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CROSSING SYMMETRY CONSTRAINTS ON PARTIAL WAVE AMPLITUDES AND THEIR PHYSICAL CONTENT

J.L. Basdevant *), G. Cohen-Tannoudji and A. Morel
CERN - Geneva

A B S T R A C T

A very simple derivation of a complete set of sum rules implied by crossing symmetry on partial wave amplitudes is given. This set is shown to be equivalent to the Balachandran-Nuyts crossing relations for partial wave amplitudes. Many applications to dynamical models are proposed and discussed, and it is shown in particular that the Padé treatment of the $\lambda\phi^4$ theory for $\pi\pi$ scattering yields extremely good agreement with crossing symmetry.

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*) On leave from: Laboratoire de Physique Théorique et Hautes Energies, Bât. 211, Faculté des Sciences, Orsay.

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I. - INTRODUCTION

The problem of putting together crossing symmetry and unitarity has led to different approaches. One of them, which has been initiated by Martin [1], and extensively developed in the recent past [2),3),4),5)], is based on the combination of the "positivity condition" imposed by unitarity with crossing symmetry. In the $\pi\pi$ case, it leads to various sets of inequalities between the values of a finite number of partial waves taken at different points of the Mandelstam triangle. Another approach, developed by Balachandran and Nuyts [6], consists in deriving a complete set of equations, each of them involving a finite number of partial waves, which ensures crossing symmetry. Unitarity can then be added as a supplementary condition on the partial wave amplitudes involved. Other developments can be found in Ref. [7]. The crossing symmetry constraints have also been extensively studied by Roskies in the $\pi\pi\pi$ case [8] *).

In this paper, we first want to show that a complete set of sum rules, each of them involving a finite number of partial wave amplitudes, for any elastic scattering, can be derived very simply, just from the (s,u) symmetry character of the Mandelstam amplitude $M(s,t,u)$. Here, complete means that the conditions obtained are necessary and sufficient to ensure crossing symmetry. This set of sum rules is shown to be equivalent to that obtained by other methods in Refs. [6] and [7]. This is achieved in Section 2. In Section 3, we discuss some physical implications of these sum rules and illustrate their usefulness in explicit dynamical theories. In particular we show that, for the $\lambda\phi^4$ theory of $\pi\pi$ scattering, Padé approximants, which verify elastic unitarity in the low energy region, seem to be also in excellent agreement with crossing symmetry.

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*) We thank Dr. Roskies for keeping us informed of his own work and bringing the paper of Ref. 7) to our attention.

II. - DERIVATION OF THE SUM RULES

1) Kinematics

Let us consider an elastic reaction

$$a + b \to a + b \qquad (s \text{ channel reaction})$$

and the two associated crossed reactions

$$\bar{b} + b \to \bar{a} + a \qquad (t \text{ channel reaction})$$

and

$$a + \bar{b} \to a + \bar{b} \qquad (u \text{ channel reaction}).$$

The kinematical configuration is described on Fig. 1. m and μ are the respective masses of a and b.

As usual we define

$$\left.\begin{aligned} s &= (q_1+q_2)^2 = (q_3+q_4)^2 \\ t &= (q_1-q_3)^2 = (q_2-q_4)^2 \\ u &= (q_1-q_4)^2 = (q_2-q_3)^2 \end{aligned}\right\} \qquad (II.1)$$

with $$s+t+u = 2m^2 + 2\mu^2$$

The physical regions for the three reactions are limited by the curve c given by

$$t\left[su - (m^2-\mu^2)^2\right] = 0$$

and shown in Fig. 2.

The scattering angles θ_s and θ_t in s and t channels are given by

$$\cos\theta_s = 1 + \frac{t}{2q^2} \qquad (II.2)$$

$$\cos\theta_t = \frac{s-u}{4 p_t k_t} \tag{II.3}$$

where

$$q^2 = \left[s - (m+\mu)^2 \right]\left[s - (m-\mu)^2 \right] / 4s \,, \tag{II.4}$$

the squared s channel centre-of-mass momentum and

$$\left.\begin{aligned} p_t &= \tfrac{1}{2}\sqrt{t-4m^2} \\ k_t &= \tfrac{1}{2}\sqrt{t-4\mu^2} \end{aligned}\right\} \tag{II.5}$$

Finally we call D the unphysical region in which $|\cos\theta_s|$ (or $|\cos\theta_t|$, or $|\cos\theta_u|$) is less than or equal to 1. We remark that, whatever a and b are, the geometrical situation is obviously symmetric under the s,u exchange.

2) Derivation of the sum rules in the spinless case

Let M(s,t,u) be an amplitude describing some charge configuration of the three reactions and which has, say, the Mandelstam analyticity domain. If $b \equiv \bar{b}$ or if $a \equiv \bar{b}$, s and u channel reactions correspond to the same process, up to spin and isospin components. We first concentrate on the spinless case. Now, according to the isospin in the t channel M(s,t,u) is either symmetric, or antisymmetric under the s,u exchange. This is the crossing symmetry property.

So, let us denote by $M^{(+)}$ (resp. $M^{(-)}$) an amplitude symmetric (resp. antisymmetric) under the s,u exchange. It is clear that for any domain Δ symmetric under this exchange :

$$\int_{\Delta} ds\, dt\; M^{(-)}(s,t,u) = 0 \qquad \text{(II.6)}$$

and

$$\int_{\Delta} ds\, dt\; (s-u)\, M^{(+)}(s,t,u) = 0 \qquad \text{(II.7)}$$

Note that if Δ is not an analyticity region for $M^{(\pm)}(s,t,u)$, Eqs. (II.6) and (II.7) must be supplemented by a prescription concerning the determinations of $M^{(\pm)}$ on their cuts.

Now if we take $\Delta = D$, the t integration reduces to an integration on $\cos\theta_s$ from -1 to $+1$ and then exhibits s channel partial wave amplitudes. We get the following sum rules for s channel S and P waves :

$$\int_{(m-\mu)^2}^{(m+\mu)^2} q^2\, ds\; f_0^{(-)}(s) = 0 \qquad \text{(II.6')}$$

$$\int_{(m-\mu)^2}^{(m+\mu)^2} q^2\, ds \left[(s-q^2-m^2-\mu^2) f_0^{(+)}(s) + q^2 f_1^{(+)}(s) \right] = 0 \qquad \text{(II.7')}$$

the partial wave amplitudes being defined by

$$f_\ell^{(\pm)}(s) = \frac{1}{2} \int_{-1}^{+1} d\cos\theta_s\; M^{(\pm)}(s,t,u)\; P_\ell(\cos\theta_s) \qquad \text{(II.8)}$$

3) A complete set of sum rules from crossing symmetry

It is also interesting to interpret the sum rules (II.6') and (II.7') in terms of t channel partial wave amplitudes,

$$g_\ell^{(\pm)} = \frac{1}{2}\int_{-1}^{+1} d\cos\theta_t \; M^{(\pm)}(s,t,u)\; P_\ell(\cos\theta_t) \qquad \text{(II.9)}$$

Taking $\Delta = D$ in (II.6) and (II.7) and changing the integration variables into s-u and t we find :

$$\int_0^{4\mu^2} dt\; p_t\, k_t\; g_0^{(-)}(t) = 0 \qquad \text{(II.10)}$$

$$\int_0^{4\mu^2} dt\; (p_t\, k_t)^2\; g_1^{(+)}(t) = 0 \qquad \text{(II.11)}$$

which are trivial consequences of

$$g_0^{(-)}(t) = 0 \;, \forall t \quad ; \quad g_1^{(+)}(t) = 0 \;, \forall t \qquad \text{(II.12)}$$

as implied by the symmetry character of $M^{(-)}$ and $M^{(+)}$.

The derivation of a complete set of sum rules is now straightforward. The crossing symmetry is verified if, and only if, for any non-negative integer ℓ ,

$$\left.\begin{aligned} & g_{2\ell}^{(-)}(t) = 0 \;, \forall t \\ \text{and} \quad & g_{2\ell+1}^{(+)}(t) = 0 \;, \forall t \end{aligned}\right\} \qquad \text{(II.13)}$$

which is equivalent to

$$\left.\begin{aligned}\int_D d(s-u)\,dt\; t^Q (s-u)^{2R} M^{(-)}(s,t,u) &= 0\\ \int_D d(s-u)\,dt\; t^Q (s-u)^{2R+1} M^{(+)}(s,t,u) &= 0\end{aligned}\right\} \qquad \text{(II.14)}$$

for any non-negative integers Q and R. Equivalence of (II.13) and (II.14) follows from the fact that if (II.14) is verified for a given R and all Q then

$$G^{(-)}_{2R} = \int_{-1}^{+1} d\cos\theta_t\; (\cos\theta_t)^{2R} (p_t k_t)^{2R+1} M^{(-)}(s,t,u) = 0, \forall t \qquad \text{(II.15)}$$

since all its moments in the interval $0 \leq t \leq 4\mu^2$ are zero. Now if Eq. (II.15) is satisfied for $0 \leq R \leq R'$, then

$$g^{(-)}_{2R}(t) = 0, \quad \forall t \quad \text{for} \quad 0 \leq R \leq R'$$

Letting R' go to infinity achieves the proof. The same is true of $g^{(+)}_{2R+1}(t)$.

Finally we notice that, given Q and R, the t integration at fixed s gives in Eq. (II.14) two sum rules involving a <u>finite</u> number of <u>s channel</u> partial wave amplitudes since the coefficients of $M^{(\pm)}$ are <u>polynomials</u> in t of degree $2R+Q$ or $2R+Q+1$.

Clearly the infinite set of sum rules just derived is equivalent to the one obtained in Ref. [7], since both are complete. They cannot differ by more than a linear transformation with numerical coefficients corresponding to different choices of sets of polynomials in s and t.

4) The π N case and the treatment of spin

We now derive the sum rules for π nucleon elastic scattering and outline the way to generalize to higher spins.

With the π N kinematics the interesting partial wave amplitudes are the helicity partial wave amplitudes which are simply related to the usual phase shifts :

$$M^{(s)}_{+\pm} = \sum_J (J+\tfrac{1}{2})\, d^J_{\pm\frac{1}{2},\frac{1}{2}}(\cos\theta_s)\; M^J_{+\pm}(s) \qquad \text{(II.16)}$$

$$\left.\begin{aligned} M^J_{+\pm}(s) &= f^+_\ell(s) \pm f^-_{\ell+1}(s), \quad \ell = J-\tfrac{1}{2} \\ f^\pm_\ell(s) &= \frac{\eta^\pm_\ell e^{2i\delta^\pm_\ell} - 1}{2i\rho}, \quad \rho = \frac{q}{8\pi\sqrt{s}} \end{aligned}\right\} \qquad \text{(II.17)}$$

$M^{(s)}_{++}$ and $M^{(s)}_{+-}$ are the non-flip and flip s channel helicity amplitudes (S.H.A.).

In order to apply the general formalism given above and to find sum rules connecting a finite number of phase shifts, we need to find linear combinations of S.H.A. with the following properties :

(i) symmetry or antisymmetry under the s, u exchange ;

(ii) the coefficients of $M^{(s)}_{++}$ (resp. $M^{(s)}_{+-}$) must be equal to $\cos\theta_s/2$ (resp. $\sin\theta_s/2$) times a polynomial in t since such coefficients can be expanded into a finite number of d^J_{11} (resp. $d^J_{-\frac{1}{2}+\frac{1}{2}}$).

The use of t channel helicity amplitudes (T.H.A.) which have obvious symmetry properties under the s, u exchange according to the isospin in the t channel, and known kinematical properties [9] provides an answer to these problems. From the crossing matrix for the helicity amplitudes we can write the T.H.A. as [9] :

$$M_{00}^{(t)} = F(t)\left[m\cos\frac{\theta_s}{2}\,M_{++}^{(s)} + E\sin\frac{\theta_s}{2}\,M_{+-}^{(s)}\right]$$

$$M_{10}^{(t)} = G(t)\left[E\sin\frac{\theta_s}{2}\,M_{++}^{(s)} - m\cos\frac{\theta_s}{2}\,M_{+-}^{(s)}\right] \qquad \text{(II.18)}$$

where E is the nucleon energy in the s channel centre-of-mass system, and F(t) and G(t) depend only on t and thus are symmetric under the s, u exchange. On the other hand the quantity $\sqrt{-t} = 2q\sin\theta_s/2$ and $\sqrt{su - (m^2-\mu^2)^2} = 2q\sqrt{s}\cos\theta_s/2$ are also symmetric under this exchange. The sum rules for πN partial wave amplitudes can then be derived from

$$\int_D ds\,dt\; t^Q (s-u)^R \left[m\cos\frac{\theta_s}{2}M_{++}^{(s)} + E\sin\frac{\theta_s}{2}M_{+-}^{(s)}\right] = 0 \qquad \text{(II.19)}$$

$$\text{for} \begin{cases} Q = 0,1,2,\ldots\ldots \\ R = \begin{cases} 1,3,5,7\ldots\ldots & \text{for isospin 0 in the t channel} \\ 0,2,4,6\ldots\ldots & \text{for isospin 1 in the t channel} \end{cases} \end{cases}$$

and

$$\int_D ds\,dt\; t^Q (s-u)^R \left[E\sin\frac{\theta_s}{2}M_{++}^{(s)} - m\cos\frac{\theta_s}{2}M_{+-}^{(s)}\right] 4q^2\sqrt{s}\sin\frac{\theta_s}{2}\cos\frac{\theta_s}{2} = 0 \qquad \text{(II.20)}$$

$$\text{for} \begin{cases} Q = 0,1,2,\ldots\ldots \\ R = \begin{cases} 1,3,5,7\ldots\ldots & \text{for isospin 1 in the t channel} \\ 0,2,4,6\ldots\ldots & \text{for isospin 0 in the t channel.} \end{cases} \end{cases}$$

It is very easy to check that for any fixed Q and R the t integration in (II.19) and (II.20) exhibits a finite number of helicity partial wave amplitudes. We postpone to Section III the explicit writing of the simplest sum rules resulting from (II.19) and (II.20). We just give the hints for the generalization to higher spins. Explicit calculations would be tedious but the principle of the method is straightforward : one would have to write the T.H.A. in terms of S.H.A. through the crossing matrix [9], to regularize them in the t variable [9] (a procedure which does not destroy the s, u symmetry properties). Then it is sufficient to multiply in by suitable powers of $\sqrt{-t}$ and $\sqrt{su - (m^2-\mu^2)^2}$ in order to get sum rules involving a finite number of partial wave helicity amplitudes.

III. - PHYSICAL CONTENT OF CROSSING SYMMETRY CONSTRAINTS

The sum rules just derived and inequalities obtained in Refs. [1)-5)] involve partial waves in an unphysical region, so they cannot be directly confronted with experiment. As remarked in Ref. [6)], the sum rules can be, if partial wave amplitudes satisfy dispersion relation, written in such a way is to exhibit physical values of the amplitudes : suppose that, in the $\pi\pi$ case, the partial wave amplitudes satisfy unsubtracted dispersion relations :

$$f_\ell(s) = \frac{1}{\pi}\int_{-\infty}^{0} \frac{\Delta f_\ell(s')\,ds'}{s'-s} + \frac{1}{\pi}\int_{4\mu^2}^{\infty} \frac{Im\, f_\ell(s')\,ds'}{s'-s} \qquad \text{(III.1)}$$

then a sum rule of the type

$$\int_0^{4\mu^2} ds\, \varphi_\ell(s)\, f_\ell(s) = 0 \qquad \text{(III.2)}$$

where $\varphi_\ell(s)$ is a known function, can be rewritten in the form

$$\int_{-\infty}^{0} \Phi_\ell(s')\, \Delta f_\ell(s')\, ds' + \int_{4\mu^2}^{\infty} \Phi_\ell(s')\, \mathrm{Im}\, f_\ell(s')\, ds' = 0 \qquad \text{(III.3)}$$

where

$$\Phi_\ell(s) = \int_0^{4\mu^2} \frac{\varphi_\ell(s')\, ds'}{s'-s} \qquad \text{(III.4)}$$

The second integral in (III.3) is expressed in terms of physical quantities while the first one exhibits the famous left-hand cut which, in the dispersion approach, describes the forces. Equation (III.3) illustrates very well the type of information which could arise from the sum rules. In the simplest case where $f_\ell(s)$ is a combination with positive coefficients of well-defined isospin amplitudes, Im $f_\ell(s)$ is positive from unitarity. Then, if $\Phi_\ell(s)$ has some positivity properties, Eq. (III.3) tells us the sign of the left-hand cut integral, which is already a non-negligible information.

It is the object of the present Section to show, on several examples, the physical implications of the crossing symmetry constraints for various theoretical approaches. In fact we shall use the sum rules in their finite version [integral from $(m-\mu)^2$ to $(m+\mu)^2$] and in the $\pi\pi$ case we will also check some of the inequalities of Refs. [1)-5)]. Of course we will not be interested in models, explicitly crossing symmetric (e.g., Veneziano model, or πN model with only t channel Regge poles). According to the model, there are two possible types of applications of the constraints :

(i) the model has free parameters or unknown functions, then the sum rules can be used as a way of determining parameters in order to verify, or at least to approach, exact crossing symmetry ;

(ii) the model has no free parameter, the sum rules can now be used to check crossing symmetry or to evaluate how much this symmetry is violated ; in this respect the example of Padé treatments of perturbation theory for $\pi\pi$ scattering will be discussed in detail below.

1) $\pi\pi$ scattering

A - Use of the sum rules

Denoting by $a_0(s)$ and $a_2(s)$ the s wave $\pi\pi$ amplitudes in isospin states $I = 0$ and $I = 2$ respectively, and by $a_1(s)$ the isospin one p wave amplitude, the three lowest order sum rules involving these amplitudes are the following [10]

$$\int_0^{4\mu^2} (s-4\mu^2)\,(2\,a_0(s) - 5\,a_2(s))\,ds = 0 \qquad \text{(III.5a)}$$

$$\int_0^{4\mu^2} (s-4\mu^2)\,(3s-4\mu^2)\,(a_0(s) + 2\,a_2(s))\,ds = 0 \qquad \text{(III.5b)}$$

$$\int_0^{4\mu^2} (s-4\mu^2)(3s-4\mu^2)\,a_2(s)\,ds = -\int_0^{4\mu^2} (s-4\mu^2)^2\,a_1(s)\,ds \qquad \text{(III.5c)}$$

where μ is the pion mass.

Note that Eq. (III.5a) reflects the fact that at the symmetry point $s = t = u = \frac{4}{3}\mu^2$ the ratio of the $I = 0$ and $I = 2$ total amplitudes is $\frac{5}{2}$, and that Eq. (III.5b) is a consequence of the complete symmetry of $\pi_0\pi_0 \to \pi_0\pi_0$ amplitude in s, t and u. Equation (III.5c) is a necessary condition for crossing symmetry to hold in $\pi\pi$ scattering.

In recent years, field theoretical calculations have been quite successful in describing meson-meson scattering when Padé approximants were used to sum the divergent strong coupling perturbation

series [11),12),13)]. Besides their nice property of being able to sum divergent series, (this was the original motivation for using them), all Padé approximants $T^{[N,M]}$ *) also have the very important property of satisfying exact unitarity [11),14)], provided that $M \geq N$. Thus, one has a unitary, and presumably convergent [15),16)], method of treating the perturbation series.

Notice, however, that the perturbation series, truncated at order p, is exactly crossing symmetric, so that it satisfies Eqs. (III.5a,b,c) identically, while it only satisfies unitarity up to order p. On the other hand, since the Padé algorithm is non-linear, unitary Padé approximants will only satisfy crossing up to order $N+M$, and Eqs. (III.5a,b,c) will only be approximate for them. The question which therefore arises is whether the violation of crossing by Padé approximants is small (which must be the case if the method actually converges) or very large as is the violation of unitarity in the perturbation series (in that case, the Padé method would not be reliable).

A simple example consists in calculating meson-meson scattering in the $g\phi^4$ theory ; this has been extensively studied in the last two years [11),12)]. It has been shown in particular that :

(i) in practice the convergence appears to be very good in that the approximants are remarkably stable as the order is increased ;

(ii) higher partial wave resonances such as the ρ, f_o, k^*, etc., are generated very well by such an interaction.

For $\pi\pi$ scattering, one obtains the ρ meson with a value of the coupling constant $g \simeq 6$. Here we have computed the perturbation series up to third order in g, so that we can build the following approximants in a given partial wave **).

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*) We recall that the $[N,M]$ approximant to the T matrix is written as a ratio of two polynomials in the coupling constant g, of degrees N and M respectively, $T^{[N,M]} = P_N(g)/Q_M(g)$, which has the same power series expansion in g as the Feynman series, up to order $N+M$.

**) The analytic expressions for the perturbation series are given in Ref. [11)].

$$T^{[1,1]} = \frac{g\, T_1(s)}{1 - g\, T_2(s)/T_1(s)} \qquad \text{(unitary)} \tag{III.6a}$$

$$T^{[2,1]} = g T_1(s) + \frac{g^2 T_2(s)}{1 - g\, T_3(s)/T_2(s)} \qquad \text{(non unitary)} \tag{III.6b}$$

$$T^{[1,2]} = g T_1(s) \Big/ \left\{ 1 - g\, T_2(s)/T_1(s) - g^2 \left[T_3(s)/T_1(s) - \left(T_2(s)/T_1(s) \right)^2 \right] \right\} \tag{III.6c}$$

(unitary)

Note that although the $[2,1]$ approximant is not unitary, the violation of unitarity turns out to be very small in practice (a few per cent), as a consequence of rapid convergence. Also, since the Born term is pure s wave in the $g\phi^4$ theory ($T_1^{\ell \neq 0} = 0$), the lowest order Padé approximant one can build for the higher partial waves is the $[2,1]$, which again was shown to be very similar to the unitary $[2,2]$ in a fourth order calculation [11].

Let us now check crossing symmetry. Defining

$$I_{(0_2)}^{[N,M]} = \int_0^{4\mu^2} (s - 4\mu^2)\, a_{(0_2)}^{[N,M]}\, ds \tag{III.7a}$$

$$J_{(0_2)}^{[N,M]} = \int_0^{4\mu^2} (s - 4\mu^2)(3s - 4\mu^2)\, a_{(0_2)}^{[N,M]}\, ds \tag{III.7b}$$

$$J_1^{[N,M]} = \int_0^{4\mu^2} (s - 4\mu^2)^2\, a_1^{[N,M]}\, ds \quad , \tag{III.7c}$$

the relations (III.5a,b,c) can be written as

$$2 I_0^{[N,M]} - 5 I_2^{[N,M]} = 0 \qquad \text{(III.8a)}$$

$$J_0^{[N,M]} + 2 J_2^{[N,M]} = 0 \qquad \text{(III.8b)}$$

$$J_2^{[N,M]} + J_1^{[N,M]} = 0 \qquad \text{(III.8c)}$$

The results are the following. In the $[1,1]$ approximation, only (III.8a) and (III.8b) can be tested since higher waves do not exist. One finds

$$I_0^{[1,1]} = -472.45$$
$$I_2^{[1,1]} = -189.06$$
$$J_0^{[1,1]} = 24.84$$
$$J_2^{[1,1]} = -12.67$$

so that Eq. (III.8a) is satisfied within 0.04% and Eq. (III.8b) within 2% *).

In the $[2,1]$ approximation, we have

$$I_0^{[2,1]} = -472.06$$
$$I_2^{[2,1]} = -189.22$$
$$J_0^{[2,1]} = 29.24$$
$$J_2^{[2,1]} = -14.97$$
$$J_1^{[2,1]} = 15.34$$

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*) The deviation from an equation $a + b = 0$ is defined as $|a+b|/|a|$.

Equation (III.8a) holds within 0.2%, Eq. (III.8b) within 2.3%, and Eq. (III.8c) within 2.4%.

Going to the [1,2] approximation we obtain

$$I_0^{[1,2]} = -472.61$$
$$I_2^{[1,2]} = -189.61$$
$$J_0^{[1,2]} = 30.09$$
$$J_2^{[1,2]} = 14.82$$

Again, for Eq. (III.8a) the deviation is 0.04%, and for Eq. (III.8b) 1.5%.

Notice that Eq. (III.8a) is always satisfied much more closely, but this is actually a consequence of the $g\phi^4$ theory in $\pi\pi$ scattering. In fact the ratio of a_0 and a_2 is close to $\frac{5}{2}$ in all the domain $0 \leqslant s \leqslant 4\mu^2$; it is somewhat larger in the region $s \sim 0$, and closer to 2 near $s = 4\mu^2$ [amplitudes in this region being damped by the factor $(s-4\mu^2)$ in Eq. (III.5a)]. Therefore, Eq. (III.8a) cannot be considered as a strong test of crossing in the $g\phi^4$ theory.

On the contrary, Eqs. (III.8b) and (III.8c) are not at all trivially satisfied and the fact that they are verified within a few percent is a very strong argument in favour of the Padé approximation.

For the s wave amplitudes this is not surprising since previous tests of crossing have been made, based on different methods [12),16)] and have yielded excellent results (in fact the $g\phi^4$ theory has strong s waves, and furthermore there exist sub-series in the Feynman series which are geometrical, so that the Padé method is well adapted to this situation). The fact that the relation between s and p waves [Eq. (III.8c)] is well satisfied is much more meaningful and indicates that the ρ resonance obtained by this method [11)12)] is a true dynamical effect and not an artifact due to the approximation.

Note in particular that with a value of the coupling constant $g \simeq 6$, the $\pi\pi$ phase shifts are very large and, thus, unitarity is very much violated by the perturbation series. Also we have varied the value of g. As g becomes smaller, the relations are satisfied better, as was to be expected. What is more interesting is that even for large values of g, for instance $g \simeq 10$ where the ρ mass goes down to $m_\rho \sim 400$ MeV, the discrepancy is still a few per cent.

The previous considerations show that, as a mathematical tool, the Padé approximation seems reliable in strong coupling field theory. However, it has been shown [16] that in the $g\phi^4$ theory, although higher partial waves are generated well, the s wave phase shifts do not agree with experiment, i.e., the short range forces are not described correctly. In other words one needs more physical information in the Lagrangian. Along these lines, it has been suggested by B.W. Lee [17] that one could incorporate the current algebra constraints in the Lagrangian by requiring that it produces the correct soft pion limit. This can also be considered as an unambiguous way of unitarizing the amplitude given by current algebra. Such a calculation of scattering has been done recently [13] in the so-called sigma model. However, although to lowest order the current algebra amplitude will satisfy the previous relations identically, since it is crossing symmetric, the unitarized amplitude will no longer satisfy them. These unitary amplitudes with the current algebra constraints yield correct s wave phase shifts together with the ρ and f_o resonances [13] in such a way as to keep the good results of the $g\phi^4$ theory and correct the bad ones (in fact, from the structure of Feynman graphs one can consider the sigma model as a sum of the ϕ^4 and ϕ^3 theories). It will therefore be very interesting to test crossing along the previous lines. This calculation is under way and will be reported elsewhere.

B - Use of the Martin inequalities

As already mentioned, a complementary test of crossing consists in verifying the inequalities given by Martin [3] for the $\pi_o \pi_o \to \pi_o \pi_o$ partial wave amplitudes. We have tested these inequalities for the various Padé approximants of the $g\phi^4$ theory. Since

we follow the notations of Ref. [11] where the convention for the unitarity convention is opposite to the usual one, ($\mathrm{Im}\, f_\ell = -\rho(s)|f_\ell(s)|^2$), the inequalities should read :

$$f_0(3.205) < f_0(0.2134) < f_0(2.9863) \qquad \text{(III.9)}$$

and

$$f_0(3.190) > f_0(0) \qquad \text{(III.10)}$$

Here $f_0(s)$ is the s wave $\pi_0\pi_0 \to \pi_0\pi_0$ amplitude, so that in the previous notations we have

$$f_0(s) = a_0(s) + 2a_2(s) \qquad \text{(III.11)}$$

The results are shown in the Table for the various approximants (remember that $[1,1]$ and $[1,2]$ are unitary, but not $[2,1]$), and for three values of the coupling constant : $g = 4$ (for which the ρ mass is $M_\rho \sim 1$ GeV), $g = 6$ ($M_\rho = 760$ MeV), $g = 10$ ($M_\rho \sim 400$ MeV).

It is remarkable that the inequalities are <u>all</u> satisfied. Notice in the Table that, at lowest order in g, $f_0(s)$ is a constant equal to $18\,g$, and it is not at all trivial that the Padé approximation, which produces small deviations from this constant, does give inequalities in the good direction. Once more we see that crossing is well satisfied by the unitary Padé approximants.

2) π-K scattering

Turning, now to the case of π-K scattering we denote by a_I^ℓ the ℓ-th partial wave amplitude in isospin I. The three lowest order relations, which do not involve the d waves, are the following

$$\int_{(m-\mu)^2}^{(m+\mu)^2} q^2 \,(a^0_{1/2} - a^0_{3/2})\, ds = 0 \qquad \text{(III.12)}$$

$$\int_{(m-\mu)^2}^{(m+\mu)^2} q^4 \,(a^0_{1/2} - a^0_{3/2})\, ds = \int_{(m-\mu)^2}^{(m+\mu)^2} q^4 \,(a^1_{1/2} - a^1_{3/2})\, ds \qquad \text{(III.13)}$$

$$\int_{(m-\mu)^2}^{(m+\mu)^2} ds\, q^2 (s - q^2 - m^2 - \mu^2)(a^0_{1/2} + 2a^0_{3/2}) = -\int_{(m-\mu)^2}^{(m+\mu)^2} q^4 (a^1_{1/2} + 2a^1_{3/2})\, ds \qquad \text{(III.14)}$$

where m and μ are the kaon and pion masses, and where q is the π-K centre-of-mass momentum.

Note that we have now only three relations between four amplitudes since we no longer have Bose statistics in the direct channel.

In a $g\phi^4$ theory involving pions and kaons one can show that the isospin $I = \frac{1}{2}$ and $I = \frac{3}{2}$ π-K amplitudes are exactly degenerate [12),18)]. This comes from the fact that at any order the crossed reaction $\pi\pi \to K\bar{K}$ is pure isospin zero. The only possible way to split these states is to introduce the η meson and the coupling $g_{\eta\pi \to K\bar{K}}$ which is pure isospin one [however, in SU(3) symmetry one has $g_{\pi\eta \to K\bar{K}} = 0$ and the degeneracy holds]. Needless

to say that Padé approximants cannot change anything in this degeneracy.

As a consequence, the first two relations will hold trivially at any order since all integrands vanish identically, and only the third relation can be used as a test of crossing [in fact, Eq. (III.12) can be compared with Eq. (III.5a) for $\pi\pi$ scattering].

On the contrary, to lowest order, the current algebra amplitude is pure isospin one in the crossed channel $\pi\pi \to K\bar{K}$ [19]. Therefore, since in the soft pion limit the Weinberg relations for π-K scattering in any partial wave are

$$a^{\ell}_{1/2} = -2\, a^{\ell}_{3/2} \qquad \text{(III.15)}$$

it is the third relation [Eq. (III.14)] which is trivially satisfied : the integrands vanish (while, of course, to lowest order, the first two relations are also satisfied by the current algebra amplitudes).

One can also do a calculation of π-K scattering with the current algebra constraints, in fact one can use broken chiral $SU(3) \times SU(3)$ dynamics [20], instead of $SU(2) \times SU(2)$ as in the sigma model. Here again the theory will appear as the sum of ϕ^4 and ϕ^3 contributions, with the correct soft pion limit, and clearly there will be great interest in testing Eqs. (III-12)-(III.14) for the Padé approximants. Finally, an interesting exercise consists in writing these sum rules in the static limit of infinite kaon mass $M \to \infty$, while the pion mass μ is kept fixed. In that case, denoting by $\omega = \sqrt{q^2 + \mu^2}$ the pion energy, we have

$$s \sim M^2 + 2M\omega \qquad (M \text{ large})$$
$$u \sim M^2 - 2M\omega \qquad \text{(III.16)}$$

and, clearly Eq. (III.13) boils down to

$$\int_{-\mu}^{+\mu} \omega(\omega^2-\mu^2)\left[a^0_{1/2}(\omega) + 2a^0_{3/2}(\omega)\right] d\omega = 0 \qquad \text{(III.17)}$$

It is easy to see that we will actually obtain for the s wave π-"K" amplitudes the set of relations

$$\int_{-\mu}^{+\mu} (\omega^2-\mu^2)\left[a^0_{1/2}(\omega) - a^0_{3/2}(\omega)\right]\omega^{2N} d\omega = 0 \qquad \text{(III.18)}$$

and

$$\int_{-\mu}^{+\mu} \omega(\omega^2-\mu^2)\left[a^0_{1/2}(\omega) + 2a^0_{3/2}(\omega)\right]\omega^{2N} d\omega = 0 \qquad \text{(III.19)}$$

valid for any N. It is also obvious that such relations are valid for any partial wave, and not just the s wave [the only difference coming from a factor $(\omega^2-\mu^2)^{2\ell}$ for the ℓ-th partial wave]. The consequence of Eqs. (III.18) and (III.19) is that the symmetric parts in ω of $a_{\frac{1}{2}}$ and $a_{\frac{3}{2}}$ are the same, while the antisymmetric parts are in the ratio -2:1, therefore we can write

$$a^\ell_{1/2}(\omega) = F^\ell(\omega^2) + 2\omega\, G^\ell(\omega^2) \qquad \text{(III.20a)}$$

$$a^\ell_{3/2}(\omega) = F^\ell(\omega^2) - \omega\, G^\ell(\omega^2) \qquad \text{(III.20b)}$$

and we thus recover the trivial $s \leftrightarrow u$ crossing relations in the static approximation.

Note that analogous relations can be obtained of course for any static target.

In other words, the partial wave sum rules due to crossing are trivially satisfied in the static limit and therefore they give information only because $s \leftrightarrow t$ crossing is possible. In fact, when the masses are finite, they relate different partial waves. In π -N scattering these sum rules will give constraints on the deviation of the amplitudes from the static approximation.

3) π N, KN and $\bar{K}$N scattering

Using the isospin crossing matrix we derive from Eq. (II.19) and Eq. (II.20) the three simplest sum rules for π N scattering which involve only $J = \frac{1}{2}$ and $J = \frac{3}{2}$ partial wave amplitudes :

$$\int_{(m-\mu)^2}^{(m+\mu)^2} ds\, q^2 \left[(E+m)(S_{11} - S_{31}) - (E-m)(P_{11} - P_{31}) \right] = 0 \qquad \text{(III.21)}$$

$$\begin{aligned} \int_{(m-\mu)^2}^{(m+\mu)^2} ds \Big\{ & (S_{11} + 2S_{31}) q^2 (E+m) \left[3(s - q^2 - m^2 - \mu^2) - (E-m)^2 \right] + \\ & + (P_{11} + 2P_{31}) q^2 (E-m) \left[-3(s - q^2 - m^2 - \mu^2) + (E+m)^2 \right] + \\ & + (P_{13} + 2P_{33}) q^4 2(E+m) - \\ & - (D_{13} + 2D_{33}) q^4 2(E-m) \Big\} = 0 \end{aligned} \qquad \text{(III.22)}$$

$$\int_{(m-\mu)^2}^{(m+\mu)^2} ds\, q^4\sqrt{s}\Big\{(S_{11}+2S_{31})(E-m)+(P_{11}+2P_{31})(E+m) -$$

$$-(P_{13}+2P_{33})(E+m)-(D_{13}+2D_{33})(E-m)\Big\} = 0 \qquad \text{(III.23)}$$

[we have used the spectroscopic notation for partial waves : $(L)_{2I,2J}$].

A few comments are in order about these sum rules :

(i) partial wave amplitudes are singular inside the integration region due to the direct and crossed nucleon poles ; these pole contributions satisfy crossing symmetry, so that the sum rules separately hold for the singular and the non-singular part of the partial waves ;

(ii) the MacDowell symmetry properties

$$f_\ell^+(+\sqrt{s}) = f_{\ell+1}^-(-\sqrt{s}) \qquad \text{(III.24)}$$

ensures that if the sum rules are satisfied in the first sheet of $\sqrt{s}$ they are also satisfied in the second sheet ;

(iii) in order to evaluate qualitatively the importance in the sum rules of each partial wave amplitude we are guided by the threshold and pseudothreshold behaviours ; with our normalization $f_\ell^{(\pm)}$ behaves like $\sqrt{s-(m+\mu)^2}^{\,2\ell}$ near threshold and as a constant near pseudothreshold ; we see then that the D wave contributions are strongly damped [particularly in Eq. (III.22)] ;

(iv) Equation (III.21) seems to be the richest in information : at threshold it is known experimentally [21] that $s_{11}-s_{31}$ is large as compared to $s_{11}+2s_{31}$ (properties of s wave scattering lengths) ; on the other hand, the P_{11} scattering length must be important due to the proximity of the Roper resonance (N^*_{1470}) ; we then expect Eq. (III.21) to be highly non-trivial.

Since for the scattering lengths $s_{11} + 2s_{31}$ is known to be small we expect Eq. (III.22) to provide essentially a correlation between the s wave effective ranges and the P wave scattering lengths.

Let us now give a rapid review of models or theoretical approaches for which the sum rules could be useful. Consider first the analyses, à la Hamilton [22),23)], of the low energy π nucleon interaction. In such analyses the nearby left-hand singularities are very carefully studied : the direct and crossed nucleon poles are put in explicitly, the t channel singularities parametrized through a few Born terms, and the crossed physical cut evaluated. The unknown far away left-hand cut contribution is described globally with the help of a few "phenomenological poles" (short range core). In this case it is clear that the sum rules will put strong constraints on the parameters of these "phenomenological poles". In the so-called method of discrepancies, we expect the sum rules to compensate the lack of information in the region between $(m-\mu)^2$ and $(m+\mu)^2$ (see Figs. 15, 16, 20 of Ref. [22)]).

We suggest also that phase shift analyses using dispersion relation as a smoothing procedure should take advantage of these sum rules. We think particularly of KN and $\bar{K}N$ phase shift analyses which are still at their very beginning and for which any theoretical information is welcomed. In the extension of the sum rules to KN and $\bar{K}N$ one would have to choose suitable combinations of KN and $\bar{K}N$ amplitudes in order to satisfy s, u symmetry properties, and to properly take into account the singularity lying inside the domain D of integration (t channel unitarity cut starting from $t = 4m_\pi^2$, annihilation singularities in the $\bar{K}N$ system).

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Approximant	g	$f_o(3.205) < f_o(0.2134) < f_o(2.9863)$			$f_o(0) < f_o(3.190)$	
[1,1]	4	70.38	70.65	70.81	70.15	70.41
	6	104.50	105.00	105.40	103.90	104.50
	10	170.80	172.00	173.10	169.10	170.90
[1,2]	4	70.37	70.69	70.80	70.22	70.40
	6	104.50	105.10	105.40	104.10	104.50
	10	171.00	172.60	173.30	170.20	171.10
[2,1]	4	70.41	70.65	70.82	70.17	70.44
	6	104.70	105.00	105.60	104.00	104.80
	10	171.3	171.5	176.2	169.0	171.2

Tests of Martin inequalities for $\pi_o\pi_o \to \pi_o\pi_o$ scattering for various values of the coupling constant g and various Padé approximants.

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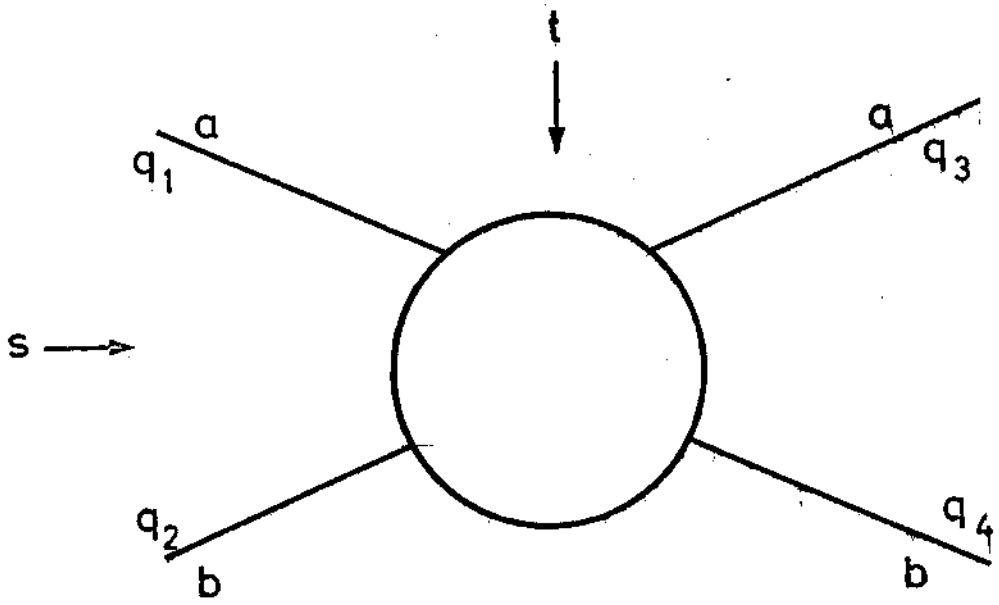

FIG.I

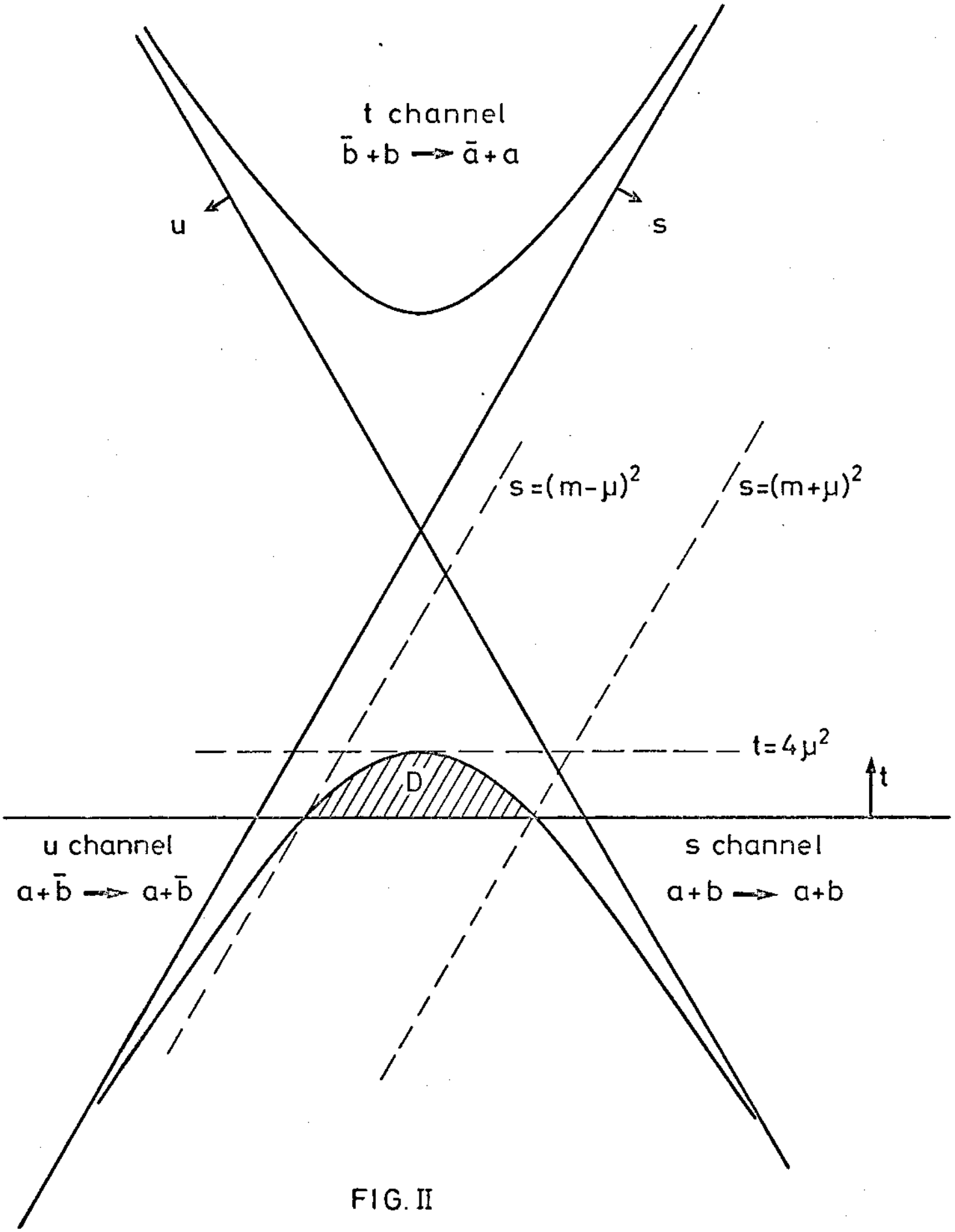
t channel
$\bar{b}+b \rightarrow \bar{a}+a$
u
s
$s=(m-\mu)^2$
$s=(m+\mu)^2$
$t=4\mu^2$
D
t
u channel
$a+\bar{b} \rightarrow a+\bar{b}$
s channel
$a+b \rightarrow a+b$

FIG. II