 4$;

ii) $f_\ell(s)$ is at least analytic in $|s| < 4$ minus the cut $s < 0$ and for $\ell \geq 2$ $|s| < 4$

$$\left| f_\ell(s) \right| < f_2(|s|) \frac{4-|s|}{|4-s|} \underset{4 < t < \infty}{\text{Max}} \frac{\left| Q_\ell\left(\frac{2t}{4-s} - 1\right) \right|}{Q_2\left(\frac{2t}{4-|s|} - 1\right)} \tag{20}$$

which implies, in particular, for $0 < s < 4$, $\ell \geq 2$

$$0 < f_\ell(s) < f_2(s) \frac{1}{\left[\frac{4+s}{4-s} + \sqrt{\left(\frac{4+s}{4-s}\right)^2 - 1} \right]^{\ell - 2}} \tag{21}$$

Then, one can show that $\sum (2\ell+1)f_\ell (s)P_\ell (\cos\Theta_s)$ converges inside the ellipse with foci at $t=0$ and $u=0$ passing through the point $t=4$. At the points where the crossing conditions are imposed [the intersection of the lines (14) with their homologues, exchanging s and t] the uncertainty, according to (18) and (21), decreases **exponentially** with L. Therefore, we have a series of sets of points populating the triangle $s>0$, $t>0$, $u>0$ where crossing symmetry is imposed with an uncertainty $\exp(-CL)$, the number of points being proportional to L^2. It is clear, from condition (20) that both $\sum (2\ell+1)f_\ell (s)P_\ell (\cos\Theta_s)$ and $\sum (2\ell+1)f_\ell (t)P_\ell (\cos\Theta_t)$ are analytic in $|s-2|<2 \otimes |t-2|<2$, which contains the triangle $s>0$, $t>0$, $u>0$. Given any point $s_0 t_0$ of the triangle, one can find another point arbitrarily close where one crossing condition for arbitrarily large L is imposed. From the uniform continuity of analytic functions and from the convergence of $F_L^s(s_0,t_0)$ towards $F_\infty^s(s_0,t_0)$ for $L\to\infty$ one can easily show that crossing symmetry is fulfilled as well as one wishes at the point s_0,t_0, and since $s_0 t_0$ is arbitrary, we have

$$F_\infty^s(s,t) \equiv F_\infty^t(s,t)$$

5. MISCELLANEOUS REMARKS

We have obtained restrictions on the pion-pion partial wave expansion which may have some importance. We have seen that from the point of view of crossing symmetry they form a kind of complete set. However, such a complete set is not unique. The ideal complete set is the one in which one has the largest number of significant independent conditions on F_L^S for L small, and we do not know how far we are from this goal. In fact we have put aside a simple consistency question which is: for a given s, what are the compatibility conditions between the $f_\ell(s)$'s which, as we know, satisfy the Froissart-Gribov formula for $\ell \geqslant 2$ $|s| < 4$? This question has been solved by Common [8] for the case of $0 < s < 4$, using again the positivity of the absorptive part in the crossed channel, and will be published shortly.

We have seen the important role of positivity through all this work. Unitarity, in the form $\operatorname{Im} f_\ell(s) \geqslant |f_\ell(s)|^2 \frac{2k}{\sqrt{s}}$ for $s > 4$ plays no significant role. As we know that it is easy to construct crossing symmetric models such that $\operatorname{Im} f_\ell(s) \geqslant 0$ for $s > 4$, we must not be astonished if such models satisfy our crossing conditions automatically. Our conditions are interesting for models in which each individual partial wave is built to be exactly unitary and where crossing symmetry is not obvious. Also let me point out that the lowest order Cini-Fubini approximation [9] to the neutral $\pi_0 \pi_0 \to \pi_0 \pi_0$ amplitude, i.e.,

$$ F = \frac{1}{\pi} \int \frac{\rho(s')ds'}{s'-s} + \frac{1}{\pi} \int \frac{\rho(t')dt'}{t'-t} + \frac{1}{\pi} \int \frac{\rho(u')du'}{u'-u} $$

where unitarity is imposed only on the S wave, satisfies exactly our conditions. However, higher order Cini-Fubini approximations will not, a priori, satisfy our conditions because they would imply that the Froissart-Gribov formula is not valid for $\ell = 2$ $0 \leqslant s < 4$.

One important problem left is that of isospin. Of course,
our results apply to the physical $\pi_0\pi_0 \to \pi_0\pi_0$ amplitude, i.e.,
$1/3(T = 0)+2/3(T = 2)$. The difficulty for other isospin states is to
have information on the sign of the absorptive parts involved. How-
ever, this generalization has been shown to be possible by
Auberson [10] and opens the very exciting possibility of having in-
equalities involving S and P waves only. In addition, we shall
show in a forthcoming paper that simple inequalities between the
values of the S and P wave amplitudes at $s = 0$ and $s = 4$
can be obtained.

It is of course not clear at all, as we said already,
that our set of inequalities is optimum. To carry this programme
it might be necessary, as suggested by Wanders, to use some systematic
approach, for instance the Balachandran-Nuyts expansion [11].

Let us discuss the relationship with Atkinson's work.
When one looks at Table II it seems hard to believe that the
$\pi\pi \to \pi\pi$ amplitude depends at least on a function of two real
variables. The fact that the low energy scattering amplitude is
sensitive only to a few parameters had already been suggested by
Wanders and seems to come out again here. The question is then to
know how far this insensitivity will propagate. Certainly not to
$s > 16$, $t > 16$, but perhaps to some rather large part of the physical
region, $s > 4$ $|\cos\theta_s| < 1$. It is certainly desirable to build
models with sufficient flexibility to investigate these questions.
It would also be useful to carry practically the Atkinson iterative
construction and see what happens when one starts with two radically
different inputs. Let me also repeat that our constraints on
individual partial waves show that it might be difficult to carry
Atkinson's programme with many subtractions, because we know that our
constraints must be satisfied and that they are necessary but most
likely not sufficient to ensure positivity.

Finally, let us conclude with one request: could people who compute $\pi\pi$ amplitudes publish not only phase shifts but also the values of their amplitudes in the unphysical interval $0 \leqslant s \leqslant 4$ to allow present and future tests of crossing symmetry and positivity.

TABLE I

$$\begin{cases} 4.067 \\ 3.061 \end{cases} \quad \begin{aligned} f_2(0.0341) &< f_0(3.839) - f_0(0.0341) \\ f_2(0.0730) &> f_0(3.654) - f_0(0.0730) \end{aligned}$$

$$\begin{cases} 1.428 \\ 1.633 \end{cases} \quad \begin{aligned} f_2(0.304) &< f_0(0.304) - f_0(2.543) \\ f_2(0.325) &> f_0(0.325) - f_0(2.463) \end{aligned}$$

$$\begin{cases} 3.061 \\ 3.252 \end{cases} \quad \begin{aligned} f_2(0.589) &> f_0(0.237) - f_0(0.589) \\ f_2(0.803) &< f_0(0.199) - f_0(0.803) \end{aligned}$$

$$\begin{cases} 1.498 \\ 2.050 \end{cases} \quad \begin{aligned} f_2(0.572) &< f_0(0.572) - f_0(1.087) \\ f_2(0.572) &> f_0(0.572) - f_0(1.294) \end{aligned}$$

$$\begin{cases} 3.267 \\ 3.061 \end{cases} \quad \begin{aligned} f_2(0.747) &< f_0(3.052) - f_0(0.747) \\ f_2(0.826) &> f_0(2.953) - f_0(0.826) \end{aligned}$$

$$\begin{cases} 1.620 \\ 1.929 \end{cases} \quad \begin{aligned} f_2(2.288) &< f_0(2.288) - f_0(1.244) \\ f_2(2.288) &> f_0(2.288) - f_0(1.091) \end{aligned}$$

$$\begin{cases} 1.633 \\ 1.630 \end{cases} \quad \begin{aligned} f_2(2.857) &> f_0(2.857) - f_0(0.377) \\ f_2(3.102) &< f_0(3.102) - f_0(0.296) \end{aligned}$$

$$\begin{cases} 3.066 \\ 3.061 \end{cases} \quad \begin{aligned} f_2(3.106) &< f_0(0.0619) - f_0(3.106) \\ f_2(3.536) &> f_0(0.0322) - f_0(3.536) \end{aligned}$$

TABLE II

3.163 $f_2(1.168)+1.374f_2(0.185) < f_0(0.185)-f_0(1.168) <$
$< 3.169f_2(1.168)+1.422f_2(0.185)$

3.147 $f_2(0.408)+1.632f_2(3.390) < f_0(3.390)-f_0(0.408) <$
$< 3.492f_2(0.408)+1.632f_2(3.390)$

3.896 $f_2(0.0758)+1.633f_2(3.770) < f_0(3.770)-f_0(0.0758) <$
$< 4.391f_2(0.0758)+1.633f_2(3.770)$

1.303 $f_2(0.0871)+3.073f_2(2.737) < f_0(0.0871)-f_0(2.737) <$
$< 1.380f_2(0.0871)+3.073f_2(2.737)$

1.494 $f_2(0.537)-1.623f_2(2.363) < f_0(0.537)-f_0(2.363) <$
$< 1.510f_2(0.573)-1.622f_2(2.363)$

$\vdots$

4.347 $f_2(0.00663)-3.061f_2(3.904) < f_0(3.904)-f_0(0.00663) <$
$< 6.503f_2(0.00663)-3.061f_2(3.904)$

R E F E R E N C E S

1) See for instance, G.F. Chew and S. Mandelstam, Phys.Rev. 119, 467 (1960).

2) D. Atkinson, Nuclear Phys. B7, 375 (1968); Nuclear Phys. B8, 377 (1968).

3) G. Auberson, O. Piguet and G. Wanders, Phys.Letters 28B, 141 (1968);

   O. Piguet and G. Wanders, University of Lausanne preprint, to appear in Nuovo Cimento.

4) A. Martin, Nuovo Cimento 42, 930 (1966);

   Y.S. Jin and A. Martin, Phys.Rev. 135, B1375 (1964).

5) H. Burkhardt, Nuovo Cimento 42, 351 (1966).

6) See for instance, A. Martin, CERN preprint TH. 968 (Appendix), submitted to Nuovo Cimento.

7) A. Martin, Nuovo Cimento 58, 303 (1968).

8) A.K. Common, Private communication, to be published.

9) M. Cini and S. Fubini, Ann.Phys. 10, 352 (1960).

10) G. Auberson, private communication.

11) A.P. Balachandran, W.J. Meggs, J. Nuyts and P. Ramond, IAEA preprint, Trieste (June 1968), to be published.