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Crossing Number and Weighted Crossing Number of Near-Planar Graphs

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Abstract A nonplanar graph G is near-planar if it contains an edge e such that $G - e$ is planar. The problem of determining the crossing number of a near-planar graph is exhibited from different combinatorial viewpoints. On the one hand, we develop min-max formulas involving efficiently computable lower and upper bounds. These min-max results are the first of their kind in the study of crossing numbers and improve the approximation factor for the approximation algorithm given by Hliněný and Salazar (Graph Drawing GD'06). On the other hand, we show that it is NP-hard to compute a weighted version of the crossing number for near-planar graphs.

Keywords Crossing number · Near-planar · Almost planar · Planar separation · Dual distance · Facial distance · NP-hardness

1 Introduction

Crossing number minimization is one of the fundamental optimization problems in the sense that it is related to various other widely used notions. Besides its mathematical interest, the concept is relevant in VLSI design [2, 10, 11], in combinatorial

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geometry [20], number theory [3, 19, 21], and for the aesthetics of drawing graphs [1, 16]. We refer to [12, 18] and to [23] for more details about diverse applications of this important notion.

A nonplanar graph G is *near-planar* if it contains an edge e such that $G - e$ is planar. Such an edge e is called a *planarizing edge*. It is easy to see that near-planar graphs can have arbitrarily large crossing number. However, it seems that understanding the crossing number of near-planar graphs should be much easier than in unrestricted cases. This is supported by a less known, but particularly interesting result of Riskin [17], who proved that the crossing number of a 3-connected cubic near-planar graph G is equal to the length of a shortest path in the geometric dual graph of the planar subgraph $G - x - y$, where $xy \in E(G)$ is a planarizing edge. It follows that the crossing number of a 3-connected cubic near-planar graph can be computed in polynomial time. Riskin asked if a similar correspondence holds in more general situations. This was disproved by Mohar [14] and Gutwenger, Mutzel, and Weiskircher [6]; see the discussion below. However, Hliněný and Salazar [7] showed that for near-planar graphs of maximum degree Δ these two values are within a factor Δ.

In this paper we show that several generalizations of Riskin's result are indeed possible. We provide efficiently computable upper and lower bounds on the crossing number of near-planar graphs in a form of min-max relations. These relations can be extended to the non-3-connected case and even to the case when graphs have weighted edges. As far as we know, these results are the first of their kind in the study of crossing numbers. It is shown that they generalize and improve some known results and we foresee that generalizations and further applications are possible.

On the other hand, we show that computing the crossing number of weighted near-planar graphs is NP-hard. This discovery is a surprise and brings more questions than answers.

2 Basic Notions

2.1 Drawings and Crossings

A *drawing* of a graph G is a representation of G in the Euclidean plane $\mathbb{R}^2$ where vertices are represented by distinct points and edges by simple polygonal arcs joining points that correspond to their endvertices. A drawing is *clean* if the interior of every arc representing an edge contains no points representing the vertices of G. If interiors of two arcs intersect or if an arc contains a vertex of G in its interior we speak about crossings of the drawing. More precisely, a *crossing* of a drawing $\mathcal{D}$ is a pair $(\{e, f\}, p)$, where e and f are distinct edges and $p \in \mathbb{R}^2$ is a point that belongs to interiors of both arcs representing e and f in $\mathcal{D}$. If the drawing is not clean, then the arc of an edge e may contain in its interior a point $p \in \mathbb{R}^2$ that represents a vertex v of G. In such a case, the pair $(\{v, e\}, p)$ is also referred to as a *crossing* of $\mathcal{D}$.

The number of crossings of $\mathcal{D}$ is denoted by $\mathtt{cr}(\mathcal{D})$ and is called the crossing number of the drawing $\mathcal{D}$. The *crossing number* $\mathtt{cr}(G)$ of a graph G is the minimum $\mathtt{cr}(\mathcal{D})$ taken over all clean drawings $\mathcal{D}$ of G. When each edge e of G has a weight

$w_e \in \mathbb{N}$, the weighted crossing number $\mathtt{wcr}(\mathcal{D})$ of a clean drawing $\mathcal{D}$ is the sum $\sum w_e \cdot w_f$ over all crossings $(\{e, f\}, p)$ in $\mathcal{D}$. The *weighted crossing number* $\mathtt{wcr}(G)$ of G is the minimum $\mathtt{wcr}(\mathcal{D})$ taken over all clean drawings $\mathcal{D}$ of G. Of course, if all edge-weights are equal to 1, then $\mathtt{wcr}(G) = \mathtt{cr}(G)$.

We shall discuss both, the weighted and unweighted crossing number. Most of the results are treated for the general weighted case. However, some results hold only in the unweighted case or are too technical to state for the weighted case. For a graph we shall assume that it is unweighted (i.e., all edge-weights are equal to 1) unless stated explicitly or when it is clear from the context that it is weighted.

A clean drawing $\mathcal{D}$ with $\mathtt{cr}(\mathcal{D}) = 0$ is also called an *embedding* of G. By a *plane graph* we refer to a planar graph together with a fixed embedding in the Euclidean plane. We shall identify a plane graph with its image in the plane.

2.2 Dual and Facial Distances

Let G_0 be a plane graph and let x, y be two of its vertices. A simple (polygonal) arc $\gamma : [0, 1] \to \mathbb{R}^2$ is an (x, y)-*arc* if $\gamma(0) = x$ and $\gamma(1) = y$. If $\gamma(t)$ is not a vertex of G_0 for every t, $0 < t < 1$, then we say that γ is *clean*. For an (x, y)-arc γ we define the crossing number of γ with G_0 as

$$\mathtt{cr}(\gamma, G_0) = |\{t \mid \gamma(t) \in G_0 \text{ and } 0 < t < 1\}|. \tag{1}$$

This definition extends to the weighted case as follows. If the graph G_0 is weighted and the edge xy realized by an (x, y)-arc γ also has weight w_{xy}, then each crossing of γ with an edge e contributes $w_{xy} \cdot w_e$ towards the value $\mathtt{cr}(\gamma, G_0)$, and each crossing $(\{v, xy\}, p)$ of xy with a vertex of G_0 contributes 1 (independently of the edge-weights).

Using this notation, we define the *dual distance*

$$d^*(x, y) = \min\{\mathtt{cr}(\gamma, G_0) \mid \gamma \text{ is a clean } (x, y)\text{-arc}\}.$$

We also introduce a similar quantity, the *facial distance* between x and y:

$$d'(x, y) = \min\{\mathtt{cr}(\gamma, G_0) \mid \gamma \text{ is an } (x, y)\text{-arc}\}.$$

It should be observed at this point that the value $d'(x, y)$ is independent of the weights—since all weights are positive integers, we can replace each crossing of an edge with a crossing through an incident vertex (without increasing $\mathtt{cr}(\gamma, G_0)$) and henceforth replace weight contributions simply by counting the number of crossings.

Let $G^*_{x,y}$ be the geometric dual graph of $G_0 - x - y$. Then $d^*(x, y)$ is equal to the distance in $G^*_{x,y}$ between the two vertices corresponding to the faces of $G_0 - x - y$ containing x and y. Of course, in the weighted case the distances are determined by the weights of their dual edges. This shows that $d^*(x, y)$ can be computed in linear time by using known shortest path algorithms for planar graphs. Similarly, one can compute $d'(x, y)$ in linear time by using the vertex-face incidence graph (see [15]).

Clearly, $d'(x, y) \le d^*(x, y)$. Note that d^* and d' depend on the embedding of G_0 in the plane. However, if G_0 is (a subdivision of) a 3-connected graph, then this dependency disappears since G_0 has essentially a unique embedding. To compensate for this dependence, we define $d_0^*(x,y)$ (and $d_0'(x,y)$) as the minimum of $d^*(x,y)$ (resp. $d'(x,y)$) taken over all embeddings of G_0 in the plane.

2.3 Overview of Results

The following proposition is clear from the definition of d^*:

Proposition 2.1 *If G_0 is a weighted planar graph and $x, y \in V(G_0)$, then $\mathtt{cr}(G_0 + xy) \leq d_0^*(x,y)$.*

This result shows that the value $d_0^*(x,y)$ is of interest. Gutwenger, Mutzel, and Weiskircher [6] provided a linear-time algorithm to compute $d_0^*(x,y)$. In Sect. 4 we study $d_0^*(x,y)$ from a combinatorial point of view and obtain a min-max expression for the value of $d_0^*(x,y)$ that turns out to be very useful.

Riskin [17] proved the following strengthening of Proposition 2.1 in a special case when G_0 is 3-connected and cubic:

Theorem 2.2 ([17]) *If G_0 is a 3-connected cubic planar graph, then*

$$\mathtt{cr}(G_0 + xy) = d_0^*(x,y).$$

Riskin also asked if Theorem 2.2 extends to arbitrary 3-connected planar graphs. One of the authors [14] has shown that this is not the case: for every integer k, there exists a 5-connected planar graph G_0 and two vertices $x, y \in V(G_0)$ such that $\mathtt{cr}(G_0 + xy) \leq 11$ and $d_0^*(x,y) \geq k$. See also Gutwenger, Mutzel, and Weiskircher [6] for an alternative construction.

However, several extensions of Theorem 2.2 are possible, and some of them are presented in this paper. In particular, we show how to deal with graphs that are not 3-connected, and what happens when we allow vertices of arbitrary degrees. In Sect. 5 we shall prove the lower bound of the following theorem:

Theorem 2.3 *If G_0 is a weighted planar graph and $x, y \in V(G_0)$, then*

$$d_0'(x,y) \leq \mathtt{cr}(G_0 + xy) \leq d_0^*(x,y).$$

If G_0 is an unweighted cubic graph, then for every planar embedding of G_0, $d'(x,y) = d^*(x,y)$. Therefore, $d_0'(x,y) = d_0^*(x,y)$, and Theorem 2.3 implies Theorem 2.2. We can also use Theorem 2.3 to improve the approximation factor in the algorithm of Hliněný and Salazar [7]; see Corollary 5.5 below.

A key idea in our results is to show that $d_0^*(x,y)$ (respectively $d_0'(x,y)$) is closely related to the maximum number of edge-disjoint (respectively vertex-disjoint) cycles that separate x and y. The notion of the separation has to be understood in a certain strong sense that is introduced in Sect. 4. This result yields a dual expression for d_0^* (respectively d_0') and is used to show that $d_0^*(x,y)$ is closely related to the crossing number of $G_0 + xy$.

Finally, we show in Sect. 6 that computing the crossing number of weighted near-planar graphs is NP-hard. Our reduction uses weights that are not polynomially bounded, and therefore it does not imply NP-hardness for unweighted graphs.

2.4 Intuition

To understand the difficulty in computing the crossing number of a near-planar graph, let us consider the graph $G_0 + xy$ shown in Fig. 1 (taken from [14]), where the subgraph inside each of the "darker" triangles is a sufficiently dense triangulation that requires many crossings when crossed by an arc. By drawing the vertex x in the outside, we see that xy is a planarizing edge. The drawing in Fig. 1 shows that its crossing number is at most 11, but it is also clear that $d^*(x, y)$ in the graph G_0 can be made as large as we want.

This construction can be generalized such that a similar redrawing as made there for x is necessary also for y (in order to bring these two vertices "close together"). At first sight this seems like the only possibility which may happen—to "flip" a part of the graph containing x and to "flip" a part containing y. And maybe some repetition of such changes may be needed. If this were the only possibility of making the crossing number smaller than the one coming from the planar drawing of G_0, this would most likely give rise to a polynomial time algorithm for computing the crossing number of near-planar graphs. However, the authors can construct examples, in which additional complications arise.

Despite these examples and despite our NP-hardness result for the weighted case, the following question may still have a positive answer:

Problem 2.4 *Is there a polynomial time algorithm which would determine the crossing number of* $G_0 + xy$ *if* G_0 *is an unweighted* 3*-connected planar graph?*

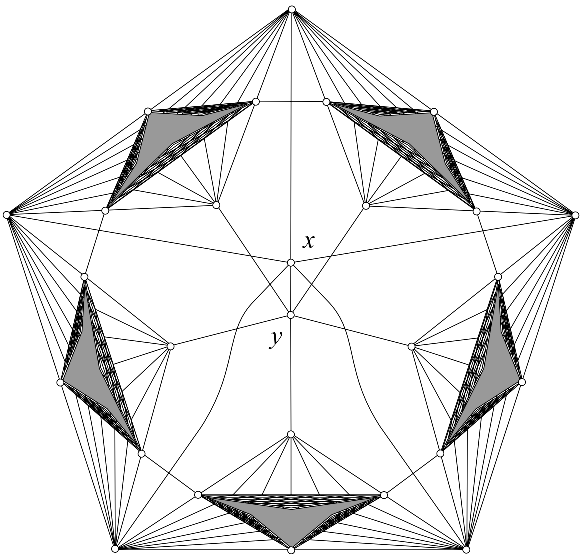

Fig. 1 A near-planar graph $G_0 + xy$ whose crossing number is unrelated to $d^*(x, y)$ in the graph G_0

3 Planar Separations and Connectivity Reductions

Let G_0 be a planar graph, x, y distinct vertices of G_0, and let Q be a subgraph of $G_0 - x - y$. We say that Q *planarly separates* vertices x and y if for every embedding of G_0 in the plane, x and y lie in the interiors of distinct faces of the induced embedding of Q. In other words, every (x, y)-arc must intersect Q.

Let Q be a subgraph of G. A *Q-bridge* in G is a subgraph of G that is either (i) an edge not in Q but with both ends in Q (and its ends also belong to the bridge), or (ii) a connected component of $G - V(Q)$ together with all edges (and their endvertices in Q) which have one end in this component and the other end in Q. Let B be a Q-bridge. Vertices of $B \cap Q$ are *vertices of attachment* of B (shortly *attachments*).

Let C be a cycle in G_0. Two C-bridges B and B' are said to *overlap* on C if either (i) C contains four vertices a, a', b, b' in this order such that a and b are attachments of B and a', b' are attachments of B', or (ii) B and B' have (at least) three vertices of attachment in common. We define the *overlap graph* $O(G_0, C)$ of C-bridges (see [15]) as the graph whose vertices are the C-bridges in G_0, and two vertices are adjacent if the two bridges overlap on C. Since G_0 is planar, the overlap graph is bipartite. Distinct C-bridges are *weakly overlapping* if they are in the same connected component of $O(G_0, C)$, and in that component they belong to distinct bipartite classes.

If $\mathcal{B}$ is the set of C-bridges in a connected component of $O(G_0, C)$, then an embedding of G_0 in the plane can be changed into another embedding by *flipping* the bridges in $\mathcal{B}$: Those that were in the interior of C are now in the exterior, and vice versa. See Fig. 2 for additional explanation of the flipping operation.

Tutte [22] characterized when $G_0 + xy$ is non-planar, i.e., when $\mathtt{cr}(G_0 + xy) \geq 1$, by proving

Theorem 3.1 (Tutte [22]) *Let x, y be vertices of a planar graph G_0. Then $G_0 + xy$ is non-planar if and only if $G_0 - x - y$ contains a cycle C such that the C-bridges of G containing x and y, respectively, are overlapping.*

The graph $G_0 + xy$ is non-planar if and only if in every embedding of G_0, x and y do not appear on a common face. This is obviously equivalent to the condition that $G_0 - x - y$ planarly separates x and y. Observe that Theorem 3.1 does not need the

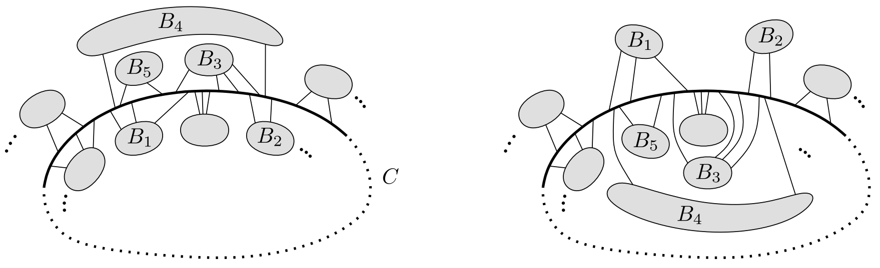

Fig. 2 Flipping a weakly-overlapping set of bridges. In this example, the bridges $B_1, B_2, \ldots, B_5$ form a connected component of $O(G_0, C)$

whole graph $G_0 - x - y$ to planarly separate x and y; it guarantees that a single cycle in $G_0 - x - y$ does. Our goal is to generalize this result to arbitrary subgraphs that planarly separate x and y. However, in this case we will only be able to say that for some cycle its bridges containing x and y weakly overlap.

If C is a cycle and z is a vertex in $V(G) \setminus V(C)$, then we denote by $B_z(C)$ the C-bridge that contains z. If C is clear from the context, we simply write B_z for $B_z(C)$. The next result follows easily from the definitions by using the flipping operation.

Lemma 3.2 *Let C be a cycle in $G_0 - x - y$. Then the cycle C planarly separates x and y if and only if $B_x(C)$ and $B_y(C)$ are distinct weakly overlapping C-bridges.*

We continue with some connectivity reductions. The first one is obvious.

Lemma 3.3 *Suppose that $G_0 = G_1 \cup G_2$, where $G_1 \cap G_2$ is either empty or a cutvertex of G_0, and suppose that $x, y \in V(G_1)$. Then a subgraph Q of $G_0 - x - y$ planarly separates x and y if and only if $Q \cap G_1$ planarly separates x and y in G_1.*

If x and y are in different components of G_0, they cannot be planarly separated, so we may assume that G_0 is connected. Our second reduction (together with the first one) will enable us to restrict our attention to 2-connected graphs.

Lemma 3.4 *Suppose that $G_0 = G_1 \cup G_2$ where $G_1 \cap G_2$ is a cutvertex v of G_0 and $x \in V(G_1)$, $y \in V(G_2)$. Then the following conditions are equivalent for every subgraph Q of $G_0 - x - y$:*

(a) *Q planarly separates x and y.*
(b) *Either $Q \cap (G_1 - v)$ or $Q \cap (G_2 - v)$ planarly separates x and y.*
(c) *Either $Q \cap (G_1 - v)$ planarly separates x and v or $Q \cap (G_2 - v)$ planarly separates y and v.*

Proof Clearly, (c) $\Rightarrow$ (b) $\Rightarrow$ (a). It remains to see that $\neg$ (c) $\Rightarrow$ $\neg$ (a). Let us therefore assume that neither $Q \cap (G_1 - v)$ planarly separates x and v nor $Q \cap (G_2 - v)$ planarly separates y and v. This means that there are embeddings of G_0 in which there is an (x, v)-arc γ_1 avoiding $Q \cap (G_1 - v)$ and a (v, y)-arc γ_2 avoiding $Q \cap (G_2 - v)$, respectively. It is clear that γ_1 and γ_2 may be chosen so that none of them intersects an edge incident with v. Let us take the induced embedding of G_1 of the first embedding, and redraw it so that γ_1 arrives to v from the outer face. Similarly, take the induced embedding of G_2 of the second embedding, and redraw it so that γ_2 arrives to v from the outer face. Now it is easy to see that these two embeddings can be combined into an embedding of G_0 and γ_1, γ_2 combined into an (x, y)-arc that avoids Q. See Fig. 3 for illustration, where Q is exhibited by using thick edges. □

Lemma 3.5 *Suppose that G_0 is 2-connected and that it can be written as $G_0 = G_1 \cup G_2$, where $G_1 \cap G_2 = \{u, v\} \subset V(G_0)$. Suppose that $x, y \in V(G_1)$, and let Q be any subgraph of $G_0 - x - y$. Let G_1^+ be the graph obtained from G_1 by adding the edge uv. If $Q \cap G_2$ contains a path from u to v, let $Q_1 = (Q \cap G_1) + uv$. Otherwise,*

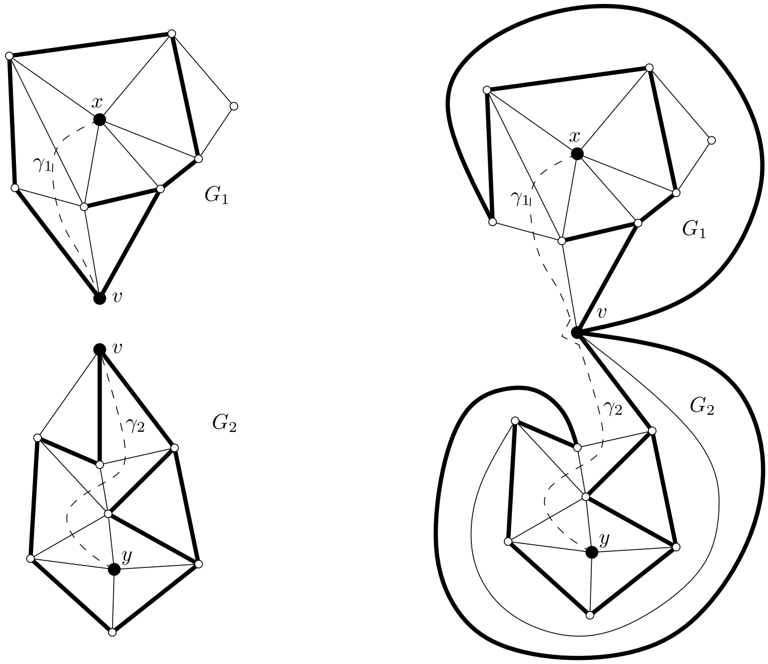

Fig. 3 Planar separations and cutvertices

let $Q_1 = Q \cap G_1$. *Then* Q *planarly separates* x *and* y *in* G_0 *if and only if* Q_1 *planarly separates* x *and* y *in* G_1^+.

The proof of Lemma 3.5 is not hard and is left to the reader.

Lemma 3.6 *Suppose that* $G_0 + xy$ *is* 3*-connected and that* G_0 *can be written as* $G_0 = G_1 \cup G_2$, *where* $G_1 \cap G_2 = \{u, v\} \subset V(G_0)$. *Suppose that* $x \in V(G_1) \setminus \{u, v\}$ *and* $y \in V(G_2) \setminus \{u, v\}$. *For* $i = 1, 2$, *let* G_i^+ *be the graph obtained from* G_i *by adding a new vertex* z_i *adjacent to* u *and* v. *Let* Q *be any subgraph of* $G_0 - x - y$ *and let* $Q_i = Q \cap G_i$. *Then* Q *planarly separates* x *and* y *in* G_0 *if and only if either* Q_1 *planarly separates* x *and* z_1 *in* G_1^+ *or* Q_2 *planarly separates* y *and* z_2 *in* G_2^+.

Proof One direction is easy. For the other one, suppose that for $i = 1$ and for $i = 2$, Q_i does not planarly separate x (or y) and z_i. Embeddings, where these pairs of vertices are not separated by Q_i, are easily combined into an embedding of G_0 showing that Q does not planarly separate x and y. □

The reduction to G_1^+ as described in Lemma 3.5 enables us to assume that the graph $G = G_0 + xy$ is 3-connected. After that, Lemma 3.6 can be used, if appropriate, to reduce planar separation problems to the case when G_0 itself is *essentially 3-connected*. By this we mean that G_0 can be obtained from a 3-connected graph by adding some edges in parallel to existing edges and by subdividing some edges. It is worth noting that all connectivity reductions discussed above can be made in linear time by using the algorithm of Hopcroft and Tarjan [8].

Our final result in this section is a generalization of Tutte's Theorem 3.1.

Theorem 3.7 *Let G_0 be a planar graph. If $Q \subseteq G_0 - x - y$ planarly separates x and y, then there is a cycle $C \subseteq Q$ that planarly separates x and y.*

Proof We may assume that Q is a minimal subgraph that planarly separates x and y. By Lemma 3.3, we may assume that $G_0 + xy$ is 2-connected. Let $B_1, B_2, \dots, B_r$ be the blocks of G_0, where $x \in V(B_1)$, $y \in V(B_r)$, and $v_i = B_i \cap B_{i+1}$ $(i = 1, \dots, r-1)$ are distinct cutvertices of G_0. For convenience, let $v_0 = x$ and $v_r = y$. Then it follows by Lemma 3.4 that Q does not contain cutvertices of G_0 and therefore, by the minimality assumption on Q, the whole subgraph Q is contained in a single block B_i in which it planarly separates the vertices v_{i-1} and v_i. By applying induction on the number r of blocks, we conclude that Q is a cycle if $r \geq 2$. Thus, we may assume henceforth that G_0 is 2-connected.

By using Lemma 3.5, it is easy to reduce the proof to the case when $G_0 + xy$ is 3-connected, which we assume henceforth.

It is easy to see that every subgraph that planarly separates two vertices contains a cycle. Let C_1 be a cycle in Q. Because of the minimality of Q, there is an embedding of G_0 in the plane such that x is in the interior of C_1 and y is in the exterior of C_1. If C_1 planarly separates x and y, we are done. Otherwise, by Lemma 3.2, the C_1-bridges $B_x(C_1)$ and $B_y(C_1)$ are in distinct components $\mathcal{O}_1, \mathcal{O}_2$ of the overlap graph $O(G_0, C_1)$. Also, since $G_0 + xy$ is 3-connected, the overlap graph has no other than these two components. This implies that C_1 can be written as the union of two paths, $C_1 = A \cup B$, where A and B have two vertices a, b in common, and all attachments of the C_1-bridges in $\mathcal{O}_1$ (resp. in $\mathcal{O}_2$) are in A (resp. B).

Since Q is a minimal separating set, for every $e \in E(C_1)$ there exists an embedding ψ_e of G_0 such that there is an (x, y)-arc γ that intersects Q only in the edge e. Let e be the edge of A incident with its endvertex a. Then it may be assumed that the initial segment γ_1 of γ from x to e does not intersect any of the bridges in $\mathcal{O}_2$. To see this, let us first observe that there is an (a, b)-arc β that is internally disjoint from G_0. Assuming that x is ψ_e-embedded in the interior of C_1, it also lies in the interior of $A \cup \beta$, while none of the bridges in $\mathcal{O}_2$ lies inside $A \cup \beta$. If γ intersected β, it would have to cross it again to return into the interior of $A \cup \beta$ before crossing the edge e. Therefore, we would be able to take the part of γ from x to its first intersection with β, then follow β until reaching the last intersection of γ with β, and then again follow γ towards e. This proves our claim.

Similarly, if we take the edge $f \in E(B)$ that is incident with a, we get an arc γ_2 from y to f that does not intersect Q or any edge in $\mathcal{O}_1$ in the corresponding embedding ψ_f.

Let ψ be an embedding of G_0 in which $A \cup \mathcal{O}_1$ is embedded as in ψ_e, and $B \cup \mathcal{O}_2$ is embedded first as in ψ_f, and then flipped, so that y ends up being embedded inside C_1. The arcs γ_1 and γ_2 can be added to this embedding so that they do not cross any edges of Q. They can be modified to come close to the endvertex a of e and f, respectively. Since Q planarly separates x and y, these two arcs cannot be joined together without intersecting Q. This means that $Q - E(C_1)$ contains a path D joining a with another vertex b' of C_1.

So far, C_1 was any cycle in Q. Let us assume henceforth that C_1 is selected such that the union of all bridges in $\mathcal{O}_1$ has minimum number of edges possible. This assumption implies that D is contained in an $\mathcal{O}_2$ bridge and $b' \in V(B)$. (If D were in a bridge in $\mathcal{O}_1$, we could replace C_2 by the cycle contained in $A \cup D$ and contradict the minimality property of $\mathcal{O}_1$.) Since γ_2 does not intersect D (as it does not intersect Q), y is contained in the interior of the unique cycle $C_2 \subseteq D \cup B$. Among all candidates for D, we select one such that the interior of C_2 (under the embedding ψ) is as large as possible.

Let us now consider the cycle $C_2 \subset Q$ instead of C_1. Observe that $B_x(C_2)$ contains all C_1-bridges in $\mathcal{O}_1$, the whole path A and the segment of B from b to b'. In particular, a and b' are vertices of attachment of $B_x(C_2)$.

Similarly, as argued above for C_1, the C_2-bridges form two components of $O(G_0, C_2)$ (or we are done). The cycle can be split into two segments A', B' such that the bridges in $\mathcal{O}'_1$ are attached to A' and the bridges in $\mathcal{O}'_2$ are attached to B'. Since $a, b' \in V(A')$, the segment B' is contained either in B or in D. In the second case we can flip $\mathcal{O}'_2$ together with the arc γ_2, and get an embedding of G_0 in which γ_1 and γ_2 can be joined without intersecting Q. (To see this, we use our assumptions that $\mathcal{O}_1$ did not contain a path in Q separating x from $\mathcal{O}_2$ and that D was such that the interior of C_2 was largest.) Thus, $B' \subseteq B$. It is now evident that the C_1-bridge B_D containing D cannot weakly overlap with the bridges in $\mathcal{O}_2$, since B_D consists of D and all C_2-bridges with an attachment on D together with a subset of $B_x(C_2)$, and all these are in $\mathcal{O}'_1$. This contradiction completes the proof. □

4 The Dual Distance

We keep using the notation and assumptions of Sect. 3. Moreover, we shall assume from now on that the vertex y lies on the outer face whenever we have an embedding of G_0 in the plane. This means that for every cycle $C \subseteq G_0 - y$, the vertex y lies in the exterior of C. Alternatively, we may consider embeddings of G_0 in the 2-sphere, and then we define the *interior* and the *exterior* of any cycle $C \subseteq G_0 - y$ such that y is in the exterior.

In some of the following results we consider a fixed embedding of G_0 in the plane. For this purpose we use the name *plane graph* to denote the graph together with its specified embedding in the plane.

For a plane graph G_0, a sequence $Q_1, \ldots, Q_k$ of edge-disjoint cycles of G_0 is *nested* if for $i = 1, \ldots, k$, all edges of the cycles Q_j $(j < i)$ lie in the interior of Q_i, while all edges of the cycles Q_j $(j > i)$ lie in the exterior of Q_i. If the embedding of G_0 is not specified, then we say that cycles $Q_1, \ldots, Q_k$ are *nested* if they are nested in some embedding of G_0 (in which y is on the boundary of the outer face).

Lemma 4.1 *Let G_0 be a plane graph, let $x, y \in V(G_0)$, and suppose that y lies on the outer face. If Q_1 and Q_2 are edge-disjoint cycles that planarly separate vertices x and y, then there exist nested edge-disjoint cycles Q'_1, Q'_2 such that $E(Q'_1) \cup E(Q'_2) \subseteq E(Q_1) \cup E(Q_2)$ and such that Q'_1, Q'_2 planarly separate x and y.*

Proof We will consider Q_1 and Q_2 as closed curves in the plane. This will enable us to classify each of their common vertices either as a *crossing* or a *touching point*. Observe that the number of crossings is even. If Q_1 and Q_2 have no crossings, then they are already nested and there is nothing to prove. Therefore, we may assume by applying Lemmas 3.3–3.4 that G_0 is 2-connected. Similarly, by applying Lemma 3.5, we may assume that $G_0 + xy$ is 3-connected. (Note that, when applying Lemma 3.5, if both Q_1 and Q_2 pass through G_2, we replace G_2 by two edges in parallel. When going back to G_0, we have to replace $(Q_1 \cup Q_2) \cap G_2$ by two paths that do not cross each other in G_2.)

If G_0 is not 3-connected, then by Lemma 3.6 any cycle that planarly separates x and y is contained in one part of any 2-separation. This enables us to reduce to the case when G_0 is essentially 3-connected.

Let us now consider the subgraph $H = Q_1 \cup Q_2$ of G_0 and its embedding in the plane. If Q_1 and Q_2 are not nested in G_0, then Q_1 and Q_2 cross an even number of times. This implies that H is 2-connected. In particular, every face of H is bounded by a cycle. Let Q_1' be the cycle bounding the face containing x, and let Q_2' be the face bounding y. Since every (x, y)-arc crosses Q_1 and Q_2, the cycles Q_1' and Q_2' cannot have an edge in common. Since G_0 is essentially 3-connected, every cycle in $G_0 - x - y$ planarly separates x and y. This shows that Q_1' and Q_2' are cycles whose existence we were to prove. □

Lemma 4.2 *Let G_0 be a plane graph. If $Q_1, \dots, Q_k$ are edge-disjoint cycles of G_0 that planarly separate vertices x and y of G_0, then there are nested edge-disjoint cycles $Q_1', \dots, Q_k'$ such that $\bigcup_{i=1}^k E(Q_i') \subseteq \bigcup_{i=1}^k E(Q_i)$ and such that $Q_1', \dots, Q_k'$ planarly separate x and y.*

Proof The proof follows rather easily by applying Lemma 4.1 consecutively on pairs of cycles Q_i, Q_j. One has to make sure that after finitely many steps we get a collection of nested cycles. This is done as follows. First we apply the lemma in such a way that one of the cycles in the family has none of the edges of the other $k - 1$ cycles in its interior. After this is done, we repeat the process with the remaining $k - 1$ cycles. □

After this preparation, we are ready to discuss a dual expression for the dual distance, both for the 3-connected and for the general case.

Theorem 4.3 *Let G_0 be a planar graph and $x, y \in V(G_0)$. If $r \geq 0$ is an integer, then the following statements are equivalent*:

(a) $r \leq d_0^*(x, y)$.
(b) *There exists a family of r edge-disjoint cycles $Q_1, \dots, Q_r$, each of which planarly separates x and y.*
(c) *For every embedding of G_0 in the plane, where y lies on the outer face, there exists a family of r nested edge-disjoint cycles $Q_1, \dots, Q_r$, each of which planarly separates x and y.*

Proof Equivalence of (b) and (c) follows from Lemma 4.2. It is also clear from the definitions (cf. Lemma 3.2) that (b) implies (a). The proof of the reverse implication that (a) yields (b) is by induction (using connectivity reductions of Lemmas 3.3–3.6) and also gives an efficient linear-time algorithm for finding $d_0^*(x,y)$ nested cycles planarly separating x and y. We will denote by $\lambda(x,y,G_0)$ the maximum number of edge-disjoint cycles in $G_0 - x - y$ that planarly separate x and y.

Our goal is to prove that $d_0^*(x,y) \le \lambda(x,y,G_0)$. By using the connectivity reduction of Lemma 3.3, we may assume that $G_0 + xy$ is 2-connected. Using the notation from the beginning of the proof of Theorem 3.7 and applying Lemma 3.4, we conclude that

$$\lambda(x,y,G_0) = \sum_{i=1}^{r} \lambda(v_{i-1}, v_i, B_i).$$

A similar formula holds for d_0^*:

$$d_0^*(x,y,G_0) = \sum_{i=1}^{r} d_0^*(v_{i-1}, v_i, B_i).$$

Therefore we may assume henceforth that G_0 is 2-connected. Moreover, by Lemma 3.5, we may assume that $G_0 + xy$ is essentially 3-connected.

If G_0 is essentially 3-connected (i.e., 3-connected up to subdivided edges and parallel edges), then it has essentially a unique embedding in the plane. Then it is easy to get a collection of $d^*(x,y) = d_0^*(x,y)$ vertex-disjoint cycles, each of which contains x in its interior and y in its exterior. Because of (essentially) unique embeddability, these cycles are planarly separating x and y, so their bridges B_x and B_y are weakly overlapping. This shows that $\lambda(x,y,G_0) \ge d_0^*(x,y)$.

For the final subcase, assume that G_0 has an "essential" 2-separation. This means that $G_0 = G_1 \cup G_2$, where $G_1 \cap G_2 = \{u,v\} \subset V(G_0)$, $x \in V(G_1) \setminus \{u,v\}$, $y \in V(G_2) \setminus \{u,v\}$, and each of G_1, G_2 has a vertex different from u,v,x,y. For $i = 1,2$, let the graph G_i^+ and its vertex z_i be as introduced in Lemma 3.6. By the induction hypothesis, $d_1 = d_0^*(x,z_1,G_1^+) = \lambda(x,z_1,G_1^+)$ and $d_2 = d_0^*(y,z_2,G_2^+) = \lambda(y,z_2,G_2^+)$. By Lemma 3.6,

$$\lambda(x,y,G_0) = \lambda(x,z_1,G_1^+) + \lambda(y,z_2,G_2^+) = d_1 + d_2. \tag{2}$$

Consider an embedding ψ_1 of G_1^+ for which $d^*(x,z_1,G_1^+) = d_1$ and an embedding ψ_2 of G_2^+ for which $d^*(y,z_2,G_2^+) = d_2$. These two embeddings can be combined into an embedding of G_0 for which $d^*(x,y,G_0) \le d_1 + d_2$. This implies that $d_0^*(x,y,G_0) \le d_1 + d_2$. After combining this inequality with (2), we conclude that $d_0^*(x,y,G_0) \le \lambda(x,y,G_0)$, which we were to prove. □

Corollary 4.4 *The value of $d_0^*(x,y)$ is equal to the maximum number of edge-disjoint cycles that planarly separate x and y.*

The above dual expression for $d_0^*(x,y)$ is a min-max relation which offers an extension to the weighted case. Suppose that the edges of $G_0 + xy$ are weighted and that all weights are positive integers. Then we can replace each edge $e \neq xy$ by w_e parallel edges (each of weight 1). Let $\tilde{G}_0$ be the resulting unweighted graph. It is easy to argue that $d_0^*(G_0, x, y)$ is equal to $d_0^*(\tilde{G}_0, x, y) \cdot w_{xy}$. By Corollary 4.4, this value can be interpreted as the maximum number of edge-disjoint cycles planarly separating x and y in $\tilde{G}_0$.

5 Facial Distance

In this section we shall prove Theorem 2.3. First, we need a dual expression for $d'(x, y)$ which can be viewed as a surface version of Menger's Theorem.

Proposition 5.1 *Let G_0 be a plane graph and $x, y \in V(G_0)$ where y lies on the boundary of the exterior face. Let r be the maximum number of vertex-disjoint cycles, $Q_1, \dots, Q_r$, contained in $G_0 - x - y$, such that for $i = 1, \dots, r$, $x \in \operatorname{int}(Q_i)$ and $y \in \operatorname{ext}(Q_i)$. Then $d'(x, y) = r$.*

Proof Since every (x, y)-arc intersects every Q_i, we conclude that $d'(x, y) \geq r$. The converse inequality is proved by induction on $d'(x, y)$. There is nothing to show if $d'(x, y) = 0$. Let F be the subgraph of G_0 containing all vertices and edges that are cofacial with x. Since $d'(x, y) \geq 1$, F contains a cycle Q such that $x \in \operatorname{int}(Q)$ and $y \in \operatorname{ext}(Q)$. Delete all vertices and edges of F except x, and let G_1 be the resulting plane graph. It is easy to see that $d'_{G_1}(x, y) = d'_{G_0}(x, y) - 1$. By the induction hypothesis, G_1 has $d'_{G_0}(x, y) - 1$ disjoint cycles that contain x in their interior and y in the exterior. By adding Q to this family, we get $d'(x, y)$ such cycles. This shows that $d'(x, y) \leq r$. □

The cycles $Q_1, \dots, Q_r$ in Proposition 5.1 all contain x in their interior and y in their exterior. Therefore, they behave essentially like cycles on a cylinder that separate the two boundary components of the cylinder. Hence they are nested cycles separating x and y.

One of the main results of this paper, Theorem 2.3, involves the minimum facial distance taken over all embeddings of G_0 in the plane. If G_0 is 3-connected, then $d'(x, y)$ is the same for every embedding of G_0, and Proposition 5.1 yields a dual expression for the facial distance. For general graphs, we need a similar concept as used in the previous section.

Let G_0 be a graph and $x, y \in V(G_0)$. Then we define $\rho(x, y, G_0)$ as the largest integer r for which there exists a collection of r vertex-disjoint cycles $Q_1, \dots, Q_r$ in $G_0 - x - y$ such that for every $i = 1, \dots, r$, x and y belong to distinct weakly overlapping bridges of Q_i (i.e., Q_i planarly separates x and y if G_0 is planar). It is convenient to realize that it may be required that the bridges containing x and y indeed overlap (not only weakly overlap), so we get an extension of Tutte's Theorem 3.1.

Lemma 5.2 *Let G_0 be a planar graph and let $r = \rho(x, y, G_0)$. Then there exists a collection of r vertex-disjoint cycles $Q_1, \dots, Q_r$ in $G_0 - x - y$ such that for every $i = 1, \dots, r$, x and y belong to distinct overlapping bridges of Q_i.*

Proof For $i = 1, \dots, r$, let B_x^i (resp. B_y^i) be the Q_i-bridge in G_0 containing x (resp. y), where $Q_1, \dots, Q_r$ are cycles from the definition of ρ. Note that every cycle Q_j ($j \neq i$) is contained either in B_x^i or in B_y^i. Therefore we can define a linear order $\prec$ on $\{Q_1, \dots, Q_r\}$ by setting $Q_i \prec Q_j$ if and only if $Q_j \subseteq B_y^i$. By adjusting indices, we may assume that $Q_1 \prec Q_2 \prec \cdots \prec Q_r$.

The proof method used in particular by Tutte in [22] is to change each cycle Q_i by rerouting it through the Q_i-bridges distinct from B_x^i and B_y^i in such a way that the two bridges with respect to the new cycle still weakly overlap, but contain more edges. The actual goal is to minimize the number t of edges that are neither on the cycle nor in one of these two bridges. If B_x^i and B_y^i do not overlap but are weakly overlapping, it is possible to decrease t. It follows that after a series of changes, that do not affect any of the other cycles, the "big" bridges B_x^i and B_y^i overlap. We refer to [9] and to [13] for an algorithmic treatment showing that these changes can be made in linear time. □

The following lemma is the analogue of Corollary 4.4.

Lemma 5.3 $d_0'(x, y) = \rho(x, y, G_0)$, *that is, the value of* $d_0'(x, y)$ *is equal to the maximum number of vertex-disjoint cycles that planarly separate* x *and* y.

Proof Clearly, $d_0'(x, y) \geq \rho(x, y, G_0)$ since the cycles from the definition of ρ planarly separate x and y and hence each of them contributes at least 1 to $d'(x, y)$ under every embedding of G_0 in the plane.

The main part of the proof, showing that $d_0'(x, y, G_0) \leq \rho(x, y, G_0)$, follows the same outline as the proof of Theorem 4.3. It is done by induction on $|G_0|$ using connectivity reductions. By Lemma 3.3 we may assume that $G_0 + xy$ is 2-connected. Using the notation from the beginning of the proof of Theorem 3.7 and applying Lemma 3.4, we conclude that

$$\rho(x, y, G_0) = \sum_{i=1}^{r} \rho(v_{i-1}, v_i, B_i).$$

A similar relation holds for d_0':

$$d_0'(x, y, G_0) \leq \sum_{i=1}^{r} d_0'(v_{i-1}, v_i, B_i).$$

By the induction hypothesis, which can be applied if $r \geq 2$, we conclude that $d_0'(x, y, G_0) \leq \rho(x, y, G_0)$. Therefore we may assume henceforth that G_0 is essentially 2-connected. Moreover, by Lemma 3.5, we may assume that $G_0 + xy$ is 3-connected.

If G_0 is essentially 3-connected, then it has essentially a unique embedding, and we can apply Proposition 5.1 to get a collection of $d'(x, y) = d_0'(x, y)$ vertex-disjoint cycles separating x and y. Because of (essentially) unique embeddability, these cycles are planarly separating x and y, so their bridges B_x and B_y are weakly overlapping. This shows that $\rho(x, y, G_0) \geq d_0'(x, y)$.

For the final subcase, assume that G_0 has an "essential" 2-separation. This means that $G_0 = G_1 \cup G_2$, where $G_1 \cap G_2 = \{u, v\} \subset V(G_0)$, $x \in V(G_1) \setminus \{u, v\}$, $y \in V(G_2) \setminus \{u, v\}$, and each of G_1, G_2 has a vertex different from u, v, x, y. For $i = 1, 2$, let the graph G_i^+ and its vertex z_i be as introduced in Lemma 3.6. By taking the 2-separation for which G_1^+ is smallest possible, G_1^+ is essentially 3-connected.

Let $d_1 = d_0'(x, z_1, G_1^+) = \rho(x, z_1, G_1^+)$. Since G_1^+ is essentially 3-connected, we may assume that a collection of d_1 disjoint nested cycles $Q_1, \ldots, Q_{d_1}$ is taken in a "greedy manner", i.e., they contain as few edges in their interior as possible. Up to symmetry between u and v, three outcomes may happen:

(a) $u, v \notin V(Q_{d_1})$,
(b) $u \in V(Q_{d_1})$ and $v \notin V(Q_{d_1})$, or
(c) $u, v \in V(Q_{d_1})$.

If (a) happens, then by Lemma 3.6

$$\rho(x, y, G_0) = \rho(x, z_1, G_1^+) + \rho(y, z_2, G_2^+).$$

By using flipping operation it is easy to see that

$$d_0'(x, y, G_0) \le d_0'(x, z_1, G_1^+) + d_0'(y, z_2, G_2^+).$$

Hence, an application of induction completes the proof. Similar proof works for cases (b) and (c). For the case (b), the recursive formula is

$$\rho(x, y, G_0) = \rho(x, v, G_1^+) + \rho(y, u, G_2^+). \tag{3}$$

In case (c) we have

$$\rho(x, y, G_0) = \rho(x, z_1, G_1^+) + \rho(y, z, G_2^+/\{uz_2, z_2v\}) \tag{4}$$

where the vertex z is obtained after contracting the edges uz_2, z_2v in the graph G_2^+, i.e. by identifying u, v, z_2 into a single vertex. Here we use the fact that

$$\rho(y, z_2, G_2^+) \le \rho(y, z, G_2^+/\{uz_2, z_2v\}) + 1$$

since the contraction of the edges u and v can intersect only the "outermost" cycle from a family of $\rho(y, z_2, G_2^+)$ disjoint cycles in G_2^+, and the other cycles planarly separate y and z in the contraction $G_2^+/\{uz_2, z_2v\}$.

Formuli (3) and (4) are easily seen to hold (as inequalities) for d_0' replacing the role of ρ. This completes the proof. □

We are ready for the proof of Theorem 2.3.

Proof of Theorem 2.3 We have already proved that $\mathrm{cr}(G_0 + xy) \le d_0^*(x, y)$. The heart of the proof is to show that $d_0'(x, y)$ is a lower bound on $\mathrm{cr}(G_0 + xy)$.

Let $r = d_0'(x, y)$. Lemmas 5.2 and 5.3 show that there are r vertex-disjoint cycles $Q_1, \ldots, Q_r$ such that for every $i = 1, \ldots, r$, vertices x and y belong to distinct overlapping bridges of Q_i. Let us denote these overlapping Q_i-bridges by B_x^i and B_y^i. To simplify the notation in the sequel, we define $Q_0 = \{x\}$ and $Q_{r+1} = \{y\}$. Since B_x^i and B_y^i overlap, one of the following cases occurs:

(i) There are paths $P_1^+, P_2^+ \subseteq B_y^i$ joining Q_i with Q_{i+1}, and there are paths $P_1^-, P_2^- \subseteq B_x^i$ joining Q_i with Q_{i-1} such that the ends of these pairs of paths on Q_i interlace.

(ii) When the bridges B_x^i and B_y^i have precisely three vertices of attachment, they may overlap only because their attachments a, b, c on Q_i coincide. In that case, we have paths P_1^+, P_2^+, P_3^+ in B_y^i (resp. paths P_1^-, P_2^-, P_3^- in B_x^i) joining a, b, c with Q_{i+1} (resp. Q_{i-1}).

If Case (i) occurs, let S^i be the union of the paths P_1^- and P_2^- and let R^i be the union of the paths P_1^+ and P_2^+. If Case (ii) occurs, we define S^i and R^i similarly, as the union of the three paths in (ii) certifying the overlapping.

Suppose that we have a clean drawing of $G_0 + xy$ in the plane. We assign types to certain crossings according to the following rules (where $1 \le i, j \le r$):

(a) If two edges of the same cycle Q_i cross, we declare such a crossing to be of type i.

(b) If two cycles Q_i and Q_j cross, where $j \ne i$, then they make at least two crossings, and we declare one of them to be a crossing of type i, and another one a crossing of type j.

(c) If the edge xy crosses Q_i, we declare such a crossing to be of type i.

(d) If there are no crossings of type i because of rules (a)–(c), then we consider the set F_i of the edges on the paths $S^1, S^2, \ldots, S^i$ and on the paths $R^i, R^{i+1}, \ldots, R^r$. If an edge in F_i crosses an edge of Q_i, we select one of such crossings and declare it to be of type i.

(e) If two edges $e \in E(S^i)$ and $f \in E(R^i)$ cross, we say that the crossing is of type i.

(f) If two edges $e \in E(S^i)$ and $f \in E(Q_{i+1})$ cross and this crossing does not have type $i+1$ assigned by rule (d), we say that this crossing is of type i. Similarly, if two edges $e \in E(R^i)$ and $f \in E(Q_{i-1})$ cross and this crossing does not have type $i-1$ assigned by rule (d), we also say that this crossing is of type i.

(g) Finally, if the cycles Q_{i-1} and Q_{i+1} intersect more than twice, we take one of the intersections that have no type assigned and declare it to be of type i.

Observe that by these rules, none of the crossings is of two different types (but for some of the crossings, the type may not have been specified).

Our goal is to show that for every $i = 1, \ldots, r$, there is a crossing of type i. This will show that there are at least r crossings, so the theorem holds.

Suppose, reductio ad absurdum, that there is no crossing of type i ($1 \le i \le r$). Then Q_i does not cross itself because of rule (a). This enables us to speak about the interior and exterior of Q_i. Both x and y are in the interior of Q_i (say) because of rule (c). Moreover, Q_i is not crossed by any of the other cycles Q_j ($j \ne i$) because of (b).

Suppose that Q_{i-1} is outside Q_i. There is a path from Q_{i-1} to x, all of whose edges are either on cycles Q_j ($j \le i-2$) or in the paths $S^1, S^2, \ldots, S^{i-1}$. Since x is in the interior of Q_i, this path crosses Q_i and gives a crossing of type i either by rule (b) or (d). A similar argument can be used to exclude the possibility that Q_{i+1} is outside Q_i. Hence, Q_{i-1} and Q_{i+1} are both inside Q_i.

Because of the rules (d) and (e), the edges in R^i cross neither Q_i nor S^i, and the edges in S^i cross neither Q_i nor R^i. However, because of overlapping, and edge in $R^i \cup Q_{i+1}$ must cross an edge in $S^i \cup Q_{i-1}$. Let us first consider the case when Q_{i-1} and Q_{i+1} cross each other. If they have more than two crossings, then we have a crossing of type i by rule (g). If there are precisely two crossings, then it is easy to see that a crossing of R^i and Q_{i-1} (or of S^i and Q_{i+1}) must occur. Note that, because rule (b) applies to $i-1$ and $i+1$, this crossing does not get type $i-1$ or $i+1$ by rule (d). So, it has type i by rule (f).

Finally, suppose that Q_{i-1} and Q_{i+1} do not cross each other. By symmetry, we may assume that the path $P_1^+ \subset R^i$ and Q_{i-1} cross. Now, Q_{i+1} is either in the interior or outside Q_{i-1}. In the former case, also the second path P_2^+ in R^i crosses Q_{i-1}, while in the latter case, P_1^+ has another crossing with Q_{i-1}. Only one of these two crossings can have type $i-1$ by rule (d), so the other one gets type i by rule (f). This excludes all possibilities and yields a contradiction. The proof is complete. □

As a corollary we get a generalization of Riskin's Theorem 2.2 by omitting the requirement about 3-connectivity and by letting x and y (and their neighbors) to have degree bigger than three (equal to four, respectively).

Corollary 5.4 *Let G_0 be a planar graph. If its subgraph $G_0 - x - y$ has maximum degree* 3, *then* $\mathrm{cr}(G_0+xy) = d_0'(x, y) = d_0^*(x, y)$. *In particular, the crossing number of $G_0 + xy$ is computable in linear time.*

Another corollary is an approximation formula for the crossing number of near-planar graphs if the maximum degree is bounded.

Corollary 5.5 *Let G_0 be a planar graph. If the graph $G_0 - x - y$ has maximum degree* Δ, *then*

$$d_0'(x, y) \le \mathrm{cr}(G_0 + xy) \le \left\lfloor \frac{\Delta}{2} \right\rfloor d_0'(x, y).$$

and

$$\left\lfloor \frac{\Delta}{2} \right\rfloor^{-1} d_0^*(x, y) \le \mathrm{cr}(G_0 + xy) \le d_0^*(x, y).$$

Proof Observe that $d_0^*(x, y) \le \lfloor \frac{\Delta}{2} \rfloor d_0'(x, y)$ because there are at most $\lfloor \frac{\Delta}{2} \rfloor$ edge-disjoint cycles through any vertex and $d_0^*(x, y)$ is defined by a collection of $d_0^*(x, y)$ nested cycles (cf. Theorem 4.3). □

Corollary 5.5 is an improvement of a theorem of Hliněný and Salazar [7] who proved the result with the factor Δ instead of $\lfloor \frac{\Delta}{2} \rfloor$.

A graph G is said to be *d-edge-apex* if G has a vertex z of degree at most $d+1$ such that $G - z$ is planar. Let us observe that every near-planar graph is essentially 1-edge-apex (subdivide the planarizing edge in order to create z).

Problem 5.6 *Is there a result similar to Corollary 5.4 for 2-edge-apex cubic graphs?*

6 NP-Hardness of $\texttt{wcr}(\cdot)$ for Near-Planar Graphs

Consider the following decision problem:

> WEIGHTED CROSSING NUMBER
> Input: G, k, where G is an edge-weighted graph and $k > 0$.
> Question: Is $\texttt{wcr}(G) \le k$?

This problem is NP-complete because it generalizes the problem CROSSING NUMBER , which is NP-complete [5]. We will see that this problem remains NP-complete when restricted to near-planar graphs. We will use the notation $[n] = \{1, \dots, n\}$.

Let $a_1, \dots, a_n$ be natural numbers, and let $S = \sum_{i \in [n]} a_i$. We define the edge-weighted graph $G(a_1, \dots, a_n)$ as follows (cf. Fig. 4):

- its vertices are $u_1, \dots, u_n$ and $v_1, \dots, v_n$;
- there is a Hamiltonian cycle $Q = u_1 u_2 \cdots u_n v_1 v_2 \cdots v_n u_1$, each edge of which has weight S^2;
- there are edges $e_i = u_i v_i$ with weight a_i for each $i \in [n]$.

It is easy to see that $G(a_1, \dots, a_n)$ is near-planar: the removal of the edge $u_1 v_n$ makes the graph planar, as can be seen in Fig. 5. For any subset of indices $I \subseteq [n]$, let $s(I) := \sum_{i \in I} a_i$.

Lemma 6.1 *It holds that*

$$2 \cdot \texttt{wcr}(G(a_1, \dots, a_n)) = \min_{I \subseteq [n]} \left\{ (s(I))^2 + (s([n] \setminus I))^2 \right\} - \sum_{i \in [n]} a_i^2.$$

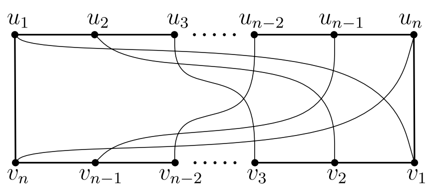

Fig. 4 The graph $G(a_1, \dots, a_n)$

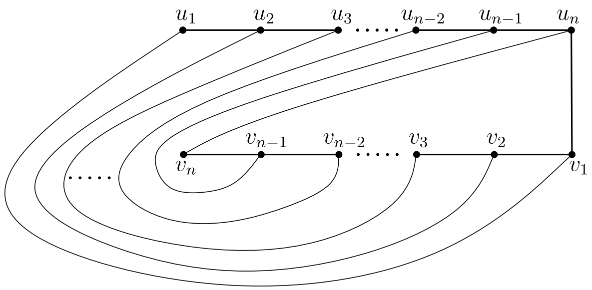

Fig. 5 The graph $G(a_1, \dots, a_n) - u_1 v_n$ is planar

Proof To simplify notation, let us take $G = G(a_1, \ldots, a_n)$ throughout this proof. Note that in the clean drawing of G given in Fig. 4 each edge e_i intersects any other edge e_j, $j \neq i$, and therefore, the weighted crossing number of that drawing is

$$\frac{1}{2}\left(\sum_{i\in[n]}\sum_{j\in[n]\setminus\{i\}} a_i \cdot a_j\right) = \frac{1}{2}\left(\sum_{i\in[n]} a_i \cdot (s([n]) - a_i)\right)$$
$$= \frac{1}{2}\left(s([n])^2 - \sum_{i\in[n]} a_i^2\right) \leq \frac{1}{2}S^2.$$

Thus $\mathtt{wcr}(G) \leq S^2/2$.

Consider a clean drawing $\mathcal{D}_0$ of G such that $\mathtt{wcr}(G) = \mathtt{wcr}(\mathcal{D}_0)$. In the drawing $\mathcal{D}_0$ cannot be that an edge of the cycle $Q = u_1u_2\cdots u_nv_1v_2\cdots v_nu_1$ participates in a crossing, because otherwise it would contribute weight over S^2 to $\mathtt{wcr}(\mathcal{D}_0)$ and $\mathcal{D}_0$ would not be optimal. Thus in the drawing $\mathcal{D}_0$ the cycle Q defines a closed Jordan curve in the plane, and each edge e_i is contained either in its interior region $\mathrm{int}(Q)$ or in its exterior region $\mathrm{ext}(Q)$. Let I_0 denote the set of indices $i \in [n]$ such that e_i is contained in $\mathrm{int}(Q)$. For any two distinct indices $i, j \in I_0$, the edges e_i, e_j cross inside $\mathrm{int}(Q)$. Symmetrically, for any two distinct indices $i, j \in [n] \setminus I_0$, the edges e_i, e_j cross in $\mathrm{ext}(Q)$. Therefore we have

$$2 \cdot \mathtt{wcr}(\mathcal{D}_0) \geq \sum_{i\in I_0}\sum_{j\in I_0\setminus\{i\}} a_i \cdot a_j + \sum_{i\in[n]\setminus I_0}\sum_{j\in[n]\setminus(I_0\cup\{i\})} a_i \cdot a_j$$
$$= \sum_{i\in I_0} a_i \cdot (s(I_0) - a_i) + \sum_{i\in[n]\setminus I_0} a_i \cdot (s([n]\setminus I_0) - a_i)$$
$$= (s(I_0))^2 + (s([n]\setminus I_0))^2 - \sum_{i\in[n]} a_i^2$$
$$\geq \min_{I\subseteq[n]}\left\{(s(I))^2 + (s([n]\setminus I))^2\right\} - \sum_{i\in[n]} a_i^2,$$

and hence

$$2 \cdot \mathtt{wcr}(G) = 2 \cdot \mathtt{wcr}(\mathcal{D}_0) \geq \min_{I\subseteq[n]}\left\{(s(I))^2 + (s([n]\setminus I))^2\right\} - \sum_{i\in[n]} a_i^2.$$

For the other inequality, consider a subset of indices I^* such that

$$\big(s(I^*)\big)^2 + \big(s([n]\setminus I^*)\big)^2 = \min_{I\subseteq[n]}\left\{(s(I))^2 + (s([n]\setminus I))^2\right\}.$$

We can make a drawing $\mathcal{D}^*$ of G where Q is drawn as a Jordan curve, the edges $e_i, i \in I^*$ are drawn in $\mathrm{int}(Q)$ with each pair crossing exactly once, and the edges $e_i, i \in [n] \setminus I^*$ are drawn in $\mathrm{ext}(Q)$ with each pair crossing exactly once. We therefore have

$$
\begin{aligned}
2 \cdot \texttt{wcr}(\mathcal{D}^*) &= \sum_{i \in I^*} a_i \cdot (s(I^*) - a_i) + \sum_{i \in [n] \setminus I^*} a_i \cdot (s([n] \setminus I^*) - a_i) \\
&= \big(s(I^*)\big)^2 + \big(s([n] \setminus I^*)\big)^2 - \sum_{i \in [n]} a_i^2 \\
&= \min_{I \subseteq [n]} \Big\{ (s(I))^2 + (s([n] \setminus I))^2 \Big\} - \sum_{i \in [n]} a_i^2
\end{aligned}
$$

and thus

$$
2 \cdot \texttt{wcr}(G) \leq 2 \cdot \texttt{wcr}(\mathcal{D}^*) = \min_{I \subseteq [n]} \Big\{ (s(I))^2 + (s([n] \setminus I))^2 \Big\} - \sum_{i \in [n]} a_i^2.
$$

□

Lemma 6.2 *The equality*

$$
\texttt{wcr}(G(a_1, \ldots, a_n)) = S^2/4 - \sum_{i \in [n]} a_i^2/2
$$

holds if and only if there exists $I \subset [n]$ such that $s(I) = s([n] \setminus I) = S/2$.

Proof Note that

$$
\min_{I \subseteq [n]} \Big\{ (s(I))^2 + (s([n] \setminus I))^2 \Big\} \geq \min\{A^2 + B^2 \mid A + B = S, A \geq 0, B \geq 0\} = S^2/2,
$$

and there is equality if and only if there is some $I \subset [n]$ such that $s(I) = s([n] \setminus I) = S/2$. The result then follows from Lemma 6.1. □

Theorem 6.3 *The problem* Weighted Crossing Number *is NP-complete for near-planar graphs.*

Proof We first show that the problem Weighted Crossing Number is in NP. In a drawing $\mathcal{D}$ of a graph G with $\texttt{wcr}(\mathcal{D}) = \texttt{wcr}(G)$ each two edges intersect at most once: if there would be two edges e, e' intersecting twice then they contain two subpaths $p \subset e$, $p' \subset e'$ with common endpoints, and we can reduce the weighted crossing number of the drawing by replacing p by a subpath “parallel” to p', or by replacing p' by a subpath parallel to p. Therefore, an optimal drawing can be guessed in $O(|V(G)|^2)$ space as a planar graph inserting additional vertices at each crossing and subdividing the edges appropriately; for subdividing the edges we also have to guess along each edge in what order the crossings appear. This shows that Weighted Crossing Number is in NP.

To show NP-hardness, consider the following NP-complete problem [4].

PARTITION
Input: natural numbers $a_1, \ldots, a_n$.
Question: is there $I \subset [n]$ such that $\sum_{i \in I} a_i = \sum_{i \in [n] \setminus I} a_i$?

Consider the function ϕ that maps the input $a_1, \ldots, a_n$ for PARTITION into the input

$$G(a_1, \ldots, a_n), S^2/4 - \sum_{i \in [n]} a_i^2/2$$

for Weighted Crossing Number. Clearly, ϕ can be computed in polynomial time. Because of Lemma 6.2 both problems have the same answer. Therefore we have a polynomial time reduction from PARTITION to WEIGHTED CROSSING NUMBER that only uses near-planar graphs. □

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