Crossing Numbers of Graphs with Rotation Systems

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Abstract

We show that computing the crossing number and the odd crossing number of a graph with a given rotation system is **NP**-complete. As a consequence we can show that many of the well-known crossing number notions are **NP**-complete even if restricted to cubic graphs (with or without rotation system). In particular, we can show that Tutte's independent odd crossing number is **NP**-complete, and we obtain a new and simpler proof of Hliněný's result that computing the crossing number of a cubic graph is **NP**-complete.

We also consider the special case of multigraphs with rotation systems on a fixed number k of vertices. For $k = 1$ we give an $O(m \log m)$ algorithm, where m is the number of edges, and for loopless multigraphs on 2 vertices we present a linear time 2-approximation algorithm. In both cases there are interesting connections to edit-distance problems on (cyclic) strings. For larger k we show how to approximate the crossing number to within a factor of $\binom{k+4}{4}/5$ in time $O(m^k \log m)$ on a graph with m edges.

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1 Introduction

Computing the crossing number is **NP**-complete, as was shown by Garey and Johnson [6]. Hliněný recently proved, using a rather complicated construction, that even determining the crossing number of a cubic graph is **NP**-complete [7], settling a long-standing open problem [1].

We take a new approach to cubic graphs through graphs with rotation systems. We show that determining the crossing number of a graph with a given rotation system is **NP**-complete, and then prove that this problem is equivalent to determining the crossing number of a cubic graph. This also gives a new and easy proof that determining the minor-monotone crossing number defined in [2] is **NP**-complete.

The constructions used in the **NP**-hardness result for graphs with rotation system and cubic graphs can be extended to work for other crossing numbers such as odd crossing number, pair crossing number and rectilinear crossing number. In the case of odd crossing number, the proof of correctness becomes more complex though, and for this proof we introduce a new problem, Minimum Tournament Arrangement, that should be of interest in its own right.

In particular, we can show that computing the independent odd crossing number of a graph is **NP**-hard; while this result is not unexpected, it does imply that the algebraic approach to crossing number through the independent odd crossing number began by Tutte [24] and continued by Székely [21, 22] will not lead to polynomial time algorithms for the independent odd crossing number (which would have allowed us to approximate the crossing number to within a square root by a recent result [19]).

Graphs with rotation systems are of interest in their own right; we have encountered them several times during recent research projects [15, 17, 18]. For example, at the core of our separation of the crossing number from the odd crossing number is a two-vertex multigraph with rotation system [18]. In Section 5 we show that the crossing number can be computed efficiently for a one-vertex multigraph with rotation system, and that crossing number can be approximated efficiently for loopless two-vertex multigraphs with rotation system (the problem is in polynomial time in this case but it requires linear programming [18]). There are unexpected connections between the two-vertex case and edit distance problems over strings. For k-vertex multigraphs with rotation system we give an approximation algorithm to compute the crossing number to within a factor of $O(k^4)$. We do not know whether this problem can be solved exactly in polynomial time, even for $k = 3$.

2 NP-hardness

Consider a graph drawn in the plane. The *rotation* at a vertex is the clockwise order of its incident edges. A *rotation system* is the list of rotations of all vertices. We are interested in drawings of a graph in the plane with a fixed rotation system. If G is equipped with a rotation system, we write $\mathrm{cr}_{rot}(G)$, as opposed to $\mathrm{cr}(G)$ to denote the fact that we only consider drawings of G that respect the given rotation system.

We also consider "flipped" rotations. *Flipping* the rotation at a vertex v means reversing the cyclic order of the edges incident to v. For a graph with rotation system we write $\mathrm{cr}_{flip}(G)$ if we restrict ourselves to drawings of G which respect the rotation of G up to allowing the rotation at each vertex to flip. Trivially, $\mathrm{cr}(G) \leq \mathrm{cr}_{flip}(G) \leq \mathrm{cr}_{rot}(G)$.

Theorem 2.1. *Computing the crossing number of a graph with rotation system is* **NP***-complete. The problem remains* **NP***-complete if we allow the rotation at each vertex to flip.*

Proof. We adapt Garey and Johnson's reduction from Minimum Linear Arrangement to Crossing Number [6]. Given a graph $G = (V, E)$, a *linear arrangement* is an injective function $\phi : V \to \{1, \ldots, |V|\}$ and the *value* of the arrangement is

$$\sum_{uv \in E} |\phi(u) - \phi(v)|.$$

Given a graph G and a number k, deciding whether G allows a linear arrangement of value at most k is **NP**-complete [6, GT42].

Let us fix a connected graph $G = (V, E)$, with $V = \{v_1, \ldots, v_n\}$, $m = |E|$, and k. We can assume that $n \leq m$ (for trees the problem can be decided in polynomial time [6, GT42]), and $k \leq m(n-1) \leq m^2$. From G we construct an edge-weighted graph H with rotation system, as shown in Figure 1. In a drawing of a weighted graph, a crossing of an edge of weight k with an edge of weight l contributes kl to the crossing number. The use of weighted edges simplifies the construction; later we will replace each weighted edge by a small unweighted graph, obtaining a simple graph H' with rotation system.

We start with a cycle $(u_1, \ldots, u_{4n})$ with edge-weights so high that it has to be embedded without any crossings. For every $1 \leq i \leq 2n$ we connect u_i to u_{4n+1-i} by a path P_i of length 2 and edges of weight w. Furthermore, we connect the midpoint a_i of P_i and the midpoint c_i of P_{2n+1-i} by a path $Q_i = a_i b_i c_i$ of length 2 with edges of weight w', but replacing $b_i c_i$ by two edges of weight $w'/2$ ($1 \leq i \leq n$).

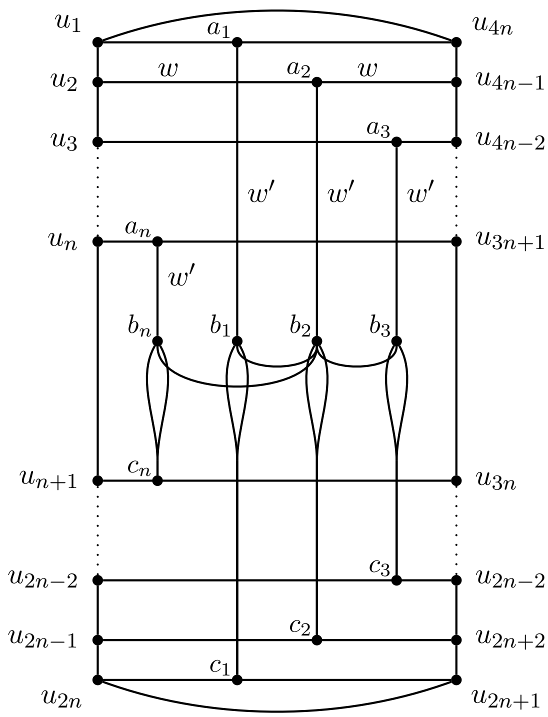

Figure 1: The graph H

Finally, we encode G as follows: for each edge $v_iv_j \in E$ we add an edge of weight 1 from b_i to b_j. The weight 1 edges incident to b_i are inserted into the rotation at b_i between the two b_ic_i-edges of weight $w'/2$; among each other these edges can otherwise be ordered arbitrarily.

This concludes the description of H with the rotation system shown in Figure 1. We let $k' = n(n-1)ww' + kw' + m^2$, where $w = 7m^4$ and $w' = 5m^2$. We claim that $\text{cr}_{\mathit{flip}}(H) \leq k'$ implies that G allows a linear arrangement of value at most k and that this in turn implies that $\text{cr}_{\mathit{rot}}(H) \leq k'$. Since $\text{cr}_{\mathit{flip}}(H) \leq \text{cr}_{\mathit{rot}}(H)$, this immediately implies that the existence of a linear arrangement of G of value at most k is equivalent to both $\text{cr}_{\mathit{rot}}(H) \leq k'$ and $\text{cr}_{\mathit{flip}}(H) \leq k'$; hence deciding either is **NP**-hard.

If G has a linear arrangement of value at most k, we can draw H with the given rotation system using the linear arrangement to order the paths Q_i; this yields a drawing of crossing number at most k' (the m^2 term in k' compensates for the potential pairwise crossings of the edges in H that represent edges in E), so $\mathrm{cr}_{rot}(G) \leq k'$.

For the other implication, consider a drawing of H with crossing number at most $k' = n(n-1)ww' + kw' + m^2$, allowing rotation flips. The heavy-weight cycle on $\{u_1, \ldots, u_{4n}\}$ is drawn without crossings, and the rest of H is connected (since G is connected) so it is drawn entirely on one side of that cycle; we may assume without loss of generality that it is on the interior of the cycle. Note that $k' < n^2ww' + m^2w' + m^2$, and by choice of w and w' this is at most $35m^8 + 5m^4 + m^2 < w^2$. Hence, in our drawing no two edges of weight w cross each other, and therefore the paths P_i ($1 \leq i \leq 2n$) are drawn as shown in Figure 1.

Next, consider the modified paths Q_i. Q_i must cross each of the paths P_{i+1} through P_{2n-i}, contributing $(2n-2i)ww'$ to the crossing number. Summing these values for $i = 1, \ldots, n$, we observe a contribution of at least $n(n-1)ww'$ from crossings between the Q_i and the P_j to the crossing number. This leaves $k' - n(n-1)ww' = kw' + m^2$ possible crossings. Since $kw' + m^2 \leq m^2w' + m^2 = (w'/5)(w'+1) < (w'/2)^2$, there cannot be any further crossings among edges from any of the paths Q_i and P_j (all of these edges have weight w, w' or $w'/2$, and $w > w' > w'/2$; so any crossing would contribute at least $(w'/2)^2$ to the crossing number). From this it follows that the rotation at each b_i is not flipped.

Finally, we want to argue that every b_i lies between P_n and P_{n+1}. We already know this for b_n. Consider any b_i. As G is connected by assumption, there is a path from b_n to b_i using edges encoding G. If this path crosses P_n or P_{n+1}, it contributes w or more to the crossing number. However, since $kw' + m^2 < m^2w' + m^2 = 5m^4 + m^2 < 7m^4 = w$, this is not possible. Therefore, every b_i is also located between P_n and P_{n+1}.

In summary, the drawing of H looks as shown in Figure 1. This drawing clearly indicates a linear arrangement ϕ of G. An edge $e = uv$ contributes at least $|\phi(u) - \phi(v)|w'$ to the crossing number of H, so $\sum_{uv \in E} |\phi(u) - \phi(v)|w' \leq kw' + m^2$. Since $m^2 < w'$, the value of the linear arrangement is at most k.

We still need to convert H to an unweighted graph. To this end, we replace each edge e of weight x by x parallel edges, and then subdivide each of these edges; the effect is that e is replaced by a copy of $K_{2,x}$ with the endpoints of e identified with the partite set of size 2. The new edges are inserted in the rotation at where e was, ordered as indicated in Figure 2. Thus we obtain an unweighted graph H' from H. Since all weights are polynomially bounded in the size of G, the unweighted graph is of size at most polynomial in the size of G.

Recall that the existence of a linear arrangement of G of value at most

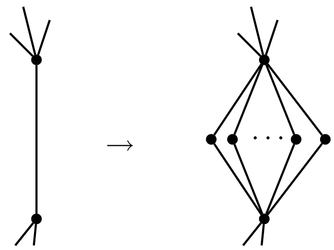

Figure 2: Replacing an edge by parallel paths

k is equivalent to both $\text{cr}_{rot}(H) \leq k'$ and $\text{cr}_{\mathit{flip}}(H) \leq k'$. Suppose that $\text{cr}_{rot}(H) \leq k'$. Since H' can be drawn like H, we have $\text{cr}_{rot}(H') \leq \text{cr}_{rot}(H)$. Also $\text{cr}_{\mathit{flip}}(H') \leq \text{cr}_{rot}(H')$, so we have $\text{cr}_{rot}(H') \leq k'$ and $\text{cr}_{\mathit{flip}}(H') \leq k'$.

To finish, since $\text{cr}_{rot}(H') \leq k'$ implies that $\text{cr}_{\mathit{flip}}(H') \leq k'$, it suffices to show that $\text{cr}_{\mathit{flip}}(H') \leq k'$ implies that $\text{cr}_{\mathit{flip}}(H) \leq k'$. Consider a drawing of H' that allows the rotation at each vertex to flip, and which has crossing number at most k'. This drawing naturally induces a rotation system of H (which corresponds to the given rotation system, except that the rotation of some vertices might have been flipped). Each edge $e = uv$ of weight x in H now corresponds to a collection $\mathcal{P}_e$ of x paths of length 2 in H'. For every edge e pick one path $P_e \in \mathcal{P}_e$ that has the smallest number of total crossings with paths in $\bigcup_{f \neq e} \mathcal{P}_f$. Then replace $\mathcal{P}_e$ by an edge following P_e of weight x. The resulting drawing has weighted crossing number at most the crossing number of the drawing of H' we started with, that is, k'. ■

In Section 4 we show that Theorem 2.1 remains true for other notions of crossing numbers.

3 Cubic Graphs

Theorem 2.1 can be used to prove that computing the crossing number of a cubic graph is **NP**-complete. This was a long-standing open question that was solved only recently by Petr Hliněný.

Theorem 3.1 (Hliněný [7])**.** *Computing the crossing number of a* 3*-connected, cubic graph is* **NP***-complete.*

Proof. Consider a graph G with rotation system. We will construct a 3-connected, cubic graph G' such that $\text{cr}_{\mathit{flip}}(G) \leq k$ if and only if $\text{cr}(G') \leq k$. This suffices, since by Theorem 2.1 deciding $\text{cr}_{\mathit{flip}}(G) \leq k$ is **NP**-complete.

We can assume that G has no vertices of degree 1 (by removing them) or 2 (by contracting an incident edge); G could become a multigraph, but the crossing number remains unchanged. Replace each vertex v by a hexagonal grid, made up of $4k+4$ rows of $d = \deg(v)$ hexagons per row. (The idea of using hexagonal grids is present in Hliněný's original proof.) Let the vertices along the top be labeled $v_1, \ldots, v_{2d+1}$, as shown in Figure 3.

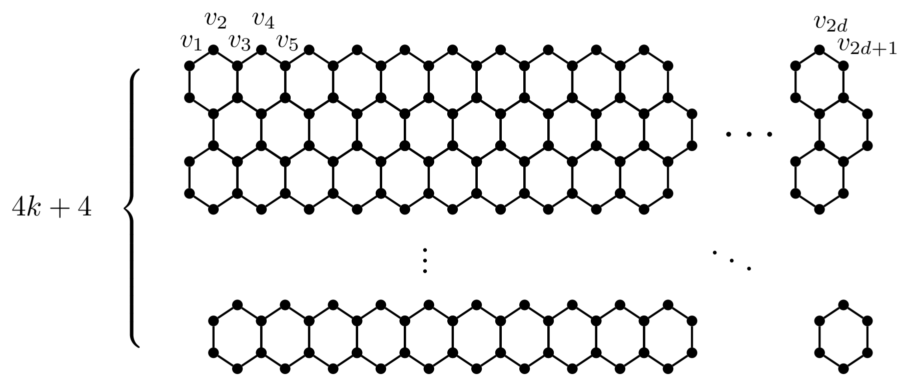

Figure 3: Hexagonal grid replacing vertex

Let us say the rotation at v lists edges in order $e_1, \ldots, e_d$ (cyclic order, so the first element is chosen arbitrarily). We make each e_i incident to v_{2i}. Repeating this for every vertex, we obtain a simple graph of maximum degree 3; the hexagonal grids still contain vertices of degree 2; we can remove these by edge contractions to obtain a simple, cubic graph G'. Let H_v be the subgraph of G' resulting from the hexagonal grid that replaced v by removing degree 2 vertices through edge contraction. Note that each edge of H_v belongs to a single row or two consecutive rows of (partially contracted) hexagons.

Any drawing of G with crossing number at most k yields a drawing of G' with at most k crossings. For the reverse direction, suppose that we have a drawing of G' with at most k crossings. Let X be the set of edges of H_v involved in crossings. Since $|X| \leq 2k$ there must be two consecutive rows in H_v that do not contain an edge of X; let R_v be the subgraph formed by these rows. R_v is a subdivision of a 3-connected graph, so R_v has a unique embedding on the sphere up to flipping the rotation. Hence, without loss of generality, R_v is drawn so that each of its hexagons, some of which are partially contracted, bounds an empty face.

We can easily find disjoint paths P_i from v_{2i} to R_v for $1 \leq i \leq d$ within

H_v. Let H'_v be the union of R_v and all P_i. Consider the restriction of the drawing of G' to the drawings of the H'_v (for all $v \in V$) and all edges between distinct H_v (the edges in E). At each H_v, consider the edges e_i extended through the paths P_i until they reach R_v. Since R_v is not involved in any crossings, the paths attach at R_v in their original order $P_1, P_2, \dots P_d$. Hence, if we contract R_v to a single point, the paths will attach in the order corresponding to the rotation at v or its flipped rotation. Contracting every R_v to a single point, we obtain a subdivision of G with the given rotation or flipped rotation at each vertex of G. Removing the subdivisions yields the desired drawing of G. Since none of the operations (restriction, contraction of crossing-free edges, removing subdivisions) increase the crossing number we have obtained a drawing of G with crossing number at most k. Thus, computing the crossing number of a simple, cubic graph is **NP**-complete.

Finally, observe that the graph G' we constructed is 3-connected: suppose that there were two vertices disconnecting G'; if both vertices belong to the same H_v, then they have to be among the labeled vertices; however, since we assumed that G has minimum degree at least 3, this is not possible. Hence the two vertices must belong to two different H_v and must be among the labeled vertices that are connected to other grids; however, each grid is attached to at least three other grids (since the original graph G has minimum degree 3), so removing two vertices from G' cannot disconnect it. ■

Remark 3.2. Theorem 3.1 remains true if the graph is given with rotation system. Indeed, the proof becomes easier since we no longer have to be concerned about the hex-grids flipping. Also note that for cubic graphs allowing vertex rotations to flip is equivalent to not specifying a rotation system since there are only two rotations at each vertex of degree 3.

As Hliněný observes, Theorem 3.1 implies that computing the minor-monotone crossing number is **NP**-complete [7]. Another result, which follows immediately (as observed in [3]) is that it is **NP**-hard to find a drawing of a directed graph in which all incoming (and therefore all outgoing) edges at a vertex are consecutive and which minimizes the crossing number.

Finally, our Theorem 2.1 is in turn derivable from Hliněný's result, as the gadget in Figure 4 shows. If we take a cubic graph and replace each vertex by the gadget, we obtain a graph with a fixed rotation system, whose crossing number differs from the crossing number of the original graph by an additive term.

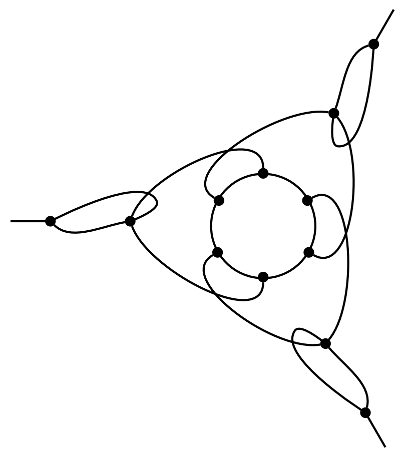

Figure 4: Rotation gadget

4 Other Crossing Numbers

There are many different ways to define a notion of crossing number and the current definition has not always been the standard one; even recently Tao and Vu in their book on additive combinatorics define crossing number to be what we would call pair crossing number, and state and use the crossing lemma for it [23]. For the historical development of the notion of crossing number and its variants, see the papers by Pach and Tóth [14] and Székely [21]. Here, we concentrate on four of the main variants: the rectilinear crossing number, the odd crossing number, the pair crossing number and the independent odd crossing number.

The *rectilinear crossing number* of G, $\mathrm{rcr}(G)$, is the minimum number of crossings in a *straight-line drawing* of G that is, a drawing in which edges are realized as straight-line segments.

A graph G has crossing number at most $\binom{m}{2}$. If we replace each crossing with a temporary vertex, the resulting graph is planar and thus has a straight-line drawing with the same rotation system by the proof of Fáry's theorem [13]; by doubling each temporary vertex (one copy corresponding to each edge of G crossing at the temporary vertex) and perturbing the temporary vertices slightly, we obtain a drawing of a subdivision of G with the same crossing number as the original drawing and in which every edge of G corresponds to a polyline with at most $\binom{m}{2} + 1$ line segments. Therefore, if G' is obtained from G by subdividing each edge of G with $\binom{m}{2}$ vertices, then

$\text{rcr}(G') = \text{cr}(G)$ and $\text{rcr}_{rot}(G') = \text{cr}_{rot}(G)$. If we start with G cubic, then G' has vertices of degree 3 and 2, but we can easily make G' cubic, by attaching the gadget in Figure 5 to each vertex of degree 2.

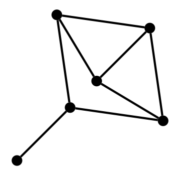

Figure 5: Gadget to attach to degree 2 vertices

Hence, by Theorem 3.1 and Remark 3.2, we obtain the next result.

Theorem 4.1. *Computing the rectilinear crossing number of a cubic graph with or without a given rotation system is* **NP***-hard.*

It is not clear whether we can maintain 3-connectivity (our proof certainly does not do so). It is not known whether rcr, with or without rotation, lies in **NP**.

The *pair crossing number* of a drawing is the number of pairs of edges that cross, counting each pair only once. The *odd crossing number* of a drawing is the number of pairs of edges that cross an odd number of times. The *independent odd crossing number* of a drawing is the number of pairs of non-adjacent edges that cross an odd number of times. Taking the minimum of each parameter over all drawings of a graph G gives the pair crossing number, $\text{pcr}(G)$, the odd crossing number, $\text{ocr}(G)$ and the independent odd crossing number, $\text{iocr}(G)$. All of these crossing numbers can be extended to weighted graphs, graphs with rotation and flipped rotations analogously to crossing number.

Computing ocr and pcr without rotation system is **NP**-complete and the problems remain in **NP** if we add rotation systems [14, 20]. In fact, the problems remain **NP**-hard as well:

Theorem 4.2. *Computing odd or pair crossing number of a cubic, 3-connected graph with or without a given rotation system is* **NP***-complete.*

While the proof of this result is based on the same basic construction as Theorem 2.1, the verification that the construction works becomes much more complex and needs new ideas. For this reason we leave the proof to Section 6.[1]

[1] The claim about pair crossing number could be established directly by modifying the proof of Theorem 2.1. Odd crossing number, however, seems to require additional work.

As a bonus, Theorem 4.2 allows us to settle the complexity of the independent odd crossing number problem simply because $\mathrm{iocr}(G) = \mathrm{ocr}(G)$ for cubic graphs, since any three edges incident to the same vertex can be redrawn so they cross each other evenly by modifying the rotation at the vertex. Since independent odd crossing number lies in **NP** [14], we obtain the following result.

Corollary 4.3. *Computing the independent odd crossing number of a graph is* **NP***-complete.*

5 Parameterization

One way to parameterize the crossing number problem is by the number of vertices of the graph; that is, we think of the number of vertices as small and fixed but allow an arbitrary number of (multiple) edges and loops. Without rotation system, this problem is equivalent to computing the crossing number of a weighted graph without multiple edges or loops: Given a graph $G = (V, E)$ with multiple edges and loops, note that in a crossing-number optimal drawing any two edges with the same endpoints can be routed in parallel. If we let G' be the complete graph on V with edge weights $w(uv)$ equal to the number of edges in E between u and v, then the weighted crossing number of G' equals $\mathrm{cr}(G)$. Note that the weights of G' can be stored using at most $\log m$ bits, where $m = |E|$.

Moreover, that weighted crossing number of G' can be computed exactly. For each edge there are $\sum_{k<\binom{n}{2}} k! \leq \binom{n}{2}!$ orderings in which other edges can cross it (where $n = |V|$), since any two edges cross at most once. Replacing each crossing with a vertex yields a planar graph with at most $O(n^4)$ vertices, so we can exhaustively try all crossing patterns and test them for planarity in time $O\left(\binom{n}{2}!^{\binom{n}{2}} n^4\right)$. For each planar drawing, we can calculate its crossing number in time $O(n^4 \log^2 m)$ by scanning each pair of crossing edges and adding the product of their weights to the crossing number. The minimum of these numbers is the weighted crossing number of G', which we had to compute. The overall running time is $O(2^{n^3 \log n} n^4 \log^2 m)$.

The problem becomes more interesting if the graph G is given with a rotation system. For example, the separation of pcr and ocr was first demonstrated via a two-vertex multigraph with rotation system [18]. In the following sections we discuss the cases of one and two vertices connecting them with well-known problems such as determining the number of inversions in a permutation and finding the edit distance of two cyclic words. We also include a weak approximation result for the general case.

5.1 One Vertex

Given a graph with a rotation system on a single vertex (with loops), it is quite straightforward to compute its crossing number in quadratic time.

In contrast, a linear time algorithm for the one-vertex case would come as a surprise, since the problem contains as a special case a well-studied problem: computing the number of inversions of a permutation. Given a permutation π over $\{1, \ldots, m\}$, an *inversion* of π is a pair (i, j) such that $i < j$ and $\pi(i) > \pi(j)$. It is well-known that the number of inversions of a permutation π equals $d_s(123\ldots m, \pi(1)\pi(2)\ldots\pi(m))$, where $d_s(u, v)$ is the number of transpositions of adjacent letters required to get from word u to word v (see, for example [9, Section 5.1.1]). The best-known algorithms for either problem run in $\Theta(m \log m)$.[2]

The inversion problem is easily encoded as a crossing number problem on a single vertex: simply let the rotation at the vertex be $12\ldots m\pi(m)\pi(m-1)\ldots\pi(2)\pi(1)$; this suggests that an algorithm for the single-vertex case that runs better than $O(m \log m)$ will be hard to come by.

We can, however, compute the crossing number of a one-vertex multigraph in time $\Theta(m \log m)$, extending the algorithm used to compute the number of inversions of a permutation.

Theorem 5.1. *The crossing number of a one-vertex multigraph with rotation system can be computed in time $O(m \log m)$.*

Proof. Let π be the rotation of the one-vertex multigraph G. If the graph has m edges, then π has length $2m$, containing each number in $1, 2, \ldots, m$ exactly twice. Split π into two halves: $\pi = \pi_0\pi_1$. If both occurrences of $i \in \{1, \ldots, m\}$ are in π_0 we say i is of type 0. If both occurrences of i are in π_1 we say i is of type 1. Otherwise i is of type 2. An edge of type 0 does not cross an edge of type 1 in a minimal drawing. Let $\mathrm{cr}_{rot}(G, i, j)$ be the number of crossings between edges of type i and type j. Then $\mathrm{cr}_{rot}(G) = \mathrm{cr}_{rot}(G, 0, 0) + \mathrm{cr}_{rot}(G, 1, 1) + \mathrm{cr}_{rot}(G, 2, 2) + \mathrm{cr}_{rot}(G, 0, 2) + \mathrm{cr}_{rot}(G, 1, 2)$.

We compute $\mathrm{cr}_{rot}(G, 0, 0)$ and $\mathrm{cr}_{rot}(G, 1, 1)$ recursively. The value of $\mathrm{cr}_{rot}(G, 0, 2)$ and, similarly, $\mathrm{cr}_{rot}(G, 1, 2)$ can be computed directly in linear time as follows: Process π_0 from left to right. Keep a counter that counts how many type 2 edges have been seen so far; initially, the counter is zero. During the loop, when we encounter a type 0 edge we store the current value

[2] See [9, Exercises 5.1.1-6 and 5.2.4.-21]. Wagner's linear time algorithm [25] for computing $d_s(u, v)$ is wrong.

of the counter at that position. At the end of the loop, we sum for each type 0 edge the difference between the two values stored at the positions where the edge begins and ends. That sum is $\mathrm{cr}_{rot}(G,0,2)$. Finally, consider $\mathrm{cr}_{rot}(G,2,2)$. All edges of type 2 begin in π_0 and end in π_1. Two type 2 edges cross if the order of their endpoints in π_0 and in π_1 is the same. Hence, we can compute $\mathrm{cr}_{rot}(G,2,2)$ by counting inversions, which can be done in time $O(m\log m)$ as we mentioned earlier.

Combining these observations, we obtain the recurrence

$$T(m) = 2T(m/2) + O(m\log m)$$

for the running time of the algorithm, which has the solution $T(m) = O(m\log^2 m)$. However, we can improve the analysis: since an edge cannot have more than one type in each step of the recursion, we really have

$$T(m) = T(m_0) + T(m_1) + O(m_2\log m_2) + O(m),$$

where m_i is the number of edges of type i, and $m_0 + m_1 + m_2 = m$. It is easy to show that $T(m) = O(m\log m)$. ■

5.2 Two Vertices

In this section we consider graphs on two vertices, allowing multiple edges, but no loops. The crossing number of a loopless two-vertex multigraph can be expressed as the solution of an integer linear program whose relaxation can be used to compute the optimal integer solution in polynomial time [18].

Here we want to give a fast and simple 2-approximation algorithm for the two-vertex case. To do so, we look at the crossing number problem as an *edit-distance* problem on words. The edit distance between two words is the smallest number of operations transforming one word into the other. There are numerous variants of this problem depending on which operations are allowed and what the associated costs are [25, 10]. There are also several papers studying objects other than words, such as trees and cyclic words (also known as necklaces) [11, 12, 8], but it seems the particular variant we find needful here—allowing only swaps (at unit cost) on cyclic words—has not been considered at all so far. A *swap* is the transposition of two adjacent letters in a word. A *cyclic word* is the equivalence class of a word under cyclic shifts. The last and first letter of a cyclic word are considered adjacent. Let $d_s(u,v)$ be the smallest number of swaps transforming u into v, where u and v are ordinary words. Similarly, let $d_s^\circ(u,v)$ be the smallest number of swaps transforming u into v allowing cyclic shifts at no cost. Then $d_s^\circ(u,v)$ is the swapping distance of the two cyclic words represented by the words u and v. E.g. $d_s^\circ(abcd, cdba) = 1$, while $d_s(abcd, cdba) = 5$.

Computing d_s is easy (see [25]). Our goal is the computation of $d_s^\circ(u, v)$.

Swapping distance of Cyclic Words
Instance: Two words u, v, integer k.
Question: Is $d_s^\circ(u, v) \leq k$?

We do not know how hard this problem is in general; however, with the restriction that the words contain each letter exactly once, we can solve the problem. Indeed, in that case it is equivalent to computing the crossing number of a loopless two-vertex multigraph G with rotation system.

This is easily seen: let the two (cyclic) words u and v represent the rotations of the two vertices p and q of G reading the edges clockwise. Draw p with edges leaving p in the order determined by u. Let v^R denote the *reverse* of v. For every swap in the sequence of $d_s^\circ(u, v^R)$ swaps we extend all edges and cross the two edges corresponding to the letters swapped. We obtain a set of curves ordered according to v^R which can then be connected without further crossings to a vertex q with rotation v.

Suppose, on the other hand, that we are given a graph G with two vertices p and q that have clockwise rotations u and v. Fix a drawing of G that respects the rotations and minimizes the crossing number. In this drawing no two edges can cross more than once, by a standard argument [21]: if they did, we could consider the segments of the edges between the crossings, and avoid one or two crossings by redrawing the segments alongside each other, following the segment with fewer crossings, reducing the total number of crossings; the redrawing move might introduce self-intersections of an edge, but those can be removed easily.

For each crossing, the arcs from p to the crossing form a closed Jordan curve; let the region bounded by this curve that does not contain q be called a *p-bigon*. For any p-bigon B, if an edge crosses its boundary exactly once, then the arc from p to that crossing must lie within B, forming a p-bigon contained within B; note that such an edge enters the rotation at p from the interior of B. Now let B be a minimal bigon (with respect to containment). Any edge that crosses B must cross each of the two arcs bounding B exactly once (it cannot cross either arc more than once since two edges cross at most once, and if it only crossed the boundary of B once then B would not be minimal). We redraw one of these arcs alongside the other, removing the p-bigon and lowering the number of crossings by one (see Figure 6). The change in the rotation at p translates into a swap of letters in u corresponding to the two curves that form the bigon. Repeating this argument, we can inductively prove that the crossing number of G equals the swapping distance of u and v^R.

Proposition 5.2. *For a loopless two-vertex multigraph G with rotations u and v^R, $\mathrm{cr}_{rot}(G) = d^\circ(u, v)$.*

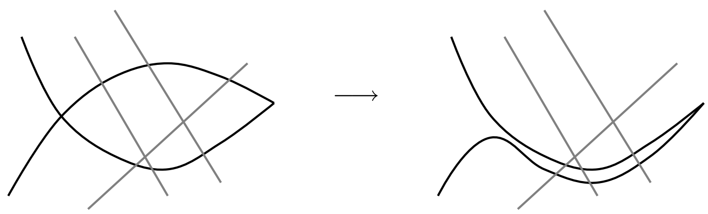

Figure 6: Removing a minimal bigon and a crossing; changing the rotation

We rephrase the restricted swapping-distance problem as follows: we can assume that $u = \sigma(1)\sigma(2)\cdots\sigma(m)$ and $v^R = 12\cdots m$ for some permutation σ of the elements of the cyclic group Z_m Define $\mathrm{cr}(\sigma) := d^\circ(u, v)$. Thinking of u as a word, we can say $\sigma(i)$ is in *position i*.

For each $i \in Z_m$, let swap_i be the permutation that switches i and $i+1$ mod m and fixes all other elements of Z_m; then for any permutation τ of Z_m, $\tau \circ \mathrm{swap}_i$ has the positions of $\tau(i)$ and $\tau(i+1)$ switched. For each $i \in Z_m$, let shift_i denote the permutation of Z_m such that $\mathrm{shift}_i(j) = j - i \mod m$ for all $j \in Z_m$; then $\tau \circ \mathrm{shift}_i$ has each $\tau(j)$ moved i positions up. By definition, $\mathrm{cr}(\sigma)$ is the minimum number of swaps in a sequence of swaps and shifts $\tau_0, \tau_1, \ldots, \tau_k$ such that $\sigma \circ \tau_0 \circ \tau_1 \circ \cdots \circ \tau_k$ is the identity permutation. Since $\mathrm{swap}_i \circ \mathrm{shift}_j = \mathrm{shift}_j \circ \mathrm{swap}_{i+j \bmod m}$ for any $i, j \in Z_m$, we can assume that τ_0 is the only shift and $\tau_1, \ldots, \tau_k$ are swaps.

We define a new function $\widetilde{\mathrm{cr}}$ on permutations that will be seen to be related to cr. For a permutation τ of Z_m and for each $i \in Z_m$, let $d_i^+(\tau) = \tau(i) - i \mod m$ and $d_i^-(\tau) = i - \tau(i) \mod m$, and let $d_i(\tau) = \min\{d_i^+(\tau), d_i^-(\tau)\}$. The latter measures the minimum number of swaps needed to move $\tau(i)$ to position $\tau(i)$. Next define $d(\tau) = \sum_{i \in Z_m} d_i(\tau)$ and

$$\widetilde{\mathrm{cr}}(\tau) = \min\{d(\tau \circ \mathrm{shift}_j) \ : \ j \in Z_m\}.$$

We claim that $\widetilde{\mathrm{cr}}$ approximates cyclic swapping distance to within a factor of 2.

Theorem 5.3. *For any permutation σ, $\mathrm{cr}(\sigma) \leq \widetilde{\mathrm{cr}}(\sigma) \leq 2\,\mathrm{cr}(\sigma)$.*

By Proposition 5.2, it immediately follows that for any two-vertex loopless multigraph G represented by a permutation σ, $\mathrm{cr}_{rot}(G) \leq \widetilde{\mathrm{cr}}(\sigma) \leq 2\,\mathrm{cr}_{rot}(G)$.

Proof. We first show that $\widetilde{\mathrm{cr}}(\sigma) \leq 2\,\mathrm{cr}(\sigma)$. For any permutation τ of Z_m and any $j \in Z_m$, $|d_i(\tau) - d_i(\tau \circ \mathrm{swap}_j)|$ is 0 unless $i \in \{j, j+1 \bmod m\}$, in which case it is 0 or 1. Therefore $|d(\tau) - d(\tau \circ \mathrm{swap}_j)| \leq 2$ for all j; if $\mathrm{cr}(G) = k$, then there is a shift τ_0 and k swaps $\tau_1, \ldots, \tau_k$ so that $\sigma \circ \tau_0 \circ \tau_1 \circ \cdots \circ \tau_k$ is the identity, so $d(\sigma \circ \tau_0 \circ \tau_1 \circ \cdots \circ \tau_k) = 0$, and, therefore, $d(\sigma \circ \tau_0) \leq 2k$. On the other hand, $d(\sigma \circ \tau_0) \geq \widetilde{\mathrm{cr}}(G)$ by definition, and we obtain $\widetilde{\mathrm{cr}}(G) \leq 2\,\mathrm{cr}(G)$.

It remains to show that $\mathrm{cr}(\sigma) \leq \widetilde{\mathrm{cr}}(\sigma)$. For this, it suffices to prove that $\mathrm{cr}(\sigma) \leq d(\sigma)$, since this implies $\mathrm{cr}(\sigma \circ \mathrm{shift}_j) \leq \widetilde{\mathrm{cr}}(\sigma)$ for some j, and $\mathrm{cr}(\sigma \circ \mathrm{shift}_j) = \mathrm{cr}(\sigma)$ (for any j).

Z_m is partitioned into cycles by σ. If all of them are trivial then $\sigma = id$ and $\mathrm{cr}(\sigma) = 0 = d(\sigma)$, but otherwise we may let $S \subseteq Z_m$ be the set of elements of a nontrivial cycle ψ in σ. (Then $\psi = \sigma|_S$, and for all $i, j \in S$ there is some k such that $\sigma^k(i) = j$.)

First we consider the case that there exist $i, j \in S$ with $d_i(\sigma) = d_i^-(\sigma)$ and $d_j(\sigma) = d_j^+(\sigma)$. Then there must exist some such i, j with $j = \sigma(i)$. We may assume that $d_i(\sigma) \leq d_j(\sigma)$; the other case is similar. We first apply $d_i^-(\sigma)$ swaps to move $\sigma(i) = j$ from position i *down* to position j, which also moves each $\sigma(j), \sigma(j+1), \ldots, \sigma(i-1)$ one position up. Then we can apply $d_i^-(\sigma) - 1$ swaps to move $\sigma(j)$ *upward* from position $j+1$ to position i, which moves each $\sigma(j+1), \sigma(j+2), \ldots, \sigma(i-1)$ back down one step to their original positions. Thus we have switched the positions of j and $\sigma(j)$ while fixing all other elements, using $2d_i(\sigma) - 1$ swaps . For the new permutation σ' we have $d_j(\sigma') = 0$, $d_i(\sigma') = d_j(\sigma) - d_i(\sigma)$, and $d_k(\sigma') = d_k(\sigma)$ for all $k \notin \{i, j\}$, so $d(\sigma') = d(\sigma) - 2d_i(\sigma)$. By induction there is a way to make σ' become the identity permutation using shifts and at most $d(\sigma')$ swaps, altogether giving us a way to change σ to the identity using at most $d(\sigma') + 2d_i(\sigma) - 1$ swaps. That is no more than $d(\sigma)$, completing this case.

In the remaining case, we may assume without loss of generality that $d_i(\sigma) = d_i^+(\sigma)$ for all $i \in S$. For each $i \in S$, let $f(i) \in S \backslash \{i\}$ be (uniquely) defined such that only the first and last elements in positions $i, i+1, \ldots f(i)$ are in S. For every $i \in S$, perform $f(i) - i - 1$ swaps that move $\sigma(i)$ upward from position i to position $f(i) - 1$; this also moves each $i \in Z_m \backslash S$ one position back. Then apply shift_1. We get a permutation σ' with $\sigma(i)$ in position $f(i)$ for each $i \in S$, and $\sigma(i) = \sigma'(i)$ for each $i \in Z_m \backslash S$. Then $d(\sigma) - d(\sigma') = \sum_{i \in S}(f(i) - i \mod m)$. Since the number of swaps used is

$\sum_{i\in S}(f(i) - i - 1 \mod m)$ which is no more than $d(\sigma) - d(\sigma')$, we finish by applying induction to σ'. ■

Remark 5.4. The bounds of Theorem 5.3 are asymptotically optimal: for $\sigma := (1\ 2)(3\ 4)\cdots(2m-1\ 2m)$ we have $\widetilde{\mathrm{cr}}(\sigma) = 2m$ and $\mathrm{cr}(\sigma) = m$; for the lower bound consider $\tau := (1\ m)$ (as a permutation of numbers $1,\ldots,2m$), then $\widetilde{\mathrm{cr}}(\tau) = 2m-2$ and $\mathrm{cr}(\tau) = 2m-3$.

Remark 5.5. We have seen that the crossing number of a loopless two-vertex multigraph equals the swapping distance of two cyclic words. If instead of cyclic words we restrict the problem to ordinary words, the swapping distance equals the crossing number of a two-vertex multigraph which has both of its vertices on the boundary of a disk and all edges within the disk. We can view this as a special case of the crossing number of a two-vertex multigraph, by replacing the boundary of the disk by many parallel edges. The fact that $\widetilde{\mathrm{cr}}(G)$ approximates $\mathrm{cr}(G)$ in this case is known as Spearman's Footrule and was first proved by Diaconis and Graham [4].

Theorem 5.3 gives us a fast and easy way to approximate $\mathrm{cr}_{rot}(G)$ for a two-vertex multigraph. Computing $\widetilde{\mathrm{cr}}(\sigma)$ from the definition can be done in quadratic time; using dynamic programming the problem can be solved in linear time: to simplify the following sketch, we assume that $|\sigma|$ is odd, and let $d_{\max} = \lfloor|\sigma|/2\rfloor$. For all $0 \le d \le d_{\max}$ and $0 \le j < |\sigma|$ let

$$\begin{aligned} n_j^d &:= |\{i : d_i^+(\sigma\circ\mathrm{shift}_j) = d_i(\sigma\circ\mathrm{shift}_j) = d\}|, \text{and} \\ n_j^{-d} &:= |\{i : d_i^-(\sigma\circ\mathrm{shift}_j) = d_i(\sigma\circ\mathrm{shift}_j) = d\}|. \end{aligned}$$

Note that $d(\sigma\circ\mathrm{shift}_{j+1}) = d(\sigma\circ\mathrm{shift}_j) - \sum_{d=1}^{d_{\max}} n_j^d + \sum_{d=0}^{d_{\max}-1} n_j^{-d}$. Also, $n_{j+1}^d = n_j^{d+1}$ for $-d_{\max} \le d < d_{\max}$, and $n_{j+1}^{d_{\max}} = n_j^{-d_{\max}}$. Let

$$X_j = -\sum_{d=1}^{d_{\max}} n_j^d + \sum_{d=0}^{d_{\max}-1} n_j^{-d}.$$

Then $X_{j+1} - X_j = -n_{j+1}^{d_{\max}} + 2n_j^1 - n_j^{-d_{\max}+1}$, and the values $n_j^{d_{\max}}, n_j^1, n_j^{-d_{\max}+1}$ can be located within $\langle n_0^d\rangle_{d=-d_{\max}}^{d=d_{\max}}$ in constant time. Thus, our algorithm is as follows: Compute $\langle n_0^d\rangle_{d=-d_{\max}}^{d=d_{\max}}$, then X_0, then $X_1,\ldots,X_{|\sigma|-1}$, then $\langle d(\sigma\circ\mathrm{shift}_j)\rangle_{j=0}^{|\sigma|-1}$, and finally take the maximum of the previous sequence. Each step takes linear time, and the last step gives us $\widetilde{\mathrm{cr}}(\sigma)$.

Corollary 5.6. *The crossing number of a loopless two-vertex multigraph with rotation system can be approximated to within a factor of* 2 *in linear time.*

5.3 Several Vertices

There is little we can say at this point about how hard it is to compute the crossing number of a graph with a rotation system on a fixed number k of vertices when $k \geq 3$. Using results from a previous paper [18], however, we can give at least an approximation result. In this section we allow both loops and multiple edges.

Theorem 5.7. *The crossing number of a multigraph $G = (V, E)$ with rotation system can be approximated to within a factor of $\binom{k+4}{4}/5$ in time $O(m^k \log m)$, where $k = |V|$ and $m = |E|$.*

In [18] we showed that $\text{cr}(G) \leq \text{ocr}(G)\binom{k+4}{4}/5$ (see Section 4 for the definition of ocr). In fact, the proof applies to a multigraph G with rotation system, yielding $\text{cr}_{rot}(G) \leq \text{ocr}_{rot}(G)\binom{k+4}{4}/5$. The proof works by choosing a particular sequence of $k-1$ edges $e_1, \ldots, e_{k-1}$ and contracting G along those edges obtaining a one-vertex multigraph G' with rotation system. For graphs on a single vertex, crossing number and odd crossing number are the same; hence, $\text{cr}_{rot}(G') = \text{ocr}_{rot}(G')$. Furthermore, the sequence of edges is chosen such that $\text{cr}_{rot}(G') \leq \text{ocr}_{rot}(G)\binom{k+4}{4}/5$. The redrawing procedure of the proof establishes that $\text{ocr}_{rot}(G) \leq \text{ocr}_{rot}(G')$. Introducing $c := \text{ocr}_{rot}(G')$ allows us to summarize the discussion as

$$c/(\binom{k+4}{4}/5) \leq \text{ocr}_{rot}(G) \leq c.$$

Since $\text{ocr}_{rot}(G) \leq \text{cr}_{rot}(G) \leq \text{ocr}_{rot}(G)\binom{k+4}{4}/5$, we conclude that

$$c/(\binom{k+4}{4}/5) \leq \text{cr}_{rot}(G) \leq c\binom{k+4}{4}/5.$$

Now c can be computed in time $O(m \log m)$ using the algorithm from Theorem 5.1 for one-vertex multigraphs with rotation system. The only remaining problem is that we do not know the sequence of edges that determines G' and π'. Thus we try all possible sequences, giving a running time of $O(m^k \log m)$.

6 Odd and Pair Crossing Numbers

The obvious strategy for proving Theorem 4.2 is to adapt the corresponding proof for cr to ocr using redrawing tools, showing that cr and ocr agree for the graphs used in the constructions or are close enough for the proof to go through.[3] We could not make this approach work and decided to replace Minimum Linear Arrangement with a different problem based on tournaments instead of linear orders. We introduce this new problem, Minimum Tournament Arrangement, and show it to be **NP**-hard in Section 6.1. With this we complete the proof of Theorem 4.2 in Section 6.2. Minimum Tournament Arrangement also leads to some combinatorial questions that are interesting in their own right.

6.1 A Tournament Problem

Our attempts to reduce Minimum Linear Arrangement to the odd crossing number problem with rotation ran into problems, since we were not able to show that an optimal drawing represents a linear arrangement. What happens if we replace the linear arrangement with the next best thing: a tournament?

A *tournament* $T = (V, F)$ is a directed graph such that for each pair of distinct vertices $u, v \in V$ exactly one of uv, vu is in F. Let $\mathrm{two}_T(u, v, w)$ be the indicator for the existence of an oriented path of length 2 between u and v, passing trough w, formally

$$\mathrm{two}_T(u,v,w) = \begin{cases} 1 & \text{if } \{uw, wv\} \subseteq F \text{ or } \{vw, wu\} \subseteq F \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Let $\mathrm{two}_T(u, v)$ be the number of $w \in V$ for which $\mathrm{two}_T(u, v, w) = 1$. We consider the following problem:

Minimum Tournament Arrangement
Given: A simple undirected graph $G = (V, E)$, number k.
Question: Does there exist a tournament $T = (V, F)$ such that

$$|E| + \sum_{uv \in E} \mathrm{two}_T(u, v) \leq k? \tag{2}$$

If G has a linear arrangement ϕ of value k then it has a tournament arrangement of value k, since one can take the linear order $T = (V, F)$ induced by ϕ: $uv \in F \Leftrightarrow \phi(u) < \phi(v)$. The value of $\mathrm{two}_T(u, v)$ in this tournament is $|\phi(u) - \phi(v)| - 1$, so $|E| + \sum_{uv\in E} \mathrm{two}_T(u, v) = \sum_{uv\in E} |\phi(u) - \phi(v)|$.

[3] Indeed, in an earlier version of the paper we claimed that Theorem 4.2 was an easy consequence of the construction for the standard crossing number [16]. We no longer believe this to be true.

We believe that the optimum of Minimum Tournament Arrangement is always attained on a tournament which is a linear order:

Conjecture 6.1. *For every graph G and every integer k the answers to* Minimum Linear Arrangement *for (G, k) and* Minimum Tournament Arrangement *for (G, k) are the same.*

We can prove Conjecture 6.1 for the complete graph. The value of the minimum linear arrangement of K_n is $\binom{n+1}{3}$, see, e. g., [5].

Lemma 6.2. *The minimum tournament arrangement of K_n has value $\binom{n+1}{3}$.*

Proof. We can count the number of paths of length 2 in T as follows:

$$A := \sum_{\{u,v\} \in \binom{V}{2}} \mathrm{two}_T(u, v) = \sum_{w \in V} d^+(w) d^-(w), \tag{3}$$

where $d^+(w)$ is the out-degree of w in T and $d^-(w)$ is the in-degree of w in T. Let $i(w) = d^+(w) - d^-(w)$ be the imbalance of w. Then

$$B := \sum_{w \in V} i(w)^2 = \left(\sum_w d^+(w)^2 \right) + \left(\sum_w d^-(w)^2 \right) - 2A. \tag{4}$$

We have

$$n(n-1)^2 = \sum_{w \in V} (d^+(w) + d^-(w))^2 = \left(\sum_w d^+(w)^2 \right) + \left(\sum_w d^-(w)^2 \right) + 2A. \tag{5}$$

From (4) and (5) we obtain $4A = n(n-1)^2 - B$. Thus the problem of minimizing A is equivalent to maximizing B.

Assume that T maximizes B. Suppose that $i(u) = i(v)$ for two distinct vertices $u, v \in V$. If we switch the orientation of $uv \in T$ then the contribution of u and v to B changes by $(x+2)^2 + (x-2)^2 - 2x^2 = 8$ where $x = i(u)$. Thus for T that maximizes B all the imbalances are different.

Now we claim that for u, v with $i(u) < i(v)$ the edge uv must be in T. Assume not. Then by reversing vu to uv the contribution of u and v changes by

$$(i(v) + 2)^2 + (i(u) - 2)^2 - i(v)^2 - i(u)^2 = 8 + 4(i(v) - i(u)) > 0.$$

Thus the optimal tournament is a linear order, for which the value agrees with the value of the linear arrangement with the corresponding ϕ. ∎

We were unable to settle Conjecture 6.1 but we will show the following result, which is sufficient for our purposes.

Theorem 6.3. Minimum Tournament Arrangement *is* **NP**-*complete.*

Proof. Minimum Tournament Arrangement is clearly in **NP**. For **NP**-hardness we will follow the **NP**-hardness proof of Minimum Linear Arrangement from [5, Theorem 1.5]. The reduction is from Max-Cut.

Given an instance $G' = (V', E')$, k' of Max-Cut, we construct an instance of Minimum Tournament Arrangement as follows. Let $n = |V'|$, $r = n^4$, and let U be a set of r vertices, $U \cap V' = \emptyset$. Let $V = V' \cup U$ and $G = (V, E)$, where E is defined by: $uv \in E \Leftrightarrow uv \notin E'$. Finally, let $k = \binom{n^4+n+1}{3} - k'n^4$. We will show that (G', k') is a positive instance of Max-Cut if and only if (G, k) is a positive instance of Minimum Tournament Arrangement.

First, suppose that (G', k') is a positive instance of Max-Cut. In this case we use the same argument as in [5]. Let S_1, S_2 be the max-cut of G'. Now consider the tournament arrangement induced by the linear order ϕ, where ϕ maps the vertices in S_1 to $\{1, \ldots, |S_1|\}$ and vertices in S_2 to $\{n^4 + n + 1 - |S_2|, \ldots, n^4 + n\}$. A quick calculation shows that the value of this tournament is at most k.

If, on the other hand, (G', k') is a negative instance of Max-Cut, then every cut in G' has size less than k'. Let T be any tournament for G. We have

$$|E| + \sum_{uv \in E} \mathrm{two}_T(u, v) = \left(\binom{n^4 + n}{2} + \sum_{\{u,v\} \in \binom{V}{2}} \mathrm{two}_T(u, v) \right) - |E'| - \sum_{uv \in E'} \mathrm{two}_T(u, v). \tag{6}$$

By Lemma 6.2, the first quantity on the right-hand side of (6) is at least $\binom{n^4+n+1}{3}$. Hence

$$|E| + \sum_{uv \in E} \mathrm{two}_T(u, v) \geq \binom{n^4 + n + 1}{3} - |E'| - \sum_{uv \in E'} \mathrm{two}_T(u, v). \tag{7}$$

Now $|E'| \leq n^2$ and

$$\sum_{uv \in E'} \mathrm{two}_T(u, v) \leq n^3 + \sum_{uv \in E'} \sum_{w \in U} \mathrm{two}_T(u, v, w), \tag{8}$$

where $\text{two}_T(u, v, w)$ is defined in (1).

Consider the right-hand side of (8). Let $w \in U$ be the vertex that contributes the most to the sum. Let $S_1 \subseteq V'$ be the vertices which point to w in T and $S_2 = V' \setminus S_1$. Then

$$\sum_{uv \in E'} \sum_{w \in U} \text{two}_T(u, v, w) \leq \sum_{uv \in E', u \in S_1, v \in S_2} n^4 \leq (k' - 1)n^4, \tag{9}$$

since the max-cut in G' has size at most $k' - 1$.

Combining (7), (8), and (9) we obtain

$$\begin{aligned} |E| + \sum_{uv \in E} \text{two}_T(u, v) &\geq \binom{n^4 + n + 1}{3} - n^2 - n^3 - (k' - 1)n^4 \\ &> \binom{n^4 + n + 1}{3} - k'n^4 = k, \end{aligned}$$

where the last inequality is true for $n \geq 2$. Hence (G, k) is a negative instance of Minimum Tournament Arrangement. ■

Conjecture 6.1 can be recast as a problem on matrices. Recall that a matrix A is *skew-symmetric* if $A^T = -A$; in particular, all diagonal elements of a skew-symmetric matrix are zero.

Conjecture 6.4. *Define the $n \times n$ skew-symmetric matrix A by $A_{ij} = -1$ for $i < j$ and let $B = AA^T$. Let $\mathcal{C}$ be the convex-hull of PBP^T where P ranges over all permutation matrices. If D is an $n \times n$ skew-symmetric matrix with entries from $[-1, 1]$, then there exists an $n \times n$ matrix F with non-negative entries so that $DD^T + F \in \mathcal{C}$.*

Proposition 6.5. *Conjecture 6.4 implies Conjecture 6.1; in the reverse direction, Conjecture 6.1 when extended to multigraphs implies Conjecture 6.4.*

Proof sketch. Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$. Let H be the adjacency matrix of G. Let T be a tournament and let D be the skew-symmetric matrix defined by $D_{ij} = +1$ if $ij \in T$ and $D_{ij} = -1$ if $ji \in T$ for $i < j$. Note that $(DD^T)_{ii} = n - 1$ and for $i \neq j$

$$(DD^T)_{ij} = \sum_k D_{ik} D_{jk} = (n - 2) - 2\text{two}_T(i, j).$$

Finally, note that

$$\langle DD^T, H \rangle = 2|E|(n - 2) - 4 \sum_{ij \in E} \text{two}_T(i, j). \tag{10}$$

(The inner product of two matrices X, Y is $\langle X, Y\rangle = \mathrm{tr}(XY^T)$.)

Comparing (2) and (10) we see that Conjecture 6.1 is equivalent to: the maximum of (10), over skew-symmetric matrices D with ± 1 off-diagonal entries, is attained for a matrix corresponding to a linear order.

First, suppose that Conjecture 6.4 is true and let D be the matrix maximizing (10). There exists F with nonnegative entries such that $DD^T + F \in \mathcal{C}$. We have

$$\max_{X\in\mathcal{C}} \langle X, H\rangle \geq \langle DD^T + F, H\rangle \geq \langle DD^T, H\rangle,$$

and the maximum of the linear function $X \mapsto \langle X, H\rangle$ on $\mathcal{C}$ is achieved on some vertex $PBP^T = (PAP^T)(PAP^T)^T$. Hence we could have taken $D = (PAP^T)$ and obtained at least as good a value for (10).

Now assume that Conjecture 6.4 is false. Let D be a counterexample to Conjecture 6.4. By linear programming duality (Farkas' Lemma) there exists a non-negative matrix Y such that

$$\langle DD^T, Y\rangle > \max_{C\in\mathcal{C}} \langle C, Y\rangle = \max_{P\in S_n} \langle (PAP^T)(PAP^T)^T, Y\rangle.$$

The space of solutions is dense, so we can choose Y with all its entries rational. Multiplying Y by the common denominator does not affect the truth of the inequality, so the entries of Y are non-negative integers. Since DD^T is symmetric, we can replace Y by $(Y + Y^T)/2$ without changing the inner product, making Y symmetric. Note that the diagonal entries of the matrices in $\mathcal{C}$ are $n-1$, and the diagonal entries of DD^T are at most $n-1$. Hence we can also assume that all diagonal entries of Y are zero. In summary, Y corresponds to a loopless multigraph. So we have found an example where the maximum of (10) with H replaced by Y is not achieved by a skew-symmetric matrix corresponding to a linear order. Thus Y is a counterexample to Conjecture 6.1 extended to multigraphs. ■

6.2 Proof of Theorem 4.2

We make use of a redrawing tool that allows us to remove crossings with even edges. An edge is called *even* if it is crossed by every other edge an even number of times (including the possibility of no crossings).

Lemma 6.6 (Pelsmajer, Schaefer, Štefankovič [15]). *If E_0 is the set of even edges in some drawing of a graph G in the plane, then G can be drawn so that no edge in E_0 is involved in any crossings and there are no new pairs of edges that cross an odd number of times. Moreover, the redrawing does not change the rotation system of G.*

We begin with the analogue of Theorem 2.1 for ocr and pcr:

Lemma 6.7. *Computing odd or pair crossing number of a graph with a given rotation system is* **NP***-complete. The problem remains* **NP***-complete if the rotation at each vertex is allowed to flip.*

Before proving Lemma 6.7, we show how to use it to complete the proof of Theorem 4.2.

Proof of Theorem 4.2. We first deal with ocr. Consider a graph G with rotation system. We construct G' from G as we did in the proof of Theorem 3.1. Any drawing of G can be turned into a drawing of G' in a natural way, which specifies a corresponding rotation system for G', such that there are corresponding pairs of oddly crossing edges in G and in G'. Hence $\text{ocr}_{\mathit{flip}}(G) \leq k$ implies $\text{ocr}_{\mathit{flip}}(G') \leq k$ and $\text{ocr}_{\mathit{rot}}(G) \leq k$ implies $\text{ocr}_{\mathit{rot}}(G') \leq k$. Since G' is cubic, $\text{ocr}_{\mathit{flip}}(G') = \text{ocr}(G')$.

For the other direction, suppose we have a drawing of G' with at most k pairs of edges that cross oddly. As we did earlier, we argue that each H_v that replaces a vertex v of G in G' must contain two consecutive rows R_v that have no edges in odd pairs. Lemma 6.6 allows us to redraw all the R_v so they are not involved in any crossings—without changing the rotation system or increasing ocr. Note that if any rotation in R_v is flipped then all of R_v is flipped. The rest of the argument now proceeds as in the original proof: we can contract each R_v to a single vertex obtaining a subdivision of the drawing of G. Removing the subdivisions does not increase ocr and leads to a drawing of G with odd crossing number at most k. If we started with a drawing realizing $\text{ocr}_{\mathit{flip}}(G') \leq k$, then this argument yields $\text{ocr}_{\mathit{flip}}(G) \leq k$. If the initial drawing realized $\text{ocr}_{\mathit{rot}}(G') \leq k$, then $\text{ocr}_{\mathit{rot}}(G) \leq k$.

Therefore $\text{ocr}_{\mathit{flip}}(G) \leq k \iff \text{ocr}_{\mathit{flip}}(G') \leq k$ and $\text{ocr}_{\mathit{rot}}(G) \leq k \iff \text{ocr}_{\mathit{rot}}(G') \leq k$, proving that $\text{ocr}_{\mathit{flip}}$ and $\text{ocr}_{\mathit{rot}}$ are **NP**-hard for 3-connected, cubic graphs.

The proof for pcr is almost the same (except that it is easier to obtain the drawing of G from the drawing of G', since the assumption that there are at most k crossing pairs of edges in the drawing of G' directly implies that there are two rows R_v in each H_v that are not involved in *any* crossings). ■

We are left with the proof of Lemma 6.7.

For embedded closed curves in the plane, the Jordan Curve Theorem tells us that the curve separates the plane into two regions. We can extend this notion to curves with self-intersections. If C is a closed curve with self-intersections we can define a notion of sides: two points not on C are *on the* *same side* of C if any curve connecting them crosses C an even number of times.[4] The notion of being on the same side of C is well-defined, since any two curves connecting two points form a closed curve and two closed curves on the plane cross an even number of times, so the two curves connecting the points must have the same parity of crossing with C.

Now any curve between two points on opposite (not the same) sides of C must cross C oddly. We will be considering curves in a drawing of a graph. Since the drawing is planar it is contained within a disk and we can arbitrarily pick a reference point outside that disk and call that point *outside*. This defines for every closed curve in the graph a notion of inside and outside.

We use G, H and parameters k, w, w' as defined in the proof of Theorem 2.1 except we now consider G as an instance of MINIMUM TOURNAMENT ARRANGEMENT rather than MINIMUM LINEAR ARRANGEMENT.

Note that the reduction from MAX-CUT to MINIMUM TOURNAMENT ARRANGEMENT in the proof of Theorem 6.3 yields a positive instance (G, k) which achieves its required value in a linear ordering ϕ. The drawing of H in Figure 1 has no edge pairs that cross more than once, so given that linear ordering ϕ of G of value at most k, we obtain a drawing of H with the given rotation system that satisfies $\text{ocr} = \text{pcr} = \text{cr} \leq k'$.

We claim that: If $\text{ocr}_{\mathit{flip}}(H) \leq k'$, then G has a tournament arrangement of value at most k. Since $\text{ocr}_{\mathit{flip}} \leq \text{pcr}_{\mathit{flip}} \leq \text{cr}_{\mathit{flip}}$, it immediately follows that if $\text{pcr}_{\mathit{flip}}(H) \leq k'$ or $\text{cr}_{\mathit{flip}}(H) \leq k'$ (or, similarly, if $\text{ocr}_{\mathit{rot}}(H) \leq k'$, $\text{pcr}_{\mathit{rot}}(H) \leq k'$, or $\text{cr}_{\mathit{rot}}(H) \leq k'$) then G has a tournament arrangement of value at most k. Thus, proving the claim establishes Lemma 6.7.

Let us assume then that $\text{ocr}_{\mathit{flip}}(H) \leq k'$. The edge weights along the cycle $(u_1, \ldots, u_{4n})$ were chosen large enough so that none of these edges can cross any other edge oddly. Hence, all of the edges on this cycle are even, and we can apply Lemma 6.6 to redraw G so that the cycle is embedded without changing the rotation system or increasing ocr. Since $G - \{u_1, \ldots, u_{4n}\}$ is connected, the whole graph must lie within the same face of the cycle, which we may assume to be the inner face. In particular, we can assume that none of the rotations at vertices $u_1, \ldots, u_{4n}$ are flipped.

As in the proof of Theorem 2.1, we know that $k' < w^2$, so no two edges of weight w can cross oddly, and, in particular, no two of the paths P_i cross each other oddly. Consider paths $S_i := u_i a_i b_i c_i u_{2n+1-i}$, for $1 \leq i \leq n$. Since the endpoints of S_i and P_j (and S_i and P_{2n+1-j}) alternate along

[4] Two curves may not touch each other and they may not cross in a self-intersection point.

the outer cycle for $1 \leq i < j \leq n$, such S_i and P_j (and S_i and P_{2n+1-j}) have to cross oddly. Since we also know that no two edges of P_i and P_j can cross oddly, the odd pair must have one edge each from Q_i and P_j (Q_i and P_{2n+1-j}). Each such pair contributes ww' to ocr. Since there are $n(n-1)$ such pairs, the overall contribution to ocr is $n(n-1)ww'$. Now $k' - n(n-1)ww' = kw' + m^2 < (w'/2)^2$, so there cannot be any further odd pairs in $Q_i \cup P_j$ since its edges have weight at least $w'/2$. Moreover, $k' - n(n-1)ww' = kw' + m^2 < m^2w' + m^2 = 5m^4 + m^2 < 7m^4 = w$, so none of the edges of weight w cross any other edge oddly (such as the b_ib_j edges) apart from the crossings between P_j and Q_i we already mentioned. Hence, the only remaining odd pairs contain an edge of type b_ib_j, and another such edge or an edge of weight w' or $w'/2$.

Claim 6.8. For $1 \leq i < j \leq n$, a_ib_i crosses P_j oddly and b_ic_i crosses P_{2n+1-j} oddly. Other than this, edges of Q_i cross edges of $P_j \cup P_{2n+1-j}$ evenly.

Proof. The second part of the claim follows from the first part since, as we argued above, there are only two odd pairs with one edge in Q_i and one in $P_j \cup P_{2n+1-j}$. To confirm the first part of the claim, by symmetry we only need to show that a_ib_i crosses P_j oddly. Since the outer cycle is crossing-free, we can instead show that a_i and b_i are on opposite sides of the closed curve $D_j := u_j \dots u_{4n+1-j}a_ju_j$.

Note that $u_{2n+1-j}c_j$ and c_jb_j cross P_j evenly. Let O be a crossing-free curve from an outer reference point to u_{2n+1-j}; then $Ou_{2n+1-j}c_jb_j$ crosses D_j oddly. Therefore b_j is inside D_j.

Since b_j is connected to b_i via edges on the "b-vertices" that each cross P_j evenly, b_i is also inside D_j. On the other hand u_ia_i is in P_i, so it crosses P_j evenly. Then, as u_i is outside of D_j, a_i is also outside of D_j. Hence a_ib_i must cross P_j oddly. ■

We define a binary relation $\lhd$ on the set $\{1, \dots, n\}$. For $1 \leq i < j \leq n$, let $i \lhd j$ if P_i crosses u_ja_j oddly, and $j \lhd i$ if P_i crosses a_ju_{4n+1-j} oddly. By Claim 6.8, exactly one of these holds true, so $\lhd$ is a tournament arrangement of G. We will see presently that the value of this arrangement is at most k.

Note that $\lhd$ is not necessarily a linear ordering. Figure 6.2 shows that it is quite possible to have $a \lhd b \lhd c \lhd a$.

Let C_i be the closed curve $u_i \dots u_{2n+1-i}c_ib_ia_iu_i$, for $1 \leq i \leq n$.

Claim 6.9. The vertex b_i is contained inside C_j if and only if $i \lhd j$ (for $i \neq j$).

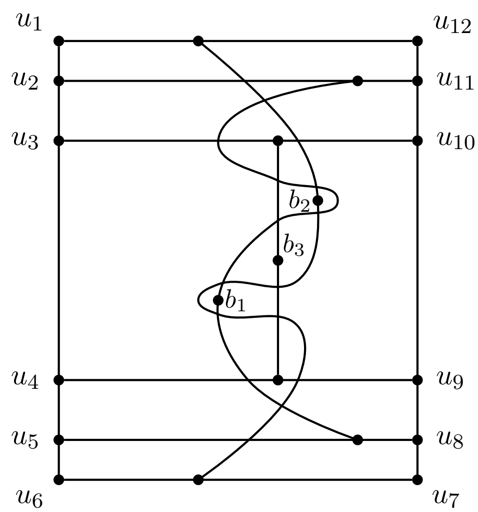

Figure 7: An example with $1 \lhd 2 \lhd 3 \lhd 1$.

Proof. Assume that $i < j$. Then in addition to the claim as stated we also need to show that b_j is inside C_i if and only if $j \lhd i$.

The edges $u_i a_i$ and $a_i b_i$ each cross all edges of C_j evenly except that $a_i b_i$ may cross $u_j a_j$ oddly. Therefore u_i and b_i are on opposite sides of C_j if and only if $a_i b_i$ crosses $u_j a_j$ oddly. Since u_i is outside C_j, it follows that b_i is inside C_j if and only if $i \lhd j$.

Let O_j be a crossing-free curve from u_j to an outer reference point. Since $O_j u_j a_j$ crosses $u_i \ldots u_{2n+1-i}$ oddly, and no edge of $u_j a_j b_j$ may cross C_i oddly except possibly $a_i b_i$ with $u_j a_j$, we conclude that $O_j u_j a_j b_j$ crosses C_i evenly if and only if $a_i b_i$ crosses $u_j a_j$ oddly. Therefore b_j is outside C_i if and only if $i \lhd j$. Thus, b_j is inside C_i if and only if $i \not\lhd j$, or $j \lhd i$. ■

Consider an edge $b_i b_j$ of G and ℓ so that $i \lhd \ell \lhd j$. By Claim 6.9, the edge $b_i b_j$ has to cross the boundary of C_ℓ oddly, contributing at least w' odd crossings to ocr. Moreover, the edge $b_i b_j$ must cross the boundary of the 2-cycle $b_i c_i$ oddly: if not, then b_j lies within that 2-cycle, but this is impossible, since none of the edges of $b_j c_j u_{2n+1-j}$ can cross that 2-cycle oddly, and u_{2n+1-j} is outside it. Hence every edge $b_i b_j$ contributes $w'/2$ odd crossings at each of its endpoints. In summary, the edges of G contribute at least

$$\sum_{b_i b_j \in E(G)} \sum_{i \lhd \ell \lhd j} w'$$

odd crossings which must be at most $k' - n(n-1)ww' = kw' + m^2$. Since $m^2 < w'$ we can conclude that

$$\sum_{b_i b_j \in E(G)} \sum_{i \lhd \ell \lhd j} 1 < k + 1$$

so the value is at most k. (Transferring the result to unweighted graphs proceeds as in the proof of Theorem 2.1.) ■

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