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Crossing Probabilities for Diffusion Processes with Piecewise Continuous Boundaries

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Abstract We propose an approach to compute the boundary crossing probabilities for a class of diffusion processes which can be expressed as piecewise monotone (not necessarily one-to-one) functionals of a standard Brownian motion. This class includes many interesting processes in real applications, e.g., Ornstein–Uhlenbeck, growth processes and geometric Brownian motion with time dependent drift. This method applies to both one-sided and two-sided general nonlinear boundaries, which may be discontinuous. Using this approach explicit formulas for boundary crossing probabilities for certain nonlinear boundaries are obtained, which are useful in evaluation and comparison of various computational algorithms. Moreover, numerical computation can be easily done by Monte Carlo integration and the approximation errors for general boundaries are automatically calculated. Some numerical examples are presented.

Keywords Boundary crossing probabilities · Brownian motion · Diffusion process · First hitting time · First passage time · Wiener process

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1 Introduction

Let $X = \{X_t, t \geq 0\}$ be a diffusion process defined on a probability space $(\Omega, \mathcal{A}, P)$ and with state space either the real space $\mathbb{R}$ or a subinterval of it. In this paper, we are concerned with the following boundary crossing probability (BCP)

$$P_X(a, b, T) = P(a(t) < X_t < b(t), \forall t \in [0, T]), \tag{1}$$

where $T > 0$ is fixed, boundaries $a(t)$ and $b(t)$ are real functions satisfying $a(t) < b(t)$ for all $0 < t \leq T$ and $a(0) < x_0 < b(0)$.

The problem of boundary crossing probabilities, or the first passage time (FPT) distributions, has drawn tremendous amount of attention in many scientific disciplines. To mention a few, for example, it arises in biology (Ricciardi et al., 1999), economics (Krämer et al., 1988), engineering reliability (Ebrahimi, 2005), epidemiology (Martin-Löf, 1998; Tuckwell and Wan, 2000; Startsev, 2001), quantitative finance (Garrido, 1989; Roberts and Shortland, 1997; Lin, 1998; Novikov et al., 2003; Borovkov and Novikov, 2005), computational genetics (Dupuis and Siegmund, 2000), seismology (Michael, 2005), and statistics (Doob, 1949; Anderson, 1960; Durbin, 1971; Sen, 1981; Siegmund, 1986; Bischoff et al., 2003; Zeileis, 2004).

Despite its importance and wide applications, explicit analytic solutions to boundary crossing problems do not exist, except for very few instances. Traditionally, the mainstream of the research of nonlinear boundary problems is based on Kolmogorov partial differential equations for the transition density function, and focuses on approximate solutions of certain integral or differential equations for the first-passage time densities. For example, to obtain the approximate solutions for Brownian motion crossing continuously differentiable boundaries, the tangent approximation and other image methods have been used by Strassen (1967); Daniels (1969, 1996); Ferebee (1982); Lerche (1986), whereas a series expansion method has been used by Durbin (1971, 1992), Ferebee (1983), Ricciardi et al. (1984); Giorno et al. (1989), Sacerdote and Tomassetti (1996) to deal with more general diffusion processes. Recently, Monte Carlo path-simulation methods have been proposed to numerically compute the FPT density for a general diffusion process crossing a one-sided constant boundary (Kloeden and Platen, 1992; Giraudo and Sacerdote, 1999; Giraudo et al., 2001).

In principle, all these FPT methods are designed for the continuously differentiable boundaries only, and they mostly deal with one boundary problems. However, two-sided and discontinuous boundary problems arise in many real applications, e.g., in problems of pricing barrier options in quantitative finance. Furthermore, the actual numerical computations in FPT methods are either intractable or give approximate solutions for which the approximation accuracies are difficult to assess (Daniels, 1996; Sacerdote and Tomassetti, 1996). The more recent path-simulation methods require heavy computation and the algorithms are typically very complicated (Giraudo et al., 2001).

In contrast to the traditional FPT methods, Wang and Pötzelberger (1997) proposed an alternative approach, which focuses on the boundary crossing probabilities (BCP) directly. Using this approach, Wang and Pötzelberger (1997) derived an explicit formula for the BCP for Brownian motion crossing a piecewise linear boundary, and then used this formula to obtain approximations of the BCP for more general nonlinear boundaries. The numerical computation can be easily done by Monte Carlo integration, and the approximation errors are automatically computed. This approach also allows one to control the approximation error before the actual computation begins. This method has been extended to two-boundary problems by Novikov et al. (1999), who use recurrent numerical integration with Gaussian quadrature, and by Pötzelberger and Wang (2001), who use Monte Carlo integration. Moreover, it has been applied in finance (Novikov et al., 2003; Borovkov and Novikov, 2005), change-point problems (Zeileis, 2004), and engineering reliability (Ebrahimi, 2005).

In this paper, we extend the method of Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001) further to a class of diffusion processes, which can be expressed as piecewise monotone (not necessarily one-to-one) functionals of a Brownian motion. This class contains many interesting processes arising in real applications, e.g., Ornstein–Uhlenbeck processes, growth processes and the geometric Brownian motion with time-dependent drift. Moreover, this approach allows us to derive explicit formulas for BCP for certain nonlinear boundaries, which are useful in evaluation and comparison of various computational methods and algorithms for boundary crossing problems.

From methodology point of view, the framework used in this paper is different from that in the previous works in the literature. Whereas the previous works are based on the Kolmogorov partial differential equations for the transition density functions, our approach is based on the stochastic differential equation (SDE) for the processes. The SDE approach becomes more and more popular in modern times, especially in mathematical finance since the publication of the stimulating works of Black and Scholes (1973) and Merton (1973). See, e.g., Garrido (1989), Roberts and Shortland (1997); Tuckwell and Wan (2000); Kou and Wang (2003); Novikov et al. (2003); Ebrahimi (2005).

The paper is organized as follows. Section 2 extends the BCP formulas of Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001) for Brownian motion to piecewise continuous boundaries. Section 3 derives the BCP formulas for more general diffusion processes. Sections 4, 5 and 6 treat the Ornstein–Uhlenbeck processes, growth processes and geometric Brownian motion respectively. Finally, Section 7 gives numerical examples and Section 8 contains conclusions and discussion.

2 Brownian Motion and Piecewise Continuous Boundaries

Let $W = \{W_t, t \geq 0\}$ be a standard Brownian motion (Wiener process) with $EW_t = 0$ and $EW_tW_s = \min(t, s)$. We consider the BCP $P_W(a, b, T) = P(a(t) < W_t < b(t), \forall t \in [0, T])$. Wang and Pötzelberger (1997); Novikov et al. (1999); Pötzelberger and Wang (2001) have derived explicit formulas for $P_W(a, b, T)$ for continuous boundaries a and b. It is straightforward to generalize their results to piecewise continuous boundaries. Throughout the paper we denote by $(t_i)_{i=1}^n$, $0 < t_1 < \cdots < t_{n-1} < t_n = T$, a partition of the interval $[0, T]$ of size $n \geq 1$. Further, let $t_0 = 0, \Delta t_i = t_i - t_{i-1}, \alpha_i = a(t_i), \beta_i = b(t_i)$ and $\delta_i = \beta_i - \alpha_i$. Then completely analog to Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001), we can establish the following results.

Theorem 1 *Given any partition $(t_i)_{i=1}^n$ of the interval $[0, T]$, suppose the boundaries a and b are linear functions in each subinterval (t_{i-1}, t_i) and are such that, at any t_i, $\lim_{t\to t_i-} a(t) \geq \lim_{t\to t_i+} a(t)$ and $\lim_{t\to t_i-} b(t) \leq \lim_{t\to t_i+} b(t)$ hold. Then*

$$P_W(a, b, T) = Eg(W_{t_1}, W_{t_2}, ..., W_{t_n}), \tag{2}$$

where the function $g(x)$, $x = (x_1, x_2, ..., x_n)'$, is defined as follows.

(1) *For one-sided BCP $P_W(-\infty, b, T)$,*

$$g(x) = \prod_{i=1}^n \mathbf{1}_{(-\infty,\beta_i)}(x_i) \left\{1 - \exp\left[-\frac{2}{\Delta t_i}(\beta_{i-1} - x_{i-1})(\beta_i - x_i)\right]\right\}, \tag{3}$$

where $\mathbf{1}(\cdot)$ is the indicator function.

(2) *For two-sided BCP $P_W(a, b, T)$,*

$$g(x) = \prod_{i=1}^n \mathbf{1}_{(\alpha_i,\beta_i)}(x_i) \left[1 - \sum_{j=1}^\infty h_{ij}(x_{i-1}, x_i)\right], \tag{4}$$

where

$$\begin{aligned} h_{ij}(x_{i-1}, x_i) = & \exp\left\{-\frac{2}{\Delta t_i}\left[j\delta_{i-1} + (\alpha_{i-1} - x_{i-1})\right]\left[j\delta_i + (\alpha_i - x_i)\right]\right\} \\ & - \exp\left\{-\frac{2j}{\Delta t_i}\left[j\delta_{i-1}\delta_i + \delta_{i-1}(\alpha_i - x_i) - \delta_i(\alpha_{i-1} - x_{i-1})\right]\right\} \\ & + \exp\left\{-\frac{2}{\Delta t_i}\left[j\delta_{i-1} - (\beta_{i-1} - x_{i-1})\right]\left[j\delta_i - (\beta_i - x_i)\right]\right\} \\ & - \exp\left\{-\frac{2j}{\Delta t_i}\left[j\delta_{i-1}\delta_i - \delta_{i-1}(\beta_i - x_i) + \delta_i(\beta_{i-1} - x_{i-1})\right]\right\}. \end{aligned}$$

Note that the function $h_{ij}(x_{i-1}, x_i)$ in Eq. 4 consists of exponential functions that decrease very rapidly with the increase of j. In all of our numerical examples, fairly accurate approximations are achieved by using six terms only. Moreover, a simplified version of Eq. 4 is given in Pötzelberger and Wang (2001).

The results of Theorem 1 can be used to approximate the BCP $P_W(a, b, T)$ for general nonlinear boundaries a and b, provided they can be sufficiently well approximated by some piecewise linear functions. More precisely, if the sequence of piecewise linear functions $a_n(t) \to a(t)$ and $b_n(t) \to b(t)$ uniformly on $[0, T]$, then it follows from the continuity property of probability measure that

$$\lim_{n\to\infty} P_W(a_n, b_n, T) = P_W(a, b, T). \tag{5}$$

The accuracy of approximation (5) depends on the partition size n. In general, larger n will give more accurate approximation. For twice continuously differentiable boundaries, Novikov et al. (1999) showed that the approximation errors converge to zero at rate $\sqrt{\log n/n^3}$, if equally-spaced piecewise linear approximation is used. Later, Pötzelberger and Wang (2001) obtained an approximation rate of $O(1/n^2)$ by using an "optimal" partition of $[0, T]$. More recently, Borovkov and Novikov (2005) obtained the same rate by using equally-spaced partitions.

In practice, one simple and straightforward way to assess the approximation accuracy is to calculate the lower and upper bounds of the BCP using two piecewise linear functions approaching a and b from each side respectively. This method will be demonstrated through numerical examples in Section 7. See Wang and Pötzelberger (1997) and Pötzelberger and Wang (2001) for more details and examples.

3 General Diffusion Processes

In this and the subsequent sections, we generalize the results of the previous section to more general diffusion processes satisfying the following stochastic differential equation (SDE)

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t, \quad X_0 = x_0, \tag{6}$$

where the drift $\mu(t, x) : [0, \infty) \times I\!R \to I\!R$ and diffusion coefficient $\sigma(t, x) : [0, \infty) \times I\!R \to I\!R_+$ are real, deterministic functions and $\{W_t, t \geq 0\}$ is the standard Brownian motion (BM) process. In order for the SDE (6) to admit unique solution, throughout this paper we assume generally that $\mu(t, x)$ and $\sigma(t, x)$ are measurable functions and satisfy the following Lipschitz and growth conditions: for any $t \geq 0$,

$$|\mu(t, x) - \mu(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq L\,|x - y|\,, \forall x, y \in I\!R, \tag{7}$$

and for any $x \in I\!R$,

$$|\mu(t, x)| + |\sigma(t, x)| \leq K(1 + |x|), \forall t \geq 0, \tag{8}$$

where $L > 0$ and $K > 0$ are constants. However, since the solution problem is not a major concern of this paper, in the subsequent derivations we will not deal with conditions (7) and (8) explicitly. Rather we refer the reader to, e.g., Karatzas and Shreve (1991, p. 289) or Lamberton and Lapeyre (1996, p. 50). See also Yamada and Watanabe (1971). In addition, throughout the paper we make the following assumption, which is necessary for the application of Itô's formula.

Assumption 1 *Let $\{X_t, t \in [0, T]\}$ denote the solution of the* SDE (6) *with state space an open or closed finite or infinite interval $\mathcal{S}$. Assume that $\sigma(t, x) > 0$ for all $t \in [0, T]$ and $x \in \mathcal{S}$. Further, assume that there exists a function $f(t, x) \in C^{1,2}([0, T] \times \mathcal{S})$ (once continuously differentiable with respect to t and twice with respect to x), such that*

(1) *for $x \in \mathcal{S}$*

$$\frac{\partial f(t, x)}{\partial x} = \frac{1}{\sigma(t, x)}$$

and

(2) *Itô's Lemma may be applied to $Y_t = f(t, X_t)$, i.e.*

$$dY_t = \frac{\partial f(t, X_t)}{\partial t}dt + \frac{\partial f(t, X_t)}{\partial x}dX_t + \frac{1}{2}\frac{\partial^2 f(t, X_t)}{\partial x^2}\sigma(t, X_t)^2 dt. \tag{9}$$

Now we consider the BCP (1) for any diffusion process $X = \{X_t, t \geq 0\}$ satisfying Eq. 6 and Assumption 1. The basic idea is that $P_X(a, b, T)$ can be computed through the BCP for the standard BM, as long as X can be expressed as a "nice" transformation of it. More precisely, given any boundaries $a(t)$, $b(t)$ and $T > 0$, let us denote

$$A(a, b, T) = \{x(t) \in C([0, \infty)) : a(t) < x(t) < b(t), \forall t \in [0, T]\}.$$

Then it is easy to see that, if there exists a standard BM $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ and a measurable functional f, such that $X = f(\tilde{W})$, then

$$P(X \in A(a, b, T)) = P(\tilde{W} \in f^{-1}A(a, b, T)). \tag{10}$$

Consequently, the BCP for X can be calculated through the BCP for $\tilde{W}$, as long as there exist boundaries $c(t)$, $d(t)$ and $S > 0$, such that $f^{-1}A(a, b, T) = A(c, d, S)$. One of such instances is described in the following theorem.

Theorem 2 *If there exists a function $f(t, x) \in C^{1,2}([0, \infty) \times I\!R)$, such that $Y_t := f(t, X_t)$ satisfies $dY_t = \tilde{\sigma}_t dW_t$, where $\tilde{\sigma}_t \in C([0, \infty))$ is a real, deterministic function satisfying $\tilde{\sigma}_t \neq 0$, $\forall t \in [0, \infty)$, then there exists a standard BM $\{\tilde{W}_s, s \geq 0\}$, such that for any boundaries $a(t), b(t)$,*

(1) $P_X(a, b, T) = P_{\tilde{W}}(c, d, S)$, *if* $\tilde{\sigma}_t > 0$, $\forall t \in [0, \infty)$;
(2) $P_X(a, b, T) = P_{\tilde{W}}(d, c, S)$, *if* $\tilde{\sigma}_t < 0$, $\forall t \in [0, \infty)$;

where

$$c(s) = f(t(s), a(t(s))) - f(0, x_0), 0 \leq s \leq S, \tag{11}$$

$$d(s) = f(t(s), b(t(s))) - f(0, x_0), 0 \leq s \leq S, \tag{12}$$

$t(s)$ is the inverse function of $s(t) = \int_0^t \tilde{\sigma}_u^2 du$ and $S = s(T)$.

Proof First, from $dY_t = \tilde{\sigma}_t dW_t$ it follows that $\{Y_t, t \geq 0\}$ is a continuous Gaussian process with variance function

$$V(Y_t) = \int_0^t \tilde{\sigma}_u^2 du = s(t), t \geq 0.$$

Because $\tilde{\sigma}_t^2 > 0$, $s(t)$ is strictly increasing for $t \geq 0$ and $s(0) = 0$. For every $s \geq 0$, define $\tilde{W}_s := Y_{t(s)} - Y_0$, where $t(s)$ is the inverse function of $s(t)$. Then since $\tilde{W}_0 = 0$ and

$$V(\tilde{W}_s) = V(Y_{t(s)}) = s(t(s)) = s,$$

$\{\tilde{W}_s, s \geq 0\}$ is a standard BM. Furthermore, for every $t \geq 0$, $Y_t = Y_0 + \tilde{W}_{s(t)}$.

On the other hand, Assumption 1 and Itô's formula (9) imply that

$$\sigma(t, x)\frac{\partial f}{\partial x} = \tilde{\sigma}_t.$$

Note that for every $t > 0$,

$$\frac{\partial f}{\partial x} = \frac{\tilde{\sigma}_t}{\sigma(t, x)}$$

is continuous in x. It follows that, for any x_0 and x in the state space of $\{X_t\}$,

$$f(t,x) = \tilde{\sigma}_t \int_{x_0}^{x} \frac{1}{\sigma(t,u)} du + h(t), \tag{13}$$

where $h(t)$ is an arbitrary function which does not depend on x. Further, since $\tilde{\sigma}_t \neq 0$ and $\sigma(t,x) > 0$, for every $t > 0$, $f(t,x)$ is a monotone function of x. For every $t > 0$, let $g(t,x)$ be the inverse function of $f(t,x)$. Then $X_t = g(t, Y_t)$.

Now we consider case (1), where $\tilde{\sigma}_t > 0, \forall t \in [0,\infty)$. In this case function $f(t,x)$ is strictly increasing in x. It follows that

$$\begin{aligned} P(a(t) < X_t < b(t), \forall t \in [0,T]) &= P\left(a(t) < g\left(t, Y_0 + \tilde{W}_{s(t)}\right) < b(t), \forall t \in [0,T]\right) \\ &= P\left(f(t,a(t)) < Y_0 + \tilde{W}_{s(t)} < f(t,b(t)), \forall t \in [0,T]\right) \\ &= P\left(c(s) < \tilde{W}_s < d(s), \forall s \in [0,S]\right), \end{aligned}$$

where $c(s)$ and $d(s)$ are given by Eqs. 11 and 12 respectively, and $S = s(T)$.

Similarly, case (2) follows from the fact that function $f(t,x)$ is strictly decreasing in x. □

Remark 3 The above derivation is readily generalized to more complicated case where $\tilde{\sigma}_t$ has finite number of zeros. Without loss of generality, suppose $\tilde{\sigma}_t$ has k zeros $0 < t_1 < t_2 < \cdots < t_k < T$. Then $\tilde{\sigma}_t$ changes signs over intervals $(t_{i-1}, t_i]$, $i = 1, 2, ..., k+1$, where $t_0 = 0$ and $t_{k+1} = T$. Denote by T_+ the union of all intervals in which $\tilde{\sigma}_t > 0$, and by T_- the union of all intervals in which $\tilde{\sigma}_t < 0$. Then by Eq. 13 function $f(t,x)$ is strictly increasing for $t \in T_+$ and decreasing for $t \in T_-$. Furthermore, let $S_+ = s(T_+)$, $S_- = s(T_-)$, where $s(t) = \int_0^t \tilde{\sigma}_u^2 du$. Then for any boundaries $a(t), b(t)$, we have $P_X(a,b,T) = P_{\tilde{W}}(c,d,S)$, where

$$c(s) = \begin{cases} f(t(s), a(t(s))) - f(0,x_0), & \text{when } s \in S_+ \\ f(t(s), b(t(s))) - f(0,x_0), & \text{when } s \in S_- \end{cases}$$

$$d(s) = \begin{cases} f(t(s), b(t(s))) - f(0,x_0), & \text{when } s \in S_+ \\ f(t(s), a(t(s))) - f(0,x_0), & \text{when } s \in S_- \end{cases}$$

and $S = s(T)$.

There remains the question that for which processes $\{X_t\}$ do functions $f(t,x)$ and $\tilde{\sigma}_t$ in Theorem 2 exist. A class of such diffusion processes is characterized by the next Theorem.

Theorem 4 *Let $\{X_t, t \geq 0\}$ satisfy Eq. 6 with drift $\mu(t,x)$ and diffusion coefficient $\sigma(t,x) \in C^{1,2}([0,\infty) \times I\!R)$. Then there exist functions $f(t,x)$ and $\tilde{\sigma}_t$ satisfying conditions of Theorem 2, if and only if $\mu_t = \mu(t,x)$ and $\sigma_t = \sigma(t,x)$ satisfy the following partial differential equation*

$$\frac{\partial}{\partial x}\left[\frac{1}{\sigma_t}\frac{\partial \sigma_t}{\partial t} + \sigma_t \frac{\partial}{\partial x}\left(\frac{1}{2}\frac{\partial \sigma_t}{\partial x} - \frac{\mu_t}{\sigma_t}\right)\right] = 0. \tag{14}$$

Proof Suppose $Y_t = f(t, X_t)$ and $dY_t = \tilde{\sigma}_t dW_t$, where $f(t, x)$ and $\tilde{\sigma}_t$ satisfy conditions in Theorem 2. Then Itô's formula (9) implies

$$\frac{\partial f}{\partial t} + \mu_t \frac{\partial f}{\partial x} + \frac{\sigma_t^2}{2} \frac{\partial^2 f}{\partial x^2} = 0 \tag{15}$$

and

$$\sigma_t \frac{\partial f}{\partial x} = \tilde{\sigma}_t. \tag{16}$$

From Eq. 16, we have

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\tilde{\sigma}_t}{\sigma_t^2} \frac{\partial \sigma_t}{\partial x}.$$

Substituting the above equation into Eq. 15, we obtain

$$\frac{\partial f}{\partial t} = \tilde{\sigma}_t \left(\frac{1}{2} \frac{\partial \sigma_t}{\partial x} - \frac{\mu_t}{\sigma_t} \right).$$

It follows that

$$f(t, x) = \int_0^t \tilde{\sigma}_u \left(\frac{1}{2} \frac{\partial \sigma_u}{\partial x} - \frac{\mu_u}{\sigma_u} \right) du + \phi(x), \tag{17}$$

where $\phi(x) \in C^2(I\!R)$. Differentiating Eq. 17 with respect to x and using Eq. 16, we have

$$\frac{\partial}{\partial x} \int_0^t \tilde{\sigma}_u \left(\frac{1}{2} \frac{\partial \sigma_u}{\partial x} - \frac{\mu_u}{\sigma_u} \right) du + \frac{d\phi(x)}{dx} = \frac{\tilde{\sigma}_t}{\sigma_t}. \tag{18}$$

Further, differentiating Eq. 18 with respect to t yields

$$\tilde{\sigma}_t \frac{\partial}{\partial x} \left(\frac{1}{2} \frac{\partial \sigma_t}{\partial x} - \frac{\mu_t}{\sigma_t} \right) = \frac{\partial}{\partial t} \frac{\tilde{\sigma}_t}{\sigma_t},$$

which is equivalent to

$$\frac{1}{\sigma_t} \frac{\partial \sigma_t}{\partial t} + \sigma_t \frac{\partial}{\partial x} \left(\frac{1}{2} \frac{\partial \sigma_t}{\partial x} - \frac{\mu_t}{\sigma_t} \right) = \frac{1}{\tilde{\sigma}_t} \frac{\partial \tilde{\sigma}_t}{\partial t}. \tag{19}$$

Since the right-hand side of Eq. 19 does not depend on x, differentiating both sides of this equation with respect to x yields Eq. 14.

Conversely, if Eq. 14 holds, then Eq. 19 defines a function $\tilde{\sigma}_t \in C^1([0, \infty))$. Further, Eq. 18 defines a function $\phi(x) \in C^2(I\!R)$, whereas Eq. 17 defines function $f(t, x)$. From Eqs. 17 and 14, it is easy to see that $f(t, x)$ satisfies condition in Theorem 2. □

Condition (14) is practical in real applications. Once a process is given, one can easily check whether condition (14) holds or not. On the other hand, Eq. 14 can also be used to characterize certain subclasses of diffusion processes, which may be of interest of researchers working on particular problems. For example, for any given type of diffusion coefficient σ_t, one can find all possible functions μ_t, such that Eq. 14 is satisfied. Some examples are given in the following Corollary.

Corollary 5 *For each of the following types of diffusion coefficient σ_t, the corresponding general solutions of Eq. 14 for the drift μ_t are given as follows, where $\alpha(t)$, $\beta(t)$ are arbitrary real functions and $\tilde{\sigma}(t) : [0, \infty) \to \mathbb{R}_+$.*

(1) (*L-class*). *For $\sigma(t, x) = \tilde{\sigma}(t)$, $\mu(t, x) = \alpha(t)x + \beta(t)$, $x \in \mathbb{R}$.*
(2) (*G-class*). *For $\sigma(t, x) = \tilde{\sigma}(t)x$, $\mu(t, x) = \alpha(t)x + \beta(t)x \log x$, $x \in \mathbb{R}_+$.*

The proof of these results is straightforward and is therefore omitted. It is easy to see that L-class corresponds to general Ornstein–Uhlenbeck processes with time-dependent coefficients, whereas G-class corresponds to growth processes which are widely used in population genetics. It is easy to see that the geometric BM process also belongs to G-class.

There remains a practical question as how to find transformations $f(t, x)$ and $\tilde{\sigma}_t$ for a given process. Obviously this depends on the functional forms of μ_t and σ_t and can only be worked out explicitly on case by case bases. Generally speaking, Eq. 19 defines $\tilde{\sigma}_t$, up to a multiplicative constant, and Eq. 13 defines $f(t, x)$ up a term $h(t)$ not depending on x, which is in turn determined by Eq. 15. In the subsequent sections, we demonstrate this for some widely used special processes and use the theoretical results of this section to derive more detailed results for these processes.

Remark 6 The problem of obtaining the transition probability of a diffusion process through one-to-one (strictly monotone) transformation of Brownian motion was first posed by Kolmogorov (1931). A class of such diffusion processes was characterized by Cherkasov (1957) and Ricciardi (1976), whereas the most general class was derived by Bluman (1980). Using this transformation method, Ricciardi et al. (1984) obtained the FPT densities for a class of diffusion processes crossing one-sided, continuously differentiable boundaries. Since the main concern of this paper is the boundary crossing probabilities, not the transformation itself, we do not intend to be most general. The class of processes characterized by Theorem 4 is large enough to contain many interesting processes arising in practice. Furthermore, the condition (14) is practical and easy to work with in real applications.

4 Ornstein–Uhlenbeck Processes

Ornstein–Uhlenbeck (O–U) processes are a class of important diffusion processes with wide applications. We start with the O–U process which is defined in state space $\mathbb{R}$ and satisfies

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t, \quad X_0 = x_0, \tag{20}$$

where $\kappa, \sigma \in \mathbb{R}_+$ and $\alpha \in \mathbb{R}$ are constants. In mathematical finance, Eq. 20 is known as Vasicek model for the short-term interest rate process (Vasicek, 1977).

In this case, special forms of transformation $f(t, x)$ and boundaries (11) and (12) can be derived. For every $t > 0$, if we define

$$Y_t = e^{\kappa t}(X_t - \alpha), \quad Y_0 = x_0 - \alpha, \tag{21}$$

then Itô's formula and Eq. 20 imply

$$\begin{aligned} dY_t &= \kappa e^{\kappa t}(X_t - \alpha)dt + e^{\kappa t}dX_t \\ &= \sigma e^{\kappa t}dW_t. \end{aligned}$$

It follows that $\{Y_t, t \geq 0\}$ is a continuous Gaussian process with variance function

$$\begin{aligned} V(Y_t) &= \sigma^2 \int_0^t e^{2\kappa u}du \\ &= \frac{\sigma^2}{2\kappa}\left(e^{2\kappa t} - 1\right) \\ &:= s(t). \end{aligned}$$

Note that the function $s(t), t \geq 0$ is strictly increasing and its inverse is given by

$$t(s) = \frac{1}{2\kappa}\log\left(1 + \frac{2\kappa s}{\sigma^2}\right), s \geq 0.$$

For every $s \geq 0$, define $\tilde{W}_s := Y_{t(s)} - Y_0$. Then, it follows from $\tilde{W}_0 = 0$ and

$$V(\tilde{W}_s) = V(Y_{t(s)}) = s(t(s)) = s,$$

that $\{\tilde{W}_s, s \geq 0\}$ is a standard BM. Furthermore, for every $t \geq 0$, $Y_t = Y_0 + \tilde{W}_{s(t)}$ and by Eq. 21 we have

$$\begin{aligned} X_t &= e^{-\kappa t}Y_t + \alpha \\ &= e^{-\kappa t}\left(x_0 - \alpha + \tilde{W}_{s(t)}\right) + \alpha. \end{aligned}$$

Therefore, we can write the one-sided BCP for $\{X_t\}$ as

$$\begin{aligned} P\left(X_t < b(t), \forall t \in [0, T]\right) &= P\left(e^{-\kappa t}\left(x_0 - \alpha + \tilde{W}_{s(t)}\right) + \alpha < b(t), \forall t \in [0, T]\right) \\ &= P\left(\tilde{W}_{s(t)} < \alpha - x_0 + e^{\kappa t}(b(t) - \alpha), \forall t \in [0, T]\right) \\ &= P\left(\tilde{W}_s < \alpha - x_0 + e^{\kappa t(s)}(b(t(s)) - \alpha), \forall s \in [0, S]\right), \end{aligned}$$

where $S = s(T) = \sigma^2\left(e^{2\kappa T} - 1\right)/2\kappa$. Obviously, similar relation for the two-sided BCP can be established in the same way. Thus, we have the following result.

Corollary 7 *Let $\{X_t, t \geq 0\}$ be an O–U process satisfying Eq. 20. Then there exists a standard BM $\{\tilde{W}_s, s \geq 0\}$, such that for any boundaries $a(t), b(t)$,*

$$P_X(a, b, T) = P_{\tilde{W}}(c, d, S), \tag{22}$$

where

$$c(s) = \alpha - x_0 + [a(t(s)) - \alpha]\left(1 + \frac{2\kappa s}{\sigma^2}\right)^{1/2}, \tag{23}$$

$$d(s) = \alpha - x_0 + [b(t(s)) - \alpha]\left(1 + \frac{2\kappa s}{\sigma^2}\right)^{1/2}, \tag{24}$$

$$t(s) = \frac{1}{2\kappa}\log\left(1 + \frac{2\kappa s}{\sigma^2}\right), s \geq 0 \tag{25}$$

and $S = \sigma^2\left(e^{2\kappa T} - 1\right)/2\kappa$.

The above derivation can be extended to more general O–U processes with time-dependent coefficients, i.e., processes which satisfy

$$dX_t = \kappa(t)(\alpha(t) - X_t)dt + \sigma(t)dW_t, \quad X_0 = x_0, \tag{26}$$

where $\kappa(t), \sigma(t) : [0, \infty) \mapsto I\!R_+$ and $\alpha(t) : [0, \infty) \mapsto I\!R$ are real, deterministic functions. Indeed, if we define, for every $t > 0$,

$$Y_t = \exp\left(\int_0^t \kappa(u)du\right)(X_t - \gamma(t)), \quad Y_0 = x_0 - \gamma(0),$$

where

$$\gamma(t) = e^{-t}\alpha(0) + e^{-t}\int_0^t e^u \alpha(u)du,$$

then by Itô's lemma, we have

$$dY_t = \exp\left(\int_0^t \kappa(v)dv\right)\sigma(t)dW_t.$$

It follows that

$$s(t) = V(Y_t) = \int_0^t \exp\left(2\int_0^u \kappa(v)dv\right)\sigma^2(u)du, \tag{27}$$

which is strictly increasing and hence admits the inverse function $t(s), s \geq 0$. Therefore,

$$\begin{aligned} X_t &= \gamma(t) + \exp\left(-\int_0^t \kappa(u)du\right)Y_t \\ &= \gamma(t) + \exp\left(-\int_0^t \kappa(u)d\right)\left(x_0 - \gamma(0) + \tilde{W}_{s(t)}\right), \end{aligned}$$

where $\tilde{W}_s = Y_{t(s)} - Y_0$ is a standard BM. Thus, we have the following result.

Corollary 8 *Let $\{X_t, t \geq 0\}$ be an O–U process satisfying Eq. 26. Then there exists a standard BM $\{\tilde{W}_s, s \geq 0\}$, such that for any boundaries $a(t), b(t)$, $P_X(a, b, T) = P_{\tilde{W}}(c, d, S)$, where*

$$c(s) = \gamma(0) - x_0 + [a(t(s)) - \gamma(t(s))] \exp\left(\int_0^{t(s)} \kappa(u) du\right),$$

$$d(s) = \gamma(0) - x_0 + [b(t(s)) - \gamma(t(s))] \exp\left(\int_0^{t(s)} \kappa(u) du\right),$$

$t(s)$ is the inverse function of $s(t)$ defined in Eq. 27 and $S = s(T)$.

Relation (22) allows us to derive exact formulas for the BCP of an O–U process $\{X_t\}$ crossing ceratin boundaries, using the existing results for Brownian motion crossing, e.g., linear or square-root boundaries. For the simplicity of notation, in the following we present three examples for one-sided BCP.

Example 9 First, consider boundaries $a(t) = -\infty$ and $b(t) = \alpha + he^{\kappa t}, 0 \leq t \leq T$, where $h \in I\!R$ is an arbitrary constant. Then by Eqs. 23 and 24, $c(s) = -\infty$ and $d(s) = \alpha - x_0 + h + 2h\kappa s/\sigma^2$ is a linear function in $s \in [0, S]$, where $S = \sigma^2(e^{2\kappa T} - 1)/2\kappa$. Therefore, by Eq. 22 and the well-known formula for the BCP for BM crossing a linear boundary (e.g., equation (3) of Wang and Pötzelberger (1997)), we have

$$P_X(a, b, T) = \Phi\left(\frac{he^{2\kappa T} + \alpha - x_0}{\sigma\sqrt{(e^{2\kappa T} - 1)/2\kappa}}\right) - \exp\left(-\frac{4h\kappa(h + \alpha - x_0)}{\sigma^2}\right) \times \Phi\left(\frac{he^{2\kappa T} - \alpha + x_0 - 2h}{\sigma\sqrt{(e^{2\kappa T} - 1)/2\kappa}}\right),$$

where Φ is the standard normal distribution function.

Example 10 Another set of boundaries are $a(t) = -\infty$ and $b(t) = \alpha + he^{-\kappa t}, 0 \leq t \leq T$. In this case $c(s) = -\infty$ and $d(s) = \alpha - x_0 + h$ is a constant. Therefore, again by Eq. 22 we have

$$P_X(a, b, T) = 2\Phi\left(\frac{\alpha - x_0 + h}{\sigma\sqrt{(e^{2\kappa T} - 1)/2\kappa}}\right) - 1.$$

Example 11 Further, consider constant boundaries $a(t) = -\infty$ and $b(t) = h, 0 \leq t \leq T$. Then $c(s) = -\infty$ and

$$d(s) = \alpha - x_0 + (h - \alpha)\left(1 + \frac{2\kappa s}{\sigma^2}\right)^{1/2}$$

is a square-root boundary, for which the "exact" BCP for BM is known (Daniels, 1996). A numerical example of this case is given in Section 7.

5 Growth Processes

Another important stochastic model in population genetics is the growth processes (Ricciardi et al., 1999), which is defined by the SDE

$$dX_t = (\alpha X_t - \beta X_t \log X_t)dt + \sigma X_t dW_t, \quad X_0 = x_0 \tag{28}$$

and has state space $\mathbb{R}_+$, where α, β and σ are positive constants. It is easy to see that the growth processes belong to the G-class described in Corollary 5.

Similar to the O–U process, again by Eqs. 15 and 16, functions $f(t,x)$ and $\tilde{\sigma}_t$ can be determined as

$$f(t,x) = \frac{e^{\beta t}}{\sigma}\left(\log x + \frac{\sigma^2 - 2\alpha}{2\beta}\right)$$

and $\tilde{\sigma}_t = e^{\beta t}$. Hence the time transformation is

$$s(t) = \int_0^t e^{2\beta u} du = \frac{1}{2\beta}\left(e^{2\beta t} - 1\right), t \geq 0.$$

Therefore, by Eqs. 11 and 12, we have the following result.

Corollary 12 *Let* $\{X_t, t \geq 0\}$ *be a growth process satisfying Eq. 28. Then there exists a standard* BM $\{\tilde{W}_s, s \geq 0\}$, *such that for any boundaries* $a(t), b(t)$, $P_X(a, b, T) = P_{\tilde{W}}(c, d, S)$, *where*

$$c(s) = \frac{\sqrt{1+2\beta s}}{\sigma}\left(\log a(t(s)) + \frac{\sigma^2 - 2\alpha}{2\beta}\right) - \frac{1}{\sigma}\left(\log x_0 + \frac{\sigma^2 - 2\alpha}{2\beta}\right), \tag{29}$$

$$d(s) = \frac{\sqrt{1+2\beta s}}{\sigma}\left(\log b(t(s)) + \frac{\sigma^2 - 2\alpha}{2\beta}\right) - \frac{1}{\sigma}\left(\log x_0 + \frac{\sigma^2 - 2\alpha}{2\beta}\right), \tag{30}$$

$$t(s) = \frac{1}{2\beta}\log(1+2\beta s) \tag{31}$$

and $S = (e^{2\beta T} - 1)/2\beta$.

Now we derive explicit formulas for the BCP for some special boundaries.

Example 13 Let $a(t) = 0$ and

$$b(t) = \exp\left(h e^{\beta t} - \frac{\sigma^2 - 2\alpha}{2\beta}\right), 0 \leq t \leq T,$$

where h is a constant. Then by Eqs. 29 and 30, $c(s) = -\infty$ and

$$d(s) = \frac{2h\beta s}{\sigma} + \frac{1}{\sigma}\left(h - \log x_0 - \frac{\sigma^2 - 2\alpha}{2\beta}\right), 0 \leq s \leq S$$

which is a linear function and $S = (e^{2\beta T} - 1)/2\beta$. Therefore,

$$P_X(a, b, T) = \Phi\left(\frac{2\beta(he^{2\beta T} - \log x_0) - \sigma^2 + 2\alpha}{\sigma\sqrt{2\beta(e^{2\beta T} - 1)}}\right) - \exp\left(\frac{4h\beta(\log x_0 - h) + 2h(\sigma^2 - 2\alpha)}{\sigma^2}\right) \times\Phi\left(\frac{2\beta(he^{2\beta T} - 2h + \log x_0) + \sigma^2 - 2\alpha}{\sigma\sqrt{2\beta(e^{2\beta T} - 1)}}\right),$$

where Φ is the standard normal distribution function.

Example 14 Let $a(t) = 0$ and

$$b(t) = \exp\left(he^{-\beta t} - \frac{\sigma^2 - 2\alpha}{2\beta}\right), 0 \le t \le T,$$

where h is a constant. Then $c(s) = -\infty$ and

$$d(s) = \frac{1}{\sigma}\left(h - \log x_0 - \frac{\sigma^2 - 2\alpha}{2\beta}\right), 0 \le s \le S.$$

Therefore,

$$P_X(a, b, T) = 2\Phi\left(\frac{2\beta(h - \log x_0) - \sigma^2 + 2\alpha}{\sigma\sqrt{2\beta(e^{2\beta T} - 1)}}\right) - 1.$$

Example 15 Again, consider the constant boundary $b(t) = h > 0, 0 \le t \le T$. Then

$$d(s) = \frac{\sqrt{1 + 2\beta s}}{\sigma}\left(\log h + \frac{\sigma^2 - 2\alpha}{2\beta}\right) - \frac{1}{\sigma}\left(\log x_0 + \frac{\sigma^2 - 2\alpha}{2\beta}\right), 0 \le s \le S$$

with $S = (e^{2\beta T} - 1)/2\beta$. A numerical example of this case is given in Section 7.

6 Geometric Brownian Motion

One popular stochastic model in mathematical finance is a generalization of the classical Black–Scholes model (Black and Scholes, 1973) to the time-dependent interest rate process, as defined by

$$dX_t = r(t)X_t dt + \sigma X_t dW_t, \quad X_0 = x_0, \tag{32}$$

where $r(t) : [0, \infty) \mapsto I\!R_+$. It is well-known that in this case there exists a risk-neutral probability measure, under which the underlying process is a geometric BM

$$X_t = X_0 \exp\left(R(t) - \frac{\sigma^2 t}{2} + \sigma\tilde{W}_t\right),$$

where $R(t) = \int_0^t r(u)du$ and $\{\tilde{W}_t\}$ is a standard BM under the same risk-neutral probability measure. In the following we use the approach of this paper to derive the BCP for $\{X_t\}$.

First, since Eq. 16 now becomes

$$\frac{\partial f}{\partial x} = \frac{\tilde{\sigma}_t}{\sigma x},$$

function $f(t,x)$ is given by

$$f(t,x) = \frac{\tilde{\sigma}_t}{\sigma}\left(\log x + h(t)\right),$$

where $h(t)$ is an arbitrary function to be determined later. Further, since

$$\frac{\partial f}{\partial t} = \frac{(\log x + h(t))}{\sigma}\frac{\partial \tilde{\sigma}_t}{\partial t} + \frac{\tilde{\sigma}_t}{\sigma}\frac{\partial h(t)}{\partial t},$$

and

$$\frac{\partial^2 f}{\partial x^2} = -\frac{\tilde{\sigma}_t}{\sigma x^2},$$

Eq. 15 becomes

$$(\log x + h(t))\frac{\partial \tilde{\sigma}_t}{\partial t} + \tilde{\sigma}_t\left(\frac{\sigma^2}{2} - r(t) - \frac{\partial h(t)}{\partial t}\right) = 0,$$

which holds, if $\tilde{\sigma}_t = 1$ and

$$\frac{\partial h(t)}{\partial t} = \frac{\sigma^2}{2} - r(t).$$

The last equation above has a particular solution

$$\begin{aligned} h(t) &= \int_0^t \left(\frac{\sigma^2}{2} - r(u)\right) du \\ &= \frac{\sigma^2 t}{2} - R(t). \end{aligned}$$

Therefore, we obtain $\tilde{\sigma}_t = 1$ and

$$f(t,x) = \frac{1}{\sigma}\left(\log x + \frac{\sigma^2 t}{2} - R(t)\right).$$

By Theorem 2 the time transformation is $s(t) = t, t \geq 0$, and by Eqs. 11 and 12, we have the following well-known result.

Corollary 16 *Let* $\{X_t, t \geq 0\}$ *be a geometric* BM *satisfying Eq. 32. Then there exists a standard BM* $\{\tilde{W}_s, s \geq 0\}$, *such that for any boundaries* $a(t), b(t)$, $P_X(a,b,T) = P_{\tilde{W}}(c,d,T)$, *where*

$$c(t) = \frac{1}{\sigma}\left(\log\left(\frac{a(t)}{x_0}\right) + \frac{\sigma^2 t}{2} - R(t)\right), 0 \leq t \leq T, \tag{33}$$

and

$$d(t) = \frac{1}{\sigma}\left(\log\left(\frac{b(t)}{x_0}\right) + \frac{\sigma^2 t}{2} - R(t)\right), 0 \leq t \leq T. \quad (34)$$

Now we consider some special cases.

Example 17 Let $a(t) = 0$ and $b(t) = \exp(pt + q + R(t))$, where p, q are constants. Then by Eqs. 33 and 34, $c(t) = -\infty$ and

$$d(t) = \frac{1}{\sigma}\left(p + \frac{\sigma^2}{2}\right)t + \frac{q - \log x_0}{\sigma}, 0 \leq t \leq T.$$

Therefore,

$$P_X(a, b, T) = \Phi\left(\frac{\left(p + \sigma^2/2\right)T + q - \log x_0}{\sigma\sqrt{T}}\right) - \exp\left(\frac{\left(2p + \sigma^2\right)(\log x_0 - q)}{\sigma^2}\right)\Phi\left(\frac{\left(p + \sigma^2/2\right)T - q + \log x_0}{\sigma\sqrt{T}}\right).$$

Example 18 Another special case is the constant interest rate $r(t) = r$, so that $R(t) = rt$. In this case, if we take constant boundaries $a(t) = 0$ and $b(t) = h > 0$, then $c(t) = -\infty$ and

$$d(t) = \frac{1}{\sigma}\left(\frac{\sigma^2}{2} - r\right)t + \frac{1}{\sigma}\log\left(\frac{h}{x_0}\right), 0 \leq t \leq T.$$

Therefore,

$$P_X(a, b, T) = \Phi\left(\frac{\left(\sigma^2/2 - r\right)T + \log(h/x_0)}{\sigma\sqrt{T}}\right) - \exp\left(\frac{\left(2r - \sigma^2\right)\log(h/x_0)}{\sigma^2}\right)\Phi\left(\frac{\left(\sigma^2/2 - r\right)T - \log(h/x_0)}{\sigma\sqrt{T}}\right).$$

More numerical examples for the geometric and standard BM are given in the next section.

7 Numerical Examples

In this section we compute some numerical examples for the BCP for various processes discussed in previous sections. For the simplicity of notation, we present examples of one-sided BCP only.

Given any diffusion process $\{X_t\}$ and a boundary, the BCP for $\{X_t\}$ can be given by the BCP for Brownian motion and a transformed boundary b, i.e., $P_W(-\infty, b, T)$. As discussed in Section 2, the BCP $P_W(-\infty, b, T)$ can be approximated by $P_W(-\infty, b_n, T)$, where b_n is a piecewise linear boundary converging to b

uniformly on the interval [0, T]. In particular, if $b_n(t)$ and $\tilde{b}_n(t)$ are such piecewise linear functions approaching $b(t)$ from below and above respectively, then

$$P_W(-\infty, b_n, T) \le P_W(-\infty, b, T) \le P_W(-\infty, \tilde{b}_n, T).$$

The two sides of the above inequalities are given by Eq. 2 with function g given in Eq. 3. The expectation in Eq. 2 can be easily computed through Monte Carlo simulation, because $W_{t_1}, W_{t_2}, ..., W_{t_n}$ have a multivariate normal distribution.

For all examples in this section, $b_n(t)$ and $\tilde{b}_n(t)$ are constructed with $n = 128$ equally-spaced nodes and lower and upper bounds for the corresponding BCP are computed. In each Monte Carlo simulation, 10^6 repetitions are carried out.

Example 19 (O–U process) Consider constant boundaries of Example 11, where $a(t) = -\infty$ and $b(t) = h, 0 \le t \le T$. Then, $c(s) = -\infty$ and $d(s)$ is a square-root boundary. In particular, we set parameters to $\alpha = x_0$, $\sigma^2 = 2\kappa = 1$, $h = x_0 + 1$ and $T = 1$. Then $d(s) = \sqrt{1+s}$, for $0 \le s \le S = e - 1$. Using the procedure described above, the computed lower and upper bounds for the BCP are $0.721463 \le P_X(a, b, T) \le 0.721464$, with a simulation standard error of 0.000440.

Example 20 (Growth process) Now consider boundaries of Example 15, where $b(t) = h, 0 \le t \le T$. Then $d(s)$ is again a square-root boundary. In particular, if we take $\sigma^2 = 2\alpha$, $\beta = 1/2$, $x_0 = 1$, $h = e^\sigma$ and $T = 1$, then $d(s) = \sqrt{1+s}$ and $S = e - 1$, which is the same as the square-root boundary of Example 19. The lower and upper bounds for the BCP are therefore given there.

Example 21 (Geometric BM) Roberts and Shortland (1997) give an example of pricing a European call option, who's underlying security X_t follows the SDE (32) with $\sigma = 0.1$, $r(t) = 0.1 + 0.05e^{-t}$ and $x_0 = 10$. The option has a knock-in boundary $b(t) = 12$ and maturity $T = 1$. Then by Corollary 16 and Eq. 34, we have $P_X(-\infty, b, T) = P_{\tilde{W}}(-\infty, d, T)$, where

$$d(t) = 10\log(1.2) - 0.5 - 0.95t + 0.5e^{-t}, 0 \le t \le T.$$

Therefore, using the approach of this paper the lower and upper bounds for the BCP are found to be $0.603728 \le P_X(-\infty, b, T) \le 0.603729$, with a simulation standard error 0.000483.

Example 22 (BM and Daniels boundary) Finally, consider the well-known Daniels' boundary for standard BM $W = \{W_t, t \ge 0\}$ (Daniels 1969, 1996), which is

$$b(t) = \frac{1}{2} - t\log\left(\frac{1}{4} + \frac{1}{4}\sqrt{1 + 8e^{-1/t}}\right), 0 \le t \le T,$$

with $T = 1$. This boundary has been used by many authors to test their computational algorithms. The "exact" BCP for this boundary is known to be $P_W(-\infty, b, T) = 0.520251$. The procedure described above gives an approximation for the corresponding BCP as $P_W(-\infty, b, T) \approx 0.520293$ with a simulation standard error 0.000490.

8 Conclusions and Discussion

We proposed a direct method for computing boundary crossing probabilities (BCP) for a class of diffusion processes which can be expressed as piecewise monotone functionals of a standard Brownian motion. This class includes many interesting processes in real applications, e.g., Ornstein–Uhlenbeck, growth processes and geometric Brownian motion with time dependent drift. This method applies to both one-sided and two-sided general nonlinear boundaries, which may be discontinuous. Using this approach explicit formulas for boundary crossing probability for certain nonlinear boundaries have been obtained, which are useful to evaluate and compare computational algorithms.

Compared to the traditional approach for the first-passage time densities, the BCP method can be applied to both one-sided and two-sided boundaries. It also applies to discontinuous boundaries. The actual computation can be easily done using Monte Carlo simulation methods, which is practical and easy to implement. Furthermore, the approximation errors are also computed automatically. The approach of this paper is based on the stochastic differential equation for the processes. It is different from the traditional methods of first-passage time densities, which is based on the partial differential equation for the transition density function.

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