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Crossing Numbers and Hard Erdős Problems in Discrete Geometry

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We show that an old but not well-known lower bound for the crossing number of a graph yields short proofs for a number of bounds in discrete plane geometry which were considered hard before: the number of incidences among points and lines, the maximum number of unit distances among n points, the minimum number of distinct distances among n points.

"A statement about curves is not interesting unless it is already interesting in the case of a circle." (H. Steinhaus)

The main aim of this paper is to derive short proofs for a number of theorems in discrete geometry from Theorem 1.

Theorem 1. (Leighton [16], Ajtai, Chvátal, Newborn and Szemerédi [1]) *For any simple graph G with n vertices and $e \geq 4n$ edges, the crossing number of G on the plane is at least $e^3/(100n^2)$.*

For many graphs, Theorem 1 is tight within a constant multiplicative factor. This result was conjectured by Erdős and Guy [10, 12], and first proved by Leighton [16], who was unaware of the conjecture, and independently by Ajtai, Chvátal, Newborn and Szemerédi [1]. The theorem is still hardly known – some distinguished mathematicians even recently thought of the Erdős–Guy conjecture as an open problem. Shahrokhi, Sýkora, Székely and Vrťo generalized Theorem 1 for compact 2-dimensional manifolds with a transparent proof (Theorem 5, [21, 22]).

The original proofs of the applications of Theorem 1 shown here (Theorems 2, 3, 4, 9) used sophisticated tools like the covering lemma [25]. Although some of those theorems were given simpler proofs and generalizations which used methods from the theory of VC dimension and extremal graph theory (see Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [6], Füredi and Pach [11] and Pach and Agarwal [18], Pach and Sharir [19]), the simpler proofs still lacked the simplicity and generality shown here. I believe that the notion of crossing number is a central one for discrete geometry, and that the right branch of graph theory to be applied to discrete geometry is 'extremal topological graph theory' rather than classical extremal graph theory. Other lower bounds for the crossing number (see [22]) may have other striking applications in discrete geometry.

A *drawing* of a graph over a surface represents edges by curves such that edges do not pass through vertices and no three edges meet in a common internal point. The *crossing number* of a graph over a surface is defined as the minimum number of crossings of edges over all drawings of the graph. It is not difficult to see that if we allow the intersection of many edges in an internal point but count crossings of pairs of edges, then we do not change the definition of the crossing number. We denote by $cr(G)$ the *planar crossing number* of a graph G. We shall always use $\#i$ to denote the number of incidences between the given sets of points and lines (or curves) in a particular situation under discussion. We shall write c for a constant; as usual, different occurences of c may denote different constants.

Theorem 2. (Szemerédi and Trotter [24]) *For n points and l lines in the Euclidean plane, the number of incidences among the points and lines is at most $c[(nl)^{2/3} + n + l]$.*

Proof. We may assume without loss of generality that all lines are incident to at least one point. Define a graph G drawn in the plane such that the vertex set of G is the set of our n given points, and join two points with an edge drawn as a straight line segment, if the points are consecutive points on one of the lines. This drawing shows that $cr(G) \le l^2$. The number of points on any of the lines is one greater than the number of edges drawn along that line. Therefore, the number of incidences among the points and the lines is at most l greater than the number of edges in G. Theorem 1 finishes the proof: either $4n \ge \#i - l$ or $cr(G) \ge c(\#i - l)^3/n^2$. □

For $n = l$ the statement of Theorem 2 was conjectured by Erdős [9]. Although Theorem 3 below is known to be a simple corollary of Theorem 2, we prove it, since we use it in the proof of Theorem 4.

Theorem 3. (Szemerédi and Trotter [24]) *Let $2 \le k \le \sqrt{n}$. For n points in the Euclidean plane, the number l of lines containing at least k of them is at most cn^2/k^3.*

Proof. Initiate the construction of the graph G from the proof of Theorem 1, taking only lines passing through at least k points. Note that G has at least $l(k-1)$ edges. Hence, we have either $l^2 > ce^3/n^2 > c[l(k-1)]^3/n^2$ or $l(k-1) < 4n$. In the first case we are at home, and so are we in the second, as $l < 4n/(k-1) < cn^2/k^3$. □

Theorem 3, which was conjectured by Erdős and Purdy [9], is known to be tight for $2 \le k \le \sqrt{n}$ for the points of the $\sqrt{n} \times \sqrt{n}$ grid [2]. In 1946, Erdős [8] conjectured that the number of unit distances among n points is at most $n^{1+o(1)}$ and proved that this number is at most $cn^{3/2}$. In 1973, Józsa and Szemerédi [13] improved this bound to $o(n^{3/2})$. In 1984, Beck and Spencer [4] further improved the bound to $n^{1.44...}$. Finally, Spencer, Szemerédi and Trotter [23] achieved the best known bound, $cn^{4/3}$.

Theorem 4. (Spencer, Szemerédi and Trotter [23]) *The number of unit distances among n points in the plane is at most $cn^{4/3}$.*

Proof. Draw a graph G in the plane in the following way. The vertex set is the set of given points. Draw a unit circle around each point; in this way, consecutive points on the unit circles are connected by circular arcs: these are the edges of a multigraph drawn in the plane. Discard the circles that contain at most two points, and for any two points joined by some circular arcs, keep precisely one of these arcs. Let G be the resulting graph. The number of edges of this graph is at most $O(n)$ less than the number of unit distances. The number of crossings of G in this drawing is at most $2n^2$, since any two circles intersect in at most 2 points. An application of Theorem 1 finishes the proof. $\square$

Based on earlier work of Kainen [14] and Kainen and White [15], Shahrokhi, Sýkora, Székely and Vrťo [21, 22] found the following generalization of Theorem 1.

Theorem 5. (Shahrokhi, Sýkora, Székely and Vrťo [21, 22]) *Let G be a simple graph G drawn on an orientable or non-orientable compact 2-manifold of genus g. Assume that G has n vertices, e edges and $e \geq 8n$. Then the number of crossings in the drawing of G is at least ce^3/n^2, if $n^2/e \geq g$, and is at least $ce^2/(g+1)$, if $n^2/e \leq g \leq e/64$.*

Theorem 5 is known to be tight within a factor of $O(\log^2(g+2))$ for some graphs [21]. We obtain the following generalization of Theorem 2 by combining Theorems 5 and 1.

Theorem 6. *Suppose that we are given l simple curves and p points on an orientable or non-orientable compact 2-manifold of genus g. Assume that any two curves intersect in at most one point. Assume, furthermore, that every curve is incident to at least one point. Then $\#i = O\big((pl)^{2/3} + p + l\big)$ if $p^2/(\#i - l) \geq g$ and $\#i = O\big(l\sqrt{g+1} + p\big)$ if $p^2/(\#i - l) \leq g \leq (\#i - l)/64$.*

Theorem 7. *Suppose G is a multigraph with n nodes, e edges and maximum edge multiplicity m. Then either $e < 5nm$ or $cr(G) \geq ce^3/(n^2m)$.*

Take any simple graph H for which Theorem 1 is tight with a drawing which shows it, and substitute each edge with m closely drawn parallel edges. For the new graph $m \cdot H$ Theorem 7 is tight. For the proof of Theorem 7 we need the following simple fact:

Proposition 1. $cr(k \cdot H) = k^2 cr(H)$.

Proof. Take any drawing of H with $cr(H)$ crossings. Substitute the edges of H with k parallel edges closely drawn to the original edge. In this way we obtain a drawing of $k \cdot H$ with $k^2 cr(H)$ edges. On the other hand, take any drawing of $k \cdot H$. Picking one representative of parallel edges in $k^{|E(H)|}$ ways, we obtain a drawing of H, which exhibits at least $cr(H)$ crossings. Any pair of crossing edges in $k \cdot H$ that arose above is counted $k^{|E(H)|-2}$ times. $\square$

Proof of Theorem 7. For $0 \leq i \leq \log_2 m$, let G_i denote the subgraph of G in which two vertices a, b are joined with $t > 0$ edges if and only if in G they are joined by exactly t edges and $2^i \leq t < 2^{i+1}$. Set

$$A = \{i \in [0, \log_2 m] \,:\, |E(G_i)| \leq 2^{i+3} n\} \quad \text{and} \quad B = [0, \log_2 m] \setminus A.$$

We may assume $e \geq 5nm$. Therefore, since $\sum_{i\in A} |E(G_i)| \leq 4nm$, we have

$$\sum_{i\in B} |E(G_i)| \geq |E(G)| - \sum_{i\in A} |E(G_i)| \geq |E(G)| - 4mn \geq |E(G)|/5.$$

Let G_i^* denote the simple graph obtained from G_i by identifying parallel edges. We have

$$\begin{aligned} cr(G) \geq \sum_{i=0}^{\log_2 m} cr(G_i) &\geq \sum_{i=0}^{\log_2 m} 2^{2i} cr(G_i^*) \geq \sum_{i\in B} c\frac{|E(G_i^*)|^3}{n^2} 2^{2i} \\ &\geq \sum_{i\in B} c\frac{|E(G_i)|^3}{n^2 2^i} = \frac{c}{n^2}\sum_{i\in B}\left(\frac{|E(G_i)|}{2^{i/3}}\right)^3, \end{aligned}$$

with the second inequality following from Proposition 1. Hence, by applying Hölder's inequality with $p = 3$ and $q = 3/2$, we find that

$$cr(G) \geq \frac{c}{n^2}\left(\sum_{i\in B}\frac{|E(G_i)|}{2^{i/3}}2^{i/3}\right)^3 \Big/ \left(\sum_{i\in B}(2^{i/3})^{3/2}\right)^2 \geq \frac{c}{n^2 m}\left(\sum_{i\in B}|E(G_i)|\right)^3 \geq \frac{ce^3}{n^2 m}. \qquad \square$$

Theorem 7 has the following immediate consequence concerning incidences between points and curves.

Theorem 8. *Given p points and l simple curves in the plane, such that any two curves intersect in at most t points and any two points belong to at most m curves, the number of incidences is at most*

$$c(lp)^{2/3}(tm)^{1/3} + l + 5mp.$$

Although Theorem 2 has been given several generalizations in the vein of Theorem 8, they seem to admit algebraic curves only [19, 6, 11].

In 1946, Erdős [8] conjectured that n points in the plane determine at least $cn/\sqrt{\log n}$ distinct distances, and showed that the minimum is at least $\sqrt{n}$. Over the years, this lower bound has been improved several times. In 1952, Moser [17] proved $n^{2/3}$, then in 1984 Chung [5] improved $n^{2/3}$ to $n^{5/7}$, and in an unpublished manuscript, Beck [3] proved $n^{58/81-\varepsilon}$. The best lower bound to date has been proved by Chung, Szemerédi and Trotter [7].

Theorem 9. (Chung, Szemerédi and Trotter [7]) *At least $n^{4/5}/(\log n)^c$ distinct distances are determined by n points in the plane.*

They proved this for a large c, and claimed that a much smaller c, perhaps even $c = 0$, can be obtained with their method. However, as they remarked, they did not prove the existence of a single point which determined many distances. Given points $x_1, \ldots, x_n$, we say that x_i determines at least t distances from the others if there are t points $x_{j_1}, \ldots, x_{j_t}$ such that the distances $d(x_i, x_{j_1}), d(x_i, x_{j_2}), \ldots, d(x_i, x_{j_t})$ are distinct and positive. Here is the best published result for this modified problem.

Theorem 10. (Clarkson, Edelsbrunner, Guibas, Sharir and Welzl [6]) *Given n points in the plane, one of them determines at least $cn^{3/4}$ distinct distances from the others.*

Theorem 11. *Given n points in the plane, one of them determines at least $cn^{4/5}$ distinct distances from the others.*

Proof. Let t be the maximum number of distinct distances measured from any point. We may assume $t = o(n/\log n)$, otherwise there is nothing to prove. Draw circles around every point with every distance as radius, which can be measured from that point. We have drawn at most t concentric circles around each point. Define a multigraph whose vertices are the points, and whose edges are arcs connecting consecutive points on the circles we have drawn. Delete edges lying in circles containing at most two points. Since $t = o(n)$, the resulting multigraph G still has cn^2 edges. Theorem 7 cannot be applied immediately, since very high edge multiplicities may occur.

Proposition 2. *The number of pairs (f, a), where f is a line with at least k points, a is an arc representing an edge of G, and f is the symmetry axis of a, is at most $ctn^2/k^2 + ctn\log n$.*

Proof. By Theorem 3, the number of lines with at least 2^i points is at most $cn^2/2^{3i}$, as far as $2^i \leq \sqrt{n}$. For each such line, the number of bisected edges is at most $2t$ times the number of points on the line. Therefore, the number of pairs (f, a) with $k \leq |f| \leq 4\sqrt{n}$ is at most

$$\sum_{i:k\leq 2^i\leq\sqrt{n}} ct\frac{n^2}{2^{3i}}2^i \leq \frac{ctn^2}{k^2}.$$

A simple and well-known inclusion-exclusion argument (see [20]) shows that the number of lines with number of points between a and $2a$ ($4\sqrt{n} \leq a$) is at most cn/a. Hence the contribution of such big lines to the number of pairs is at most

$$\sum_{i:\sqrt{n}<2^i<n} ct\frac{n}{2^i}2^i < ctn\log n. \qquad \square$$

Now the number of pairs in Proposition 2 is just the number of edges joining pairs of points which are joined by at least k edges. Thus, deleting all such edges with $k = K\sqrt{t}$ for a suitable constant K, we arrive at a multigraph G_1 still having cn^2 edges. The crossing number of G_1 is at most $2n^2t^2$. Thus, using Theorem 7 we have

$$2n^2t^2 \geq cr(G_1) \geq \frac{c|E(G_1)|^3}{n^2K\sqrt{t}} \geq \frac{cn^6}{n^2K\sqrt{t}}$$

and the theorem follows. $\square$

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