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Crystalizing the q-Analogue of Universal Enveloping Algebras

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Abstract. For an irreducible representation of the q-analogue of a universal enveloping algebra, one can find a canonical base at $q = 0$, named crystal base (conjectured in a general case and proven for A_n, B_n, C_n and D_n). The crystal base has a structure of a colored oriented graph, named crystal graph. The crystal base of the tensor product (respectively the direct sum) is the tensor product (respectively the union) of the crystal base. The crystal graph of the tensor product is also explicitly described. This gives a combinatorial description of the decomposition of the tensor product into irreducible components.

0. Introduction

The q-analogue of a universal enveloping algebra introduced by Drinfeld [2] and Jimbo [3] is a deformation of the universal enveloping algebra at $q = 1$. Since $q = 0$ corresponds to the absolute temperature zero in the lattice model defined by the R-matrix, we can expect that the q-analogue has a simple structure in that case. Some indications have been already observed in Date–Jimbo–Miwa [1], where the Gelfand–Tsetlin bases become monomes in the tensor algebra of the fundamental representation when $q = 0$. In this note, we shall clarify this phenomenon. For an irreducible representation of the q-analogue, we can find a canonical base at $q = 0$, named crystal base (conjectured in a general case and proven in A_n, B_n, C_n, D_n). The crystal base of the tensor product is the tensor product of the crystal bases. Moreover the crystal base has a structure of colored oriented graph (called crystal graph). The crystal graph of the tensor product is explicitly described.

We shall state our results more precisely. We introduce operators $\tilde{e}_i$ and $\tilde{f}_i$ by modifying the simple root vectors e_i and f_i of the q-analogue U_q (see Sect. 3). Let M be an integrable representation of U_q defined over $\mathbf{Q}(q)$. We consider a pair (L, B) of a lattice L of M defined over the ring of rational functions in q regular at $q = 0$ and a base B of the $\mathbf{Q}$-vector space L/qL. Such a pair (L, B) is called crystal base if it satisfies certain axioms (see Sect. 4). Although we do not write them here, we only note the most important axiom: $\tilde{e}_i L \subset L$, $\tilde{f}_i L \subset L$, $\tilde{e}_i B \subset B \cup \{0\}$ and $\tilde{f}_i B \subset B \cup \{0\}$. Our conjecture is the existence and the uniqueness of the crystal base and our main result is that this conjecture is true in the case of A_n, B_n, C_n and D_n.

1. The q-Analogue of a Universal Enveloping Algebra

Let $\mathfrak{t}$ be a finite-dimensional vector space over $\mathbf{Q}$, and I a finite index set. Let $\{h_i \in \mathfrak{t}; i \in I\}$ and $\{\alpha_i \in \mathfrak{t}^*; i \in I\}$ be linearly independent sets, such that $\{\langle h_i, \alpha_j \rangle\}_{i,j}$ is a symmetrizable generalized Cartan matrix. Let us take an inner product $(\ ,\ )$ of $\mathfrak{t}^*$ such that $2(\alpha_i, \lambda) = (\alpha_i, \alpha_i)\langle h_i, \lambda \rangle$ and (α_i, α_i) is a strictly positive integer for any $i \in I$ and any $\lambda \in \mathfrak{t}^*$. Let $P \subset \mathfrak{t}^*$ be a lattice such that $\langle h_i, P \rangle \in \mathbf{Z}$ for any i and $P \supset Q \underset{\text{def}}{=} \sum_i \mathbf{Z}\alpha_i$. Then, we have

$$(P, Q) \in \mathbf{Z}/2 \quad \text{and} \quad (\lambda, \lambda) - (\mu, \mu) \in \mathbf{Z} \quad \text{if} \quad \lambda, \mu \in P \quad \text{and} \quad \lambda - \mu \in Q. \tag{1.0}$$

Let $P^* \subset \mathfrak{t}$ denote the dual lattice of P.

Let U_q be the algebra over $\mathbf{Q}(q)$ generated by the symbols q^h ($h \in P^*$), e_i, f_i ($i \in I$) satisfying the following fundamental relations.

$$q^{h+h'} = q^h \cdot q^{h'} \quad \text{for} \quad h, h' \in P^* \quad \text{and} \quad q^0 = 1. \tag{1.1}$$

$$q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i \quad \text{and} \quad q^h f_i q^{-h} = q^{\langle h, \alpha_i \rangle} f_i \quad \text{for} \quad h \in P^* \quad \text{and} \quad i \in I. \tag{1.2}$$

Setting $t_i = q^{(\alpha_i, \alpha_i) h_i}$,

$$[e_i, f_j] = \delta_{ij} \frac{t_i - t_i^{-1}}{q^{(\alpha_i, \alpha_i)} - q^{-(\alpha_i, \alpha_i)}}. \tag{1.3}$$

We have therefore

$$t_i e_j t_i^{-1} = q^{2(\alpha_i, \alpha_j)} e_j$$

and

$$t_i f_j t_i^{-1} = q^{-2(\alpha_i, \alpha_j)} f_j. \tag{1.4}$$

The comultiplication $\Delta: U_q \to U_q \otimes U_q$ is given by

$$\Delta(q^h) = q^h \otimes q^h,$$

$$\Delta(e_i) = e_i \otimes 1 + t_i \otimes e_i$$

and

$$\Delta(f_i) = f_i \otimes t_i^{-1} + 1 \otimes f_i. \tag{1.5}$$

By this, the tensor product of U_q-modules becomes a U_q-module.

A U_q-module M is called *integrable* if $M_\lambda = \{u \in M; q^h u = q^{\langle h, \lambda \rangle} u\}$ is finite-dimensional for any $\lambda \in P$, $M = \bigoplus_{\lambda \in P} M_\lambda$ and, for any i, M is a union of finite-dimensional representations over the subalgebra generated by e_i and f_i. An element of M_λ is called a weight vector of weight λ. An integrable U_q-module M is called *with highest weights*, if there is a finite set F of P such that

$$M = \bigoplus_{\lambda \in F + Q_-} M_\lambda,$$

where $Q_- = \{\sum m_i\alpha_i;\ m_i\in\mathbf{Z}_{\leqq 0}\}$. It is known (Rosso [5], Lusztig [4]) that the category of integrable U_q-modules with highest weights is semi-simple. For $\lambda\in P_+ = \{\lambda\in P;\ \langle h_i,\lambda\rangle \geqq 0\}$, let $\mathcal{M}(\lambda)$ denote an irreducible U_q-module with highest weight λ and u_λ a highest weight vector of $\mathcal{M}(\lambda)$.

2. sl_2-Case

Let us review the sl_2 case. Let $U_q(sl_2)$ be the algebra over $\mathbf{Q}(q)$ generated by t, t^{-1}, e, f with the fundamental relation: $tet^{-1} = q^2 e$; $tft^{-1} = q^{-2}f$, $[e,f] = (t - t^{-1})/(q - q^{-1})$. Then $\Delta = q^{-1}t + qt^{-1} + (q-q^{-1})^2 ef - 2 = qt + q^{-1}t^{-1} + (q-q^{-1})^2 fe - 2$ belongs to the center of $U_q(sl_2)$.

An $(l+1)$-dimensional irreducible representation V_l has a basis $\{u_k\}_{0\leqq k\leqq l}$ with

$$tu_k = q^{l-2k}u_k,$$

$$eu_k = [k]u_{k-1}$$

and

$$fu_k = [l-k]u_{k+1}, \tag{2.1}$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$. Then $\Delta|_{V_l} = q^{l+1} - 2 + q^{-l-1}$. Hence $\sqrt{qt\Delta}$ operates on V_l by

$$\sqrt{qt\Delta}u_k = q^{-k}(1 - q^{l+1})u_k. \tag{2.2}$$

Now define $\tilde{e}$ and $\tilde{f}$ by

$$\tilde{e} = (qt\Delta)^{-1/2}e \quad\text{and}\quad \tilde{f} = (qt^{-1}\Delta)^{-1/2}f. \tag{2.3}$$

Then we have

$$\tilde{e}u_k = (1 - q^{2k})(1 - q^2)^{-1}(1 - q^{l+1})^{-1}u_{k-1},$$

$$\tilde{f}u_k = (1 - q^{2(l-k)})(1 - q^2)^{-1}(1 - q^{l+1})^{-1}u_{k+1}. \tag{2.4}$$

Since $\tilde{e}$ and $\tilde{f}$ operate on V_l, $\tilde{e}$ and $\tilde{f}$ operate on any integrable representation of $U_q(sl_2)$.

Let A be the ring of rational functions in q regular at $q = 0$ and $L = \oplus Au_k$. Then we have $\tilde{e}L \subset L$ and $\tilde{f}L \subset L$. Furthermore, $\tilde{e}$ and $\tilde{f}$ have the property:

$$\tilde{e}u_k \equiv u_{k-1} \bmod qL \quad\text{for}\quad 0 < k \leqq l$$

and

$$\tilde{f}u_k \equiv u_{k+1} \bmod qL \quad\text{for}\quad 0 \leqq k < l. \tag{2.5}$$

3. Operators $\tilde{e}_i$ and $\tilde{f}_i$

Now let us come back to the general situation in Sect. 1. For $i\in I$, set $q_i = q^{(\alpha_i,\alpha_i)}$, $\Delta_i = q_i^{-1}t_i + q_i t_i^{-1} + (q_i - q_i^{-1})^2 e_i f_i - 2$, $\tilde{e}_i = (q_i t_i \Delta_i)^{-1/2}e_i$ and $\tilde{f}_i = (q_i t_i^{-1}\Delta_i)^{-1/2}f_i$. Since the subalgebra generated by e_i, f_i and t_i is isomorphic to $U_q(sl_2)$ by $q\mapsto q_i$, $e\mapsto e_i$, $f\mapsto f_i$ and $t\mapsto t_i$, the operators $\tilde{e}_i$, $\tilde{f}_i$ operate on any integrable U_q-module.

4. Crystal Base

Set $K=\mathbf{Q}(q)$ and let A be the ring of rational functions in q without pole at $q=0$. Then A is a discrete valuation ring and K is its fraction field. For a K-vector space V, a *lattice* of V is, by definition, a free A-module L such that $K\bigotimes_A L\cong V$.

Let M be an integrable U_q-module. A *crystal base* (L,B) of M is, by definition, a pair of a lattice L of M and a base B of the $\mathbf{Q}$-vector space L/qL satisfying the following conditions (4.1)–(4.5):

$$L=\bigoplus_{\lambda\in P}L_\lambda,\quad \textit{where}\quad L_\lambda=L\cap M_\lambda. \tag{4.1}$$

$$B=\bigsqcup_{\lambda}B_\lambda,\quad \textit{where}\quad B_\lambda=B\cap(L_\lambda/qL_\lambda). \tag{4.2}$$

$$\tilde{e}_iL\subset L\quad \textit{and}\quad \tilde{f}_iL\subset L\quad \textit{for any}\quad i\in I. \tag{4.3}$$

$$\tilde{e}_iB\subset B\cup\{0\}\quad \textit{and}\quad \tilde{f}_iB\subset B\cup\{0\}\quad \textit{for any}\quad i\in I. \tag{4.4}$$

$$\textit{For } u,v\in B \textit{ and } i\in I,\ u=\tilde{e}_iv \textit{ if and only if } v=\tilde{f}_iu. \tag{4.5}$$

Now, we can state the conjecture.

Conjecture 1. *Any integrable U_q-module has a crystal base.*

More precisely, the crystal base of irreducible representations is described as follows. For $\lambda\in P_+$, we set

$$\mathscr{L}(\lambda)=\sum A\tilde{f}_{i_1}\cdots\tilde{f}_{i_k}u_\lambda\subset\mathscr{M}(\lambda)$$

and

$$\mathscr{B}(\lambda)=\{v=\tilde{f}_{i_1}\cdots\tilde{f}_{i_k}u_\lambda \bmod q\mathscr{L}(\lambda); v\neq 0\}\subset\mathscr{L}(\lambda)/q\mathscr{L}(\lambda)$$

(see Sect. 1 for $\mathscr{M}(\lambda)$ and u_λ). Here $(i_1,\dots,i_k)(0\leqq k)$ ranges over the set of sequences in I.

Conjecture 2. *For any $\lambda\in P_+$ the following property holds:*

$$C(\lambda){:}(\mathscr{L}(\lambda),\mathscr{B}(\lambda))\ \textit{is a crystal base of}\ \mathscr{M}(\lambda).$$

As we shall see in Lemma 2 and Proposition 4, the following conjecture is a consequence of Conjectures 1 and 2.

Conjecture 3. *For any crystal base (L,B) of any integrable U_q-module with highest weights, we have*

$C(L,B)$: *for any $u\in B$ such that $\tilde{e}_iu=0$ for any i, there exists*
$u'\in L$ such that $u=u' \bmod qL$ and $e_iu'=0$ for any i.

Note that $C(L,B)$ is equivalent to the following condition:

$C(L)$: *for any $u\in L/qL$ such that $\tilde{e}_iu=0$ for any i, there exists*
$u'\in L$ such that $u=u' \bmod qL$ and $e_iu'=0$ for any i.

In the sl_2-case, a crystal base of V_l is given by $L=\bigoplus_{j=0}^{l}Au_j$ and $B=\{u_j \bmod qL;$

$j=0,\dots,l\}$ under the notation in Sect. 2. Therefore, Conjectures 1 and 2 are true in the sl_2-case.

One of our main results is the following theorem.

Theorem. *Conjectures 1 and 2 are true for* A_n, B_n, C_n *and* D_n.

We shall start by the following elementary property of crystal bases:

Lemma 1. *If* (L_j, B_j) *is a crystal base of* M_j $(j=1,2,\dots,r)$, *then* $\oplus(L_j, B_j)$ *is a crystal base of* $\oplus M_j$. *Here* $\oplus(L_j,B_j)=(L,B)$ *with* $L=\oplus L_j$ *and* $B=\bigsqcup_j B_j$.

In the sequel, $(L,B)\oplus\cdots\oplus(L,B)$ (m-times) will be denoted by $(L,B)^{\oplus m}$.

Lemma 2. *Let* $\lambda\in P_+$. *If* $(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ *is a crystal base of* $\mathscr{M}(\lambda)$, *then* $C(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ *is true.*

Proof. Assume $u\in\mathscr{B}(\lambda)_\mu$ and $\tilde{e}_i u=0$ for any i. If $\lambda=\mu$, then the assertion on u in $C(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ is evident. Otherwise, there exist $v\in\mathscr{B}(\lambda)$ and i such that $u=\tilde{f}_i v$, and hence $\tilde{e}_i u=v\neq 0$.

5. Polarization

Let (L,B) be a crystal base of an integrable U_q-module M. A *polarization* of (L,B) is a K-valued inner product of M satisfying the following conditions (5.1), (5.2) and (5.3):

$$(e_i u, v)=(u, q_i t_i f_i v), (f_i u, v)=(u, q_i^{-1} e_i t_i^{-1} v)$$

and

$$(q^h u, v)=(u, q^h v) \quad \text{for any} \quad h\in P^*, i\in I \quad \text{and} \quad u,v\in M. \tag{5.1}$$

Hence $(M_\lambda, M_\mu)=0$ if $\lambda\neq\mu$,

$$(L,L)\subset A. \tag{5.2}$$

Let $(\ ,\ )_0$ be the $\mathbf{Q}$-valued inner product on L/qL induced by $(\ ,\ )$.

$$(u,v)_0=\delta_{u,v} \quad \text{for any} \quad u,v\in B. \tag{5.3}$$

By (5.1), we have

$$(\tilde{e}_i u, v)=(u, \tilde{f}_i v). \tag{5.4}$$

Remark that by (5.3), we have

$$L=\{u\in M; (L,u)\subset A\}. \tag{5.5}$$

If we define $\langle u,v\rangle=q^{(\lambda,\lambda)}(u,v)$ for $u,v\in M_\lambda$ and extend this to the inner product of M, then one has

$$\langle e_i u, v\rangle=\langle u, f_i v\rangle \quad \text{and} \quad \langle q^h u, v\rangle=\langle u, q^h v\rangle$$
$$\text{for} \quad h\in P^*, i\in I \quad \text{and} \quad u,v\in M. \tag{5.6}$$

Remark also that, for any $\lambda\in P_+$, $\mathscr{M}(\lambda)$ has always an inner product satisfying (5.1).

Lemma 3. *For $\lambda \in P_+$, assume $C(\lambda)$. Then $(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ is polarizable.*

Proof. Let us take an inner product (,) satisfying (5.1) and $(u_\lambda, u_\lambda) = 1$. We shall show

$$(\mathscr{L}(\lambda)_\mu, \mathscr{L}(\lambda)_\mu) \subset A, \tag{5.7}$$

$$(u, v)_0 = \delta_{u,v} \quad \text{for} \quad u, v \in \mathscr{B}(\lambda)_\mu \tag{5.8}$$

by the induction on $\mu \in P$.

We may assume $\mu \neq \lambda$ and (5.7) and (5.8) are true for $\mu + \alpha_i$ for any i. Then, one has

$$(\tilde{f}_i \mathscr{L}(\lambda)_{\mu+\alpha_i}, \mathscr{L}(\lambda)_\mu) \subset (\mathscr{L}(\lambda)_{\mu+\alpha_i}, \tilde{e}_i \mathscr{L}(\lambda)_\mu) \subset (\mathscr{L}(\lambda)_{\mu+\alpha_i}, \mathscr{L}(\lambda)_{\mu+\alpha_i}) \subset A$$

and hence $\mathscr{L}(\lambda)_\mu = \sum_i \tilde{f}_i \mathscr{L}(\lambda)_{\mu+\alpha_i}$ implies (5.7).

Similarly, for $u \in \mathscr{B}(\lambda)_\mu$, there is i such that $\tilde{e}_i u \neq 0$. Hence $u = \tilde{f}_i \tilde{e}_i u$ and $(u, v)_0 = (\tilde{f}_i \tilde{e}_i u, v)_0 = (\tilde{e}_i u, \tilde{e}_i v)_0 = \delta_{\tilde{e}_i u, \tilde{e}_i v} = \delta_{u,v}$. Q.E.D.

The following proposition asserts that any crystal base is a direct sum of $(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ when Conjecture 2 is true.

Proposition 4. *Let $\lambda \in P_+$ and assume $C(\lambda)$. Let M be a U_q-module isomorphic to $\mathscr{M}(\lambda)^{\oplus m}$ and N an integrable U_q-module such that $N_\lambda = 0$. Let (L, B) be a crystal base of $M \oplus N$. Set $L_M = L \cap M, L_N = L \cap N, B_M = B \cap (L_M/qL_M)$ and $B_N = B \cap (L_N/qL_N)$. Then we have*

(i) *$L = L_M \oplus L_N$ and $B = B_M \sqcup B_N$.*
(ii) *(L_N, B_N) and (L_M, B_M) are a crystal base of N and M, respectively.*
(iii) *There is an isomorphism $M \cong \mathscr{M}(\lambda)^{\oplus m}$ by which $(L_M, B_M) \cong (\mathscr{L}(\lambda), \mathscr{B}(\lambda))^{\oplus m}$.*

Proof. Since $L_N \cap qL = qL_N$, we have $L_N/qL_N \subset L/qL$. Similarly, $L_M/qL_M \subset L/qL$. Setting $(L_0, B_0) = (\mathscr{L}(\lambda), \mathscr{B}(\lambda)^{\oplus m}$, we may choose an isomorphism $M \cong \mathscr{M}(\lambda)^{\oplus m}$ such that $L_\lambda = L_{0\lambda}$ and $B_\lambda = B_{0\lambda}$. By the preceding lemma, (L_0, B_0) admits a polarization (,). We shall show first that for any $\mu \in P$

$$(L_{0\mu}, L_{M\mu}) \subset A. \tag{5.9}$$

By the induction we may assume $\mu \neq \lambda$ and (5.9) holds for all $\mu + \alpha_i$. Then $(\tilde{f}_i L_{0\mu+\alpha_i}, L_{M\mu}) \subset (L_{0\mu+\alpha_i}, \tilde{e}_i L_{M\mu+\alpha_i}) \subset A$, and hence $L_{0\mu} = \Sigma \tilde{f}_i L_{0\mu+\alpha_i}$ implies (5.9). Since $L_0 \subset L_M$ and that $L_M \supset L_0$ by (5.5), we obtain $L_0 = L_M$. Since $B_0 \subset B_m$, we have $B_0 = B_M$. This shows (iii). Next, we shall show

$$L_\mu = (L_M)_\mu + (L_N)_\mu. \tag{5.10}$$

In order to show this, we shall prove

$$qL_\mu \cap ((L_M)_\mu + (L_N)_\mu) \subset q(L_M)_\mu + q(L_N)_\mu. \tag{5.11}$$

Let $u \in (L_M)_\mu$, $v \in (L_N)_\mu$ satisfies $u + v \in qL$. Let $\bar{u} \in L_M/qL_M$ be $u \bmod qL_M$. Then there exist a set J of sequences $\sigma = (\sigma(1), \ldots, \sigma(p))$ in I and $u_\sigma \in (L_M)_\lambda$ such that

$$\bar{u} \equiv \sum_{\sigma \in J} \tilde{f}^\sigma \bar{u}_\sigma,$$

where $\bar{u}_\sigma$ is $u_\sigma \bmod qL_M$ and $\tilde{f}^\sigma = \tilde{f}_{\sigma(1)} \cdots \tilde{f}_{\sigma(p)}$. We may assume further that $|J|$ is minimal among such expressions. In particular $\{\tilde{f}^\sigma \bar{u}_\sigma\}_{\sigma \in J}$ is linearly independent.

Hence $\tilde{f}^\sigma \bar{u}_\lambda \in \mathscr{B}(\lambda)$, where $\bar{u}_\lambda$ is $u_\lambda \bmod q\mathscr{L}(\lambda)$. Hence, setting $\tilde{e}^\sigma = \tilde{e}_{\sigma(p)} \cdots \tilde{e}_{\sigma(1)}$, we have $\tilde{e}^\sigma \tilde{f}^\sigma \bar{u}_\lambda = \bar{u}_\lambda$, which implies

$$\tilde{e}^\sigma \tilde{f}^\sigma \bar{u}_\sigma = \bar{u}_\sigma. \tag{5.12}$$

We have also

$$\tilde{e}^\tau \tilde{f}^\sigma \bar{u}_\sigma = 0 \quad \text{for} \quad \sigma \neq \tau. \tag{5.13}$$

In fact, otherwise $\tilde{e}^\tau \tilde{f}^\sigma \bar{u}_\lambda = \bar{u}_\lambda$ and hence $\tilde{f}^\sigma \bar{u}_\lambda = \tilde{f}^\tau \bar{u}_\lambda$, which implies $\tilde{f}^\sigma \bar{u}_\sigma = \tilde{f}^\tau \bar{u}_\sigma$. Therefore, $\tilde{f}^\sigma \bar{u}_\sigma + \tilde{f}^\tau \bar{u}_\tau = \tilde{f}^\tau (\bar{u}_\sigma + \bar{u}_\tau)$ and this contradicts the minimality condition on J. Thus, we have

$$\sum_{\sigma \in J} \tilde{f}^\sigma \tilde{e}^\sigma \bar{u} = \sum_{\sigma \in J} \sum_{\tau \in J} \tilde{f}^\sigma \tilde{e}^\sigma \tilde{f}^\tau \bar{u}_\tau = \sum_\sigma \tilde{f}^\sigma \bar{u}_\sigma = \bar{u}.$$

Therefore $\sum \tilde{f}^\sigma \tilde{e}^\sigma u \equiv u \bmod qL$. Because $u + v \in qL$, one has

$$u \equiv -\sum \tilde{f}^\sigma \tilde{e}^\sigma v \bmod qL.$$

Now $N_\lambda = 0$ implies $\tilde{e}^\sigma v = 0$, and we conclude $u \in qL$ and $v \in qL$. Thus we obtain (5.11) and hence (5.10).

Now it remains to prove

$$B \subset B_M \cup B_N. \tag{5.14}$$

Let us prove, for any $\mu \in P$

$$B_\mu \subset (B_M)_\mu \cup (B_N)_\mu. \tag{5.15}$$

We may assume that μ is a weight of $\mathscr{M}(\lambda)$. By the induction, we may assume further that (5.15) holds for all $\mu + \alpha_i$. If $\mu = \lambda$, (5.15) is trivial, and therefore we may assume $\mu \neq \lambda$. For $u \in B_\mu$, if there is i such that $\tilde{e}_i u \neq 0$, then $u = \tilde{f}_i \tilde{e}_i u \in \tilde{f}_i ((B_M)_{\mu+\alpha_i} \cup (B_N)_{\mu+\alpha_i}) \backslash \{0\} \subset (B_M)_\mu \cup (B_N)_\mu$. Otherwise we have $\tilde{e}_i u = 0$ for any i. Write $u = u_1 + u_2$ with $u_1 \in L_M/qL_M$ and $u_2 \in L_N/qL_N$. Then $\tilde{e}_i u_1 = 0$. The condition $C(\mathscr{L}(\lambda), \mathscr{B}(\lambda))$ and $\mu \neq \lambda$ implies $u_1 = 0$. Finally we conclude $u \in (B_N)_\mu$. Q.E.D.

Proposition 5. *Let M_j be an integrable U_q-module and (L, B) a crystal base of $\bigoplus_j M_j$ with a polarization (,). If $(M_j, M_k) = 0$ for $j \neq k$, then $L = \bigoplus_j (L \cap M_j)$.*

Proof. Setting $L_j = L \cap M_j$, it is enough to show

$$qL \cap \oplus L_j \subset \oplus qL_j. \tag{5.16}$$

Let $u_j \in L_j$ and assume $\sum_j u_j \in qL$. Then we have $\left(\sum_j u_j, \sum_j u_j \right) = \sum_j (u_j, u_j) \in qA$. Hence $(u_j, u_j)|_0 = 0$ because $(\ , \ )_0$ is positive definite on L/qL. Therefore $u_j \in qL$. Q.E.D.

6. Tensor Product of Crystal Bases

Let (L, B) be a crystal base of an integrable U_q-module M with highest weights. Then B has a structure of colored oriented graph. The colors are labelled by I. For u, $v \in B$, $u \xrightarrow{i} v$ when $v = \tilde{f}_i u$. We shall call this graph the *crystal graph* of M. Remark that, for any i, the crystal graph with only arrows colored by i has no branch points and hence a disjoint union of sequences of arrows. The following proposition describes the crystal graph of the tensor product.

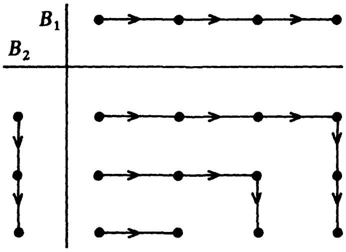
Proposition 6. *Let* (L_j, B_j) *be a crystal base of an integrable* U_q*-module* M_j $(j = 1, 2)$.
(a) *Then* $(L_1, B_1) \otimes (L_2, B_2) \underset{\text{def}}{=} (L_1 \otimes L_2, B_1 \times B_2)$ *is a crystal base of* $V_1 \otimes V_2$. *Here* $B_1 \times B_2 \hookrightarrow L_1 \otimes L_2/q(L_1 \otimes L_2) \cong (L_1/qL_1) \otimes (L_2/qL_2)$ *is given by* $(u, v) \mapsto u \otimes v$.
(b) *For* $u \in B_1, v \in B_2$, *we have*

$$\tilde{f}_i(u \otimes v) = \begin{cases} \tilde{f}_i u \otimes v & \text{if there exists } n \geqq 1 \text{ such that } \tilde{f}_i^n u \neq 0 \text{ and } \tilde{e}_i^n v = 0, \\ u \otimes \tilde{f}_i v & \text{otherwise.} \end{cases} \tag{6.1}$$

$$\tilde{e}_i(u \otimes v) = \begin{cases} u \otimes \tilde{e}_i v & \text{if there exists } n \geqq 1 \text{ such that } \tilde{e}_i^n v \neq 0 \text{ and } \tilde{f}_i^n u = 0, \\ \tilde{e}_i u \otimes v & \text{otherwise.} \end{cases} \tag{6.2}$$

(c) *If* $(\ ,\ )_j$ *is a polarization of* (L_j, B_j), *then* $(u_1 \otimes u_2, v_1 \otimes v_2) = (u_1, v_1)_1 \cdot (u_2, v_2)_2$ *gives a polarization of* $(L_1, B_1) \otimes (L_2, B_2)$.

This can be visualized as follows.



7. Proof of Proposition 6

Since (c) follows easily from (a), we shall prove (a) and (b). Since it is enough to check for each i, we can reduce to the sl_2-case. Then we can reduce to the irreducible case by Proposition 4. Note that Conjecture 2 is true in the sl_2-case. Set $M_1 = V_m$ and $M_2 = V_l$ (see Sect. 2). Let $u_0, \ldots, u_m$ and $v_0, \ldots, v_l$ be the base of M_1 and M_2 given in Sect. 2. Then $L_1 = \bigoplus_{j=0}^{m} Au_j$, $L_2 = \bigoplus_{k=0}^{l} Av_k$ and $B_1 = \{u_j \bmod qL_1\}$, $B_2 = \{v_k \bmod qL_2\}$. Set $L = L_1 \otimes L_2$ and $M = M_1 \otimes M_2$.
(i) First let us prove the case when $m = 1$. Then $M = N_1 \oplus N_2$, where N_1 is generated by $w = u_0 \otimes v_0$ and N_2 is generated by $z = u_0 \otimes v_1 - qu_1 \otimes v_0$. Setting

$$\begin{aligned} w_k &= \frac{1}{[l+1]}([l-k+1]u_0 \otimes v_k + q^{k-1-l}[k]u_1 \otimes v_{k-1}) \\ &= q^k \frac{1-q^{2l-2k+2}}{1-q^{2l-2}} u_0 \otimes v_k + \frac{1-q^{2k}}{1-q^{2l-2}} u_1 \otimes v_{k-1} \end{aligned}$$

for $0 \leqq k \leqq l+1$ (with $v_{l+1} = 0$), we have

$$w_0 = w$$

$$f w_k = [l+1-k] w_{k+1}.$$

Hence $L_1 = \bigoplus_{k=0}^{l+1} Aw_k$ and $B_1 = \{w_k \bmod qL_1; 0 \leqq k \leqq l+1\}$ form a crystal base of N_1.

Setting

$$z_k = u_0 \otimes v_{k+1} - q^{k+1} u_1 \otimes v_k$$

for $0 \leqq k \leqq l-1$, we have

$$z_0 = z,$$
$$fz_k = [l-1-k] z_{k+1}.$$

Hence $L_2 = \bigoplus_{k=0}^{l-1} Az_k$ and $B_2 = \{z_k \bmod qL_2; 0 \leqq k \leqq l-1\}$ from a crystal base. Since $w_0 \equiv u_0 \otimes v_0, w_k \equiv u_1 \otimes v_{k-1}$ and $z_k \equiv u_0 \otimes v_{k+1}$ modulo qL, we obtain the desired result.

(ii) The general case. Assuming that the statement is proven for $V_{m-1} \otimes V_l$, we shall prove the statement for $V_m \otimes V_l$ for $m \geqq 2$. Let (L_l, B_l) be a crystal base of M_l. By the hypothesis, $(L_{m-1}, B_{m-1}) \otimes (L_l, B_l)$ is a crystal base of $V_{m-1} \otimes V_l$. Then by (i), $(L_1, B_1) \otimes (L_{m-1}, B_{m-1}) \otimes (L_l, B_l)$ is a crystal base of $V_1 \otimes V_{m-1} \otimes V_l$. By (i) and Proposition 4, we have $(L_1, B_1) \otimes (L_{m-1}, B_{m-1}) \simeq (L_m, B_m) \oplus (L_{m-2}, B_{m-2})$. Therefor, $(L_m, B_m) \otimes (L_l, B_l) \oplus (L_{m-2}, B_{m-2}) \otimes (L_l, B_l)$ is a crystal base of $V_m \otimes V_l \oplus V_{m-2} \otimes V_l$. Hence, its direct summand $(L_m, B_m) \otimes (L_l, B_l)$ is a crystal base of $V_m \otimes V_l$. This shows (i). Since we know the actions of $\tilde{e}$ and $\tilde{f}$ on $B_{m-1} \times B_l$ and hence those on $B_1 \times B_{m-1} \times B_l$ and those on $B_m \times B_l \subset B_1 \times B_{m-1} \times B_l$. Then explicit calculations show (ii).

8. Proof of Theorem

In order to prove Theorem, we shall prepare

Lemma 7. *Let $\lambda_0, \lambda_1 \in P_+$. We assume*

$$C(\lambda_0) \quad \text{and} \quad C(\lambda_1) \quad \text{hold.} \tag{8.1}$$

$$\dim \mathscr{M}(\lambda_0)_\lambda = 1 \quad \text{for any weight } \lambda \text{ of } \mathscr{M}(\lambda_0), \tag{8.2}$$

$$\mathscr{M}(\lambda_1) \otimes \mathscr{M}(\lambda_0) = \sum_{\mu \in S} \mathscr{M}(\lambda_1 + \mu), \tag{8.3}$$

where S is the set of weights μ of $\mathscr{M}(\lambda_0)$ such that $u \in \mathscr{B}(\lambda_0)_\mu$ satisfies $\tilde{e}_i^{1+\langle h_i, \lambda_1 \rangle} u = 0$ for any i.

Then $C(\lambda_1 + \mu)$ is true for any $\mu \in S$.

Remark. (i) (8.3) is a consequence of Conjecture even without (8.2) (cf. the proof below).

(ii) We have $\lambda_1 + S \subset P_+$ by the observation below.

(iii) Let (L, B) be a crystal base of an integrable U_q-module M_0. Then, for any $i \in I, \mu \in P$ and $n \geqq 1, e_i^n M_\mu = 0$ if and only if $\tilde{e}_i^n B_\mu = 0$. This can be easily checked by reducing it to the irreducible representations of $U_q(sl_2)$.

Proof. Let $M = \mathscr{M}(\lambda_1) \otimes \mathscr{M}(\lambda_0)$, $L = \mathscr{L}(\lambda_1) \otimes \mathscr{L}(\lambda_0)$ and $B = \mathscr{B}(\lambda_1) \times \mathscr{B}(\lambda_0)$. Since $(\mathscr{L}(\lambda_\nu), \mathscr{B}(\lambda_\nu))$ is a polarizable crystal base of $\mathscr{M}(\lambda_\nu)$ $(\nu = 0, 1)$, (L, B) is a polarizable crystal base of M. Let $M = \oplus M_j$ be an irreducible decomposition of M. Set $L_j = L \cap M_j$ and $B_j = B \cap (L_j/qL_j)$. Then, by Proposition 5 we have $L = \oplus L_j$. We shall show

$$B = \bigcup_j B_j. \tag{8.4}$$

In order to prove this it is enough to show that for $\mu \in P$,

$$\text{if } u \in B_\mu \text{ satisfies } \tilde{e}_i u = 0 \text{ for any } i\text{, then } u \in B_j \text{ for some } j. \tag{8.5}$$

Write $u = u_1 \otimes u_0$ with $u_\nu \in \mathscr{B}(\lambda_\nu)$. Then, by Proposition 6, we have $\tilde{e}_i u_1 = 0$ for any i and $\tilde{e}_i^{1+\langle h_i, \lambda_1 \rangle} u_2 = 0$. Hence $u_1 = \bar{u}_{\lambda_1}$, where $\bar{u}_{\lambda_1} = u_{\lambda_1} \bmod q\mathscr{L}(\lambda_1)$. We have therefore

$$\{u \in B; \tilde{e}_i u = 0 \text{ for any } i\} = \{\bar{u}_{\lambda_1}\} \times S. \tag{8.6}$$

This implies that for any $\mu \in P$,

$$\{v \in (L/qL)_\mu; \tilde{e}_i v = 0 \text{ for any } i\} \subset \bar{u}(\lambda_1) \otimes \mathscr{L}(\lambda_0)_{\mu - \lambda_1},$$

and hence it is one-dimensional. Since we have

$$\{v \in (L/qL)_\mu; \tilde{e}_i v = 0 \text{ for any } i\} = \bigoplus_j \{v \in (L_j/qL_j)_\mu; \tilde{e}_i v = 0 \text{ for any } i\},$$
$$\{v \in (L_j/qL_j)_\mu; \tilde{e}_i v = 0 \text{ for any } i\} = 0$$

except one j. Thus we obtain $u \in B_j$ for some j. This shows (8.5) and hence (8.4). This implies that (L_j, B_j) is a crystal base of M_j.

Since $\sum_j \#\{u \in B_j; \tilde{e}_i u = 0 \text{ for any } i\} = \#S$ and that this coincides with the number of irreducible components, we can conclude that, for each j, there exists only one $u \in B_j$ such that $\tilde{e}_i u = 0$ for any i. Q.E.D.

Lemma 8. *Let $\lambda_0, \lambda_1 \in P_+$ and assume (8.2) and*

$$e_i^2 \mathscr{M}(\lambda_0) = 0 \quad \textit{for any } i. \tag{8.7}$$

Let us set

$$S = \{\mu \in P; \mu \textit{ is a weight of } \mathscr{M}(\lambda_0) \textit{ such that } e_i^{1+\langle h_i, \lambda_1 \rangle} \mathscr{M}(\lambda_0)_\mu = 0\}.$$

Then, (8.3) holds.

Proof. By Weyl's character formula, it is enough to show

$$\left(\sum_\xi e^\xi\right)\left(\sum_{w \in W} \operatorname{sgn} e^{w(\lambda_1 + \rho)}\right) = \sum_\mu \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda_1 + \mu + \rho)}. \tag{8.8}$$

Here, ξ ranges over the set P_0 of weights of $\mathscr{M}(\lambda_0)$ and μ ranges over S. Since the left-hand side of (8.8) equals $\sum_{\xi \in P_0} \sum_{w \in W} \operatorname{sgn}(w) e^{w(\lambda_1 + \rho + \xi)}$, it is enough to show that

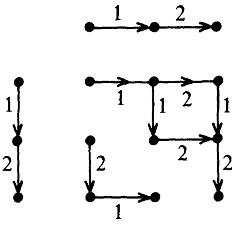
$$\sum_{w \in W} \operatorname{sgn}(w) e^{(\lambda_1 + \xi + \rho)} = 0 \tag{8.9}$$

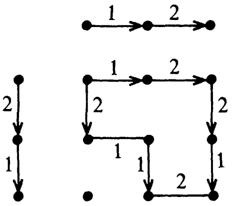
for any $\xi \in P_0 \backslash S$. For such a ξ, there exists i such that $e^{1+\langle h_i, \lambda_1 \rangle} u \neq 0$ where u is the weight vector with weight ξ. Then $\langle h_i, \lambda_1 \rangle = 0$ and $\langle h_i, \xi \rangle = -1$. They imply $s_i(\lambda_1 + \xi + \rho) = 0$ and hence we obtain (8.9). Here s_i denotes the simple reflection $s_i(\lambda) = \lambda - \langle h_i, \lambda \rangle \alpha_i$. Q.E.D.

Let us finish the proof of the theorem. One can check easily the conjecture for the fundamental representation or the spin representations. In fact, the usual base of the fundamental representation or the base of the spin representation in Reshetikhin [6] gives a crystal base. Now, we can apply successively Lemma 7 with the fundamental representation or spin representations as $\mathcal{M}(\lambda_0)$. Note that the fundamental representation or the spin representations satisfy (8.2) and (8.7) and hence (8.3) follows from Lemma 8 and Remark (iii) after Lemma 7.

9. Decomposition

Assume that Conjectures 1 and 2 are true. Let M be an integrable U_q-module with highest weights. Then M is irreducible if its crystal graph (forgetting colors and directions) is connected. Since the crystal graph of the tensor product is described by Proposition 6, we can describe combinatorially the decomposition of the tensor products. The following figures describe $\square \otimes \square = \square\square \oplus \begin{smallmatrix}\square\\\square\end{smallmatrix}$ and $\square \otimes \begin{smallmatrix}\square\\\square\end{smallmatrix} = \begin{smallmatrix}\square\square\\\square\end{smallmatrix} \oplus \phi$ in the sl_3-case.





10. Final Remark

Let (L, B) be a crystal base of an integrable U_q-module M. Then one has

Proposition 9. *Assume that* $\lambda \in P$, $i \in I$ *satisfy* $l = \langle h_i, \lambda \rangle > 0$. *Then* $\tilde{f}_i^l : L_\lambda \to L_{\lambda - l\alpha_i}$ *and* $\tilde{e}_i^l : L_{\lambda - l\alpha_i} \to L_\lambda$ *are isomorphisms.*

This is proven by reducing it to the sl_2-case.

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