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Crystalline mean curvature flow of convex sets

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Abstract

We prove a local existence and uniqueness result of crystalline mean curvature flow starting from a compact convex admissible set in $\mathbb{R}^N$. This theorem can handle the facet breaking/bending phenomena, and can be generalized to any anisotropic mean curvature flow. The method provides also a generalized geometric evolution starting from any compact convex set, existing up to the extinction time, satisfying a comparison principle, and defining a continuous semigroup in time. We prove that, when the initial set is convex, our evolution coincides with the flat ϕ-curvature flow in the sense of Almgren-Taylor-Wang. As a by-product, it turns out that the flat ϕ-curvature flow starting from a compact convex set is unique.

Key words: crystalline mean curvature, ϕ-regular flows, flat flows, convex bodies

AMS (MOS) subject classification: 53C44 35J60 49N60

1 Introduction

In this paper we deal with the anisotropic mean curvature motion, which is defined as the gradient flow of the surface energy functional P_ϕ defined as

$$P_\phi(E) := \int_{\partial E} \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1}, \qquad E \subset \mathbb{R}^N,$$

where ν^E is the outward unit normal to the boundary ∂E of E and ϕ° (the surface tension) is a positively one-homogeneous and even function such that $\{\phi^\circ \le 1\}$ is a compact convex set with nonempty interior. We are particularly interested in the case when $N \ge 3$ and $\{\phi^\circ \le 1\}$ is not smooth; in this respect, we say that the anisotropy ϕ° is *crystalline* if $\{\phi^\circ \le 1\}$ is a polyhedron.

Anisotropic mean curvature flow and its generalizations are used to describe several phenomena in material science and crystal growth, see for instance [22], [51], [42]. From the mathematical point of view, the analysis was initiated by J. Taylor [51], [52] and developed further in [2], [21], [1], [39], [54]. In comparison with more familiar geometric evolutions such as mean curvature flow (corresponding to

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the choice $\phi^\circ(x) = |x|$), it presents additional difficulties, since both the involved differential operators and the flowing interfaces may be nonsmooth. We recall that, for mean curvature flow, a short time existence theorem of a smooth solution is known [35], [33] as well as long time existence starting from special initial data [43], [29]. In addition, singularities may appear during the evolution, and have been partially classified (see for instance [44, 45, 53, 3] and references therein). A comparison theorem is valid, but uniqueness cannot be expected in general [11], due to the so-called fattening phenomenon.

Concerning anisotropic mean curvature flow, if no further assumptions on ϕ° are required (such as regularity and strict convexity of $\{\phi^\circ \leq 1\}$) even the notions of "smooth surface" and "regular evolution" are not immediate in $N \geq 3$ dimensions; for instance $\mathcal{C}^1$ regularity for small times cannot be expected in the crystalline case. The notions of ϕ-regular set and Lipschitz ϕ-regular set, introduced in [16], [14] (see Section 2.3), are two candidates for substituting the notion of smooth surface; ϕ-regular sets admit, by definition, a (selection of the) Cahn-Hoffman vector field with bounded divergence in a neighbourhood of the surface, while Lipschitz ϕ-regular sets have such a selection which is Lipschitz.

We are interested in proving an existence theorem for anisotropic mean curvature flow with no restriction on the anisotropy; we will discuss both a result for small times in the class of ϕ-regular sets starting from special initial data, and some qualitative results for long times. We shall always restrict our discussion to compact and *convex* initial data.

In the effort of proving (even local in time) existence theorems, we looked for weak solutions. As far as we know, the viscosity theory has not yet been adapted to this type of evolutions (see however [36] for recent developments in this direction), at least for nonsmooth (such as crystalline) surface tensions in three dimensions. The only notion of global solution that we know to be available at the present moment is the one given by the method of Almgren-Taylor-Wang [2], referred here as the *flat ϕ-curvature flow* (see also [48] for a similar notion). We point out that in this paper we choose the mobility function accordingly to [23], which is different from the choice in [2]; however, the results of this paper (as well as the results of [2]) can be easily adapted to the case of a different mobility (see [23, Appendix D]). Such a solution is constructed via a time-step minimization method, and provides a global solution in the class of Ahlfors regular finite perimeter sets (and no more regularity is known for this evolution). We remark that no uniqueness and semigroup in time are known for flat ϕ-curvature flow in dimension higher than two. In the two-dimensional case, at least for purely crystalline evolutions, the authors in [1] proved that the flat ϕ-curvature flow coincides with the solution given by the ODE method [39], hence it is unique.

To state the main result of the present paper let us introduce some notation. Given a compact convex set $C \subset \mathbb{R}^N$ we denote by d_ϕ^C the ϕ-signed distance function from ∂C negative inside C, see (2.3). Let Ω be a sufficiently large ball containing C.

Let $\mathcal{G}: \ (0,1) \times \big(L^2(\Omega) \cap BV(\Omega)\big) \times \big(L^2(\Omega) \cap BV(\Omega)\big) \to [0, +\infty]$ be the functional defined as

$$\mathcal{G}(h, v, w) := \int_\Omega \phi^\circ(Du) \ + \frac{1}{2h} \int_\Omega (u - w)^2 \, dx.$$

Let us define recursively the functions d_h^i and the sets C_h^i as follows: for any $h \in (0,1)$ and any $i \in \mathbb{R} \cup \{0\}$,

$$C_h^0 := C, \qquad d_h^0 := d_\phi^C,$$

$$\mathcal{G}(h, u, d_h^i) = \min\{\mathcal{G}(h, v, d_h^i) : v \in L^2(\Omega) \cap BV(\Omega)\} \tag{1.1}$$

and

$$C_h^{i+1} := \{u \le 0\}, \qquad d_h^{i+1} = d_\phi^{C_h^{i+1}}. \tag{1.2}$$

Given a real number a, we indicate by $[a]$ the integer part of a. Our results can be summarized in the following two statements.

Theorem 1.1. *Assume that $\{\phi^\circ \le 1\} \subset \mathbb{R}^N$ $(N \ge 2)$ is a compact convex set with nonempty interior and symmetric with respect to the origin. Let $C \subset \mathbb{R}^N$ be a compact convex ϕ-regular set. Then there is $T > 0$ such that*

$$\text{there exists } \lim_{h\to 0} C_h^{[t/h]} =: C(t) \qquad \text{for any } t \in [0,T]$$

in the Hausdorff distance, and $C(0) = C$. Each set $C(t)$ is compact, convex and ϕ-regular, and the map $t \in [0,T] \to C(t)$ is the unique local in time ϕ-regular flow starting from C.

Theorem 1.2. *Assume that $\{\phi^\circ \le 1\} \subset \mathbb{R}^N$ $(N \ge 2)$ is a compact convex set with nonempty interior and symmetric with respect to the origin. Let $C \subset \mathbb{R}^N$ be a compact convex set. Then there exists a finite time $t_C > T$, bounded by $c_{N,\phi^\circ}|C|$, where c_{N,ϕ° is an explicit (and optimal) constant depending only on ϕ° and N, such that the following properties hold:*

(i) There is a subsequence $\{h_k\}$ such that

$$\text{there exists } \lim_{h\to 0} C_{h_k}^{[t/h_k]} =: C(t) \qquad \text{for a.e. } t \in [0,t_C]$$

in the Hausdorff distance, $C(0) = C$ and t_C is the extinction time of $C(t)$.

(ii) Each set $C(t)$ is compact and convex, and the map $t \in [0,t_C] \to C(t)$ satisfies the comparison principle and defines a continuous semigroup in time.

Of course, the two evolutions given by Theorems 1.1 and 1.2 coincide on $[0,T]$.

Concerning Theorems 1.1, 1.2, proved respectively in Section 6 and in Section 8, some comments are in order. First of all, Theorem 1.1 is a local in time existence (and uniqueness) result, and shows the central rôle played by ϕ-regular sets in our theory. The main obstacle in the proof of such a kind of result is perhaps represented by the fact that, in general even a polyhedral (convex) initial datum may develop, under anisotropic curvature flow, the facet breaking phenomenon; similarly, a facet can bend during the evolution, see [16]. Therefore, unlike the two-dimensional case, in general crystalline mean curvature flow (even for short times and for convex initial data) is not easy to describe in terms of systems of differential equations governing the evolution of each facet. The examples of [16] (which turn out to be related to the calibrability of facets and their classification, see [14]) were also useful to devise reasonable classes of sets were looking for existence and uniqueness results, namely ϕ-regular sets and Lipschitz ϕ-regular sets. We do not know whether these two classes coincide. This seems not to be immediate even in the first explicit example in [16], and is related to the regularity properties of minimizers to the variational problem considered in [14]. A definition of ϕ-regular flow (for short times) where a uniqueness result is available was given in [18], using a reaction-diffusion inclusion to approximate the evolution problem. In any case, we point out that in both these two classes of flows the evolving hypersurfaces are Lipschitz, hence they are far more regular than the flat ϕ-curvature flow.

Theorem 1.1 is based on the time-step minimization procedure (1.1)-(1.2), which was introduced in [24], and can handle the facet breaking/bending phenomena. Such a method can be viewed as a combination of the ideas in [2] and of the restarting heat-equation method considered in [19] and [30]. The advantage of the method of [24] is that it provides uniqueness of each minimizer at each discrete step, which in general is not the case in the flat ϕ-curvature flow procedure; this is essentially the reason for which the corresponding flow is, in general, also a flat ϕ-curvature flow. In addition, it reduces the geometric problem to the study of a nonlinear partial differential inclusion, and one can consider functions in place of boundaries. For this latter PDE problem a comparison principle is available, which directly implies an inclusion principle for the corresponding geometric evolution.

From the technical point of view, we can mention that the proof is based on: i) the properties of those convex sets satisying the $r\mathcal{W}_\phi$-ball condition (see Definition 2.8), namely those sets which contain a tangent unit ball (in the intrinsic norm) of fixed radius r; ii) a smoothing argument (Lemma 3.8), which allows to regularize $\{\phi^\circ \leq 1\}$ with strictly convex bodies of class $\mathcal{C}^\infty$ and, at the same time, the initial convex set C with smoother convex sets.

Theorem 1.2 is, instead, a result for long times, up to the extinction. We remark that two of the authors in [23] showed that the flow $C(t)$ starting from a compact convex set C is convex for all times (and contained in C). Here, we improve the result by proving that such a flow is unique for all times, satisfies a comparison principle and defines a continuous semigroup in time. As a by-product of our analysis, we deduce that the flat ϕ-curvature flow starting from C is unique (and coincides with $C(t)$).

The evolution of convex sets by mean curvature has been considered from a classical viewpoint by Huisken [43], who proved the existence of a smooth evolution of an initial uniformly convex set until its extinction as a point. The extinction time was identified as the time at which the second fundamental form explodes. The asymptotic profile of extinction was identified as a sphere. The smooth evolution of a convex set was also recovered by Evans-Spruck in [32, 34] using different methods. The convexity preserving properties of viscosity geometric evolutions were studied in [38]. Theorems 1.1, 1.2 can be interpreted as a partial extension of Huisken's results, since we obtain the existence of a ϕ-regular solution in some time interval $[0, T)$, and the existence of a flow before the extinction. We do not know, however, if this flow remains ϕ-regular up to the extinction time, as it happens in the case of mean curvature flow. In addition, we miss the identification of the extinction profile.

The structure of the paper is the following. In Section 2 we give some notation. In particular, anisotropies, ϕ-distances and ϕ-normal vectors are defined in subsection 2.1; ϕ-regularity and the $r\mathcal{W}_\phi$-condition are defined in subsection 2.3; ϕ-regular flows are defined in subsection 2.4. In Section 3 we prove that for a convex set C and a strictly convex anisotropy of class $\mathcal{C}^{1,1}$, the maximal neighbourhood where we have smoothness of the ϕ-signed distance is controlled by the sup-norm of divergence of the Cahn-Hoffman vector field n_ϕ^C on the boundary of C and we prove a bound on $\operatorname{div} n_\phi^C$ in such a neighobourhood similar to the classical one for mean curvature. Moreover we give a characterization of convex ϕ-regular sets for general anisotropies in terms of the $r\mathcal{W}_\phi$-condition which is stable by approximation of the convex sets and the anisotropies. In Subsection 3.2 we recall how this approximations can be performed by using some results in [50]. In Section 4 we study the elliptic problem corresponding to (1.1), whose limit solution (as $h \to 0$) embeds the solution $C(t)$ inside the flat ϕ-curvature flows (this connection is precisely recalled in Subsection 4.2). We also recall some results for this elliptic problem proved in [23]. In Section 5 we prove that the iterates of the Almgren-Taylor-Wang algorithm satisfy a uniform $r\mathcal{W}_\phi$-estimate for a certain number of iterations which amounts to a positive time of evolution and this time is related to the explosion of our estimate on the anisotropic mean curvature. For that we have to prove a basic estimate for the anisotropic mean curvature of the flat ϕ-curvature flow of a compact convex set C (satisfying the $r\mathcal{W}_\phi$-condition) in neighborhoods of ∂C. This curvature estimate has to be iterated and may become worse during iteration but holds for a positive time. The results of Section 5 are proved for smooth anisotropies. In Section 6 we pass to the limit in the above iterations and in the anisotropies to obtain a local in time existence theorem for ϕ-regular evolutions of initial compact convex sets for general anisotropies ϕ. We also give a lower bound for the existence time T and show that as $t \to T$, the $r\mathcal{W}_\phi$-condition is lost, which can be interpreted as the explosion of the ϕ-mean curvature. In Section 7 we prove an estimate for the rate of decrease of the volume of the evolving convex set. In Section 8 we prove a comparison principle (hence uniqueness) for our flow of convex sets. We also prove the stability of the extinction times of the flows under convex approximations of the initial convex set in the Hausdorff distance.

2 Notation

Given an open set $A \subseteq \mathbb{R}^N$ and a function $f : A \to \mathbb{R}$, we write $f \in \mathcal{C}^{1,1}(A)$ (resp. $f \in \mathcal{C}^{1,1}_{\rm loc}(A)$) if $f \in \mathcal{C}^1(A)$ and $\nabla f \in {\rm Lip}(A;\mathbb{R}^N)$ (resp. $f \in \mathcal{C}^1_{\rm loc}(A)$ and $\nabla f \in {\rm Lip}_{\rm loc}(A;\mathbb{R}^N)$). Let $B \subset \mathbb{R}^N$ be a set; we say that B (or that ∂B) is of class $\mathcal{C}^{1,1}$ (resp. Lipschitz) if ∂B can be written, locally around each point, as the graph (with respect to a suitable orthogonal coordinate system) of a function f of $(N-1)$ variables, of class $\mathcal{C}^{1,1}$ (resp. Lipschitz), and B can be written (locally) as the epigraph of f.

Given two nonempty sets A, B we denote by $d_{\mathcal{H}}(A,B)$ the Hausdorff distance between A and B. 1_A stands for the characteristic function of A. $\overline{A}$ (resp. ${\rm int}(A)$) is the closure (resp. the interior part) of A.

We let $S^{N-1} := \{\xi \in \mathbb{R}^N : |\xi| = 1\}$ and for $\rho > 0$ we let $B_\rho := \{x \in \mathbb{R}^N : |x| < \rho\}$.

We denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure in $\mathbb{R}^N$, and by $|\cdot|$ the Lebesgue measure. Given a function f defined on the boundary ∂C of a convex set C, we set $\|f\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C)}$ to be the $\mathcal{H}^{N-1}$-essential supremum of f on ∂C.

If $a, b \in \mathbb{R}$, we let $a \wedge b := \min(a,b)$ and $a \vee b := \max(a,b)$. If $\xi, \eta \in \mathbb{R}^N$, by $\xi \otimes \eta$ we indicate the $(N \times N)$-matrix whose ij-entry is $\xi_i \eta_j$, and by $\xi \cdot \eta$ we denote the canonical scalar product between ξ and η. If M_1 and M_2 are two symmetric $(N \times N)$ matrices, by $M_1 \preccurlyeq M_2$ we mean that $M_1 - M_2$ is nonpositive definite. If M is a $(N \times N)$-matrix, $M = (m_{ij})$, we set $|M|^2 := \sum_{i,j}(m_{ij})^2$.

Remark 2.1. Observe that if M_1, M_2 are two nonnegative definite symmetric $(N \times N)$-matrices and $M_1 \leq M_2$, then $|M_1| \leq |M_2|$. Indeed, $\sqrt{M_2}$ is still a nonnegative definite symmetric matrix, and one can check that $\sqrt{M_2} M_1 \sqrt{M_2} \leq M_2 M_2$. Taking the trace we deduce ${\rm tr}(M_1 M_2) = {\rm tr}(\sqrt{M_2} M_1 \sqrt{M_2}) \leq |M_2|^2$. Similarly ${\rm tr}(M_1 M_2) = {\rm tr}(\sqrt{M_1} M_2 \sqrt{M_1}) \geq |M_1|^2$. Hence $|M_1| \leq |M_2|$. Furthermore, if M_3 is another $(N \times N)$-matrix, then $|M_1 M_3| \leq |M_2 M_3|$.

In the following div (resp. ∇) will always indicate the divergence (resp. the gradient, which is understood as a row vector) with respect to the space variables.

Finally, we will always identify a set with its Lebesgue equivalence class.

2.1 Anisotropies and distance functions

Let $\phi : \mathbb{R}^N \to [0, \infty)$ be a positively one-homogeneous convex function on $\mathbb{R}^N$ satisfying

$$m|\xi| \le \phi(\xi) \qquad \forall \xi \in \mathbb{R}^N, \tag{2.1}$$

for some $m > 0$. Observe that there exists $M \in [m, +\infty)$ such that $\phi(\xi) \le M|\xi|$ for all $\xi \in \mathbb{R}^N$. We let $\mathcal{W}_\phi := \{\phi \le 1\}$. The dual function ϕ° of ϕ (called surface tension) is defined as $\phi^o(\xi) := \sup\{\eta \cdot \xi : \phi(\eta) \le 1\}$ for any $\xi \in \mathbb{R}^N$, and turns out to be a positively one-homogeneous function satisfying (2.1) and $(\phi^\circ)^\circ = \phi$.

By a convex body we mean a compact convex set whose interior contains the origin. If K is a convex body, the function $h_K(\xi) := \sup_{\eta \in K} \eta \cdot \xi$ is called the support function of K; notice that $\{(h_K)^o \le 1\} = K$.

In the sequel of the paper, the function ϕ will always denote an anisotropy, i.e., a function ϕ satisfying (2.1) and

$$\phi(t\xi) = |t|\phi(\xi) \qquad \forall \xi \in \mathbb{R}^N, \ \forall t \in \mathbb{R}. \tag{2.2}$$

In particular $\phi(\xi) = \phi(-\xi)$ for any $\xi \in \mathbb{R}^N$. If ϕ is an anisotropy, then ϕ^o is an anisotropy. A convex body is said to be centrally symmetric if it is symmetric with respect to the origin. If ϕ is an anisotropy, then $\mathcal{W}_\phi$ (sometimes called Wulff shape) is a centrally symmetric convex body.

As usual, we shall denote by $\partial\phi(\xi)$ the subdifferential of ϕ at $\xi \in \mathbb{R}^N$. If ϕ is differentiable at ξ, we write $\nabla\phi(\xi)$ in place of $\partial\phi(\xi)$. If Φ is a convex function defined on a Hilbert space, we still denote by $\partial\Phi$ the subdifferential of Φ.

Given a nonempty set $E \subseteq \mathbb{R}^N$, we let

$$d_\phi(x, E) := \inf_{y \in E} \phi(x - y), \qquad x \in \mathbb{R}^N.$$

We denote by d_ϕ^E the signed ϕ–distance function to ∂E negative inside E, that is

$$d_\phi^E(x) := d_\phi(x, E) - d_\phi(x, \mathbb{R}^N \setminus E), \qquad x \in \mathbb{R}^N. \tag{2.3}$$

Observe that $|d_\phi^E(x)| = d_\phi(x, \partial E)$.

The function d_ϕ^E is Lipschitz and at each point x where it is differentiable we have $\phi^\circ(\nabla d_\phi^E(x)) = 1$. We set

$$\nu_\phi^E := \nabla d_\phi^E \qquad \text{on } \partial E, \tag{2.4}$$

at those points where ∇d_ϕ^E exists. When ϕ is the euclidean norm, i.e., $\phi(\xi) = |\xi|$, we set $\nu^E = \nu_{|\cdot|}^E$ and $B_1 = \mathcal{W}_{|\cdot|}$. Vector fields which are selections in $\partial\phi^\circ(\nabla d_\phi^E)$ are sometimes called Cahn-Hoffman vector fields.

Observe that the signed ϕ-distance d_ϕ^C from a compact set C is convex if and only if C is convex.

For $A, B \subseteq \mathbb{R}^N$ we let $d_\phi(A, B) = \inf\{\phi(x - y) : x \in A, y \in B\}$ the ϕ-distance between A and B.

Definition 2.2. *We say that $\phi \in \mathcal{C}_+^{1,1}$ (resp., $\mathcal{C}_+^\infty$) if ϕ^2 is of class $\mathcal{C}^{1,1}(\mathbb{R}^N)$ (resp., $\mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$) and there exists a constant $c > 0$ such that $\nabla^2(\phi^2) \ge c\,\mathrm{Id}$ almost everywhere (resp. in $\mathbb{R}^N \setminus \{0\}$). We say that a centrally symmetric convex body is of class $\mathcal{C}_+^{1,1}$ (resp. $\mathcal{C}_+^\infty$) if it is the unit ball of an anisotropy of class $\mathcal{C}_+^{1,1}$ (resp., $\mathcal{C}_+^\infty$).*

Definition 2.3. *We say that ϕ is crystalline if the unit ball $\mathcal{W}_\phi$ of ϕ is a polytope.*

Remark 2.4. Observe that

(a) $\phi \in \mathcal{C}^{1,1}_+$ (resp., $\mathcal{C}^\infty_+$) if and only if $\phi^\circ \in \mathcal{C}^{1,1}_+$ (resp., $\mathcal{C}^\infty_+$) [50, p. 111];

(b) ϕ is cystalline if and only if ϕ° is crystalline;

(c) if $\phi \in \mathcal{C}^{1,1}_+$, then there exist $0 < \lambda \le \Lambda < +\infty$ such that

$$\lambda \mathrm{Id} \preccurlyeq \phi^\circ(\xi)\nabla^2\phi^\circ(\xi) + \nabla\phi^\circ(\xi) \otimes \nabla\phi^\circ(\xi) \preccurlyeq \Lambda \mathrm{Id}, \qquad \text{a.e. } \xi \in \mathbb{R}^N. \tag{2.5}$$

Finally, we recall that a convex function on $\mathbb{R}^N$ is locally Lipschitz.

2.2 BV functions, ϕ-total variation and generalized Green formula

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation $|Du|(\Omega)$ in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. We denote by $BV_{\mathrm{loc}}(\Omega)$ the space of functions $w \in L^1_{\mathrm{loc}}(\Omega)$ such that $w\varphi \in BV(\Omega)$ for all $\varphi \in \mathcal{C}^\infty_c(\Omega)$. Concerning all properties and notation relatively to functions of bounded variation we will follow [8].

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in Ω if $|D1_E|(\Omega) < \infty$. The perimeter of E in Ω is defined as $P(E,\Omega) := |D1_E|(\Omega)$, and $P(E,\Omega) = P(\mathbb{R}^N \setminus E, \Omega)$. We shall use the notation $P(E) := P(E, \mathbb{R}^N)$.

Let $u \in BV(\Omega)$. We define the anisotropic total variation of u with respect to ϕ in Ω [5] as

$$\int_\Omega \phi^\circ(Du) = \sup\left\{ \int_\Omega u \,\mathrm{div}\,\sigma \; dx : \sigma \in \mathcal{C}^1_c(\Omega;\mathbb{R}^N), \phi(\sigma) \le 1 \right\}. \tag{2.6}$$

If $E \subseteq \mathbb{R}^N$ has finite perimeter in Ω, we set

$$P_\phi(E,\Omega) := \int_\Omega \phi^\circ(D1_E)$$

and we have [5]

$$P_\phi(E,\Omega) = \int_{\Omega\cap\partial^* E} \phi^\circ(\nu^E)\, d\mathcal{H}^{N-1}. \tag{2.7}$$

where $\partial^* E$ is the reduced boundary of E and ν^E the (generalized) outer unit normal to E at points of $\partial^* E$. We sometimes use the notation $d\mathcal{P}_\phi$ to indicate the density of the measure in (2.7), i.e., if $B \subseteq \mathbb{R}^N$ is a Borel set, we let

$$d\mathcal{P}_\phi(B) := \int_{B\cap\partial^* E} \phi^o(\nu^E) \; d\mathcal{H}^{N-1}.$$

Recall that, since ϕ° is homogeneous, $\phi^\circ(Du)$ coincides with the nonnegative Radon measure in $\mathbb{R}^N$ given by $\phi^\circ(Du) = \phi^\circ(\nabla u(x))\, dx + \phi^\circ\left(\frac{D^s u}{|D^s u|}\right) |D^s u|$, where $\nabla u(x)\, dx$ is the absolutely continuous part of Du, and $D^s u$ its singular part.

2.3 ϕ-regularity and Lipschitz ϕ-regularity. The $r\mathcal{W}_\phi$-condition

Following [18, 16, 17, 14] we define the class of ϕ-regular sets and Lipschitz ϕ-regular sets (these latter are a generalization of sets of class $\mathcal{C}^{1,1}$).

Definition 2.5. *Let $E \subset \mathbb{R}^N$ be a set. We say that E is ϕ-regular if ∂E is a compact Lipschitz hypersurface and there exist an open set $U \supset \partial E$ and a vector field $n \in L^\infty(U;\mathbb{R}^N)$ such that $\mathrm{div}\, n \in L^\infty(U)$, and $n \in \partial\phi^\circ(\nabla d_\phi^E)$ almost everywhere in U. We say that E is Lipschitz ϕ-regular if E is ϕ-regular and $n \in \mathrm{Lip}(U;\mathbb{R}^N)$.*

It is clear that a Lipschitz ϕ-regular set is ϕ-regular. With a little abuse of notation, sometimes we will denote by (E, n), or by (E, U) or by (E, U, n), a ϕ-regular set.

Observe that, in general, vector fields n are not unique, unless $\phi \in \mathcal{C}_+^{1,1}$. When $\phi \in \mathcal{C}_+^{1,1}$ the inclusion $n \in \partial\phi^\circ(\nabla d_\phi^E)$ becomes an equality; in this respect we give the following definition.

Definition 2.6. *Let $\phi \in \mathcal{C}_+^{1,1}$ and (E, U) be a Lipschitz ϕ-regular set. Let $x \in U$ be a point where there exists $\nabla d_\phi^E(x)$. We set*

$$n_\phi^E(x) := \nabla\phi^\circ(\nabla d_\phi^E(x)). \tag{2.8}$$

Remark 2.7. Observe that $(\mathcal{W}_\phi, n)$, with $n(x) := x/\phi(x)$, is Lipschitz ϕ-regular, and $\mathrm{div}\, n(x) = N - 1$ for $\mathcal{H}^{N-1}$ almost every $x \in \partial\mathcal{W}_\phi$.

The next definition will play an important rôle in the sequel.

Definition 2.8. *Let $E \subset \mathbb{R}^N$ be a set with nonempty interior and $r > 0$. We say that E satisfies the $r\mathcal{W}_\phi$-condition if, for any $x \in \partial E$, there exists $y \in \mathbb{R}^d$ such that*

$$r\mathcal{W}_\phi + y \subseteq \overline{E} \qquad \text{and} \qquad x \in \partial\left(r\mathcal{W}_\phi + y\right).$$

The following result is proved in [14, Lemmata 3.4, 3.5].

Lemma 2.9. *Let E be a Lipschitz ϕ-regular set. Then E and $\mathbb{R}^N \setminus E$ satisfy the $r\mathcal{W}_\phi$-condition for some $r > 0$.*

In Proposition 3.9 below we will prove that a convex set satisfying the $r\mathcal{W}_\phi$-condition is ϕ-regular when ϕ is an arbitrary anisotropy. On the other hand, if $\phi \in \mathcal{C}_+^{1,1}$, the relations between ϕ-regularity and Definition 2.8 are listed in the next observation.

Remark 2.10. Assume that $\phi \in \mathcal{C}_+^{1,1}$. The following assertions hold.

(a) E is Lipschitz ϕ-regular if and only if E is of class $\mathcal{C}^{1,1}$.

(b) Let C be a compact convex which satisfies the $r\mathcal{W}_\phi$-condition for some $r > 0$. Then C is Lipschitz ϕ-regular (hence C is of class $\mathcal{C}^{1,1}$ by (a)). Let us briefly comment on the proof. Observe that there exists $C' \subset C$ such that $C = C' + r\mathcal{W}_\phi$. Then $C_t := \{d_\phi^C \le t\} = C' + (r+t)\mathcal{W}_\phi$ for $t \in (-r, r)$. Thus C_t satisfies the $(r+t)\mathcal{W}_\phi$-condition. Since $\phi \in \mathcal{C}_+^{1,1}$, C_t also satisfies the $r'(r+t)B_1$-condition for some $r' > 0$, hence the mean curvature of C_t is bounded. Thus $\Delta d_{|\cdot|}^C \in L^\infty_{\mathrm{loc}}(|d_\phi^C| < r)$. Since $d_{|\cdot|}^C$ is a convex function, we obtain that $d_{|\cdot|}^C \in W^{2,\infty}_{\mathrm{loc}}(|d_\phi^C| < r)$. Then the result follows since we may take $n = \nabla\phi^\circ(\frac{\nabla d_\phi^C}{\phi^\circ(\nabla d_\phi^C)})$ in $\{|d_\phi^C| < r\}$ (in particular, ∂C is $\mathcal{C}^{1,1}$ [27, Theorem 5.5]).

(c) E is Lipschitz ϕ-regular if and only if E and $\mathbb{R}^N \setminus E$ satisfy the $r\mathcal{W}_\phi$-condition for some $r > 0$ (in this case, E and $\mathbb{R}^N \setminus E$ satisfy also the $r' B_1$-condition for some $r' > 0$, hence E is of class $\mathcal{C}^{1,1}$ by standard regularity results [28]).

(d) Let E be Lipschitz ϕ-regular. Then [14, Lemmata 3.4, 3.5, 4.5]

  (d1) there exists a neighbourhood V of ∂E, depending on the Lipschitz norm of n_ϕ^E, such that $d_\phi^E \in \mathcal{C}^{1,1}(V)$;

  (d2) there exists a uniquely defined projection map $\pi_\phi^E : V \to \partial E$, $\pi_\phi^E(x) := x - d_\phi^E(x) n_\phi^E(x)$, which satisfies $n_\phi^E = n_\phi^E(\pi_\phi^E)$ in V. Moreover $\nabla n_\phi^E n_\phi^E = 0$ and $\nabla^2 d_\phi^E n_\phi^E = 0$ almost everywhere in V;

  (d3) the trace of $\operatorname{div} n_\phi^E$ (still denoted by $\operatorname{div} n_\phi^E$) is defined $\mathcal{H}^{N-1}$-almost everywhere on ∂E and coincides on ∂E with the tangential divergence of n_ϕ^E.

As already observed in the introduction, our existence and uniqueness theorem will be, roughly speaking, in the class of flows $t \to E(t)$ such that each $E(t)$ is ϕ-regular. In general we do not know under which assumptions a ϕ-regular set is also Lipschitz ϕ-regular. Proving an existence and uniqueness result for crystalline mean curvature flow in the class of Lipschitz ϕ-regular sets is an open problem.

Definition 2.11. *Let $\phi \in \mathcal{C}_+^{1,1}$ and E be a Lipschitz ϕ-regular set. We define*

$$\kappa_\phi^E := \operatorname{div} n_\phi^E \qquad \mathcal{H}^{N-1} - \text{a.e. on } \partial E. \tag{2.9}$$

2.4 ϕ-regular flows

The following definition is essentially the one given in [18, Definition 2.2].

Definition 2.12. *Let $T > 0$. A ϕ-regular flow in $[0, T)$ is a map $t \in [0, T) \to E(t)$ satisfying the following properties:*

(i) $\partial E(t)$ is a compact Lipschitz hypersurface;

(ii) there exists an open set $A \subseteq \mathbb{R}^N \times [0, T)$ such that $\bigcup_{t \in [0,T)} \partial E(t) \times \{t\} \subseteq A$, and the function $d(x, t) := d_\phi^{E(t)}(x)$ is locally Lipschitz in A;

(iii) there exists a vector field $n \in L^\infty(A; \mathbb{R}^N)$ such that $\operatorname{div} n \in L^\infty(A)$ and $n \in \partial\phi^\circ(\nabla d)$ almost everywhere in A;

(iv) for any $\bar{t} < T$, there exists $c = c(\bar{t}) > 0$ with $|\partial_t d(x, t) - \operatorname{div} n(x, t)| \le c|d(x, t)|$ for almost any $(t, x) \in A$ with $t \le \bar{t}$.

With a little abuse of notation, sometimes we will denote by $(E(t), n)$, or by $(E(t), A, n)$, a ϕ-regular flow.

The following result, proved in [18], provides uniqueness of a ϕ-regular flow.

Theorem 2.13. *Let $E_1(t)$, $E_2(t)$ be two ϕ-regular flows in $[0, T)$. Then*

$$E_1(0) \subseteq E_2(0) \Rightarrow E_1(t) \subseteq E_2(t), \qquad t \in [0, T). \tag{2.10}$$

In particular if $E_1(0) = E_2(0)$ then $E_1(t) = E_2(t)$ for any $t \in [0, T)$, i.e. the ϕ-regular flow starting from a compact set E is unique.

Remark 2.14. The self-similar evolution of the Wulff shape is a ϕ-regular flow.

3 On convex ϕ-regular sets

In this section we want to analyze the relations between ϕ-regularity and the $r\mathcal{W}_\phi$-condition for a convex set, with no restrictions on the anisotropy (see Proposition 3.9 below). We will first consider the case of a smooth anisotropy, the general case will be studied by means of a suitable approximation argument.

3.1 The case $\phi \in \mathcal{C}^\infty_+$

In this subsection we will assume that $\phi \in \mathcal{C}^\infty_+$ (equivalently, $\phi^\circ \in \mathcal{C}^\infty_+$, *cfr.* (a) of Remark 2.4), however, all results will still be true if $\phi \in \mathcal{C}^{1,1}_+$, see Remark 3.5 below.

The following theorem shows that, for a convex set C of class $\mathcal{C}^{1,1}$, the maximal neighbourhood where we have smoothness of the ϕ-signed distance d^C_ϕ is controlled by the sup-norm of the divergence of the Cahn-Hoffman vector field n^C_ϕ (defined in (2.8)) on the boundary of C (see (d3) of Remark 2.10). In addition, on such a neighbourhood we have an expansion of $\operatorname{div} n^C_\phi$.

Theorem 3.1. *Let $\phi \in \mathcal{C}^\infty_+$, and let C be a compact convex set which satisfies the $r\mathcal{W}_\phi$-condition for some $r > 0$. Then, setting*

$$\mathcal{K} := \|\kappa^C_\phi\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C)},$$

we have

$$d^C_\phi \in \mathcal{C}^{1,1}_{\rm loc}\left(\{|d^C_\phi| < \mathcal{K}^{-1}\}\right). \tag{3.1}$$

Moreover, π^C_ϕ is well defined on $\{|d^C_\phi| < \mathcal{K}^{-1}\}$, and

$$0 \le \operatorname{div} n^C_\phi \le \frac{\kappa^C_\phi(\pi^C_\phi)}{1 - |d^C_\phi|\kappa^C_\phi(\pi^C_\phi)} \qquad \text{a.e. in } \left\{|d^C_\phi| < \mathcal{K}^{-1}\right\}. \tag{3.2}$$

Proof. Given $t \in \mathbb{R}$ let

$$U_t := \{|d^C_\phi| < t\}.$$

By (b) and (d) of Remark 2.10 it follows that C is of class $\mathcal{C}^{1,1}$, hence it is Lipschitz ϕ-regular, and there exist $\epsilon_0 > 0$ such that $d^C_\phi \in \mathcal{C}^{1,1}(U_{\epsilon_0})$ and a projection map $\pi^C_\phi : U_{\epsilon_0} \to \partial C$, $\pi^C_\phi(x) := x - d^C_\phi(x) n^C_\phi(x)$, which satisfy $n^C_\phi = n^C_\phi(\pi^C_\phi)$ in U_{ϵ_0}. In addition κ^C_ϕ is well defined by (d3) of Remark 2.10. Possibly reducing ϵ_0, we can assume that

$$\left|d^C_\phi \nabla n^C_\phi(\pi^C_\phi)\right| < 1 \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.3}$$

Moreover, let us observe that ϵ_0 only depends on λ, Λ and $\|\kappa^C_\phi\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C)}$, see [14].

We divide the proof into five steps.

Step 1. We have

$$|\nabla n^C_\phi| \le \frac{\Lambda}{\lambda} \operatorname{div} n^C_\phi \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.4}$$

Indeed, from $n^C_\phi = \nabla\phi^\circ(\nabla d^C_\phi)$, we get

$$\nabla n^C_\phi = \nabla^2\phi^\circ(\nabla d^C_\phi)\nabla^2 d^C_\phi \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.5}$$

Therefore, from the convexity of d_ϕ^C it follows that for almost every $x \in U_{\epsilon_0}$ the matrix $\nabla n_\phi^C(x)$ is a product of two symmetric nonnegative definite matrices.

Hence, recalling that $\phi^\circ(\nabla d_\phi^C) = 1$, using (2.5), the fact that $0 \preccurlyeq \nabla\phi^\circ(\xi) \otimes \nabla\phi^\circ(\xi)$, and Remark 2.1 (applied with $M_1 = \nabla^2\phi^\circ(\nabla d_\phi^C)(x))$ and $M_3 = \nabla^2 d_\phi^C(x))$ we get

$$|\nabla n_\phi^C| = |\nabla^2\phi^\circ(\nabla d_\phi^C)\nabla^2 d_\phi^C| \leq \Lambda|\nabla^2 d_\phi^C| \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.6}$$

Letting $\{e_1, \ldots, e_N\}$ be the canonical basis of $\mathbb{R}^N$, using (2.5) and the relation $\nabla^2 d_\phi^C n_\phi^C = 0$ valid almost everywhere in U_{ϵ_0}, it follows

$$\begin{aligned} \operatorname{div} n_\phi^C &= \operatorname{tr}\left(\nabla^2\phi^\circ(\nabla d_\phi^C)\nabla^2 d_\phi^C\right) = \operatorname{tr}\left(\sqrt{\nabla^2 d_\phi^C}\,\nabla^2\phi^\circ(\nabla d_\phi^C)\sqrt{\nabla^2 d_\phi^C}\right) \\ &= \sum_{i=1}^N \nabla^2\phi^\circ(\nabla d_\phi^C)\sqrt{\nabla^2 d_\phi^C}\; e_i \cdot \sqrt{\nabla^2 d_\phi^C} e_i \geq \lambda \sum_{i=1}^N \nabla^2 d_\phi^C\; e_i \cdot e_i \\ &= \lambda \operatorname{tr}(\nabla^2 d_\phi^C) \qquad \text{a.e. in } U_{\epsilon_0}. \end{aligned} \tag{3.7}$$

Since the convexity of the function d_ϕ^C implies $\operatorname{tr}\left(\nabla^2 d_\phi^C\right) \geq |\nabla^2 d_\phi^C|$, from (3.6) and (3.7) we then get

$$\operatorname{div} n_\phi^C \geq \lambda \operatorname{tr}\left(\nabla^2 d_\phi^C\right) \geq \lambda|\nabla^2 d_\phi^C| \geq \frac{\lambda}{\Lambda}|\nabla n_\phi^C| \qquad \text{a.e. in } U_{\epsilon_0},$$

which is (3.4).

Step 2. We have

$$\nabla n_\phi^C = \nabla n_\phi^C(\pi_\phi^C)\left(\mathrm{Id} + d_\phi^C \nabla n_\phi^C(\pi_\phi^C)\right)^{-1} \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.8}$$

Differentiating the equality $\pi_\phi^C(x) = x - d_\phi^C(x) n_\phi^C(\pi_\phi^C(x))$ we get

$$\nabla\pi_\phi^C = \left(\mathrm{Id} + d_\phi^C \nabla n_\phi^C(\pi_\phi^C)\right)^{-1}\left[\mathrm{Id} - n_\phi^C(\pi_\phi^C) \otimes \nabla d_\phi^C\right] \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.9}$$

At almost every point $x \in U_{\epsilon_0}$ assumption (3.3) ensures

$$\left(\mathrm{Id} + d_\phi^C(x)\nabla n_\phi^C(\pi_\phi^C(x))\right)^{-1} = \sum_{k=0}^\infty (-d_\phi^C(x))^k (\nabla n_\phi^C(\pi_\phi^C(x)))^k. \tag{3.10}$$

Therefore, from (3.9) and $\nabla n_\phi^C(\pi_\phi^C) n_\phi^C(\pi_\phi^C) = 0$ we deduce

$$\nabla\pi_\phi^C = (\mathrm{Id} + d_\phi^C \nabla n_\phi^C(\pi_\phi^C))^{-1} - n_\phi^C(\pi_\phi^C) \otimes \nabla d_\phi^C \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.11}$$

Assertion (3.8) then follows from (3.11) by differentiating the relation $n_\phi^C(x) = n_\phi^C(\pi_\phi^C(x))$, and using once more $\nabla n_\phi^C(\pi_\phi^C) n_\phi^C(\pi_\phi^C) = 0$.

Step 3. For any $k \in \mathbb{N}$ we have

$$0 \leq \operatorname{tr}\left((\nabla n_\phi^C)^k\right) \leq \left(\operatorname{div} n_\phi^C\right)^k \qquad \text{a.e. in } U_{\epsilon_0}. \tag{3.12}$$

Write $A := \nabla^2\phi^\circ(\nabla d_\phi^C)$ and $B := \nabla^2 d_\phi^C$; in view of (3.5) we have to prove that

$$0 \leq \operatorname{tr}\left((AB)^k\right) \leq (\operatorname{tr}(AB))^k, \qquad k \in \mathbb{N}. \tag{3.13}$$

If $D := \sqrt{B}A\sqrt{B}$, then D is symmetric and nonnegative definite. Hence

$$0 \le \mathrm{tr}(D^k) \le (\mathrm{tr}(D))^k.$$

On the other hand $\mathrm{tr}\left((AB)^k\right) = \mathrm{tr}(D^k)$ for any $k \in \mathbb{N}$, and (3.13) follows.

Step 4. We have

$$0 \le \mathrm{div}\, n_\phi^C \le \frac{\kappa_\phi^C(\pi_\phi^C)}{1 - |d_\phi^C|\kappa_\phi^C(\pi_\phi^C)} \qquad \text{a.e. in } U_{\epsilon_0 \wedge \mathcal{K}^{-1}}. \tag{3.14}$$

Using (3.8) and (3.10) we get

$$\nabla n_\phi^C = \nabla n_\phi^C(\pi_\phi^C)\left(\mathrm{Id} + d_\phi^C \nabla n_\phi^C(\pi_\phi^C)\right)^{-1} = \sum_{k=0}^{\infty} (-d_\phi^C)^k (\nabla n_\phi^C(\pi_\phi^C))^{k+1} \qquad \text{a.e. in } U_{\epsilon_0 \wedge \mathcal{K}^{-1}}. \tag{3.15}$$

Then (3.14) follows by taking the trace of both sides in (3.15), passing to the absolute values and by using the definition of $\mathcal{K}$.

Observe that if $\epsilon_0 \ge \mathcal{K}^{-1}$ the proof is concluded. Therefore we can assume $\epsilon_0 < \mathcal{K}^{-1}$. From (3.14) it follows

$$\sup_{U_{\epsilon_0}} \mathrm{div}\, n_\phi^C \le \frac{\mathcal{K}}{1 - \epsilon_0 \mathcal{K}}. \tag{3.16}$$

Step 5. We have

$$d_\phi^C \in \mathcal{C}^{1,1}(U_t) \quad \text{and} \quad 0 \le \mathrm{div}\, n_\phi^C \le \frac{\kappa_\phi^C(\pi_\phi^C)}{1 - |d_\phi^C|\kappa_\phi^C(\pi_\phi^C)} \qquad \text{in } U_t, \quad t \in (0, \mathcal{K}^{-1}). \tag{3.17}$$

Let us fix $t \in (0, \mathcal{K}^{-1})$ for which (3.17) holds in U_t (for instance, for $t = \epsilon_0$). Observe that, in this case, (3.16) holds in U_t, with t in place of ϵ_0. Given $s \in \mathbb{R}$ let $C_s := \{d_\phi^C < s\}$. We will assume $0 < s < t$. If $\partial C_s \subset U_t$, we have $d_\phi^{C_s} = d_\phi^C - s \in \mathcal{C}^{1,1}(U_t)$, and $n_\phi^{C_s} = n_\phi^C$ in U_t. Observe that

$$\|\mathrm{div}\, n_\phi^{C_s}\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C_s)} \le \mathcal{K}/(1 - t\mathcal{K}).$$

Applying (d) of Remark 2.10, using *step 1*, and (3.16) with C_s in place of C, we obtain that there exists a constant $\epsilon > 0$ depending on λ, Λ and $\mathcal{K}/(1 - t\mathcal{K})$ such that

$$d_\phi^{C_s} \in \mathcal{C}^{1,1}(\{|d_\phi^{C_s}| < \epsilon\}) \quad \text{and} \quad 0 \le \kappa_\phi^{C_s} \le \frac{\kappa_\phi^{C_s}(\pi_\phi^s)}{1 - |d_\phi^{C_s}|\mathrm{div}\, n_\phi^{C_s}(\pi_\phi^s)} \qquad \text{a.e. in } \left\{|d_\phi^{C_s}| < \epsilon\right\}. \tag{3.18}$$

We want to prove that (3.17) holds in $U_t \cup \{|d_\phi^{C_s}| < \epsilon\}$. Thanks to (3.18), it is enough to prove (3.2) in $\{|d_\phi^{C_s}| < \epsilon\} \setminus U_t$. Observe that if $\xi, \eta \in \mathbb{R}$, $\xi \le \eta$, and if $\alpha \in \mathbb{R}$, then $\frac{\xi}{1+\alpha\xi} \le \frac{\eta}{1+\alpha\eta}$. Choosing $\xi = \mathrm{div}\, n_\phi^C(\pi_\phi^{C_s})$, $\eta = \frac{\mathrm{div}\, n_\phi^C(\pi_\phi^C)}{1 - |d_\phi^C(\pi_\phi^{C_s})|\mathrm{div}\, n_\phi^C(\pi_\phi^C)}$ and $\alpha = -|d_\phi^C - s|$, using $n_\phi^C = n_\phi^{C_s}$, $d_\phi^C(\pi_\phi^{C_s}) = s$, (3.18) and (3.17) on ∂C_s, we obtain

$$\begin{aligned}
0 \quad &\le \quad \mathrm{div}\, n_\phi^C \le \frac{\mathrm{div}\, n_\phi^C(\pi_\phi^{C_s})}{1 - |d_\phi^C - s|\mathrm{div}\, n_\phi^C(\pi_\phi^{C_s})} \\
&\le \quad \frac{\mathrm{div}\, n_\phi^C(\pi_\phi^C)}{1 - |d_\phi^C(\pi_\phi^{C_s})|\mathrm{div}\, n_\phi^C(\pi_\phi^C)} \, \frac{1 - |d_\phi^C(\pi_\phi^{C_s})|\mathrm{div}\, n_\phi^C(\pi_\phi^C)}{1 - |d_\phi^C - s|\mathrm{div}\, n_\phi^C(\pi_\phi^C) - |d_\phi^C(\pi_\phi^{C_s})|\mathrm{div}\, n_\phi^C(\pi_\phi^C)} \\
&= \quad \frac{\mathrm{div}\, n_\phi^C(\pi_\phi^C)}{1 - |d_\phi^C|\mathrm{div}\, n_\phi^C(\pi_\phi^C)}.
\end{aligned}$$

Arguing in the same way for the case $-t < s < 0$, we deduce that (3.17) hold in $\{|d_\phi^C| < |t| + \epsilon\}$.

Let now

$$t^* := \sup\left\{t \in (0, \mathcal{K}^{-1}) : (3.17) \text{ holds in } U_t\right\}.$$

Since (3.2) holds in U_{t^*} and, by the previous proof, it also holds in $U_{t^*+\epsilon}$ for some $\epsilon > 0$ as soon as $t^* < \mathcal{K}^{-1}$, we deduce that $t^* = \mathcal{K}^{-1}$. □

Proposition 3.2. *Let $\phi \in \mathcal{C}_+^\infty$. Assume that C satisfies the $r\mathcal{W}_\phi$-condition for some $r > 0$. Then $d_\phi^C \in \mathcal{C}^{1,1}_{\mathrm{loc}}(\{|d_\phi^C| < r\})$ and*

$$0 \le \operatorname{div} n_\phi^C \le \frac{N-1}{d_\phi^C + r} \qquad \text{a.e. in } \{|d_\phi^C| < r\}. \tag{3.19}$$

Proof. From Theorem 3.1 we have that $d_\phi^C \in \mathcal{C}^{1,1}_{\mathrm{loc}}(\{|d_\phi^C| < \mathcal{K}^{-1}\})$. For any $x \in \partial C$, let $W_r(y) := r\mathcal{W}_\phi + y \subseteq C$ be such that $x \in \partial W_r(y)$. Notice that

$$d_\phi^C(z) \le d_\phi^{W_r(y)}(z) = \phi(z-y) - r, \qquad z \in \mathbb{R}^N,$$

and

$$d_\phi^C(x + tn_\phi^C(x)) = d_\phi^{W_r(y)}(x + tn_\phi^C(x)), \qquad x \in \partial C,\ |t| < r \wedge \mathcal{K}^{-1}.$$

Therefore, observing that $n_\phi^C(x) = n_\phi^{W_r(y)}(x)$, using Remark 2.7 we deduce

$$\operatorname{div} n_\phi^C(x + tn_\phi^C(x)) \le \operatorname{div} n_\phi^{W_r(y)}(x + tn_\phi^C(x)) = \frac{N-1}{r+t} \tag{3.20}$$

for $\mathcal{H}^{N-1}$-almost every $x \in \partial C$ and for $|t| < r \wedge \mathcal{K}^{-1}$. Therefore the inequalities in (3.19) hold in the set $\{|d_\phi^C| < r \wedge \mathcal{K}^{-1}\}$. Iterating this argument similarly as in *Step 5* of the proof of Theorem 3.1, assertion (3.19) follows. □

The following result specifies for which $r > 0$ a compact convex ϕ-regular set satisfies the $r\mathcal{W}_\phi$-condition (compare with Lemma 2.9).

Corollary 3.3. *Let $\phi \in \mathcal{C}_+^\infty$, and let C be a compact convex set of class $\mathcal{C}^{1,1}$. Let $r > 0$ be such that $n \in L^\infty(\{|d_\phi^C| < r\}; \mathbb{R}^N)$ and $\operatorname{div} n \in L^\infty(\{|d_\phi^C| < r\})$, where $n := \partial\phi^\circ(\nabla d_\phi^E)$ (see (a) of Remark 2.10). Then C satisfies the $r\mathcal{W}_\phi$-condition.*

Proof. Reasoning as in *step 5* of the proof of Theorem 3.1, we have $d_\phi^C \in \mathcal{C}^{1,1}_{\mathrm{loc}}(\{|d_\phi^C| < r\})$. Therefore for any $x \in \{|d_\phi^C| < r\}$ there exists a unique point in ∂C minimizing the ϕ-distance from ∂C, which gives the assertion. □

Corollary 3.4. *Let $\phi \in \mathcal{C}_+^\infty$, and let C be a compact convex set satisfying the $r\mathcal{W}_\phi$-condition for some $r > 0$. Let $\mathcal{K} := \|\operatorname{div} n_\phi^C\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C)}$. Then*

$$\mathcal{K}^{-1} \le \sup\{\rho > 0 : C \text{ satisfies the } \rho\mathcal{W}_\phi - \text{condition}\} \le (N-1)\mathcal{K}^{-1}.$$

Proof. It follows from Theorem 3.1 and Proposition 3.2. □

Remark 3.5. Theorem 3.1, Corollary 3.3, and Proposition 3.2 still hold if ϕ belongs to $\mathcal{C}^{1,1}_+$. This can be shown either by using the appropriate chain rule when computing the derivative of the composition of two Lipschitz maps $\nabla\phi^\circ(\nabla d^C_\phi)$ (see [8, p. 193]), or by smoothing in an appropriate way ϕ, ϕ° and C, using the method of Subsection 3.2 below.

The following lemma will be useful in the proof of Theorem 6.1.

Lemma 3.6. *Let $\{\phi_\epsilon\}$ be a sequence of anisotropies uniformly converging to an anisotropy ϕ. Let $\{C_\epsilon\}$ be a sequence of compact convex sets satisfying the $r\mathcal{W}_{\phi_\epsilon}$-condition for some $r > 0$ independent of ϵ. If C is a compact convex set and $\lim_{\epsilon\to 0} d_{\mathcal{H}}(\partial C_\epsilon, \partial C) = 0$, then C satisfies the $r\mathcal{W}_\phi$-condition.*

Proof. Since C_ϵ satisfy the $r\mathcal{W}_{\phi_\epsilon}$-condition, there exist convex sets $C'_\epsilon \subset C_\epsilon$ such that

$$C_\epsilon = C'_\epsilon + r\mathcal{W}_{\phi_\epsilon}, \qquad \epsilon > 0. \tag{3.21}$$

By compactness, and possibly passing to a subsequence, there exists a convex set $C' \subset C$ such that $\lim_{\epsilon\to 0} d_{\mathcal{H}}(C'_\epsilon, C') = 0$. Passing to the limit in (3.21) and using the continuity of the sum in the Hausdorff metric ([50, p. 51]), we deduce $C = C' + r\mathcal{W}_\phi$. ☐

3.2 The case of a generic anisotropy ϕ

The following result is proved in [50, Theorem 3.3.1 and pp. 111, 160].

Theorem 3.7. *Let $\epsilon > 0$ and let $\eta : [0, \infty) \to [0, \infty)$ be a function of class $\mathcal{C}^\infty$ with support in $[\frac{\epsilon}{2}, \epsilon]$ and with $\int_{\mathbb{R}^N} \eta(|x|)\, dx = 1$. If $\phi^\circ : \mathbb{R}^N \to [0, +\infty)$ is an anisotropy, then the function $\widetilde{\phi^\circ}$ defined by*

$$\widetilde{\phi^\circ}(\xi) := \int_{\mathbb{R}^N} \phi^\circ(\xi + |\xi| z)\eta(|z|)\, dz, \qquad \xi \in \mathbb{R}^N \tag{3.22}$$

is an anisotropy of class $\mathcal{C}^\infty(\mathbb{R}^N \setminus \{0\})$.

Similarly, given a convex body K, define the map $K \mapsto \mathcal{T}(K)$ as follows: taking $\widetilde{h}_K(\xi) := \int_{\mathbb{R}^N} h_K(\xi + |\xi|z)\eta(|z|)\, dz$ for any $\xi \in \mathbb{R}^N$, Then $\widetilde{h}_K$ is the support function $h_{\mathcal{T}(K)}$ of $\mathcal{T}(K)$. The map $\mathcal{T}$ has the following properties: if K_1 and K_2 are two convex bodies, then

(a) $\mathcal{T}(K_1 + K_2) = \mathcal{T}(K_1) + \mathcal{T}(K_2)$ and $\mathcal{T}(\alpha K_1) = \alpha\mathcal{T}(K_1)$ for any $\alpha > 0$;

(b) if K_1 is contained in B_R, then $d_{\mathcal{H}}(K_1, \mathcal{T}(K_1)) \le R\epsilon$;

(c) $d_{\mathcal{H}}(\mathcal{T}(K_1), \mathcal{T}(K_2)) \le (1 + \epsilon) d_{\mathcal{H}}(K_1, K_2)$;

(d) $\mathcal{T}(K_1) + B_\epsilon$ is of class $\mathcal{C}^\infty_+$.

Theorem 3.7 provides a way to regularize a generic anisotropy with $\mathcal{C}^\infty_+$ anisotropies and, at the same time, to regularize a convex set with convex sets which are more regular (with respect to the regularized metrics). Indeed the following result holds.

Lemma 3.8. *Let ϕ be an anisotropy, and let C be a compact convex set satisfying the $r\mathcal{W}_\phi$-condition for some $r > 0$. Then there exist a sequence $\{\phi_\epsilon\}$ of anisotropies and a sequence $\{C_\epsilon\}$ of compact convex sets satisfying the following properties:*

(i) $\{\phi_\epsilon\}$ *converges to* ϕ *uniformly on* $\mathbb{R}^N$ *as* $\epsilon \to 0$*;*

(ii) $\{C_\epsilon\}$ *converges to* C *in the Hausdorff distance as* $\epsilon \to 0$*;*

(iii) ϕ_ϵ, $\phi_\epsilon^\circ \in \mathcal{C}^\infty_+$ *and* C_ϵ *is of class* $\mathcal{C}^\infty_+$ *for any* $\epsilon > 0$*;*

(iv) C_ϵ *satisfies the* $r\mathcal{W}_{\phi_\epsilon}$*-condition for any* $\epsilon > 0$.

Proof. Let $\mathcal{T}$ be the map defined in Theorem 3.7. Let ϕ_ϵ be the anisotropy such that $\mathcal{W}_{\phi_\epsilon} = \mathcal{T}(\mathcal{W}_\phi)+B_\epsilon$; then $\phi_\epsilon \in \mathcal{C}^\infty_+$ by (d) of Theorem 3.7, hence also $\phi_\epsilon^\circ \in \mathcal{C}^\infty_+$ by (a) of Remark 2.4. Then (b) of Theorem 3.7 yields (i). Let $C_\epsilon := \mathcal{T}(C)+B_{r\epsilon}$. It is clear that (ii) is satisfied. Since C satisfies the $r\mathcal{W}_\phi$-condition, there exists $C' \subset C$ such that $C = C' + r\mathcal{W}_\phi$. By (a) in Theorem 3.7 we have

$$C_\epsilon = \mathcal{T}(C) + B_{r\epsilon} = \mathcal{T}(C') + r\mathcal{T}(\mathcal{W}_\phi) + B_{r\epsilon} = \mathcal{T}(C') + r\left(\mathcal{T}(\mathcal{W}_\phi) + B_\epsilon\right) = \mathcal{T}(C') + r\mathcal{W}_{\phi_\epsilon}, \tag{3.23}$$

hence (iv) follows. □

Observe that

$$\phi_\epsilon^\circ(\xi) \;=\; \sup_{x\in\mathcal{T}(\mathcal{W}_\phi)+B_\epsilon} x\cdot\xi \;=\; \sup_{y\in\mathcal{T}(\mathcal{W}_\phi)}\ \sup_{z\in B_\epsilon} (y+z)\cdot\xi \;=\; \widetilde{\phi^\circ}(\xi) + \epsilon|\xi|\,.$$

Proposition 3.9. *Let* ϕ *be an anisotropy and let* C *be a compact convex set satisfying the* $r\mathcal{W}_\phi$*-condition for some* $r > 0$*. Then* C *is* ϕ*-regular.*

Proof. For each $\epsilon > 0$ let ϕ_ϵ and C_ϵ be the regularizations of ϕ and C constructed in Lemma 3.8. Let $V_\epsilon := \{|d_{\phi_\epsilon}^{C_\epsilon}| < r/2\}$. Recalling that $\phi_\epsilon \in \mathcal{C}^\infty_+$, $C_\epsilon \in \mathcal{C}^\infty$ and (iv) of Lemma 3.8, by Corollary 3.4 it follows that

$$0 \le \operatorname{div} n_{\phi_\epsilon}^{C_\epsilon} \le \frac{2(N-1)}{r} \qquad \text{a.e. in } V_\epsilon,$$

where we recall that $n_{\phi_\epsilon}^{C_\epsilon} = \nabla\phi_\epsilon^\circ(\nabla d_{\phi_\epsilon}^{C_\epsilon})$. Letting $\epsilon \to 0^+$ and possibly passing to a suitable subsequence, we can assume that $n_{\phi_\epsilon}^{C_\epsilon} \rightharpoonup n$ weakly in $L^2(U;\mathbb{R}^N)$ and $\operatorname{div} n_{\phi_\epsilon}^{C_\epsilon} \rightharpoonup \operatorname{div} n$ weakly in $L^2(U)$, where $U := \{|d_\phi^C| < r/2\}$, $n \in L^\infty(U;\mathbb{R}^N)$ and

$$0 \le \operatorname{div} n \le \frac{2(N-1)}{r} \qquad \text{a.e. in } U.$$

To conclude the proof that (C,U,n) is ϕ-regular, it remains to show that $n \in \partial\phi^\circ(\nabla d_\phi^C)$ almost everywhere in U. We observe that, since $|d_{\phi_\epsilon}^{C_\epsilon} - d_\phi^C| \le |d_{\phi_\epsilon}^{C_\epsilon} - d_\phi^{C_\epsilon}| + |d_\phi^{C_\epsilon} - d_\phi^C|$, by (i) and (ii) of Lemma 3.8 we have $d_{\phi_\epsilon}^{C_\epsilon} \to d_\phi^C$ uniformly in $\mathbb{R}^N$. Hence, by convexity, $d_{\phi_\epsilon}^{C_\epsilon} \to d_\phi^C$ in $H^1_{\rm loc}(\mathbb{R}^N)$, and $\nabla d_{\phi_\epsilon}^{C_\epsilon} \to \nabla d_\phi^C$ almost everywhere in $\mathbb{R}^N$. It follows that $\operatorname{dist}(n_{\phi_\epsilon}^{C_\epsilon}, \partial\phi^\circ(\nabla d_\phi^C)) \to 0$ as $\epsilon \to 0^+$ almost everywhere in $\mathbb{R}^N$, and the assertion follows from the convexity and closedness of $\partial\phi^\circ(\nabla d_\phi^C(x))$. □

4 The approximating elliptic problem

Let $g \in L^2_{\rm loc}(\mathbb{R}^N)$. In this section we will consider the following partial differential inclusion

$$u - \operatorname{div}\partial\phi^\circ(\nabla u) \ni g \quad \text{in } \mathbb{R}^N. \tag{4.1}$$

We remark that, if $\phi \in \mathcal{C}^{1,1}_+$ and $\nabla u \neq 0$ in an open set, then (4.1) becomes an equality (in that open set).

We will then turn our attention to a particular case of (4.1), namely to the partial differential inclusion

$$u - h \operatorname{div} \partial\phi^\circ(\nabla u) \ni d_\phi^C \quad \text{in } \mathbb{R}^N, \tag{4.2}$$

where $h > 0$ and C is a compact convex set. We begin with some preliminary notation.

Following [12], let

$$X_2(\Omega) := \{z \in L^\infty(\Omega; \mathbb{R}^N) : \operatorname{div} z \in L^2(\Omega)\}.$$

If $z \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap BV(\Omega)$ we define the functional $(z, Dw) : \mathcal{C}_c^\infty(\Omega) \to \mathbb{R}$ by the formula

$$< (z, Dw), \varphi >:= -\int_\Omega w\, \varphi \operatorname{div} z \, dx - \int_\Omega w\, z \cdot \nabla\varphi \, dx.$$

Then (z, Dw) is a Radon measure in Ω,

$$\int_\Omega (z, Dw) = \int_\Omega z \cdot \nabla w \, dx \qquad \forall w \in L^2(\Omega) \cap W^{1,1}(\Omega),$$

and

$$\left| \int_B (z, Dw) \right| \le \int_B |(z, Dw)| \le \|z\|_\infty \int_B |Dw| \qquad \forall B \text{ Borel set } \subseteq \Omega.$$

We recall the following result proved in [12].

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $u \in BV(\Omega) \cap L^2(\Omega)$ and $z \in X_2(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty_{\mathcal{H}^{N-1}}(\partial\Omega)$ such that $\|[z \cdot \nu^\Omega]\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial\Omega)} \le \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and*

$$\int_\Omega u \operatorname{div} z \, dx + \int_\Omega (z, Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u \, d\mathcal{H}^{N-1}.$$

When $\Omega = \mathbb{R}^N$ we have the following integration by parts formula [12], for $z \in X_2(\mathbb{R}^N)$ and $w \in L^2(\mathbb{R}^N) \cap BV(\mathbb{R}^N)$:

$$\int_{\mathbb{R}^N} w \operatorname{div} z \, dx + \int_{\mathbb{R}^N} (z, Dw) = 0. \tag{4.3}$$

4.1 Existence and uniqueness of solutions to the elliptic problem

We are now in the position to define solutions to (4.1) and to study existence and comparison properties.

Definition 4.2. *We say that $u \in BV_{\rm loc}(\mathbb{R}^N) \cap L^2_{\rm loc}(\mathbb{R}^N)$ is a solution of (4.1) if there exists a vector field $z \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that*

(a) $z \in \partial\phi^\circ(\nabla u)$ almost everywhere;

(b) $\operatorname{div} z \in L^2_{\rm loc}(\mathbb{R}^N)$ and $u - \operatorname{div} z = g$ in $\mathcal{D}'(\mathbb{R}^N)$;

(c) $(z, Du)(\psi) = \phi^\circ(Du)(\psi)$ for any $\psi \in \mathcal{C}_c(\mathbb{R}^N)$.

With a little abuse of notation, sometimes we will denote by (u, z) a solution to (4.1). Even if u is unique, observe that, for a generic nonsmooth anisotropy ϕ, vector fields z are not unique.

Definition 4.3. *We say that* $u \in BV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2_{\mathrm{loc}}(\mathbb{R}^N)$ *is a minimizing solution to* (4.1) *if for any* $v \in \mathcal{C}^1_c(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \phi^\circ(Du) \ + \frac{1}{2}\int_{\mathbb{R}^N} (u-g)^2\,dx \leq \int_{\mathbb{R}^N} \phi^\circ(D(u+v)) \ + \frac{1}{2}\int_{\mathbb{R}^N} (u+v-g)^2\,dx. \tag{4.4}$$

Proposition 4.4. *The following assertions are equivalent.*

(i) u is a solution of (4.1).

(ii) u is a minimizing solution of (4.1).

(iii) For any $R > 0$ *the function* $u \in BV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2_{\mathrm{loc}}(\mathbb{R}^N)$ *is a solution of*

$$\min\left\{\int_{B_R} \phi^\circ(Dw) + \frac{1}{2}\int_{B_R} (w-g)^2\ dx : w \in BV(B_R), w|_{\partial B_R} = u|_{\partial B_R}\right\}. \tag{4.5}$$

(iv) For any $R > 0$ *the function* $u \in BV_{\mathrm{loc}}(\mathbb{R}^N) \cap L^2_{\mathrm{loc}}(\mathbb{R}^N)$ *is a solution of*

$$\min\left\{\int_{B_R} \phi^\circ(Dw) + \frac{1}{2}\int_{B_R} (w-g)^2\ dx + \int_{\partial B_R} |w-u|\phi^\circ(\nu^{B_R})\ d\mathcal{H}^{N-1} : w \in BV(B_R)\right\}. \tag{4.6}$$

Proof. See [23, Proposition 3.1]. □

Proposition 4.5. *Let* $u, \overline{u}$ *be two solutions of* (4.1) *corresponding to the right hand sides* $g, \overline{g} \in L^\alpha_{\mathrm{loc}}(\mathbb{R}^N)$ *respectively, where* $\alpha > \max(N, 2)$. *Then*

$$g \leq \overline{g} \Longrightarrow u \leq \overline{u}. \tag{4.7}$$

Proof. See [23, Appendix C]. □

Lemma 4.6. *If* (u, z) *is a solution of* (4.1), *then*

$$-(z, D1_{\{u\leq s\}})(\psi) = \phi^\circ(D1_{\{u\leq s\}})(\psi) \qquad \forall \psi \in \mathcal{C}_c(\mathbb{R}^N),$$

for almost any $s \in \mathbb{R}$.

Proof. See [23, proof of Lemma 5.1]. □

The following theorem applies to inclusion (4.2).

Theorem 4.7. *Assume that* $g : \mathbb{R}^N \to \mathbb{R}$ *is a convex function with* $\limsup_{|x|\to\infty} g(x)/|x| = L < +\infty$ *(in particular g is L–Lipschitz). Then there exists a unique solution u of* (4.1). *Moreover, we have*

(i) u is convex and L–Lipschitz;

(ii) $u \geq g$.

Proof. See [4] or [23, Theorem 3]. □

Remark 4.8. For any $g \in L^2_{\rm loc}(\mathbb{R}^N)$, problem (4.1) admits a unique solution u. Since this result is not used in the present paper, we just sketch the proof. Let $\Phi(v) := \int_{\mathbb{R}^N} \phi^\circ(Dv)$ if $v \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, $\Phi(v) := +\infty$ if $v \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N)$. Then [20] there exists a unique solution u of $u - \partial\Phi(u) \ni g$ for any $g \in L^2(\mathbb{R}^N)$. By Proposition 4.4 u is a solution of (4.1). If $g \in L^2_{\rm loc}(\mathbb{R}^N)$ we approximate it in $L^2_{\rm loc}(\mathbb{R}^N)$ with functions $g_n \in L^2(\mathbb{R}^N)$, find u_n the solution of $u - \partial\Phi(u) \ni g_n$, and observe that u_n converges in $L^2_{\rm loc}(\mathbb{R}^N)$ to a solution u of (4.1). The (full) convergence of u_n is a consequence of an estimate related to the comparison principle [23, Theorem 6]. As in [13] we may prove that u is a solution of (4.1).

Definition 4.9. *Let $C \subset \mathbb{R}^N$ be a compact convex set and $h > 0$. Let u be the solution of* (4.2)*. We define*

$$T_{\phi,h}(C) := \{u \le 0\}. \tag{4.8}$$

Remark 4.10. Thanks to Theorem 4.7, the set $T_{\phi,h}(C)$ is compact and convex. Moreover $u \ge d^C_\phi$, hence $T_{\phi,h}(C) \subseteq C$. By the convexity of u, if $\min u < 0$ (which will be true as soon as h is small enough), then $\{u < 0\}$ is a bounded open convex set with closure equal to $\{u \le 0\}$.

Definition 4.11. *Given $i \in \mathbb{N}$, we recursively define u_{i+1} to be the solution of*

$$u - h \,\mathrm{div}\, \partial\phi^\circ(\nabla u) \ni d^{C_i}_\phi, \tag{4.9}$$

where $C_i := \{u_i \le 0\} = T_{\phi,h}(C_{i-1})$, $u_0 := u$ and $C_0 := C$.

Recall that $u_i \ge u_{i-1}$, hence $C_i \subseteq C_{i-1}$.

Lemma 4.12. *For any $r > 0$ let v_r be the solution of* (4.2) *with C replaced by $r\mathcal{W}_\phi$. Then*

$$v_r(\xi) = \begin{cases} \phi(\xi) + \dfrac{N-1}{\phi(\xi)} h - r & \text{if } \phi(\xi) \ge \sqrt{(N+1)h}, \\[2ex] \dfrac{2N}{\sqrt{N+1}}\sqrt{h} - r & \text{otherwise.} \end{cases}$$

Proof. See [23, Section 6]. □

Corollary 4.13. *For any $r' \in (0, r)$ and any $h \in (0, \frac{(r')^2}{N+1})$ we have*

$$v_r(\xi) \le \big(\phi(\xi) \vee r'\big) + \frac{N-1}{r'} h - r, \qquad \xi \in \mathbb{R}^N. \tag{4.10}$$

Proof. Our choice of h implies that $r' > \sqrt{(N+1)h}$. If $\phi(\xi) \ge r'$ inequality (4.10) is immediate. Since the right hand side of (4.10) is constant on $\{\phi < r'\}$ and v_r is convex, the assertion follows. □

4.2 Comparison with the flat ϕ-curvature flow minimizers at each step

The following result, proved in [24, 23], shows that the sublevels of a solution of (4.2) solves a variational problem of the type considered in [2].

Proposition 4.14. *Let u be a solution of* (4.2)*. Then* *(a) for any $s \in \mathbb{R}$, the set $\{u \leq s\}$ is a solution of the variational problem*

$$\min\left\{P_\phi(F) \ + \ \frac{1}{h}\int_F (d_\phi^C - s)\, dx : \chi_F \in BV(\mathbb{R}^N)\right\}. \tag{4.11}$$

(b) For any $s > \min u$, the set $\{u \leq s\}$ is the unique minimizer of (4.11)*.*

(c) If $\{u < 0\} \neq \emptyset$, then $\{u \leq 0\}$ is the unique solution of

$$\min\left\{P_\phi(F) + \frac{1}{h}\int_{F\triangle C} d_\phi(x, \partial C)\, dx : \chi_F \in BV(\mathbb{R}^N)\right\}. \tag{4.12}$$

Proof. Let λ be a constant such that $g := d_\phi^C + \lambda \geq 0$. Then the corresponding solution of (4.2) is $v := u + \lambda \geq d_\phi^C + \lambda \geq 0$. By the coarea formula we have, for any $R > 0$,

$$\int_{B_R} \phi^\circ(Dv) = \int_0^{+\infty}\int_{B_R\cap\partial\{v\leq s\}} \phi^\circ(\nu^{\{v\leq s\}})\ d\mathcal{H}^{N-1} ds.$$

In addition, we recall that, if μ is a measure on $\mathbb{R}^N$ and $p \geq 0$ and $q \geq 0$ are two μ-integrable functions, then the Cavalieri formula yields $\int_{\mathbb{R}^N} pq\ d\mu = \int_0^{+\infty}\int_{\{q\leq s\}} p\ d\mu ds$. Hence

$$\int_{B_R} gv\ dx = \int_0^{+\infty}\int_{B_R\cap\{v\leq s\}} g\ dxds.$$

Similary, making the change of variable $\sqrt{s} = \tau$,

$$\int_{B_R} v^2\ dx = \int_0^{+\infty} |B_R \cap \{v^2 \leq s\}|\ ds = 2\int_0^{+\infty} s|B_R \cap \{v \leq s\}|ds.$$

Hence

$$\int_{B_R} \phi^\circ(Dv) \ + \frac{1}{2h}\int_{B_R} (v-g)^2\, dx = \int_0^{+\infty}\int_{B_R\cap\partial\{v\leq s\}} \phi^\circ(\nu^{\{v\leq s\}})\ d\mathcal{H}^{N-1}ds \tag{4.13}$$

$$+\frac{1}{h}\int_{\mathbb{R}}\int_{B_R\cap\{v>s\}} (s-g)\ dx\ ds + c, \tag{4.14}$$

where $c := \frac{1}{2h}\int_{B_R} g^2\ dx$. Subtracting λ and using standard arguments [7], [8] it follows that $\{u \leq s\}$ is a minimizer of $\int_{B_R\cap\partial F} \phi^\circ(\nu^F) + \frac{1}{h}\int_{B_R\setminus F}(s - d_C^\phi)\, dx$ for almost every $s \in \mathbb{R}$. The same statement holds for any $s \in \mathbb{R}$, as it follows by approximation. Therefore, being C bounded, letting $R \to \infty$ implies that $\{u \leq s\}$ is a minimizer of $P_\phi(F) + \frac{1}{h}\int_F(d_C^\phi - s)\ dx$. Uniqueness follows from [23, Lemma 5.3]. Assertion (c) follows by letting $s = 0$ and recalling that $T_{\phi,h}(C) \subseteq C$. □

Notice that, when $\{u < 0\} = \emptyset$, the sets $\{u < 0\}$ and $\{u \leq 0\}$ could provide two different solutions of (4.12).

5 Iteration procedure for $\phi \in \mathcal{C}_+^\infty$

If $C = C_0$ satisfies the $r\mathcal{W}_\phi$-condition, then in general it is not true that C_1 satisfies the $r\mathcal{W}_\phi$-condition. The aim of this section is to prove the following result, which specifies for which $r' > 0$ the set C_1 (and all C_i) satisfies the $r'\mathcal{W}_\phi$-condition, and estimates, roughly speaking, the existence time of the discretized evolutions.

Theorem 5.1. *Let $\phi \in \mathcal{C}_+^\infty$ and let C be a compact convex set satisfying the $r\mathcal{W}_\phi$-condition for some $r > 0$. Let $h > 0$ be small enough in such a way that*

$$2\sqrt{Nh} < \frac{r}{N-1}.$$

Fix $r' \in \left(2\sqrt{Nh}, \frac{r}{N-1}\right)$. Then

$$C_i \text{ satisfies the } r'\mathcal{W}_\phi - \text{condition for any } i \le \frac{\ln\left(\frac{(N-1)r'}{r}\right)}{\ln\left(1 - \frac{Nh}{r'^2}\right)}. \tag{5.1}$$

To prove Theorem 5.1 we need some preliminaries. We begin with the following lemma.

Lemma 5.2. *Assume that $\phi \in \mathcal{C}_+^{1,1}$ and let $h > 0$. Let (u, z) be the solution of (4.2). Let C be a compact convex set with nonempty interior. Then $u \ge d_\phi^C$. Moreover, if*

$$R_C := \sup\left\{\rho > 0 : \exists y \in \mathbb{R}^N : \rho\mathcal{W}_\phi + y \subseteq C\right\}$$

is the radius of a maximal Wulff shape contained in C and

$$h < \frac{N+1}{4N^2} R_C^2, \tag{5.2}$$

then

$$\nabla u \neq 0 \quad \text{and} \quad \phi(z) = 1 \qquad \text{a.e. in } \left\{|d_\phi^C| < R_C - \frac{2N}{\sqrt{N+1}}\sqrt{h}\right\}.$$

Proof. The inequality $u \ge d_\phi^C$ follows from Proposition 4.5. Therefore, to prove the assertion it is sufficient to check that $\nabla u \neq 0$ almost everywhere in $\{|d_\phi^C| < \sigma\}$, where $\sigma := R_C - \frac{2N}{\sqrt{N+1}}\sqrt{h}$, since, by (a) of Definition 4.2, at almost all points x where $\nabla u(x) \neq 0$ we have $\phi(z(x)) = 1$. Let p be the center of a maximal Wulff shape contained in C. From Proposition 4.5 we have

$$u(x) \le v_{R_C}(x - p) \qquad \forall x \in \mathbb{R}^N,$$

where v_{R_C} is explicitly given in Lemma 4.12. Hence $u(p) \le \frac{2N}{\sqrt{N+1}}\sqrt{h} - R_C$ and $u \ge d_\phi^C > -\sigma$ on $\{|d_\phi^C| < \sigma\}$. Since u is convex, it follows $\nabla u \neq 0$ almost everywhere in $\{|d_\phi^C| < \sigma\}$. □

Lemma 5.3. *Assume that $\phi \in \mathcal{C}_+^\infty$ and let $h > 0$. Let C be a compact convex set satisfying the $\delta\mathcal{W}_\phi$-condition for some $\delta > 0$. Let (u, z) be the solution of (4.2). Then for any $\delta' \in (0, \delta)$ and for any $h \in (0, \frac{(\delta-\delta')^2}{N+1})$ we have*

$$u \le d_\phi^C + \frac{(N-1)h}{\delta - \delta'} \qquad \text{in } \{|d_\phi^C| \le \delta'\}. \tag{5.3}$$

Proof. Let δ' and h be as in the statement, and let C' be such that $C = C' + \delta\mathcal{W}_\phi$. We define $\alpha := \frac{\delta-\delta'}{\delta} \in (0,1)$, and $r(y) := d_\phi(y, \partial C)$ for any $y \in C$. Observe that the assumption $h \in (0, \frac{(\delta-\delta')^2}{N+1})$ implies $\sqrt{(N+1)h} \le \alpha r(y)$ for any $y \in C' = \{y \in C : r(y) > \delta\}$. Hence, from Corollary 4.13 (applied with $r = r(y)$, $r' = \alpha r(y) \in (0, r(y))$, $y \in C'$), comparing the solution u with a Wulff shape centered at $y \in C'$ of radius $r(y)$, we have

$$u(x) \le \min_{y\in C'} \left\{ \phi(x-y) + \frac{N-1}{\alpha r(y)} h - r(y) \right\} \qquad \forall x \in \{|d_\phi^C| \le \delta'\}. \tag{5.4}$$

Let $x \in \{0 \le d_\phi^C \le \delta'\}$. We have

$$\min_{y\in C'} \left\{ \phi(x-y) - r(y) + \frac{N-1}{\alpha r(y)} h \right\} \le d_\phi^C(x) + (N-1)h \min_{y\in C'} \frac{1}{\alpha r(y)},$$

where the last inequality follows from $\phi(x-y) \le d_\phi^C(x) + r(y)$ which, in turn, is a consequence of the triangular property of d_ϕ. From (5.4) and $\alpha r(y) \ge \alpha\delta = \delta - \delta'$, we deduce

$$u(x) \le d_\phi^C(x) + \frac{(N-1)h}{\delta - \delta'}.$$

Let now $x \in \{-\delta' \le d_\phi^C < 0\}$. If we set $\overline{y} := x - (d_\phi^C(x) + \delta) n_\phi^C(x) \in C'$, then we have $r(\overline{y}) = \delta$, $d_\phi^C(x) = \phi(x - \overline{y}) - \delta \le 0$, and $d_{r(\overline{y})}(x - \overline{y}) = d_\phi^C(x)$. Therefore from (5.4) we deduce

$$u(x) \le \phi(x - \overline{y}) + \frac{N-1}{\alpha r(\overline{y})} h - r(\overline{y}) \le d_\phi^C(x) + \frac{N-1}{\delta - \delta'} h.$$

This concludes the proof. □

Lemma 5.4. *Assume that $\phi \in \mathcal{C}_+^\infty$ and let $h > 0$. Let C be a compact convex set satisfying the $\delta\mathcal{W}_\phi$-condition for some $\delta > 0$. Let (u, z) be the solution of (4.2). Let $a < b$ be such that $X_{a,b} := \{u \ge a\} \cap \{d_\phi^C \le b\} \subseteq \{|d_\phi^C| < \delta\}$. Then $\operatorname{div} z \in L^\infty(X_{a,b})$ and*

$$\|\operatorname{div} z\|_{L^\infty(X_{a,b})} \le \|\operatorname{div} n_\phi^C\|_{L^\infty(X_{a,b})}. \tag{5.5}$$

Proof. Let $p : \mathbb{R} \to [0, +\infty)$ be a smooth increasing function. Since (u, z) solves (4.2), we find

$$\begin{aligned} h^{-1} \int_{X_{a,b}} (u - d_\phi^C)\; p(u - d_\phi^C)\; dx &= \int_{X_{a,b}} \operatorname{div} z\; p(u - d_\phi^C)\; dx \\ = \int_{X_{a,b}} (\operatorname{div} z - \operatorname{div} n_\phi^C)\, p(u - d_\phi^C)\; dx &+ \int_{X_{a,b}} \operatorname{div} n_\phi^C\; p(u - d_\phi^C)\; dx =: \mathrm{I} + \mathrm{II}. \end{aligned}$$

We have, observing that $X_{a,b}$ has Lipschitz boundary,

$$\begin{aligned} \mathrm{I} = \; & - \int_{X_{a,b}} (z - n_\phi^C) \cdot \nabla(p(u - d_\phi^C))\; dx \\ & + \int_{\partial X_{a,b}} (z \cdot \nu^{X_{a,b}} - n_\phi^C \cdot \nu^{X_{a,b}})\; p(u - d_\phi^C)\; d\mathcal{H}^{N-1} =: \mathrm{I}_1 + \mathrm{I}_2. \end{aligned}$$

First, observe that from Definition 4.2

$$\begin{aligned} \mathrm{I}_1 &= -\int_{X_{a,b}} p'(u - d_\phi^C)(z - n_\phi^C) \cdot \nabla(u - d_\phi^C)\; dx \\ &= -\int_{X_{a,b}} p'(u - d_\phi^C)(\phi^\circ(\nabla u) - n_\phi^C \cdot \nabla u + \phi^\circ(\nabla d_\phi^C) - z \cdot \nabla d_\phi^C)\; dx \le 0. \end{aligned}$$

We claim also that $\mathrm{I}_2 \le 0$. Observe that since $\nu^{X_{a,b}} = \nu^{\{d_\phi^C \le b\}} = \nabla d_\phi^C/|\nabla d_\phi^C|$, $\mathcal{H}^{N-1}$-almost everywhere on $\{d_\phi^C = b\}$ and $\nu^{X_{a,b}} = \nu^{\{u \ge a\}} = -\nabla u/|\nabla u|$, $\mathcal{H}^{N-1}$-almost everywhere on $\{u = a\}$, we have

$$n_\phi^C \cdot \nu^{\{d_\phi^C \le b\}} = \phi^\circ(\nu^{\{d_\phi^C \le b\}}) \qquad \mathcal{H}^{N-1} - \text{a.e. on } \{d_\phi^C = b\}$$

because $n_\phi^C \in \partial\phi^\circ(\nu^{\{d_\phi^C \le b\}})$. On the other hand,

$$z \cdot \nu^{X_{a,b}} \le \phi^\circ(\nu^{\{d_\phi^C \le b\}}) \qquad \mathcal{H}^{N-1} - \text{a.e. on } \{d_\phi^C = b\},$$

and

$$z \cdot \nu^{\{u \ge a\}} = -\phi^\circ(\nu^{\{u \ge a\}}) \qquad \mathcal{H}^{N-1} - \text{a.e. on } \{u = a\},$$
$$n_\phi^C \cdot \nu^{\{u \ge a\}} \ge -\phi^\circ(\nu^{\{u \ge a\}}) \qquad \mathcal{H}^{N-1} - \text{a.e. on } \{u = a\}.$$

Hence

$$\begin{aligned} \mathrm{I}_2 = \quad & \int_{\{u=a\}} (z \cdot \nu^{\{u \ge a\}} - n_\phi^C \cdot \nu^{\{u \ge a\}}) p(u - d_\phi^C)\ d\mathcal{H}^{N-1} \\ + \quad & \int_{\{d_\phi^C = b\}} (z \cdot \nu^{\{d_\phi^C \le b\}} - n_\phi^C \cdot \nu^{\{d_\phi^C \le b\}}) p(u - d_\phi^C)\ d\mathcal{H}^{N-1} \le 0. \end{aligned}$$

We conclude that $\mathrm{I} \le 0$, hence

$$\int_{X_{a,b}} (u - d_\phi^C) p(u - d_\phi^C)\ dx \le h \int_{X_{a,b}} \mathrm{div}\, n_\phi^C\, p(u - d_\phi^C)\ dx. \tag{5.6}$$

Let $q > 2$, let $r^+ := r \vee 0$, and let $\{p_n\}$ be a sequence of smooth increasing nonnegative functions such that $p_n(r) \to r^{+(q-1)}$ uniformly as $n \to \infty$. From (5.6) we obtain

$$\int_{X_{a,b}} ((u - d_\phi^C)^+)^q\ dx \le h \int_{X_{a,b}} \mathrm{div}\, n_\phi^C\, ((u - d_\phi^C)^+)^{q-1}\ dx.$$

Applying Young's inequality we obtain

$$\|(u - d_\phi^C)^+\|_{L^q(X_{a,b})} \le h \|\mathrm{div}\, n_\phi^C\|_{L^q(X_{a,b})}.$$

Dividing by $h > 0$ and using $u \ge d_\phi^C$ and (4.2) we get

$$\|\mathrm{div}\, z\|_{L^q(X_{a,b})} \le \|\mathrm{div}\, n_\phi^C\|_{L^q(X_{a,b})}.$$

Letting $q \to \infty$ we obtain (5.5). □

The following lemma proves some regularity properties of solutions to (4.2); the statement does not imply the (true) assertion that a convex ϕ-regular set is Lipschitz ϕ-regular, since in general $z \in \partial\phi^\circ(\nabla u)$ (instead of $z \in \partial\phi^\circ(\nabla d_\phi^C)$).

Lemma 5.5. *Assume $\phi \in \mathcal{C}_+^\infty$ and let C be a compact convex set with nonempty interior. Let R_C and h be as in Lemma 5.2. Let (u, z) be the solution of (4.2), and let U be an open set with $\partial C \subset U \subseteq \left\{|d_\phi^C| < R_C - \frac{2N}{\sqrt{N+1}}\sqrt{h}\right\}$. Then $z \in \mathrm{Lip}(U; \mathbb{R}^N)$ and*

$$|\nabla z| \le \frac{\Lambda}{\lambda} \mathrm{div}\, z \qquad \text{a.e. in } U, \tag{5.7}$$

where λ and Λ are as in (2.5).

Proof. Set $\psi_\epsilon := \sqrt{\epsilon^2 + (\phi^\circ)^2}$, so that differentiating $\psi_\epsilon = \sqrt{\epsilon^2 + (\phi^\circ)^2}$ we get

$$\nabla^2 \psi_\epsilon - \epsilon^2 \frac{\nabla \phi^\circ \otimes \nabla \phi^\circ}{(\epsilon^2 + (\phi^\circ)^2)^{3/2}} = \frac{\phi^\circ}{\sqrt{\epsilon^2 + (\phi^\circ)^2}} \nabla^2 \phi^\circ.$$

From (2.5) we obtain

$$\frac{1}{\sqrt{\epsilon^2 + (\phi^\circ)^2}} (\lambda \mathrm{Id} - \nabla \phi^\circ \otimes \nabla \phi^\circ) \preccurlyeq \nabla^2 \psi_\epsilon - \epsilon^2 \frac{\nabla \phi^\circ \otimes \nabla \phi^\circ}{(\epsilon^2 + (\phi^\circ)^2)^{3/2}} \tag{5.8}$$

$$\preccurlyeq \frac{1}{\sqrt{\epsilon^2 + (\phi^\circ)^2}} (\Lambda \mathrm{Id} - \nabla \phi^\circ \otimes \nabla \phi^\circ). \tag{5.9}$$

Let $u_\epsilon \in \mathcal{C}^{1,1}(\mathbb{R}^N)$ be the solution of

$$u_\epsilon - h \operatorname{div} \nabla \psi_\epsilon(\nabla u_\epsilon) - \epsilon \Delta u_\epsilon = d_\phi^C,$$

and set $z_\epsilon := \nabla \psi_\epsilon(\nabla u_\epsilon)$. By the convexity of u_ϵ [38, 4, 23] and (5.8) it follows

$$\begin{aligned} |\nabla z_\epsilon| &= |\sqrt{\nabla^2 u_\epsilon} \nabla^2 \psi_\epsilon(\nabla u_\epsilon) \sqrt{\nabla^2 u_\epsilon}| \leq \operatorname{tr}\left(\sqrt{\nabla^2 u_\epsilon} \nabla^2 \psi_\epsilon(\nabla u_\epsilon) \sqrt{\nabla^2 u_\epsilon}\right) \\ &\leq \frac{1}{\sqrt{\epsilon^2 + \phi^\circ(\nabla u_\epsilon)^2}} \left(\Lambda \Delta u_\epsilon - \nabla^2 u_\epsilon \nabla \phi^\circ(\nabla u_\epsilon) \cdot \nabla \phi^\circ(\nabla u_\epsilon)\right) \\ &\quad + \frac{\epsilon^2}{(\epsilon^2 + \phi^\circ(\nabla u_\epsilon)^2)^{3/2}} \nabla^2 u_\epsilon \nabla \phi^\circ(\nabla u_\epsilon) \cdot \nabla \phi^\circ(\nabla u_\epsilon) \end{aligned}$$

On the other hand, using again (5.8),

$$\begin{aligned} \operatorname{div} z_\epsilon &= \sum_{i=1}^N (\nabla^2 \psi_\epsilon(\nabla u_\epsilon) \nabla^2 u_\epsilon e_i) \cdot e_i \\ &\geq \frac{1}{\sqrt{\epsilon^2 + \phi^\circ(\nabla u_\epsilon)^2}} \left(\lambda \Delta u_\epsilon - (\nabla^2 u_\epsilon \nabla \phi^\circ(\nabla u_\epsilon)) \cdot \nabla \phi^\circ(\nabla u_\epsilon)\right) \\ &\quad + \frac{\epsilon^2}{(\epsilon^2 + \phi^\circ(\nabla u_\epsilon)^2)^{3/2}} (\nabla^2 u_\epsilon \nabla \phi^\circ(\nabla u_\epsilon)) \cdot \nabla \phi^\circ(\nabla u_\epsilon) \end{aligned}$$

We deduce

$$\operatorname{div} z_\epsilon \geq \frac{\lambda}{\Lambda} |\nabla z_\epsilon| \qquad \text{a.e. in } \mathbb{R}^N. \tag{5.10}$$

As proved in [23] the sequence $\{u_\epsilon\}$ converges to u uniformly on compact subsets of $\mathbb{R}^N$, and $\nabla u_\epsilon \to \nabla u$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$ as $\epsilon \to 0^+$. Moreover, there exists $\tilde{z} \in L^\infty(\mathbb{R}^N; \mathbb{R}^N)$ such that, possibly extracting a subsequence, $z_\epsilon \rightharpoonup \tilde{z}$ weakly* in $L^\infty(\mathbb{R}^N; \mathbb{R}^N)$. From [23, Theorem 5]), we have $u - h \operatorname{div} \tilde{z} = d_\phi^C$, hence $\operatorname{div} \tilde{z} = \operatorname{div} z$ and $\tilde{z} \in \nabla \phi^\circ(\nabla u)$ almost everywhere in $\mathbb{R}^N$. Since $\phi \in \mathcal{C}^\infty_+$, and $\{\nabla u \neq 0\}$ almost everywhere in U by Lemma 5.2, we deduce that $\tilde{z} = z$ in U. Recalling that $\operatorname{div} z \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, and letting $\epsilon \to 0^+$ in (5.10), we obtain (5.7). □

We are now in the position of proving Theorem 5.1.

Proof of Theorem 5.1. Let (u_i, z_i) be the solution to (4.9). Recall that (Proposition 3.9) $C_0 = C$ is ϕ-regular, hence by (d) Remark 2.10, C is Lipschitz ϕ-regular, and therefore, by (d3) of Remark 2.10, $\operatorname{div} n_\phi^C$ is $\mathcal{H}^{N-1}$-almost everywhere defined on ∂C. We divide the proof into four steps.

Step 1. Let $i \in \mathbb{N}$. Assume that C_i is Lipschitz ϕ-regular and define

$$\mathcal{K}_i := \|\kappa_\phi^{C_i}\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C_i)}. \tag{5.11}$$

Then C_i satisfies the $\mathcal{K}_i^{-1}\mathcal{W}_\phi$-condition.

This is a consequence of Corollary 3.3.

Step 2. Let $i \in \mathbb{N}$. Assume that C_i is Lipschitz ϕ-regular. Then

$$h < \frac{1}{4N^2\mathcal{K}_i^2} \Longrightarrow d_\phi(\partial C_i, \partial C_{i+1}) \le N\mathcal{K}_i h. \tag{5.12}$$

Indeed, (5.12) follows from our assumptions on h, using Lemma 5.3 (applied with $\delta = \mathcal{K}_i^{-1}$ and $\delta' = \frac{\delta}{N}$), observing that $h < \frac{1}{4N^2\mathcal{K}_i^2}$ implies $h < \frac{(\delta-\delta')^2}{N+1}$ and $\frac{(N-1)h}{\delta-\delta'} < \delta'$.

Step 3. Let $i \in \mathbb{N}$. Assume that C_i is Lipschitz ϕ-regular. If $h < \frac{1}{4N^2\mathcal{K}_i^2}$, then C_{i+1} is Lipschitz ϕ-regular and

$$\mathcal{K}_{i+1} \le \frac{\mathcal{K}_i}{1 - N\mathcal{K}_i^2 h}, \tag{5.13}$$

where $\mathcal{K}_{i+1} := \|\kappa_\phi^{C_{i+1}}\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C_{i+1})}$.

The requirement $h < \frac{1}{4N^2\mathcal{K}_i^2}$ implies that $4N\mathcal{K}_i h < \mathcal{K}_i^{-1}$; in particular, using (5.12),

$$\partial C_{i+1} \subset \{|d_\phi^{C_i}| < \mathcal{K}_i^{-1} - N\mathcal{K}_i h\}.$$

Using Lemma 5.4 with $b = -a = \mathcal{K}_i^{-1} - N\mathcal{K}_i h$ and $\delta = \mathcal{K}_i^{-1}$, we have $\operatorname{div} z_i \in L^\infty(X_{a,b})$. Since $\mathcal{K}_i^{-1}/4 \le \mathcal{K}_i^{-1} - \frac{2N}{\sqrt{N+1}}\sqrt{h}$ and $R_C \ge \mathcal{K}_i^{-1}$, by Lemma 5.2, we deduce that $\nabla u_i \neq 0$ a.e. on $\{|d_\phi^{C_i}| < \mathcal{K}_i^{-1}/4\} \subseteq X_{a,b})$. Now, by Lemma 5.5, we have that $z_i \in \mathrm{Lip}(\{|d_\phi^{C_i}| < \frac{\mathcal{K}_i^{-1}}{4}\})$. Now, observing that $\partial C_{i+1} \subset \{|d_\phi^{C_i}| < \frac{\mathcal{K}_i^{-1}}{4}\}$, we deduce $n_\phi^{C_{i+1}} = z_i$ on ∂C_{i+1}, and therefore C_{i+1} is Lipschitz ϕ-regular.

It remains to prove (5.13). Recalling that $\mathcal{K}_{i+1} = \|\operatorname{div} z_i\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C_{i+1})}$, from Lemma 5.4, applied with $a = 0 < b = N\mathcal{K}_i h < \delta = \mathcal{K}_i^{-1}$, we get

$$\mathcal{K}_{i+1} \le \|\operatorname{div} z_i\|_{L^\infty(\{u_i \ge 0\}\cap\{d_\phi^{C_i} \le N\mathcal{K}_i h\})} \le \|\operatorname{div} n_\phi^{C_i}\|_{L^\infty(\{u_i \ge 0\}\cap\{d_\phi^{C_i} \le N\mathcal{K}_i h\})} \le \|\operatorname{div} n_\phi^{C_i}\|_{L^\infty(\{|d_\phi^{C_i}| \le N\mathcal{K}_i h\})}, \tag{5.14}$$

where the last inequality follows from (5.12). Applying now (3.2) of Theorem 3.1 we obtain

$$\|\operatorname{div} n_\phi^{C_i}\|_{L^\infty(\{|d_\phi^{C_i}| \le N\mathcal{K}_i h\})} \le \frac{\mathcal{K}_i}{1 - N\mathcal{K}_i^2 h}, \tag{5.15}$$

which inserted in (5.15) concludes the proof of *step 3*.

Step 4. For any M such that $\mathcal{K}_0 < M < \frac{1}{2\sqrt{Nh}}$, we have

$$\mathcal{K}_i \le M \quad \text{for any} \quad i \le \frac{\ln\left(\frac{\mathcal{K}_0}{M}\right)}{\ln(1 - NM^2h)}. \tag{5.16}$$

Let us prove (5.16) by induction and assume that C_i is Lipschitz ϕ-regular and $\mathcal{K}_i \le M$ for any $i \le \bar{i}$, for some $\bar{i} \le \frac{\ln\left(\frac{\mathcal{K}_0}{M}\right)}{\ln(1-NM^2h)} - 1$. Then, by *step 3*, we have that $C_{\bar{i}+1}$ is also Lipschitz ϕ-regular and

$$\mathcal{K}_{i+1} \le \frac{\mathcal{K}_i}{1 - N\mathcal{K}_i^2 h} \le \frac{\mathcal{K}_i}{1 - NM^2 h} \le \frac{\mathcal{K}_0}{(1 - NM^2h)^{i+1}}, \qquad i \le \bar{i}.$$

Since $\bar{i}+1 \le \frac{\ln\left(\frac{\mathcal{K}_0}{M}\right)}{\ln(1-NM^2h)}$, we have $(1-NM^2h)^{\bar{i}+1} \ge \frac{\mathcal{K}_0}{M}$, which implies

$$\mathcal{K}_{\bar{i}+1} \le \frac{\mathcal{K}_0}{(1-NM^2h)^{\bar{i}+1}} \le M,$$

and this concludes the proof of *step 4*.

To conclude the proof of the theorem, it is enough to apply (5.16) with $M = 1/r'$, observing that, from Corollary 3.3, we have $\mathcal{K}_0 \le \frac{N-1}{r}$. □

Remark 5.6. Observe that (5.12) can be refined into

$$d_\phi(\partial C_i, \partial C_{i+1}) \le \frac{2(N-1)h}{\delta + \sqrt{\delta^2 - 4(N-1)h}} = \frac{(N-1)h}{\delta} + O(h^2), \tag{5.17}$$

which is obtained with the choice of δ' so that $\delta - \delta' = \frac{(N-1)h}{\delta'}$.

6 Existence of ϕ-regular flows of convex sets for a generic ϕ

In this section we prove the following result (Theorem 1.1).

Theorem 6.1. *Let ϕ be an anisotropy and let C be a compact convex set satisfying the $r\mathcal{W}_\phi$-condition for some $r > 0$. Then there exist $T > 0$ and a ϕ-regular flow $C(t)$ for $t \in [0, T]$, such that $C(0) = C$.*

Proof. Define

$$r' := \frac{r}{2(N-1)}, \qquad U := \{|d_C^\phi| < r'/2\}, \qquad T := \frac{r'^2}{8N}. \tag{6.1}$$

Let C_ϵ and ϕ_ϵ be as in Lemma 3.8 and recall that C_ϵ satisfies the $r\mathcal{W}_{\phi_\epsilon}$-condition for any $\epsilon > 0$. For $\epsilon > 0$ small enough, we have by Lemma 3.8,

$$\left\{|d_{\phi_\epsilon}^{C_\epsilon}| < 3r'/8\right\} \subseteq U \subseteq \left\{|d_{\phi_\epsilon}^{C_\epsilon}| < 5r'/8\right\}. \tag{6.2}$$

For any $i \in \mathbb{N}$, let $C_{\epsilon,i}$ be defined as in Definition 4.11, with $C_{\epsilon,0} = C_\epsilon$. For simplicity of notation, set $d_i^\epsilon := d_{\phi_\epsilon}^{C_{\epsilon,i}}$, $n_i^\epsilon := n_{\phi_\epsilon}^{C_{\epsilon,i}}$, $\pi_i^\epsilon := \pi_{\phi_\epsilon}^{C_{\epsilon,i}}$, $\mathcal{K}_i^\epsilon := \|\mathrm{div}\, n_i^\epsilon\|_{L^\infty_{\mathcal{H}^{N-1}}(\partial C_{\epsilon,i})}$. Denote by $(u_i^\epsilon, z_i^\epsilon)$ the solution of (4.2) with d_ϕ^C replaced by d_i^ϵ.

By Theorem 5.1 we have that $C_{\epsilon,i}$ satisfies the $r'\mathcal{W}_{\phi_\epsilon}$-condition, provided $h < \frac{(r')^2}{4N}$ and

$$i \le -\frac{\ln 2}{\ln\left(1 - \frac{Nh}{(r')^2}\right)}. \tag{6.3}$$

As $h < \frac{(r')^2}{4N}$, one can check that, if T is as in (6.1), $i \le T/h$ implies (6.3). Therefore, from now on we shall assume $i \le T/h$.

Since C_ϵ satisfies the $r\mathcal{W}_{\phi_\epsilon}$-condition, from (b) of Corollary 3.3 it follows that $\mathcal{K}_0^\epsilon \le \frac{N-1}{r}$. Therefore, from (5.16) applied with $M = \frac{2(N-1)}{r} = \frac{1}{r'}$, we get

$$\mathcal{K}_i^\epsilon \le \frac{1}{r'}, \qquad i \le T/h. \tag{6.4}$$

From (5.12) it follows $|d_{\phi_\epsilon}^{C_{\epsilon,i}} - d_{\phi_\epsilon}^{C_\epsilon}| \le \frac{Nih}{r'} < \frac{r'}{8}$, hence using (6.2) we get

$$\left\{|d_{\phi_\epsilon}^{C_{\epsilon,i}}| < r'/4\right\} \subseteq U \subseteq \left\{|d_{\phi_\epsilon}^{C_{\epsilon,i}}| < 3r'/4\right\}, \qquad i \le T/h. \tag{6.5}$$

Step 1. Let v be a convex function such that

$$|v - d_{i+1}^\epsilon| \le c \qquad \text{in } U, \tag{6.6}$$

for some constant $c > 0$. Then

$$|v - d_{i+1}^\epsilon - (v(\pi_{i+1}^\epsilon) - d_{i+1}^\epsilon(\pi_{i+1}^\epsilon))| \le \frac{16c}{r'}|d_{i+1}^\epsilon| \qquad \text{in } U. \tag{6.7}$$

Fix $x \in \partial C_{\epsilon,i+1}$. Then, the restriction of the function $v - d_{i+1}^\epsilon$ to the segment $\{x + sn_{i+1}^\epsilon(x) : \ |s| < r'/4\}$ (which is contained in U by (6.5)) is convex. Hence, using (6.6), such a restriction is Lipschitz continuous with constant $\frac{16c}{r'|n_{i+1}^\epsilon(x)|}$ on the segment $\{x + sn_{i+1}^\epsilon(x) : \ |s| \le r'/8\}$. Thus, for any $y \in \{x + sn_{i+1}^\epsilon(x) : \ |s| \le r'/8\}$, we get

$$|v(y) - d_{i+1}^\epsilon(y) - (v(x) - d_{i+1}^\epsilon(x))| \le \frac{16c}{r'|n_{i+1}^\epsilon(x)|}|y - x| = \frac{16c}{r'}|d_{i+1}^\epsilon(y)|.$$

Hence (6.7) holds in $\{|d_{i+1}^\epsilon| \le \frac{r'}{8}\}$.

If $y \in U$ and $|d_{i+1}^\epsilon(y)| > r'/8$, using (6.6)

$$|v(y) - d_{i+1}^\epsilon(y) - (v(x) - d_{i+1}^\epsilon(x))| \le 2c \le \frac{16c}{r'}|d_{i+1}^\epsilon(y)|,$$

which gives (6.7) and concludes the proof of *step 1.*

Step 2. For any $i \le T/h$ we have

$$|\operatorname{div} n_i^\epsilon| \le \frac{4}{r'} \quad \text{in } U, \tag{6.8}$$

and

$$\left|\frac{d_{i+1}^\epsilon - d_i^\epsilon}{h} - \operatorname{div} n_{i+1}^\epsilon\right| \le c\ |d_{i+1}^\epsilon| \quad \text{in } U, \tag{6.9}$$

where $c = \frac{16N+4}{(r')^2}$.

Inequality (6.8) follows from (3.2), (6.5) and (6.4). Let us prove (6.9). Recall that

$$\frac{u_i^\epsilon - d_i^\epsilon}{h} - \operatorname{div} z_i^\epsilon = 0 \quad \text{in } \mathbb{R}^N. \tag{6.10}$$

Moreover, z_i^ϵ and n_{i+1}^ϵ are Lipschitz continuous, and coincide on $\partial C_{\epsilon,i+1}$; hence by Lemma 5.2 their divergences (which equal their tangential divergences) coincide on $\partial C_{\epsilon,i+1}$, i.e. $\operatorname{div} z_i^\epsilon = \operatorname{div} n_{i+1}^\epsilon\ \mathcal{H}^{N-1}$-almost everywhere on $\partial C_{\epsilon,i+1}$. Hence, recalling that $\partial C_{\epsilon,i+1} = \{u_i^\epsilon = 0\}$, we have

$$\frac{d_{i+1}^\epsilon - d_i^\epsilon}{h} - \operatorname{div} n_{i+1}^\epsilon = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial C_{\epsilon,i+1}. \tag{6.11}$$

Recalling (6.4), from (5.12) we have

$$|d_{i+1}^\epsilon - d_i^\epsilon| \le \frac{N}{r'}h \qquad \text{in } \mathbb{R}^N. \tag{6.12}$$

By *step 1* applied with $v = d_i^\epsilon$ and $c = \frac{N}{r'}h$, we have

$$|d_{i+1}^\epsilon - d_i^\epsilon - (d_{i+1}^\epsilon(\pi_{i+1}^\epsilon) - d_i^\epsilon(\pi_{i+1}^\epsilon))| \leq \frac{16N}{(r')^2} h |d_{i+1}^\epsilon| \qquad \text{in } U. \tag{6.13}$$

On the other hand, by (3.2) and (6.4) we have

$$|\operatorname{div} n_{i+1}^\epsilon - \operatorname{div} n_{i+1}^\epsilon(\pi_{i+1}^\epsilon)| \leq \frac{|\operatorname{div} n_{i+1}^\epsilon(\pi_{i+1}^\epsilon)|^2}{1 - |d_{i+1}^\epsilon \operatorname{div} n_{i+1}^\epsilon(\pi_{i+1}^\epsilon)|} |d_{i+1}^\epsilon| \leq \frac{4}{(r')^2} |d_{i+1}^\epsilon| \qquad \text{in } U, \tag{6.14}$$

where the last inequality follows using (6.5). Therefore, from (6.11), (6.13), and (6.14), we deduce (6.9). This concludes the proof of *step 2*.

Step 3. Passing to the limit as $\epsilon \to 0$ for fixed h.

Letting $\epsilon \to 0$ and possibly passing to a suitable subsequence, we can assume that $n_i^\epsilon \rightharpoonup n_i$ weakly in $L^2(U; \mathbb{R}^N)$ and $\operatorname{div} n_i^\epsilon \rightharpoonup \operatorname{div} n_i$ weakly in $L^2(U)$. By compactness, for any $i \leq T/h$ there exist compact convex sets C_i such that $\lim_{\epsilon \to 0} d_{\mathcal{H}}(C_{\epsilon,i}, C_i) = 0$. Hence $d_i^\epsilon \to d_\phi^{C_i}$ uniformly in $\mathbb{R}^N$ as $\epsilon \to 0$. As in the proof of Proposition 3.9, this implies that

$$n_i \in \partial\phi^\circ(\nabla d_\phi^{C_i}) \qquad \text{a.e. in } U,\ i \leq T/h. \tag{6.15}$$

Taking the limits in (6.8) and (6.9) as $\epsilon \to 0$, we obtain

$$|\operatorname{div} n_i| \leq \frac{4}{r'} \qquad \text{in } U,\ i \leq T/h, \tag{6.16}$$

$$\left| \frac{d_\phi^{C_{i+1}} - d_\phi^{C_i}}{h} - \operatorname{div} n_{i+1} \right| \leq c |d_\phi^{C_{i+1}}| \quad \text{in } U,\ i \leq T/h. \tag{6.17}$$

Moreover, from Lemma 3.6 we have that

$$C_i \text{ satisfies the } r'\mathcal{W}_\phi\text{-condition} \tag{6.18}$$

and from (6.12)

$$|d_\phi^{C_{i+1}} - d_\phi^{C_i}| \leq \frac{Nh}{r'} \qquad \text{in } U. \tag{6.19}$$

Let us define

$$C_h(t) := C_{[t/h]}, \qquad n_h(x,t) := n_{[t/h]}(x), \qquad t \in [0,T],\ x \in U, \tag{6.20}$$

where $[t/h]$ denotes the integer part of t/h. We also define

$$d_h(x,t) := d_\phi^{C_h(t)}(x), \qquad x \in \mathbb{R}^N,\ t \in [0,T].$$

Step 4. Passing to the limit as $h \to 0$.

Using [23, Lemmata 7.1, 7.2], letting $h \to 0$, there exists a map $t \in [0,T] \to C(t)$ which is Hausdorff continuous, $C(t)$ are compact convex sets, with $C(0) = C$, such that (up to a subsequence) $\lim_{h \to 0} \sup_{t \in [0,T]} d_{\mathcal{H}}(C_h(t), C(t)) = 0$. Hence $d_h \to d$ uniformly in $\mathbb{R}^N \times [0,T]$ as $h \to 0$, where $d(x,t) = d_\phi^{C(t)}(x)$ for any $x \in \mathbb{R}^N$ and $t \in [0,T]$.

Observe that, as a consequence of (6.19), we have $d \in \mathrm{Lip}(\mathbb{R}^N \times [0,T])$, and $\frac{\partial d}{\partial t} \leq \frac{N}{r'}$ almost everywhere in $U \times (0,T)$. Therefore, we may also assume that $d_h \rightharpoonup d$ weakly in $H^1(U \times (0,T))$ as $h \to 0$.

Possibly passing to a further subsequence, using (6.15) and (6.16), we may assume that $n_h \rightharpoonup n$ weakly in $L^2(U \times (0,T)); \mathbb{R}^N)$ and $\mathrm{div}\, n_h \rightharpoonup \mathrm{div}\, n$ weakly in $L^2(U \times (0,T))$ as $h \to 0$. Recalling (6.17) we conclude

$$\left| \frac{\partial d}{\partial t} - \mathrm{div}\, n \right| \leq c|d| \quad \text{in } U \times (0,T), \tag{6.21}$$

$$|\mathrm{div}\, n| \leq \frac{4}{r'} \quad \text{in } U \times (0,T). \tag{6.22}$$

Moreover, by Lemma 3.6 $C(t)$ satisfies the $r'\mathcal{W}_\phi$-condition for any $t \in [0,T]$, hence, by Proposition 3.9, $C(t)$ is Lipschitz ϕ-regular for any $t \in [0,T]$. The map $t \to C(t)$ is therefore a ϕ-regular flor on $[0,T]$ starting from C, and this concludes the proof of the theorem. □

Remark 6.2.

(i) Since ϕ-regular flows are unique by Theorem 2.13, we get that $\lim_{h\to 0} d_{\mathcal{H}}(C_h(t), C(t)) = 0$ (without extracting a subsequence).

(ii) From (3.2) we obtain $\mathrm{div}\, n \geq 0$ almost everywhere in $U \times [0,T]$.

Remark 6.3. (i) Arguing as in [23, Theorem 3] and using the uniqueness of solutions of (4.2), it follows that the convex sets C_i constructed in the proof of Theorem 6.1 can be written as $C_{i+1} = T_{\phi,h}(C_i)$, for any $i \leq T/h$, where the corresponding functions u_i are the uniform limits in $\mathbb{R}^N$ as $\epsilon \to 0$, of the functions $u_{\epsilon,i}$.

(ii) Let u_i be as in (i), and let us define $u_h(x,t) := u_{[t/h]}(x)$ for any $x \in \mathbb{R}^N$ and $t \in [0,T]$ and $z_h(x,t) := z_{[t/h]}(x)$ for almost every $(x,t) \in \mathbb{R}^N \times (0,T)$. Using Lemma 5.2 it follows

$$\left| \frac{d_\phi^{C_{i+1}} - d_\phi^{C_i}}{h} - \mathrm{div}\, z_i \right| \leq c|d_\phi^{C_{i+1}}| \quad \text{in } U,\ i \leq T/h. \tag{6.23}$$

Indeed, using Lemma 5.3 with $\delta' = \frac{\delta}{N}$ and $\delta = r'$, we have $d_\phi^{C_i} \leq u_i \leq d_\phi^{C_i} + ch$ and $d_\phi^{C_i} \leq d_\phi^{C_{i+1}} \leq d_\phi^{C_i} + ch$ in U, where $c = \frac{N}{r'}$. Hence $|u_i - d_\phi^{C_{i+1}}| \leq ch$ in U. Then, by (6.7) with $v = u_i$, we have

$$|u_i - d_\phi^{C_{i+1}}| \leq c|d_\phi^{C_{i+1}} h| \qquad \text{in } U. \tag{6.24}$$

Hence (6.23) follows from (6.10).

(iii) Reasoning as in the proof of Theorem 6.1 and letting $h \to 0$, from (6.23) and (6.24) we obtain that $u_h \to d$ uniformly, $z_h \rightharpoonup z$ (up to a subsequence) weakly in $L^2(U \times (0,T); \mathbb{R}^N)$ and $\mathrm{div}\, z_h \rightharpoonup \mathrm{div}\, z$ weakly in $L^2(U \times (0,T))$, with z satisfying the same properties as n, in particular

$$\left| \frac{\partial d}{\partial t} - \mathrm{div}\, z \right| \leq c|d| \quad \text{in } U \times (0,T), \tag{6.25}$$

$$|\mathrm{div}\, z| \leq \frac{4}{r'} \quad \text{in } U \times (0,T). \tag{6.26}$$

Moreover, from $\operatorname{div} z_h \geq 0$ almost everywhere in $\mathbb{R}^N \times [0,T]$, we obtain $\operatorname{div} z \geq 0$ almost everywhere in $U \times [0,T]$. It follows that, even if in general n and z may not coincide, we have

$$|\operatorname{div} z - \operatorname{div} n| \leq c|d| \qquad \text{in } U \times (0,T).$$

Remark 6.4. If we define $C_{\epsilon,h}(t) := C_{\epsilon,[t/h]}$, for $t \in [0,T]$, for fixed $\epsilon > 0$ we can pass to the limit as $h \to 0$ and get the (unique) ϕ_ϵ-regular flow $t \to [0,T] \to C_\epsilon(t)$ on $[0,T]$ starting from $C_{0,\epsilon}$. Then, since our estimates are independent of ϵ, we deduce $\lim_{\epsilon\to 0} d_{\mathcal{H}}(C_\epsilon(t), C(t)) = 0$ for any $t \in [0,T]$, i.e. we can approximate the ϕ-regular flow starting from C with ϕ_ϵ-regular flows.

Iterating the construction in the proof of Theorem 6.1, we can extend the flow starting from C to a maximal time interval $[0, T_{\max})$, with $\lim_{t\uparrow T_{\max}} r(t) = 0$, where

$$r(t) := \sup\{r > 0 : C(t) \text{ satisfies the } r\mathcal{W}_\phi - \text{condition}\}.$$

Corollary 6.5. *We have*

$$\sqrt{T_{\max} - t} \geq \frac{r(t)}{4\sqrt{2N}(N-1)}, \qquad t \in [0, T_{\max}).$$

Proof. It follows by iterating the proof of Theorem 6.1, recalling that $T = \frac{r^2}{32(N-1)^2 N}$, see (6.1). □

The following proposition shows that n and z are in some sense canonical.

Proposition 6.6. *For almost every $t \in (0,T)$ the vector fields $n(\cdot,t)$ and $z(\cdot,t)$ solve the following minimum problem:*

$$\liminf_{\delta\to 0^+} \ \min \left\{ \frac{1}{2\delta} \int_{\{|d(\cdot,t)|<\delta\}} (\operatorname{div} Z)^2 \, dx : Z \in X_\delta(t) \right\}, \tag{6.27}$$

where

$$X_\delta(t) := \Big\{ Z : \{|d(\cdot,t)| < \delta\} \to \mathbb{R}^N, \ Z \in \partial\phi^\circ(\nabla d(\cdot,t)) \text{ a.e. in } \{|d(\cdot,t)| < \delta\} \Big\}.$$

Proof. Let $d(t) := d(\cdot,t)$ and let $\delta > 0$ be such that $U_\delta(t) := \{|d(t)| < \delta\} \subset\subset U$. Let $f \in L^\infty(U \times (0,T))$ be such that

$$\frac{\partial d}{\partial t} - \operatorname{div} n = fd \qquad \text{in } U \times (0,T). \tag{6.28}$$

Let us prove that for almost every $t \in (0,T)$ we have

$$\int_{U_\delta(t)} (\operatorname{div} n + fd)^2 \, dx = \min \left\{ \int_{U_\delta(t)} (\operatorname{div} Z + fd)^2 \, dx : Z \in X_\delta(t) \right\}. \tag{6.29}$$

We proceed along the lines of the proof of [20, Theorem 3.5]. We take $t_0 \in (0,T)$ such that $\frac{\partial d}{\partial t}$ and $\operatorname{div} n$ both exist at t_0 as functions of $L^2(U_\delta(t_0))$, and such that t_0 is a Lebesgue point of $f(\cdot,t)1_{U_\delta(t)}$ as a function of t with values in $L^2(\mathbb{R}^N)$. Let us denote by $f(t_0)$ the Lebesgue value of $f(\cdot,t)$ at $t = t_0$. Let us denote by $\tilde{n}$ a solution of

$$\min \left\{ \int_{U_\delta(t_0)} (\operatorname{div} Z + f(t_0)d(t_0))^2 \, dx : Z \in X_\delta(t_0) \right\}.$$

Observe that there exists $h > 0$ such that

$$\frac{\partial}{\partial t}(d(t) - d(t_0)) - (\operatorname{div} n(t) - \operatorname{div} \tilde{n}) = f(t)d(t) + \operatorname{div} \tilde{n} \qquad \text{in } \{(x,t) : t \in]t_0, t_0 + h[,\, x \in U_\delta(t)\}. \quad (6.30)$$

Possibly reducing h, we can find $\delta' \in (0, \delta)$ such that $U_{\delta'}(t_0) \subseteq U_\delta(t)$ for any $t \in]t_0, t_0 + h[$. Let $\epsilon, \alpha > 0$, and let

$$0 \le \eta(r) := \begin{cases} 0 & \text{if } r \le -\delta\,, \\ \frac{(r+\delta)^2}{\epsilon^2} & \text{if } -\delta \le r \le -\delta + \epsilon\,, \\ 1 & \text{if } r \ge -\delta + \epsilon. \end{cases}$$

Define $A_{\delta'}(t_0) := \{d(t) > -\delta\} \cap \{d(t_0) < \delta'\}$. Let us multiply (6.30) by $(d(t) - d(t_0))\eta(d(t))$, for $t \in]t_0, t_0 + h[$, and integrate in $A_{\delta'}(t_0)$. We have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{A_{\delta'}(t_0)} (d(t) - d(t_0))^2 \eta(d(t))\, dx &= \int_{A_{\delta'}(t_0)} \frac{\partial}{\partial t}(d(t) - d(t_0))(d(t) - d(t_0))\eta(d(t))\, dx \\ &\quad + \int_{A_{\delta'}(t_0)} (d(t) - d(t_0))^2 \eta'(d(t)) \frac{\partial d}{\partial t}(t)\, dx \\ &= \int_{A_{\delta'}(t_0)} (\operatorname{div} n - \operatorname{div} \tilde{n})(d(t) - d(t_0))\eta(d(t))\, dx \\ &\quad + \int_{A_{\delta'}(t_0)} (f(t)d(t) + \operatorname{div} \tilde{n})(d(t) - d(t_0))\eta(d(t))\, dx \\ &\quad + \int_{A_{\delta'}(t_0)} (d(t) - d(t_0))^2 \eta'(d(t)) \frac{\partial d}{\partial t}(t)\, dx =: \mathrm{I} + \mathrm{II} + \mathrm{III}. \end{aligned}$$

Let us prove that $\mathrm{I} \le 0$. We have

$$\begin{aligned} \mathrm{I} = \; & - \int_{A_{\delta'}(t_0)} (n - \tilde{n}) \cdot \nabla(d(t) - d(t_0))\, \eta(d(t))\, dx \\ & - \int_{A_{\delta'}(t_0)} (n - \tilde{n}) \cdot \nabla d(t)\, (d(t) - d(t_0))\eta'(d(t))\, dx \\ & + \int_{\partial A_{\delta'}(t_0)} (n - \tilde{n}) \cdot \nu^{A_{\delta'}(t_0)} (d(t) - d(t_0))\eta(d(t))\, d\mathcal{H}^{N-1} \le 0, \end{aligned}$$

since the three expressions on the right hand side are negative, recalling that $d(t) \ge d(t_0)$ for $t \ge t_0$, $\eta' \ge 0$, $n \cdot \nabla d(t) = \phi^\circ(\nabla d(t)) \ge \tilde{n} \cdot \nabla d(t)$, $\tilde{n} \cdot \nabla d(t_0) = \phi^\circ(\nabla d(t_0))$. The third expression is negative since $\eta(d(t)) = 0$ on $\{d(t) = -\delta\}$ and we may use the same argument as in Lemma 5.4 when dealing with $\{d(t_0) = \delta'\}$. Thus, we have

$$\frac{1}{2}\frac{d}{dt}\int_{A_{\delta'}(t_0)} (d(t) - d(t_0))^2 \eta(d(t))\, dx \le \mathrm{II} + \mathrm{III}.$$

Integrating on $]t_0, t_0 + h[$, we obtain

$$\frac{1}{2}\int_{A_{\delta'}(t_0)} (d(t_0 + h) - d(t_0))^2 \eta(d(t_0 + h))dx \le \int_{t_0}^{t_0+h} (\mathrm{II} + \mathrm{III})\; dt.$$

By [20, Lemma A.5] we obtain that

$$
\begin{aligned}
&\left(\int_{A_{\delta'}(t_0)} (d(t_0+h)-d(t_0))^2 \eta(d(t_0+h))\,dx\right)^{1/2} \\
\leq\ & \int_{t_0}^{t_0+h} \left(\int_{A_{\delta'}(t_0)} (f(t)d(t)+\operatorname{div}\tilde{n})^2 \eta(d(t))\,dx\right)^{1/2} dt \\
&+ \int_{t_0}^{t_0+h} \left(\int_{A_{\delta'}(t_0)} (f(t)d(t)+\operatorname{div} n)^2 \frac{(\eta'(d(t)))^2}{\eta(d(t))} (d(t)-d(t_0))^2\,dx\right)^{1/2} dt
\end{aligned}
$$

Since $\frac{(\eta'(d(t)))^2}{\eta(d(t))} \leq \frac{4}{\epsilon^2}$ in $A_{\delta'}(t_0)$, dividing by $h>0$ and letting $h \to 0^+$, we obtain

$$
\int_{A_{\delta'}(t_0)} \left(\frac{\partial d}{\partial t}(t_0)\right)^2 \eta(d(t_0))\,dx \leq \int_{A_{\delta'}(t_0)} (f(t_0)d(t_0)+\operatorname{div}\tilde{n})^2 \eta(d(t_0))\,dx
$$

Now, we let $\epsilon \to 0$, and $\delta' \to \delta$, to obtain

$$
\int_{U_\delta(t_0)} \left(\frac{\partial d}{\partial t}(t_0)\right)^2 dx \leq \int_{U_\delta(t_0)} (f(t_0)d(t_0)+\operatorname{div}\tilde{n})^2\,dx.
$$

Using (6.28), this gives (6.29).

To prove (6.27), fix $Z \in X_\delta(t)$. Using Hölder inequality and $2ab \leq \delta a^2 + \frac{b^2}{\delta}$, it follows

$$
\begin{aligned}
\int_{U_\delta(t)} (\operatorname{div} n)^2\,dx &\leq \int_{U_\delta(t)} (\operatorname{div} Z + fd)^2\,dx + c(r')\delta^2 \\
&\leq \int_{U_\delta(t)} (\operatorname{div} Z)^2\,dx + 2\int_{U_\delta(t)} fd\,\operatorname{div} Z\,dx + c(r')\delta^2 \\
&\leq (1+\delta)\int_{U_\delta(t)} (\operatorname{div} Z)^2\,dx + c(r')\delta^2,
\end{aligned}
$$

for a constant $c = c(r')$ independent of δ (which may vary from line to line). The assertion on the minimality of n follows by dividing by 2δ and letting $\delta \to 0^+$. Finally, since the vector field z satisfies (6.25), its minimality follows as above. □

We expect that, if E is a Lipschitz ϕ-regular set, the constant defined by the minimum problem in (6.27) (with $d(\cdot,t)$ replaced by d_ϕ^E) coincides with the square of the L^2-norm of the ϕ-mean curvature of ∂E as defined in [14].

7 A volume estimate in time

In this section we prove estimate (7.2) that involves the transformation $T_{\phi,h}$ introduced in (4.8), and deduce an estimate for the decay of the volume of a convex ϕ-regular flow. Before proving Lemma 7.2, let us show the following result on which it is based. For nonsmooth anisotropies ϕ (such as in the crystalline case), given a Lipschitz ϕ-regular set (E,n) the quantity $\int_{\partial E} \operatorname{div} n\, d\mathcal{P}_\phi$ does not depend on the choice of the vector field n [14, Lemma 4.4].

Lemma 7.1. *Let* $\lambda > 0$ *and set* $\rho := (\lambda/|\mathcal{W}_\phi|)^{1/N}$*. Then* $\rho\mathcal{W}_\phi$ *is a minimizer of*

$$\min\left\{\int_{\partial E} \operatorname{div} n\, d\mathcal{P}_\phi : E \text{ compact, convex}, (E,n) \text{ Lipschitz } \phi-\text{regular},\ |E| = \lambda\right\}. \tag{7.1}$$

Proof. For any family of convex bodies $K_1, \ldots, K_N \subseteq \mathbb{R}^N$, let $V(K_1, \ldots, K_N)$ denote the Minkowski mixed volume. If K_1, K_2 are two convex bodies, let $V_j(K_1, K_2) = V(K_1, \ldots, K_1, K_2, \ldots, K_2)$ where K_1 appears j times and K_2 appears $(N-j)$ times.

Let E be a compact convex Lipschitz ϕ-regular set such that $|E| = \lambda$. Observe that using [50, (6.8.8)] we have

$$V_{N-2}(E, \mathcal{W}_\phi) \geq |\mathcal{W}_\phi|^{2/N}|E|^{(N-2)/N}.$$

Hence, setting $W := \rho\mathcal{W}_\phi$ and using [14, Theorem 5.1], we have

$$\begin{aligned}
\int_{\partial E} \operatorname{div} n\, d\mathcal{P}_\phi &= N(N-1)V_{N-2}(E, \mathcal{W}_\phi) \geq N(N-1)|\mathcal{W}_\phi|^{2/N}|E|^{(N-2)/N} \\
&= N(N-1)|\mathcal{W}_\phi|^{2/N}|W|^{(N-2)/N} = N(N-1)|\mathcal{W}_\phi|\rho^{N-2} = \int_{\partial W} \operatorname{div} \hat{n}\, d\mathcal{P}_\phi,
\end{aligned}$$

where $\hat{n}(x) := x/\phi(x)$. □

Lemma 7.2. *Let* ϕ *be an anisotropy. Let* C *be a compact convex set with nonempty interior, and let* $h < (N+1)r^2/(4N^2)$*, where* r *is the radius of a Wulff shape contained in* C*. Then*

$$|C| - |T_{\phi,h}(C)| \geq hN(N-1)|\mathcal{W}_\phi|^{2/N}|T_{\phi,h}(C)|^{(N-2)/N} \tag{7.2}$$

Proof. Assume first $\phi \in \mathcal{C}^\infty_+$. If $h < (N+1)r^2/(4N^2)$, then by Lemma 5.5, the convex set $\widehat{C} := T_{\phi,h}(C)$ has $\mathcal{C}^{1,1}$ boundary. As in Lemma 5.5 we denote by (u, z) the solution of (4.2). We first estimate $|C| - |\widehat{C}|$ with an integral on the boundary $\partial\widehat{C}$. The field $z = \nabla\phi^\circ(\nabla u)$ is Lipschitz near $\partial\widehat{C}$ and coincides with $n^{\widehat{C}}_\phi$ on $\partial\widehat{C}$. Let us define the one-to-one map $F : \partial\widehat{C} \times [0, \infty) \to \mathbb{R}^N \setminus \operatorname{int}(\widehat{C})$ by $F(y, s) = y + sn^{\widehat{C}}_\phi(y)$. Then the inverse map of F is given by $G(x) = (\pi^{\widehat{C}}_\phi(x), d_\phi(x, \partial\widehat{C}))$. We observe that ∂C can be written, in the (y, s) coordinates, as the graph of the map $f : \partial\widehat{C} \to [0, +\infty)$ such that $\pi^{\widehat{C}}_\phi(y + f(y)n^{\widehat{C}}_\phi(y)) = y$. In particular, we have $f(y) \geq d_\phi(y, \partial C) = -d^C_\phi(y)$. Hence

$$|C| - |\widehat{C}| = \int_{\partial\widehat{C}} \int_0^{f(y)} J(y, s)\, ds\, d\mathcal{H}^{N-1}(y) \tag{7.3}$$

where $J(y, s)$ is the Jacobian of the map F. Notice that, letting I the $(n \times (n-1))$-matrix with elements $\mathrm{I}_{ij} = \delta_{ij}$, we have

$$J(y, s) = \det\Big(\mathrm{I} + s\nabla n^{\widehat{C}}_\phi(y) \bigm| n^{\widehat{C}}_\phi(y)\Big),$$

where $\Big(\mathrm{I} + s\nabla n^{\widehat{C}}_\phi(y) \bigm| n^{\widehat{C}}_\phi(y)\Big)$ is the $(n \times n)$-matrix composed by the $(n \times (n-1))$-matrix $\mathrm{I} + s\nabla n^{\widehat{C}}_\phi(y)$ and having $n^{\widehat{C}}_\phi(y)$ as last column. Recalling (3.8), a direct computation gives

$$\frac{d}{ds}\, J(y, s) = J(y, s)\ \operatorname{div} n^{\widehat{C}}_\phi\left(y + sn^{\widehat{C}}_\phi(y)\right),$$

for any s and for almost every $y \in \partial\widehat{C}$. Since $\operatorname{div} n_\phi^{\widehat{C}} \geq 0$ on $\mathbb{R}^N \setminus \operatorname{int}(\widehat{C})$, we obtain that $J(y,s)$ is increasing in s, which implies $J(y,s) \geq J(y,0) = \phi^\circ(\nu^{\widehat{C}}(y))$. From (7.3), we deduce that

$$|C| - |\widehat{C}| \ \geq\ -\int_{\partial\widehat{C}} d_\phi^C \phi^\circ(\nu^{\widehat{C}})\, d\mathcal{H}^{N-1}.$$

Since $-d_\phi^C(y) = \operatorname{div} z(y) = \operatorname{div} n_\phi^{\widehat{C}}(y)$ at any $y \in \partial\widehat{C}$, we obtain

$$|C| - |\widehat{C}| \ \geq\ \int_{\partial\widehat{C}} \operatorname{div} n_\phi^{\widehat{C}} \phi^\circ(\nu^{\widehat{C}})\, d\mathcal{H}^{N-1} \ \geq\ N(N-1)|\mathcal{W}_\phi|^{2/N}|\widehat{C}|^{(N-2)/N},$$

where the last inequality follows from Lemma 7.1. This shows (7.2) when $\phi \in \mathcal{C}_+^\infty$. In the general case, we can approximate ϕ as in Lemma 3.8 by smooth anisotropies ϕ_ϵ (it is not necessary here to smoothen as well C). We then observe that (7.2) is stable under the limit $\epsilon \to 0$, as well as the condition $h < (N+1)r^2/(4N^2)$. $\square$

We deduce the following result.

Theorem 7.3. *Let ϕ be an anisotropy. Let $C = C(0)$ be a compact convex set satisfying the $r\mathcal{W}_\phi$-condition for some $r > 0$. Let $C(t)$ be the ϕ-regular flow on $[0,T]$ starting from C constructed in Theorem 6.1. Then*

$$|C(t_2)|^{2/N} \leq |C(t_1)|^{2/N} - 2(N-1)|\mathcal{W}_\phi|^{2/N}(t_2 - t_1) \qquad 0 \leq t_1 \leq t_2 \leq T. \tag{7.4}$$

Proof. Let i, (u_i, z_i) and C_i be as in the proof of Theorem 6.1. Let $c := 2(N-1)|\mathcal{W}_\phi|^{2/N}$. If h is small enough, then (7.2) is valid for C_i and $C_{i+1} = T_{h,\phi}(C_i)$ as long as $hi < T$. By Taylor expansion, we obtain for any $\epsilon > 0$,

$$|C_{i+1}|^{2/N} \leq |C_i|^{2/N} - (1+\epsilon)ch \quad \text{if } |C_{i+1}|^{2/N} \geq c(\epsilon)h \tag{7.5}$$

for some positive constant $c(\epsilon)$. Iterating (7.5), we may write

$$|C_h(t_2)|^{2/N} \leq |C_h(t_1)|^{2/N} - (1+\epsilon)c\left(\left[\frac{t_2}{h}\right] - \left[\frac{t_2}{h}\right]\right)h \quad \text{if } |C_h(t_2)|^{2/N} \geq c(\epsilon)h. \tag{7.6}$$

Passing to the limit as $h \to 0$ we show that $C(t)$ satisfies (7.4). $\square$

8 Evolution from an arbitrary convex initial data

In this section we prove Theorem 1.2; we will use our construction of ϕ-regular convex flows to define in a unique way the ϕ-curvature flow starting from any compact convex initial set. Thanks to Proposition 4.14, this also proves the uniqueness of the flat ϕ-curvature flow of Almgren, Taylor and Wang [2] (with a different mobility).

Let us recall some results proved in [23]. Given a compact convex set C, let $C_h(t) := T_{\phi,h}^{[t/h]}(C)$ and $C_h := \{(x,t) : t \geq 0, x \in C_h(t)\}$ (compare with Definition 4.11 and (6.20)). Up to a subsequence $\{h_k\}$, C_{h_k} converges in the Hausdorff sense to a set $C^* \subset \mathbb{R}^N \times [0,+\infty)$, while $(\mathbb{R}^N \times [0,+\infty)) \setminus C_{h_k}$ converges in the Hausdorff sense to $(\mathbb{R}^N \times [0,+\infty)) \setminus C_*$, and clearly $C_* \subseteq C^*$. For any $t \geq 0$ let

$$C_*(t) := \{x : (x,t) \in C_*\}, \qquad C^*(t) := \{x : (x,t) \in C^*\}.$$

It is shown that $C_* = \text{int}(C^*)$ and $C_*(t)$ and $C^*(t)$ are both convex sets (respectively open and closed) for any $t \geq 0$. Define

$$t_C := \inf\{t \geq 0 : C_*(t) = \emptyset\}, \qquad \bar{t}_C := \inf\{t \geq 0 : C^*(t) = \emptyset\}.$$

We clearly have $\bar{t}_C \geq t_C$, moreover, since the set C^* is close and C_* is open, we have $C^*(\bar{t}_C) \neq \emptyset$ whereas $C_*(t_C) = \emptyset$.

The following result is proved in [23, Lemma 7.1].

Lemma 8.1. *For any $t \in [0, t_C)$ we have*

$$\lim_{k\to+\infty} d_{\mathcal{H}}(C_{h_k}(t), C^*(t)) = 0, \qquad \lim_{k\to+\infty} d_{\mathcal{H}}(\mathbb{R}^N \setminus C_{h_k}(t), \mathbb{R}^N \setminus C_*(t)) = 0,$$

$C^*(t) = \overline{C_*(t)}$ *and* $C_*(t) = \text{int}(C^*(t))$. *In particular,* $\lim_{k\to+\infty} d_{\mathcal{H}}(\partial C_{h_k}(t), \partial C(t)) = 0$.

This means that up to t_C, $C^*(t)$ and $C_*(t)$ are essentially the same convex set, whereas after t_C, $C_*(t)$ is empty and $C^*(t)$ could still exist as a convex set of lower dimension (up to $\bar{t}_C$). In fact, we will show that $\bar{t}_C = t_C$, so that both sets vanish simultaneously. Observe that, if the initial set C satisfies the $r\mathcal{W}_\phi$-condition for some $r > 0$, then the convergence of the subsequence $\{C_{h_k}(t)\}$ implies that the evolution $C(t)$ of Theorem 6.1 and the evolution $C^*(t)$ coincide on $[0, T)$. With a slight abuse of notation, from now on the evolution $C^*(t)$, for $t \in [0, +\infty)$, will be denoted by $C(t)$ (notice that this is consistent with the notation of Theorems 1.1, 1.2), and will be addressed as a ϕ-flow.

The flat ϕ-curvature flow of Almgren, Taylor and Wang corresponds to the $L^\infty([0, +\infty); L^1(\mathbb{R}^N))$ limit of a subsequence of $(\chi_{C_h(t)})_{h>0}$. However it is clear that $\chi_{C_{h_k}(t)}(x) \to 1$ in $C_*(t)$ whereas $\chi_{C_{h_k}(t)}(x) \to 0$ out of $C(t)$, so that, since $|C(t) \setminus C_*(t)| = 0$ for any t, both C^* and C_* are flat ϕ-curvature flows in the sense of [2]. Conversely, one checks that given a flat ϕ-curvature flow of [2], there exists a corresponding pair (C_*, C^*) of sets which coincides with this flat ϕ-curvature flow up to a negligible set.

A first important observation is that estimate (7.4) in Theorem 7.3 also holds for a flow $C(t)$, up to t_C: indeed, the proof is the same, based on estimate (7.2). In particular, we have that $t_C \leq |C|^{2/N}/(2(N-1)|\mathcal{W}_\phi|^{2/N})$.

Let us now show the following comparison lemma for two ϕ-flows starting from two convex sets satisfying a strict inclusion.

Lemma 8.2. *Let C_1, C_2 be two compact convex sets with $C_1 \subset C_2$. Assume that*

$$\delta := d_\phi(\partial C_1, \partial C_2) > 0.$$

Let $C_1(t)$ and $C_2(t)$ be two flows (as described above) starting respectively from C_1 and C_2, and let $\delta(t) := d_\phi(\partial C_1(t), \partial C_2(t))$, which is well-defined for $t \in [0, \bar{t}_{C_1}]$. Then $\delta(t)$ is nondecreasing on $[0, \bar{t}_{C_1}]$.

Proof. Since $\delta > 0$, the set C_2 has nonempty interior. Let us show that if for some $t \geq 0$ it happens that $\delta(t) \geq \delta$, then for any $\tau \in [0, \tau_\delta]$, with $\tau_\delta := \delta^2/(64N(N-1)^2)$, we have $\delta(t+\tau) \geq \delta(t)$ if $t + \tau \leq \bar{t}_{C_1}$. By induction, this gives the thesis of the lemma.

We let $Q := C_1(t) + (\delta(t)/2)\mathcal{W}_\phi$. By construction, $d_\phi(C_1(t), \partial Q) = \min\{\phi(x-y) : x \in C_1(t), y \in \partial Q\} = \delta(t)/2$, whereas $d_\phi(\partial Q, \mathbb{R}^N \setminus C_2(t)) \geq \delta(t)/2$. If $x \in \partial C_1(t)$ and $y \in \partial C_2(t)$ are such that

$\phi(x-y) = d_\phi(C_1(t), \mathbb{R}^N \setminus C_2(t)) = \delta(t)$, then $(x+y)/2 \in Q$ and $\phi(y-(x+y)/2) = \delta(t)/2$, showing that in fact $d_\phi(\partial Q, \mathbb{R}^N \setminus C_2(t)) = \delta(t)/2$.

The set Q satisfies the $(\delta/2)\mathcal{W}_\phi$-condition set, hence its ϕ-regular flow $Q(\tau)$ exists for $\tau \le \tau_\delta$ and coincides, in this time interval, with any ϕ-flow starting from Q (hence the ϕ-flow starting from Q is uniquely defined for $\tau \le \tau_\delta$). Let us consider a sequence $\{h_k\}$ such that the set $(C_1)_{h_k} := \{(x,t) : x \in T_{\phi,h_k}^{[t/h_k]} C_1\}$ converges to C_1^* in the Hausdorff sense in $\mathbb{R}^N \times [0,+\infty)$. Choose ξ such that $\phi(\xi) < \delta(t)/2$. Then $\xi + C_1(t) \subset\subset Q$, so that for k large enough, $\xi + T_{\phi,h_k}^{[t/h_k]}(C_1) \subseteq Q$. For any $\tau \in [0,\tau_\delta]$, it follows

$$\xi + T_{\phi,h_k}^{[(t+\tau)/h_k]}(C_1) \ \subseteq \ T_{\phi,h_k}^{[(t+\tau)/h_k]-[t/h_k]}(Q).$$

Since $h_k\left([(t+\tau)/h_k] - [t/h_k]\right) = \tau + O(h_k)$, one checks that (*cfr.* the proof of Theorem 6.1) the set $\{(x,\tau) : 0 \le \tau \le \tau_\delta, x \in T_{\phi,h_k}^{[(t+\tau)/h_k]-[t/h_k]} Q\}$ converges in the Hausdorff sense in $\mathbb{R}^N \times [0,\tau_\delta]$ to $\{(x,\tau) : x \in Q(\tau)\}$. On the other hand, it is possible to check that the Hausdorff limit of any subsequence of $((\xi,0) + (C_1)_{h_k}) \cap (\mathbb{R}^N \times [t, t+\tau_\delta])$ coincides with $(\xi,0) + C_1^*$ in $\mathbb{R}^N \times (t, t+\tau_\delta)$. One deduces that as long as $\tau \le \tau_\delta$,

$$\xi + C_1(t+\tau) \subseteq Q(\tau). \tag{8.1}$$

Since this is true for any ξ with $\phi(\xi) < \delta(t)/2$, we deduce that $d_\phi(\partial C_1(t+\tau), \partial Q(\tau)) \ge \delta(t)/2$, as long as $\partial C_1(t+\tau) \ne \emptyset$, that is, if $t + \tau \le \bar{t}_{C_1}$. A similar proof will shows that for any $\tau \in [0,\tau_\delta]$, $d_\phi(\partial Q(\tau), \partial C_2(t+\tau)) \ge \delta(t)/2$. Since $\partial Q(\tau)$ separates $C_1(t+\tau)$ and $C_2(t+\tau)$, we deduce $d_\phi(\partial C_1(t+\tau), \partial C_2(t+\tau)) \ge \delta(t)$, as long as $t+\tau \le \bar{t}_{C_1}$.

It follows that if $\delta = \delta(0) > 0$ the function $\delta(t)$ is nondecreasing in $[0, \bar{t}_{C_1}]$. In particular, we have $C_1(t) \subseteq \mathrm{int}(C_2(t))$ for any $t \ge 0$. □

Lemma 8.3. *Let $C \subset \mathbb{R}^N$ be a compact convex set and let $C(t)$, $t \ge 0$, be a ϕ- flow starting from C. Let $\theta > 0$. Then $t \mapsto \theta C(t/\theta^2)$ is a ϕ-flow starting from θC.*

Proof. Notice that the signed distance to θC is given by $d_\phi^{\theta C}(x) = \theta d_\phi^C(x/\theta)$. Let $h > 0$ and u solve

$$-h \mathrm{div}\, \partial\phi^\circ(\nabla u) \,+\, u \,\ni\, d_\phi^C$$

in $\mathbb{R}^N$. Then $u_\theta(x) := \theta u(x/\theta)$ solves

$$-\theta^2 h \mathrm{div}\, \partial\phi^\circ(\nabla u_\theta) \,+\, u_\theta \,\ni\, d_\phi^{\theta C}\,.$$

Hence, $\theta T_{\phi,h}(C) = T_{\phi,\theta^2 h}(\theta C)$, and the lemma follows. □

Theorem 8.4. *Let C_1, C_2 be two compact convex sets, and assume that $C_1 \subseteq C_2$. Let $C_1(t)$ and $C_2(t)$ be two ϕ-flows starting from C_1 and C_2 respectively. Then*

$$C_1(t) \subseteq C_2(t) \qquad \forall t \ge 0.$$

Proof. Let $\delta := d_\phi(\partial C_1, \partial C_2) > 0$. If $\delta > 0$ then the thesis follows from Lemma 8.2, hence we can suppose $\delta = 0$. Let us first consider the case $|C_2| > 0$. Since C_2 has non empty interior, we may assume without loss of generality that 0 is in the interior of C_2. Then $C_1 \subset\subset \theta C_2$ for any $\theta > 1$. From Lemmata 8.3 and 8.2, we deduce that $C_1(t) \subseteq \mathrm{int}(\theta C_2(t/\theta^2))$ for all $t \ge 0$. The left continuity of $\partial C_2(\cdot)$ at any $t \ge 0$ [23, Lemma 7.2] implies that $C_1(t) \subseteq C_2(t)$ for any $t \ge 0$.

Let $t' \in (t_{C_2}, \bar{t}_{C_2}]$ and $t > t'$. Observe that $C_1(t) = \emptyset$. Indeed, since $C_2(t')$ has empty interior, then $\theta C_2(t/\theta^2)$ has empty interior for $\theta^2 < t/t'$. Therefore $C_1(t) \subseteq \mathrm{int}(\theta C_2(t/\theta^2))$ must be empty.

In particular, taking $C := C_1 = C_2$, we find that if t is larger than the extinction time of the interior of C, then $C(t) = \emptyset$, i.e. the extinction time of the interior of the ϕ-flow $C(t)$ is the same as the extinction time of the flow itself (i.e., the flow can not proceed for a while with empty interior). In other words $t_C = \bar{t}_C$, provided $|C(0)| > 0$.

Now we assume that $|C_2| = 0$. As we observed in Lemma 8.2, for any compact convex set $C' \supset\supset C_2$, if $C'(t)$ is a ϕ-flow starting from C' we have that $C_2(t) \subseteq \mathrm{int}(C'(t))$. Since (7.4) also holds for a ϕ-flow up to the extinction time of its interior, we have that $\mathrm{int}(C'(t)) = \emptyset$ for any $t \geq |C'|^{2/N}/(2(N-1)|\mathcal{W}_\phi|^{2/N})$, and, by the previous statement, also $C_2(t) = \emptyset$ for any $t \geq |C'|^{2/N}/(2(N-1)|\mathcal{W}_\phi|^{2/N})$. Since the volume $|C'|$ can be taken arbitrarily small, we deduce $\bar{t}_{C_2} = 0$ and the proof is complete. □

Corollary 8.5. *The ϕ-flow starting from a compact convex set C is unique.*

As a byproduct of the proof of Theorem 8.4, we also have shown the following result.

Corollary 8.6. *For any convex ϕ-flow $C(t)$, if $|C(t)| = 0$ for some $t \geq 0$, then $C(s) = \emptyset$ for any $s > t$. In other words, $t_C = \bar{t}_C$.*

Corollary 8.6 and the estimate (7.4) establish that any ϕ-flow $C(t)$ starting from a compact convex set C vanishes beyond the extinction time t_C of its interior, which is estimated with

$$t_C \ \leq \ \frac{|C|^{\frac{2}{N}}}{2(N-1)|\mathcal{W}_\phi|^{\frac{2}{N}}}. \tag{8.2}$$

Observe that if $C = \lambda \mathcal{W}_\phi$, then this estimate is optimal.

Remark 8.7. Observe that, for a generic anisotropy $\phi \notin \mathcal{C}^\infty_+$, the assumption $C_1 \subsetneq C_2$ does not necessarily imply $\partial C_1(t) \cap \partial C_2(t) = \emptyset$ for all $t > 0$ for which both $\partial C_1(t)$ and $\partial C_2(t)$ are nonempty.

Theorem 8.8. *Let $\{C_n\}$ be a sequence of uniformly bounded compact convex sets. Assume that $\{C_n\}$ converges in the Hausdorff distance to a set C. Let t_{C_n} and t_C be the extinction times of the ϕ-flows $C_n(t)$ and $C(t)$, starting respectively from C_n and C. Then $\lim_{n\to\infty} t_{C_n} = t_C$ and*

$$\lim_{n\to+\infty} d_{\mathcal{H}}(C_n(t), C(t)) = 0 \qquad t < t_C.$$

Proof. If C has empty interior, then the assertion follows from estimate (8.2). Otherwise, we may assume 0 is in the interior of C. Let $\theta < 1$: then, if n is large enough, $\theta C \subseteq C_n \subseteq (1/\theta)C$. We deduce that $\theta C(t/\theta^2) \subseteq C_n(t) \subseteq (1/\theta)C(\theta^2 t)$ for any t. In particular $\theta^2 t_C \leq t_{C_n} \leq t_C/\theta^2$, hence $\{t_{C_n}\}$ must converge to t_C. On the other hand, if $t < t_C$, then both $\theta C(t/\theta^2)$ and $(1/\theta)C(\theta^2 t)$ converge to $C(t)$ as $\theta \to 1$ in the Hausdorff distance, so that also $C_n(t)$ must converge to $C(t)$. □

Using Theorems 8.4 and 8.8, we deduce the following result.

Corollary 8.9. *The ϕ-flow defines a continuous and monotone semigroup on compact convex sets.*

Remark 8.10. Theorem 1.1 allows to prove rigorously that the initial set C considered in the second example in [16] develops the bending phenomenon. Indeed, if by contradiction there exists a ϕ-regular flow starting from C whose facets do not bend, then the subsequent evolution must be governed by a system of ODEs, the velocity of each facet F being the quotient of the (anisotropic) perimeter of F and its area. However, such an evolution cannot satisfy the comparison principle, if the frontal facet of C is sufficiently elongated. This is obtained by comparing the evolution with the evolution of a suitable Wulff shape inside C (and using Remark 2.14). It follows that the ODE evolution cannot be the ϕ-regular flow of C.

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