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Crystallization of Random Matrix Orbits

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Three operations on eigenvalues of real/complex/quaternion (corresponding to $\beta = 1, 2, 4$) matrices, obtained from cutting out principal corners, adding, and multiplying matrices, can be extrapolated to general values of $\beta > 0$ through associated special functions. We show that the $\beta \to \infty$ limit for these operations leads to the finite free projection, additive convolution, and multiplicative convolution, respectively. The limit is the most transparent for cutting out the corners, where the joint distribution of the eigenvalues of principal corners of a uniformly-random general β self-adjoint matrix with fixed eigenvalues is known as the β-corners process. We show that as $\beta \to \infty$ these eigenvalues crystallize on an irregular lattice consisting of the roots of derivatives of a single polynomial. In the second order, we observe a version of the discrete Gaussian Free Field put on top of this lattice, which provides a new explanation as to why the (continuous) Gaussian Free Field governs the global asymptotics of random matrix ensembles.

1 Matrix Operations at General β

Fix N and consider two $N \times N$ self-adjoint matrices A_N, B_N with either real, or complex, or quaternion entries. In this article we are mostly interested in the (real) *eigenvalues* of these matrices. In particular, we consider three natural matrix operations, which have

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non-trivial influence on the eigenvalues.

(1) We can cut out the principal top-left $k \times k$ corner of the matrix, $A_N \mapsto A_k$, where A_k is the projection of A_N.
(2) We can add the matrices $(A_N, B_N) \mapsto A_N + B_N$.
(3) We can multiply the matrices $(A_N, B_N) \mapsto A_N B_N$.

In principle, all three operations can be expressed in terms of multiplication, since for small ε, $(1 + \varepsilon A_N)(1 + \varepsilon B_N) \approx 1 + \varepsilon(A_N + B_N)$, thus reducing addition to multiplication (all three operations can be expressed in terms of addition as well, but in a less obvious way, see (7)). On the other hand, if the eigenvalues of B_N are positive, then $A_N B_N$ and $(B_N)^{1/2} A_N (B_N)^{1/2}$ differ by conjugation, and therefore have the same spectrum. If now B_N is the matrix of the projector onto the first k basis vectors, then $(B_N)^{1/2} A_N (B_N)^{1/2}$ is precisely the $k \times k$ corner of A_N, that is, the projection A_k. Nevertheless, we will consider all three operations, as this will provide more insights.

For deterministic matrices, the relations between the spectra of A_N and A_k are folklore; it is given by simple interlacing conditions (cf. [44] and Definition 1.3). The result of the operation $(A_N, B_N) \to A_N + B_N$ on the spectrum is the subject of the celebrated *Horn's (ex-)conjecture*, and similar results are now known for $(A_N, B_N) \to A_N B_N$, see [22] for a review.

Our point of view is different, as we consider *random* A_N, B_N with *invariant* distributions, which means that given the eigenvalues of a matrix, its eigenvectors are conditionally uniform. This is the same as declaring the distribution to be invariant under the action of orthogonal/unitary/symplectic group (depending on the base field) by conjugations—hence, the name.

It suffices to study the case when the eigenvalues of A_N and B_N are *deterministic*, since other cases can be obtained as mixtures. We therefore fix two N-tuples of reals $\mathbf{a} = (a_1, \dots, a_N)$, $\mathbf{b} = (b_1, \dots, b_N)$ and define A_N, B_N, to be *uniformly random independent* matrices with corresponding prescribed eigenvalues.

For each of the three operations on matrices, we arrive at an operation of the eigenvalues, whose result is a *random spectrum*.

(1) $\mathbf{a} \mapsto \pi^{\beta}_{N \to k}(\mathbf{a}) = (a_1^{(k)}, \dots, a_k^{(k)})$, where the latter is the random k (real) eigenvalues of A_k, the corner of A_N;
(2) $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \boxplus_{\beta} \mathbf{b}$, where the latter is the random N (real) eigenvalues of $A_N + B_N$
(3) Assume that all eigenvalues in $\mathbf{a}$, $\mathbf{b}$ are *positive*, and define $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \boxtimes_{\beta} \mathbf{b}$, where the latter is the random N (real) eigenvalues of either $A_N B_N$, or $B_N A_N$, or $(B_N)^{1/2} A_N (B_N)^{1/2}$, or $(A_N)^{1/2} B_N (A_N)^{1/2}$. (The eigenvalues of all four

matrices are the same, and the last two matrices are self-adjoint, which shows that these eigenvalues are real).

The subscript β in the operations $\pi^{\beta}_{N\to k}$, $\boxplus_\beta$, $\boxtimes_\beta$ serves as an indication that they *depend* on whether we deal with real/complex/quaternion matrices, corresponding to $\beta = 1, 2, 4$. More generally, these operations can be *extrapolated* to general values of the real parameter $\beta > 0$. For the projection $\pi^{\beta}_{N\to k}$ the result is known as β-corners process, cf. [9, 25, 44]. For addition $\boxplus_\beta$ and multiplication $\boxtimes_\beta$ this is done by identifying the random eigenvalues with their Laplace-type integral transforms related to multivariate Bessel functions and Heckman–Opdam hypergeometric functions, respectively. The operations then turn into simple multiplication of these special functions and re-expansion of the result in terms of the same functions (There is a tricky point in the definition: the positivity of the coefficients in the re-expansion is a well-known conjecture, which is still open. In the event that the positivity is not true for some values of β, the distributions of $\mathbf{a} \boxplus_\beta \mathbf{b}$, $\mathbf{a} \boxtimes_\beta \mathbf{b}$ might fail to be probability measures, but rather be tempered distributions, i.e., continuous linear functionals on smooth test functions. For $\pi^{\beta}_{N\to k}$ the situation is simpler, as the available explicit formulas make positivity immediate.), see Section 2 for the details.

An alternative, yet conjectural, approach to the operations on random matrices at general values of β has been proposed in [18] where the framework of *β–ghosts and shadows* was developed. The idea there is to treat arbitrary $\beta > 0$ as the dimension of a (typically non-existent) real-division algebra, by expressing all probabilistic properties of interest through Dirichlet distributions. We do not know whether the technique of [18] can be pushed through to the point of reproducing the operations $\boxplus_\beta$, $\boxtimes_\beta$.

Our first result concerns the dependence on β of such operations.

Theorem 1.1. Let z be a formal variable. For fixed $\mathbf{a}$ and $\mathbf{b}$, define the polynomials $Q^{N\to k}(z)$, $Q^{\boxplus}(z)$, $Q^{\boxtimes}(z)$ of degrees k, N, N, respectively, as expected characteristic polynomials of the corresponding matrices, that is,

$$Q^{N\to k}(z) = \mathbb{E} \prod_{\alpha \in \pi^{\beta}_{N\to k}(\mathbf{a})} (z - \alpha), \tag{1}$$

$$Q^{\boxplus}(z) = \mathbb{E} \prod_{\alpha \in \mathbf{a} \boxplus_\beta \mathbf{b}} (z - \alpha) \tag{2}$$

$$Q^{\boxtimes}(z) = \mathbb{E} \prod_{\alpha \in \mathbf{a} \boxtimes_\beta \mathbf{b}} (z - \alpha). \tag{3}$$

Then the polynomials $Q^{N\to k}(z)$, $Q^{\boxplus}(z)$, $Q^{\boxtimes}(z)$ (i.e., their coefficients) do not depend on the choice of $\beta > 0$. They can be computed as follows:

$$Q^{N\to k}(z) = \frac{1}{N(N-1)\cdots(k+1)} \left(\frac{\partial}{\partial z}\right)^{N-k} \prod_{i=1}^{N}(z-a_i) \tag{4}$$

$$Q^{\boxplus}(z) = \frac{1}{N!} \sum_{\sigma\in S_N} \prod_{i=1}^{N}(z-a_i-b_{\sigma(i)}) \tag{5}$$

$$Q^{\boxtimes}(z) = \frac{1}{N!} \sum_{\sigma\in S_N} \prod_{i=1}^{N}(z-a_i b_{\sigma(i)}). \tag{6}$$

The proof of Theorem 1.1 is given in Section 3.2. Note that Theorem 1.1 includes the fact that the expectations of the elementary symmetric functions in the variables $\pi^{\beta}_{N\to k}(\mathbf{a})$, $\mathbf{a} \boxplus_\beta \mathbf{b}$, $\mathbf{a} \boxtimes_\beta \mathbf{b}$ do not depend on β. A similar observation was recently used in [6] in the context of the Macdonald measures (following an earlier observation [35, p. 318] that the application of the Macdonald q-difference operator to the Macdonald reproducing kernel produces an independent in q factor). We believe that this is more than a coincidence, but rather a manifestation of a general phenomenon. Indeed, Macdonald polynomials can be degenerated to Heckman–Opdam and multivariate Bessel functions that we rely on, cf. [9, 62] and discussion in the next section. A parallel degeneration leads to a class of Gibbs probability measures on (continuous) interlacing particle configurations, as studied in [47]. On the other hand, the same article (mostly for $\beta = 2$, see however the very last paragraph there) explains that such measures can be also obtained from β-corners processes with fixed top rows. An extension of Theorem 1.1 to Macdonald polynomials is also explained in Section 4.

For classical Gaussian/Laguerre/Jacobi ensembles of random matrices the independence on β of the expectation of the characteristic polynomial was also previously noticed by some authors, see for example, [9, Lemma 5.3]. Theorem 1.1 in fact holds for a much larger class of distributions. In [37], it is shown that Theorem 1.1 holds for any distribution that is invariant under conjugation by signed permutation matrices, and it has been noted in [53] that the same holds for any distribution that is invariant under conjugation by matrices in the standard representation of S_{N+1}.

The polynomial operations defined by (5) and (6) have a long history in the literature (dating at least back to [68, p. 176]) due to an interest in understanding operations that preserve real rootedness of polynomials. These ideas were more recently extended to the realm of *stable* polynomials (a multivariate version of real rootedness) in a series of works by Borcea and Brändén, see, for example, [4]. One of their results implies that any convolution that treats the coefficients of polynomials linearly can be written using the additive convolution (sometimes at the cost of needing to use extra variables). In particular, for fixed **a** and **b**, one has

$$\sum_{\sigma\in S_N}\prod_{i=1}^{N}(z-a_ib_{\sigma(i)})=\left(\prod_{i=1}^{N}b_i\right)\sum_{\sigma\in S_N}\prod_{i=1}^{N}(y-a_i-z/b_{\sigma(i)})\Big|_{y=0} \tag{7}$$

which, in theory, would allow one to compute $Q^{\boxtimes}(z)$ using $Q^{\boxplus}(z)$. We have not found any advantage to treating both as $Q^{\boxplus}(z)$ (or treating both as $Q^{\boxtimes}(z)$ as was mentioned earlier).

Recent interest in such operations has come from new techniques involving the expected characteristic polynomials of certain combinations of random matrices. In particular, Theorem 1.1 links us directly to the finite free probability developed in [36, 38], where the operations producing $Q^{\boxplus}(z)$, $Q^{\boxtimes}(z)$ from **a**, **b** are called finite free additive and multiplicative convolutions. These convolutions are a useful tool in the "method of interlacing polynomials" as introduced in [39]—in particular, the additive convolution plays an important role in the same authors' proof of the existence of Ramanujan (multi)graphs of any degree and any size [40].

The second main result deals with $\beta\to\infty$ limit, a proof of which appears in Section 3.3.

Theorem 1.2. Fix $k\le N$, **a** and **b**. The distributions of $\pi^{\beta}_{N\to k}(\mathbf{a})$, $\mathbf{a}\boxplus_{\beta} b$, $\mathbf{a}\boxtimes_{\beta} b$ (in k, N, N dimensional spaces, respectively) weakly converge on polynomial test-functions as $\beta\to\infty$ to δ-measures on the roots of polynomials $Q^{N\to k}(z)$, $Q^{\boxplus}(z)$, $Q^{\boxtimes}(z)$ of Theorem 1.1, respectively.

Theorem 1.2 implies that $Q^{N\to k}(z)$, $Q^{\boxplus}(z)$, $Q^{\boxtimes}(z)$ are real-rooted polynomials. For $Q^{N\to k}(z)$ this is easy to prove directly; proofs of the other two appear in [38] using the techniques of [4], but also follow directly from the (vastly more general) main result in [10].

There is a remarkable link of Theorem 1.2 for $\pi^{\beta}_{N\to k}(\mathbf{a})$ to classical ensembles of random matrices. It is well-known (cf. [9, 16, 32]) that the eigenvalues in Hermite/Laguerre/Jacobi N-particle ensembles (i.e., GβE/LβE/JβE) concentrate as $\beta\to\infty$ near the roots of the corresponding orthogonal polynomials. Theorem 1.2 predicts that varying N should be the same as taking derivatives (see [44] and [63] for an explanation why classical ensembles agree with the operation $\pi^{\beta}_{N\to k}$), and indeed, the derivatives of Hermite/Laguerre/Jacobi orthogonal polynomials are again such orthogonal polynomials of smaller degree—this is a relation known as the forward shift operator, see for example, [33].

For the β-corners processes, capturing $\pi_\beta^{N\to k}(\mathbf{a})$ simultaneously for all $k = 1,\dots,N$ gives a more precise $\beta \to \infty$ asymptotic theorem, which we now present. For each $N = 1,2,\dots$, let $\mathcal{G}_N$ be the set of all Gelfand–Tsetlin patterns of rank N, which are arrays $\{x_i^k\}_{1\le i\le k\le N}$ satisfying $x_i^{k+1} \le x_i^k \le x_{i+1}^{k+1}$. We refer to the coordinate x_i^k as the position of the ith particle in the kth row.

Definition 1.3. The β-corners process with top row $y_1 < \cdots < y_N$ is the unique probability distribution on the arrays $\{x_i^k\}_{1\le i\le k\le N} \in \mathcal{G}_N$, such that $x_i^N = y_i$, $i = 1,\dots,N$, and the remaining $N(N-1)/2$ particles have the density

$$\frac{1}{Z_N}\prod_{k=1}^{N-1}\left[\prod_{1\le i<j\le k}\left(x_j^k - x_i^k\right)^{2-\beta}\right]\cdot\left[\prod_{a=1}^{k}\prod_{b=1}^{k+1}\left|x_a^k - x_b^{k+1}\right|^{\beta/2-1}\right], \tag{8}$$

where Z_N is the normalizing constant computed as

$$Z_N = \prod_{k=1}^{N}\frac{\Gamma(\beta/2)^k}{\Gamma(k\beta/2)}\cdot\prod_{1\le i<j\le N}(y_j - y_i)^{\beta-1}. \tag{9}$$

Remark 1.4. The inductive (in N) computation of Z_N in (9) is the Dixon–Anderson integration formula, see [1, 15], [21, chapter 4].

When $\beta = 1,2,4$, the β-corners process admits a random matrix realization. In this setting, one considers a uniformly random self-adjoint real/complex/quaternion (corresponding to $\beta = 1,2,4$, respectively) $N \times N$ matrix $[A_{i,j}]$ with fixed eigenvalues $a_1,\dots,a_N$. Neretin shows in [44] (Neretin remarks that the statement is a folklore going back at least to the work of Gelfand and Naimark in 1950s; see also [3] for $\beta = 2$ case) that x_i^k can be identified with k eigenvalues of the $k \times k$ principal corner $[A_{i,j}]_{i,j=1}^k$.

Definition 1.5. The ∞-corners process with top row $\mathbf{a} = (a_1 < \cdots < a_N)$, is a deterministic array of particles $\{x_i^k\}_{1\le i\le k\le N} \in \mathcal{G}_N$ such that for each $k = 1,2\dots,N$, $x_1^k < x_2^k < \cdots < x_k^k$ are k roots of $Q^{N\to k}(z)$, which means

$$\prod_{i=1}^{k}\left(z - x_i^k\right) = \frac{1}{N(N-1)\cdots(k+1)}\left(\frac{\partial}{\partial z}\right)^{N-k}\left[\prod_{i=1}^{N}(z-a_i)\right], \tag{10}$$

and equipped with a Gaussian field $\{\xi_i^k\}_{1\le i\le k\le N}$, such that $\xi_1^N = \xi_2^N = \cdots = \xi_N^N = 0$ and the remaining coordinates have density proportional to

$$\exp\left(\sum_{k=1}^{N-1}\left[\sum_{1\le i<j\le k}\frac{\left(\xi_i^k-\xi_j^k\right)^2}{2\left(x_i^k-x_j^k\right)^2}-\sum_{a=1}^{k}\sum_{b=1}^{k+1}\frac{\left(\xi_a^k-\xi_b^{k+1}\right)^2}{4\left(x_a^k-x_b^{k+1}\right)^2}\right]\right). \tag{11}$$

We call $\{x_i^k\}_{1\le i\le k\le N}\in\mathcal{G}_N$ the *deterministic part* of the ∞-corners process and $\{\xi_i^k\}_{1\le i\le k\le N}$ the *discrete Gaussian Free Field* (dGFF) on top of it. The fact that the ∞-corners process captures $\beta\to\infty$ behavior of the β-corners processes is proved in Section 3.4.

Theorem 1.6. Fix $y_1<\cdots<y_N$, take a β-corners process $\{x_i^k(\beta)\}_{1\le i\le k\le N}$ with top row $y_1<\cdots<y_N$, and ∞-corners process $\{\tilde{x}_i^k,\xi_i^k\}_{1\le i\le k\le N}$ with the same top row. Then we have the convergence in distribution:

$$\tilde{x}_i^k=\lim_{\beta\to\infty}x_i^k(\beta),\qquad \xi_i^k=\lim_{\beta\to\infty}\sqrt{\beta}\left(x_i^k(\beta)-\tilde{x}_i^k\right),\qquad 1\le i\le k\le N. \tag{12}$$

The proof of Theorem 1.6 is given in Section 3. The difference between Theorem 1.2 for $\pi_\beta^{N\to k}(\mathbf{a})$ and Theorem 1.6 is that the latter captures not only a deterministic limit (Law of Large Numbers), but also Gaussian fluctuations around it. It would be interesting to do the same for $\mathbf{a}\boxplus_\beta\mathbf{b}$ and $\mathbf{a}\boxtimes_\beta\mathbf{b}$ as $\beta\to\infty$, but we do not have any theorems in this direction so far.

There is a significant amount of literature studying the fluctuations of β-corners processes as $N\to\infty$ with β being fixed. When the top row $\mathbf{a}$ is *random* with specific distribution (rather than deterministically fixed in our setting), then the asymptotic centered fluctuations were identified with a pullback of the 2d Gaussian Free Field (GFF) with Dirichlet boundary conditions in [5, 9, 26]. For a discrete analog of $\beta=2$ case (cf. [12, Section 1.3] for the link between discrete and continuous setting) similar results are also known [13, 52] in the deterministic $\mathbf{a}$ case. Despite these works, the conceptual reasons for the appearance of the GFF in β-random matrices have remained somewhat unclear.

On the other hand, the $\beta=\infty$ case, which is the joint distribution of $\{\xi_i^k\}$ in Definition 1.3, is a version of the dGFF. In many examples the convergence of dGFF to GFF is known, cf. [58, 59, 69], which provides a new insight on why the continuous GFF should show up in $N\to\infty$ limit of the random matrices. Let us however emphasize that the convergence of the ∞-corners process toward GFF does not follow from any known results, since the points $\{x_i^k\}$, where dGFF lives, vary in a very non-trivial way as $N \to \infty$, and, in addition, the exponent in (11) has both positive and negative terms.

At $\beta = 1, 2, 4$ the $N \to \infty$ limit of the operations $\mathbf{a} \boxplus_\beta \mathbf{b}$ and $\mathbf{a} \boxtimes_\beta \mathbf{b}$ is also well-studied. In particular, the first-order behavior ("Law of Large Numbers") is linked to the free probability theory cf. [45, 67], and the limiting operations are known as free (additive and multiplicative) convolutions, see for example, [64–66]. Analogous statements for $\beta = +\infty$ are proven in [36]. Heuristically, one would like to be able to claim that the β-independence of the expectations of Theorem 1.1 should imply the β-independence of the $N \to \infty$ limit for $\mathbf{a} \boxplus_\beta \mathbf{b}$ and $\mathbf{a} \boxtimes_\beta \mathbf{b}$; however, we do not know how to produce a rigorous proof along this line, and the problem of $N \to \infty$ limits for general values of $\beta > 0$ remains open.

The global Gaussian fluctuations as $N \to \infty$ for $\mathbf{a} \boxplus_\beta \mathbf{b}$ and $\mathbf{a} \boxtimes_\beta \mathbf{b}$ have been addressed for $\beta = 2$ in a free probability context using "second order freeness" [42, 43].

There are additional questions concerning the asymptotics as $N \to \infty$. One recent topic is the study of the global fluctuations for the difference between two adjacent levels in the corners processes (again at $\beta = 1, 2, 4$) [11, 19, 26, 32, 60]. The link with taking derivatives of a polynomial that appears in Theorem 1.2 is somewhat visible in the results of [60].

Another popular theme is to study *local limits*, for example, the spacings between individual particles (eigenvalues) in the bulk. At $\beta = 2$ the local limits of the corners-processes were shown to coincide with the sine process in [41], and similar local asymptotics for $\mathbf{a} \boxplus_{\beta=2} \mathbf{b}$ was proven in [14]. At general values of $\beta > 0$ one would expect an appearance of the β-Sine process, but this is not proven rigorously. What is known is that as $\beta \to \infty$ the β-Sine processes crystallize on a lattice, see [34], [56], and indeed for $\beta = \infty$ version of the corners process such limit theorems leading to a lattice are known, see [20].

The rest of the article is organized as follows. In Section 2 we give formal definitions of the operations $\pi_\beta^{N\to k}(\mathbf{a})$, $\mathbf{a} \boxtimes_\beta \mathbf{b}$, $\mathbf{a} \boxplus_\beta \mathbf{b}$. Proofs of Theorems 1.1, 1.2, and 1.6 are given in Section 3. Section 4 discusses extensions to discrete (q, t)–setting related to Macdonald polynomials.

2 Special Functions and Operations

We use several classes of symmetric functions of representation-theoretic origin, which are degenerations of Macdonald polynomials. We refer to [35] for general information about these polynomials and only summarize here the required facts.

Let Λ_N denote the algebra of symmetric polynomials in N-variables $x_1, \dots, x_N$. Define a difference operator $D_{q,t}$ acting in Λ_N through

$$D_{q,t} f = \sum_{i=1}^{N} \left[\prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} \right] T_{q,x_i},$$

where T_{q,x_i} is the q-shift operator acting as

$$\left[T_{q,x_i} f \right] (x_1, \dots, x_N) = f(x_1, \dots, x_{i-1}, q x_i, x_{i+1}, \dots, x_N).$$

$D_{q,t}$ possesses a complete set of eigenfunctions in Λ_N, which are the Macdonald polynomials $P_\lambda(x_1, \dots, x_N; q, t)$ parameterized by N-tuples of non-negative integers $\lambda = (\lambda_1 \geq \cdots \geq \lambda_N)$, which are called non-negative *signatures* (of rank N):

$$D_{q,t} P_\lambda(\cdot; q, t) = \left[\sum_{i=1}^{N} q^{\lambda_i} t^{N-i} \right] P_\lambda(\cdot; q, t). \tag{13}$$

$P_\lambda(\cdot;\, q, t)$ is a homogeneous polynomial of degree $|\lambda| := \lambda_1 + \cdots + \lambda_N$. The leading monomial (with respect to the lexicographic ordering) of $P_\lambda(\cdot; q, t)$ is $x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_N^{\lambda_N}$. We normalize the polynomials by declaring the coefficient of this monomial to be 1.

The definition (13) readily implies the shift property

$$P_{\lambda+1}(x_1, \dots, x_N;\, q, t) = (x_1 x_2 \cdots x_N) P_\lambda(x_1, \dots, x_N;\, q, t), \quad \lambda + 1 = (\lambda_1 + 1, \lambda_2 + 1, \dots, \lambda_N + 1),$$

which allows one to extend the definition of P_λ to general signatures λ with (possibly) negative integer coordinates λ_i. The result is a Laurent polynomial.

Another property that links negative and positive signatures is

$$P_\lambda\left(x_1^{-1}, \dots, x_N^{-1};\, q, t\right) = P_{-\lambda}(x_1, \dots, x_N;\, q, t), \quad -\lambda = (-\lambda_N, -\lambda_{N-1}, \dots, -\lambda_1). \tag{14}$$

The property (14) is somewhat hard to locate in standard references in the explicit form, but it readily follows from the combinatorial formula for Macdonald polynomials and [35, Example VI.6.2 (b)].

If $q = t$, then Macdonald polynomials coincide with Schur polynomials, and so can be given by an explicit determinantal formula

$$P_\lambda(x_1, \ldots, x_N;\, q, q) = s_\lambda(x_1, \ldots, x_N) = \frac{\det\left[x_i^{\lambda_j+N-j}\right]_{i,j=1}^N}{\prod_{1\le i<j\le N}(x_i - x_j)}.$$

We mostly do not use Macdonald polynomials directly, instead relying on three different degenerations. The Jack symmetric polynomials J_λ are defined for $\theta > 0$ through (cf. [35, Chapter VI, Section 10]):

$$J_\lambda(x_1, \ldots, x_N;\, \theta) = \lim_{q\to 1} P_\lambda\left(x_1, \ldots, x_N;\, q, q^\theta\right).$$

The Heckman–Opdam hypergeometric (for root system of type A, cf. [28, 29, 50]) functions $\mathcal{F}_r$ are defined for $\theta > 0$ and N-tuples of *distinct real* labels $r = (r_1 > r_2 > \ldots > r_N)$ through (cf. [9, Section 6], [62], and references therein)

$$\begin{gathered} t = q^\theta, \quad q = \exp(-\varepsilon), \quad \lambda = \lfloor \varepsilon^{-1}(r_1, \ldots, r_N)\rfloor, \quad x_i = \exp(\varepsilon y_i), \\ \mathcal{F}_r\left(y_1, \ldots, y_N;\, \theta\right) = \lim_{\varepsilon\to 0} \varepsilon^{\theta N(N-1)/2} P_\lambda\left(x_1, \ldots, x_N;\, q, t\right). \end{gathered} \tag{15}$$

The multivariate Bessel functions $\mathcal{B}_r$ (cf. [17, 27, 30, 46, 51]) are defined for $\theta > 0$ and N-tuples of distinct real labels $r = (r_1 > r_2 > \ldots > r_N)$ through

$$\mathcal{B}_r\left(z_1, \ldots, z_N;\, \theta\right) = \lim_{\varepsilon\to 0} \mathcal{F}_{\varepsilon r}\left(\varepsilon^{-1} z_1, \ldots, \varepsilon^{-1} z_N;\, \theta\right).$$

Or, equivalently (cf. [46, Section 4]),

$$\begin{gathered} \lambda = \lfloor \varepsilon^{-1}\left(r_1, \ldots, r_N\right)\rfloor, \quad x_i = \exp\left(\varepsilon z_i\right), \\ \mathcal{B}_r\left(z_1, \ldots, z_N;\, \theta\right) = \lim_{\varepsilon\to 0} \varepsilon^{\theta N(N-1)/2} J_\lambda\left(x_1, \ldots, x_N;\, \theta\right). \end{gathered} \tag{16}$$

We also need the *normalized* versions of all these symmetric functions:

$$\hat{P}_\lambda(\cdot;\, q, t) = \frac{P_\lambda(\cdot;\, q, t)}{P_\lambda(1, t, \ldots, t^{N-1};\, q, t)}, \quad \hat{J}_\lambda(\cdot;\, \theta) = \frac{J_\lambda(\cdot;\, \theta)}{J_\lambda(1, 1, \ldots, 1;\, \theta)},$$

$$\hat{\mathcal{F}}_r(\cdot;\, \theta) = \frac{\mathcal{F}(\cdot;\, \theta)}{\mathcal{F}_r(0, -\theta, \ldots, (1-N)\theta;\, \theta)}, \quad \hat{\mathcal{B}}_r(\cdot;\, \theta) = \frac{\mathcal{B}_r(\cdot;\, \theta)}{\mathcal{B}_r(0, \ldots, 0;\, \theta)}.$$

One advantage of the normalized functions $\hat{\mathcal{F}}$ and $\hat{\mathcal{B}}$ is that one can now extend their definition to labels $r = (r_1 \geq \cdots \geq r_N)$ with possibly coinciding coordinates through limit transitions.

The following property is the label-variable symmetry of the Macdonald (Laurent) polynomials, see [35, Chapter VI, Section 6]:

$$\hat{P}_\mu\left(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \ldots, q^{\lambda_N};\, q, t\right) = \hat{P}_\lambda\left(q^{\mu_1}t^{N-1}, q^{\mu_2}t^{N-2}, \ldots, q^{\mu_N};\, q, t\right). \tag{17}$$

It implies the following two symmetries for the degenerations:

$$\hat{\mathcal{F}}_r(-\mu_1 - (N-1)\theta, -\mu_2 - (N-2)\theta, \ldots, -\mu_N;\, \theta) = \hat{J}_\mu(\exp(-r_1), \exp(-r_2), \ldots, \exp(-r_N);\, \theta) \tag{18}$$

$$\hat{\mathcal{B}}_r\left(\ell_1, \ldots, \ell_N;\, \theta\right) = \hat{\mathcal{B}}_\ell\left(r_1, \ldots, r_N;\, \theta\right). \tag{19}$$

The Bessel function $\hat{\mathcal{B}}_r$ coincides with a (partial) Laplace tranform of the $\beta = 2\theta$–corners process $\{x_i^k\}_{1\leq i\leq k\leq N}$ with top row $r = (r_1, \ldots, r_N)$ of Definition 1.3, given by

$$\hat{\mathcal{B}}_r(z_1, \ldots, z_N;\, \theta) = \mathbb{E}_{\{x_i^k\}}\left[\exp\left(\sum_{k=1}^{N} z_k \cdot \left(\sum_{i=1}^{k} x_i^k - \sum_{j=1}^{k-1} x_j^{k-1}\right)\right)\right]. \tag{20}$$

This connection is known as the *combinatorial formula* for Bessel functions; (20) is a limit of similar combinatorial formulas for Macdonald and Jack polynomials, Heckman–Opdam hypergeometric functions.

We will now discuss the connection of the degenerations of Macdonald polynomials to the addition and multiplication of random matrices. For that we define four types of connection coefficients, which are the variants/generalizations of the *Littlewood–Richardson coefficients*. They are determined by the following decompositions:

$$\hat{P}_\lambda(\cdot;\, q, t)\hat{P}_\mu(\cdot;\, q, t) = \sum_\nu c^\nu_{\lambda,\mu}(\hat{P};\, q, t)\hat{P}_\nu(\cdot;\, q, t), \tag{21}$$

$$\hat{J}_\lambda(\cdot;\, \theta)\hat{J}_\mu(\cdot;\, \theta) = \sum_\nu c^\nu_{\lambda,\mu}(\hat{J};\, \theta)\hat{J}_\nu(\cdot;\, \theta), \tag{22}$$

$$\hat{\mathcal{F}}_r(\cdot;\, \theta)\hat{\mathcal{F}}_\ell(\cdot;\, \theta) = \int_s c^s_{\ell,r}(\hat{\mathcal{F}};\, \theta)\hat{\mathcal{F}}_s(\cdot;\, \theta)\,\mathrm{d}s, \tag{23}$$

$$\hat{\mathcal{B}}_r(\cdot;\, \theta)\hat{\mathcal{B}}_\ell(\cdot;\, \theta) = \int_s c^s_{\ell,r}(\hat{\mathcal{B}};\, \theta)\hat{\mathcal{B}}_s(\cdot;\, \theta)\,\mathrm{d}s. \tag{24}$$

The meaning of the decomposition in (21), (22) is straightforward, as this is an expansion of a symmetric (Laurent) polynomial on the left-hand side by functions forming a linear basis of the space of all such polynomials. The sums in (21), (22) have only finitely many non-zero terms; in more details, the coefficients $c^{\nu}_{\lambda,\mu}(\hat{P};\, q, t)$ and $c^{\nu}_{\lambda,\mu}(\hat{J};\, \theta)$ are non-zero only if

$$\nu_1 + \cdots + \nu_N = (\lambda_1 + \mu_1) + \cdots + (\lambda_N + \mu_N), \quad \lambda_N + \mu_N \le \nu_i \le \lambda_1 + \mu_1,\ 1 \le i \le N. \tag{25}$$

Although, in principle, $c^{\nu}_{\lambda,\mu}(\hat{J};\, \theta)$, $c^{s}_{\ell,r}(\hat{\mathcal{F}};\, \theta)$, and $c^{s}_{\ell,r}(\hat{\mathcal{B}};\, \theta)$ can be all obtained from $c^{\nu}_{\lambda,\mu}(\hat{P};\, \theta)$ by limit transitions, the exact mathematical meaning of (23), (24) is a bit more delicate. There is a solid amount of literature devoted to the expansions of functions in appropriate spaces into the integrals of $\mathcal{F}$ and $\mathcal{B}$ functions; this is a far-reaching generalization of the conventional Fourier transform, and it is known under the name Cherednik transform for $\mathcal{F}$ and Dunkl transform for $\mathcal{B}$. We refer to [2] for a recent brief review with many references to the original articles and more detailed treatments. In particular, it is rigorously known that

$$f \mapsto \int_s c^{s}_{\ell,r}(\hat{\mathcal{F}};\, \theta) \cdot f \, \mathrm{d}s, \quad f \mapsto \int_s c^{s}_{\ell,r}(\hat{\mathcal{B}};\, \theta) \cdot f \, \mathrm{d}s,$$

are well-defined as distributions with compact support. The support of $c^{s}_{\ell,r}(\hat{\mathcal{F}};\, \theta)$, $c^{s}_{\ell,r}(\hat{\mathcal{B}};\, \theta)$, is a subset of those s which satisfy

$$s_1 + \cdots + s_N = (r_1 + \ell_1) + \cdots + (r_N + \ell_N), \quad r_N + \ell_N \le s_i \le r_1 + \ell_1,\ 1 \le i \le N. \tag{26}$$

However, much more is conjectured, as we discuss below.

Conjecture 2.1. All the coefficients $c^{\nu}_{\lambda,\mu}(\hat{P};\, q, t)$ for $0 < q,\ t < 1$, (and hence also $c^{\nu}_{\lambda,\mu}(\hat{J};\, \theta)$, $c^{s}_{\ell,r}(\hat{\mathcal{F}};\, \theta)$, $c^{s}_{\ell,r}(\hat{\mathcal{B}};\, \theta)$ for $\theta > 0$) are non-negative.

In the $q = t$ case, $c^{\nu}_{\lambda,\mu}(\hat{P};\, q, q)$ are (up to a normalization) conventional Littlewood–Richardson coefficients, which are known to be non-negative due to either representation-theoretic interpretations or combinatorial formulas, see for example, [35, Chapter I, Section 10]. For $\theta = 1/2, 1, 2$, the coefficients $c^{\nu}_{\lambda,\mu}(\hat{J};\, \theta)$, $c^{s}_{\ell,r}(\hat{\mathcal{F}};\, \theta)$, $c^{s}_{\ell,r}(\hat{\mathcal{B}};\, \theta)$ are again known to be non-negative due to representation-theoretic interpretations, cf. [35, Chapter VII]. When $q = 0$, the non-negativity of the coefficients $c^{\nu}_{\lambda,\mu}(\hat{P};\, q, t)$ is due to known combinatorial formulas, see [54, Theorem 4.9], [57, Theorem 1.3], [31], and references therein. The non-negativity of $c^{\nu}_{\lambda,\mu}(\hat{J};\theta)$ would also follow from a (still open) conjecture of Stanley [61, Conjecture 8.3]. We refer to [55] for progress concerning the non-negativity of $c^{s}_{\ell,r}(\hat{\mathcal{B}};\theta)$.

Since our definitions of $\hat{P}$, $\hat{J}$, $\hat{\mathcal{F}}$, $\hat{B}$ imply that

$$\sum_{\nu} c^{\nu}_{\lambda,\mu}(\hat{P};\, q,t) = \sum_{\nu} c^{\nu}_{\lambda,\mu}(\hat{J};\,\theta) = \int_s c^{s}_{\ell,r}(\hat{\mathcal{F}};\,\theta)\, \mathrm{d}s = \int_s c^{s}_{\ell,r}(\hat{\mathcal{B}};\,\theta)\, \mathrm{d}s = 1,$$

Conjecture 2.1 would then allow for these coefficients (with fixed λ,μ or r, ℓ, and varying ν or s) can be treated as *probability measures*. Without Conjecture 2.1 the coefficients in (21), (22) define *signed* measures of total mass 1, while the coefficients in (23), (24) have only distributional meaning.

The connection of these definitions to random matrix operations is explained in the next two propositions. For $r = (r_1,\dots,r_N)$, let $\exp(r)$ denote the vector $(\exp(r_1),\dots,\exp(r_N))$.

Proposition 2.2. Choose $\theta = \frac{1}{2}$, 1, or 2, and let $\beta = 2\theta$. Take $r = (r_1,\dots,r_N)$, $\ell = (\ell_1,\dots,\ell_N)$ and let $s = (s_1,\dots,s_N)$ be $c^{s}_{r,\ell}(\hat{\mathcal{F}};\theta)$–distributed random vector. Then in the notations of Section 1

$$\exp(s) \overset{d}{=} \exp(r) \boxtimes_{\beta} \exp(\ell). \tag{27}$$

Proof. Combining (23) with (18) we conclude that for $c^{s}_{r,\ell}(\hat{\mathcal{F}};\theta)$–distributed random vector and each integral vector $\lambda = (\lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N)$, we have

$$\begin{aligned}&\mathbb{E}\hat{J}_{\lambda}(\exp(-s_1),\dots,\exp(-s_N);\,\theta)\\ &\qquad = \hat{J}_{\lambda}(\exp(-r_1),\dots,\exp(-r_N);\,\theta)\hat{J}_{\lambda}(\exp(-\ell_1),\dots,\exp(-\ell_N);\,\theta).\end{aligned}$$

Taking into account the Jack limit of (14), we can equivalently write

$$\mathbb{E}\hat{J}_{\lambda}(\exp(s_1),\dots,\exp(s_N);\,\theta) = \hat{J}_{\lambda}(\exp(r_1),\dots,\exp(r_N);\,\theta)\hat{J}_{\lambda}(\exp(\ell_1),\dots,\exp(\ell_N);\,\theta). \tag{28}$$

Note that $\hat{J}_{\lambda}$ form a basis in symmetric Laurent polynomials, and therefore, the expectations of the form (28) uniquely define the distribution of $\exp(s_1),\dots,\exp(s_N)$. On the other hand, the identity (28) is well-known to hold with $\exp(s)$ replaced by $\exp(r) \boxtimes_{\beta} \exp(\ell)$, see [35, Chapter VII], [21, Section 13.4.3], and references therein. ■

Proposition 2.3. Choose $\theta = \frac{1}{2}$, 1, or 2, and let $\beta = 2\theta$. Take $r = (r_1, \dots, r_N)$, $\ell = (\ell_1, \dots, \ell_N)$ and let $s = (s_1, \dots, s_N)$ be $c^s_{r,\ell}(\hat{\mathcal{B}};\, \theta)$–distributed random vector. Then in the notations of Section 1

$$s \stackrel{d}{=} r \boxplus_\beta \ell. \tag{29}$$

Proof. The formula (20) identifies the Bessel functions with Laplace transforms of the β-corners processes. At $\theta = 1/2, 1, 2$, this implies an identification with Laplace transforms of real/complex/quaternion random matrices, see the discussion after Definition 1.3. Laplace transform of the sum of independent random variables (matrices) is a product of Laplace transforms, hence the definition of $r \boxplus_\beta \ell$ coincides with that of $c^s_{r,\ell}(\hat{\mathcal{B}};\, \theta)$–distributed s. ∎

Propositions 2.2, 2.3 motivate extrapolation of matrix operations to general $\beta > 0$.

Definition 2.4. Choose $\theta > 0$, and let $\beta = 2\theta$. Take $\mathbf{a} = (a_1, \dots, a_N)$, $\mathbf{b} = (b_1, \dots, b_N)$ and assume $\mathbf{a} = \exp(r)$, $\mathbf{b} = \exp(\ell)$. Let $s = (s_1, \dots, s_N)$ be $c^s_{r,\ell}(\hat{\mathcal{F}};\, \theta)$–distributed random vector. Then in the notations of Section 1

$$\mathbf{a} \boxtimes_\beta \mathbf{b} := \exp(s). \tag{30}$$

Definition 2.5. Choose $\theta > 0$, and let $\beta = 2\theta$. Take $\mathbf{a} = (a_1, \dots, a_N)$, $\mathbf{b} = (b_1, \dots, b_N)$ and let $s = (s_1, \dots, s_N)$ be $c^s_{\mathbf{a},\mathbf{b}}(\hat{\mathcal{B}};\, \theta)$–distributed random vector. Then in the notations of Section 1

$$\mathbf{a} \boxplus_\beta \mathbf{b} := s. \tag{31}$$

Definition 2.6. Choose $\theta > 0$, and let $\beta = 2\theta$. Take $\mathbf{a} = (a_1, \dots, a_N)$, $k \le N$ and let $s = (s_1, \dots, s_k)$ be the kth row of β-corners process with top row $\mathbf{a}$, that is, the particles $x^k_1, x^k_2, \dots, x^k_k$ in the notations of Definition 1.3. Then

$$\pi^{N\to k}_\beta(\mathbf{a}) := s. \tag{32}$$

Comparing with (20), we see the following property (which can be taken as an equivalent definition of $\pi^{N\to k}_\beta(\mathbf{a})$):

$$\mathbb{E}\mathcal{B}_{\pi^{N\to k}_\beta(\mathbf{a})}(z_1, \dots, z_k;\, \beta/2) = \mathcal{B}_{\mathbf{a}}\left(z_1, \dots, z_k, \underbrace{0, \dots, 0}_{N-k}\right). \tag{33}$$

3 Proofs

3.1 The expectation identities

The proofs of Theorems 1.1, 1.2 are based on the evaluations of the expectations of Jack polynomials summarized in the next three propositions.

For a finite integer vector $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq 0)$, we define

$$(t)_{\lambda;\theta} = \prod_{\substack{(i,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\\ j\leq\lambda_i}} (t + (j-1) - \theta(i-1)).$$

We need to introduce yet another normalization of Jack polynomials. The dual Jack polynomials $J_\lambda^{\mathrm{dual}}$ differ from J_λ by multiplication by explicit constants independent of N and $x_1, \ldots, x_N$ (see [35, Chapter VI, Section 10]) in such a way that makes the following identity true:

$$\sum_{\lambda_1\geq\cdots\geq\lambda_N\geq 0} J_\lambda(x_1,\ldots,x_N;\,\theta) J_\lambda^{\mathrm{dual}}(y_1,\ldots,y_N;\,\theta) = \prod_{i,j=1}^{N} (1 - x_i y_j)^{-\theta}.$$

We also need corresponding Jack–Littlewood–Richardson coefficients $c^\lambda_{\mu,\nu}(J^{\mathrm{dual}}, \theta)$, defined by

$$J_\mu^{\mathrm{dual}}(x_1,\ldots,x_N;\,\theta)\cdot J_\nu^{\mathrm{dual}}(x_1,\ldots,x_N;\,\theta) = \sum_\lambda c^\lambda_{\mu,\nu}(J^{\mathrm{dual}};\,\theta)\cdot J_\lambda^{\mathrm{dual}}(x_1,\ldots,x_N;\,\theta).$$

Proposition 3.1. Fix positive vectors of eigenvalues $\mathbf{a}$ and $\mathbf{b}$, and let $\mathbf{a}\boxtimes_\beta\mathbf{b}$ denote the corresponding N-dimensional random vector in the notations of Section 1. For each $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0$ we have

$$\mathbb{E}\, J_\lambda(\mathbf{a}\boxtimes_\beta\mathbf{b};\,\beta/2) = \frac{J_\lambda(\mathbf{a};\,\beta/2)\cdot J_\lambda(\mathbf{b};\,\beta/2)}{J_\lambda(1,\ldots,1;\,\beta/2)}. \tag{34}$$

Proof. This is equivalent to (28), whose proof, in fact, did not use $\theta = 1/2, 1, 2$. ■

Proposition 3.2. Fix $k \leq N$ and $\mathbf{a}$, and let $\pi^\beta_{N\to k}(\mathbf{a})$ denote the corresponding k-dimensional random vector in the notations of Section 1. For each $\lambda = \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 0$ we have

$$\mathbb{E}\, J_\lambda\left(\pi^\beta_{N\to k}(\mathbf{a});\,\beta/2\right) = J_\lambda(\mathbf{a};\,\beta/2)\cdot\frac{(k\beta/2)_{\lambda;\beta/2}}{(N\beta/2)_{\lambda;\beta/2}}, \tag{35}$$

where in the right-hand side we view λ as a signature of rank N by adding $N-k$ zeros.

Proof. We use the series expansion of the Bessel functions, see [46, Section 4]:

$$\mathcal{B}_{a_1,\dots,a_N}(x_1,\dots,x_N;\,\beta) = \sum_{\lambda=(\lambda_1\ge\dots\ge\lambda_N\ge 0)} \frac{J_\lambda(x_1,\dots,x_N;\,\beta/2)J_\lambda^{\mathrm{dual}}(a_1,\dots,a_N;\,\beta/2)}{(N\beta/2)_{\lambda;\beta/2}}. \tag{36}$$

We plug (36) into both sides of (33) and use the stability property of Jack polynomials valid for all $\lambda = (\lambda_1 \ge \dots \ge \lambda_N \ge 0)$:

$$J_\lambda\Bigg(z_1,\dots,z_k,\underbrace{0,\dots,0}_{N-k};\theta\Bigg) = \begin{cases} J_\lambda(z_1,\dots,z_k;\theta), & \lambda_{k+1}=\dots=\lambda_N=0,\\ 0, & \text{otherwise,}\end{cases}$$

where in the right-hand side we treat λ as a signature of rank k by removing $N-k$ zero coordinates. We get

$$\mathbb{E}\sum_{\lambda=(\lambda_1\ge\dots\ge\lambda_k\ge 0)} \frac{J_\lambda\left(\pi_\beta^{N\to k}(\mathbf{a});\,\beta/2\right)J_\lambda^{\mathrm{dual}}(z_1,\dots,z_k;\,\beta/2)}{(k\beta/2)_{\lambda;\beta/2}},$$
$$= \sum_{\lambda=(\lambda_1\ge\dots\ge\lambda_k\ge 0)} \frac{J_\lambda(\mathbf{a};\,\beta/2)\,J_\lambda^{\mathrm{dual}}(z_1,\dots,z_k;\,\beta/2)}{(N\beta/2)_{\lambda;\beta/2}}. \tag{37}$$

Comparing the coefficient of $J_\lambda^{\mathrm{dual}}(z_1,\dots,z_k;\,\beta/2)$ in both sides of (37) we arrive at the desired statement. ■

Proposition 3.3. Fix $\mathbf{a}$ and $\mathbf{b}$, and let $\mathbf{a}\boxplus_\beta\mathbf{b}$ denote the corresponding N-dimensional random vectors (in k, N, N dimensional spaces, respectively) in the notations of Section 1. For each $\lambda = \lambda_1 \ge \lambda_2 \ge \dots \ge \lambda_N \ge 0$ we have

$$\mathbb{E}\,J_\lambda(\mathbf{a}\boxplus_\beta\mathbf{b};\,\beta/2) = (N\beta/2)_{\lambda;\beta/2}\sum_{\nu,\mu} c^\lambda_{\mu,\nu}(J^{\mathrm{dual}};\,\beta/2)\cdot\frac{J_\mu(\mathbf{a};\,\beta/2)}{(N\beta/2)_{\mu;\beta/2}}\cdot\frac{J_\nu(\mathbf{b};\,\beta/2)}{(N\beta/2)_{\nu;\beta/2}}. \tag{38}$$

Remark 3.4 The sum in (38) is finite, since $c^\lambda_{\mu,\nu}(J^{\mathrm{dual}};\,\beta/2)$ vanishes unless $0\le\mu_i\le\lambda_i$, $0\le\nu_i\le\lambda_i$, $\sum_{i=1}^N(\mu_i+\nu_i-\lambda_i)=0$.

Proof of Proposition 3.3. The proof is similar to that of Proposition 3.2. We rewrite (23) as

$$\mathbb{E}\mathcal{B}_{\mathbf{a}\boxplus_\beta\mathbf{b}}(z_1,\dots,z_N;\,\beta/2) = \mathcal{B}_{\mathbf{a}}(z_1,\dots,z_N;\,\beta/2)\cdot\mathcal{B}_{\mathbf{b}}(z_1,\dots,z_N;\,\beta/2),$$

then plug (36) into both sides, and compare the coefficients of $J_\lambda^{\mathrm{dual}}(z_1,\dots,z_N;\,\beta/2)$. ■

3.2 Proof of Theorem 1.1

We use Propositions 3.1, 3.2, and 3.3 with a particular choice of λ: $\lambda_1 = \cdots = \lambda_\ell = 1$, $\lambda_{\ell+1} = \lambda_{\ell+2} = \cdots = 0$. For such λ, Jack polynomial coincides with elementary symmetric function, cf. [35, Chapter VI]:

$$J_{1^\ell,0^{N-\ell}}(x_1,\ldots,x_N;\theta) = e_\ell(x_1,\ldots,x_N).$$

Then (34) becomes

$$\mathbb{E}e_\ell(\mathbf{a} \boxtimes_\beta \mathbf{b}) = \frac{e_\ell(\mathbf{a})e_\ell(\mathbf{b})}{\binom{N}{\ell}}. \tag{39}$$

On the other hand, evaluating the coefficient of $z^{N-\ell}$, (6) is equivalent to

$$\mathbb{E}e_\ell(\mathbf{a} \boxtimes_\beta \mathbf{b}) = \frac{1}{N!}\sum_{\sigma\in S_N} e_\ell\left(a_1 b_{\sigma(1)}, a_2 b_{\sigma(2)}, \ldots, a_N b_{\sigma(N)}\right). \tag{40}$$

It is straightforward to check that the right-hand sides of (39) and (40) are the same.

Further, (35) becomes

$$\mathbb{E}e_\ell\left(\pi_\beta^{N\to k}(\mathbf{a})\right) = e_\ell(\mathbf{a}) \cdot \frac{k(k-1)\cdots(k-\ell+1)}{N(N-1)\cdots(N-\ell+1)}, \tag{41}$$

which is the same expression as the coefficient of $z^{k-\ell}$ in (4).

We finally apply Proposition 3.3 with $\lambda_1 = \lambda_2 \cdots = \lambda_\ell = 1$, $\lambda_{\ell+1} = \cdots = \lambda_N = 0$. Using Remark 3.4 we conclude that the right-hand side has $\ell+1$ term and the $(p+1)$st term has $\mu_1 = \mu_2 \cdots = \mu_p = 1$, $\mu_{p+1} = \cdots = \mu_N = 0$; $\nu_1 = \nu_2 \cdots = \nu_{\ell-p} = 1$, $\nu_{\ell-p+1} = \cdots = \nu_N = 0$. We get

$$\mathbb{E}e_\ell(\mathbf{a} \boxplus_\beta \mathbf{b}) = N(N-1)\cdots(N-\ell+1)\sum_{p=0}^{\ell} c^{1^\ell}_{1^p,1^{\ell-p}}(J^{\text{dual}};\beta/2) \times \frac{e_p(\mathbf{a})}{N(N-1)\cdots(N-p+1)} \cdot \frac{e_{\ell-p}(\mathbf{b})}{N(N-1)\cdots(N-\ell+p+1)}. \tag{42}$$

It remains to find the value of the constant $c^{1^\ell}_{1^p,1^{\ell-p}}(J^{\text{dual}};\beta/2)$. For that we use the automorphism ω_θ of the algebra of symmetric functions in *infinitely many variables* (cf. [35, Chapter VI, Section 10]) with $\theta = \beta/2$. It has the following action on e_ℓ:

$$\omega_\theta\left(J^{\text{dual}}_{1^\ell}(\cdot;\theta)\right) = J_{(\ell,0,0,\ldots)}\left(\cdot;\theta^{-1}\right),$$

and therefore $c^{1^\ell}_{1^p,1^{\ell-p}}(J^{\mathrm{dual}};\ \beta/2)$ is the coefficient of $\lambda=(\ell,0,0,\dots)$ in the expansion

$$J_{(p,0,\dots)}(\cdot;\ 2/\beta)\cdot J_{(\ell-p,0,\dots)}(\cdot;\ 2/\beta)=\sum_{\lambda} c^{\lambda}_{(p,0,\dots),(\ell-p,0,\dots)}(J;\ 2/\beta)\cdot J_\lambda(\cdot;\ 2/\beta). \tag{43}$$

This coefficient is readily found by comparing the coefficient of the leading monomial x_1^ℓ in both sides of (43), and therefore $c^{1^\ell}_{1^p,1^{\ell-p}}(J^{\mathrm{dual}};\ \beta/2)=1$. We conclude that

$$\mathbb{E}e_\ell(\mathbf{a}\boxplus_\beta \mathbf{b})=\sum_{p=0}^{\ell} e_p(\mathbf{a})e_{\ell-p}(\mathbf{b})\frac{N(N-1)\cdots(N-\ell+1)}{N(N-1)\cdots(N-p+1)\cdot N(N-1)\cdots(N-\ell+p+1)}. \tag{44}$$

On the other hand, the coefficient of $z^{N-\ell}$ in (5) gives

$$\mathbb{E}e_\ell(\mathbf{a}\boxplus_\beta \mathbf{b})=\frac{1}{N!}\sum_{\sigma\in S(N)} e_\ell(a_1+b_{\sigma(1)},a_2+b_{\sigma(2)},\dots,a_N+b_{\sigma(N)}). \tag{45}$$

One readily checks that the right-hand sides of (44) and (45) give the same expression.

3.3 Proof of Theorem 1.2

The proof is based on the following two limit relations, which can be found for example, in [61, Proposition 7.6] (The parameter α of [61] is θ^{-1}.):

$$\lim_{\theta\to\infty} J_\lambda(x_1,\dots,x_N;\ \theta)=\prod_{i=1}^{N}\left[e_i(x_1,\dots,x_N)\right]^{\lambda_i-\lambda_{i+1}}. \tag{46}$$

$$\lim_{\theta\to 0} J_\lambda(x_1,\dots,x_N;\ \theta)=m_\lambda(x_1,\dots,x_N), \tag{47}$$

where e_k is the elementary symmetric function and m_λ is the monomial symmetric function.

We start with $\lim_{\beta\to\infty} \mathbf{a} \boxtimes_\beta \mathbf{b}$. Take any collection of polynomials $f_1, \ldots, f_m$ in N variables. We aim to prove that the following limits exist and satisfy

$$\lim_{\beta\to\infty} \mathbb{E}\left[\prod_{i=1}^{m} f_i(\mathbf{a} \boxtimes_\beta \mathbf{b})\right] = \lim_{\beta\to\infty} \prod_{i=1}^{m} [\mathbb{E} f_i(\mathbf{a} \boxtimes_\beta \mathbf{b})]. \tag{48}$$

This precisely means that $\lim_{\beta\to\infty} \mathbf{a} \boxtimes_\beta \mathbf{b}$ is a delta function at a certain point, and then Theorem 1.1 would identify this point with roots of $Q^{\boxtimes}(z)$. Note that under Conjecture 2.1, $\mathbf{a} \boxtimes_\beta b$ is a bona-fide random variable, and therefore, convergence to a delta-function for polynomial test functions f_i would imply a similar convergence for arbitrary continuous test-functions (without Conjecture 2.1, formally, there might be no such implication).

Since $\mathbf{a} \boxtimes_\beta \mathbf{b}$ is supported on *ordered* N-tuples of reals, it suffices to consider *symmetric* polynomials f_i in (48). Further, since (48) is multi-linear in f_i, it suffices to check it on an arbitrary basis of the algebra Λ_N of symmetric polynomials. This algebra is generated by elementary symmetric functions $e_1, \ldots, e_N$, and so we can choose functions of the form

$$e_{(\lambda)} := e_1^{\lambda_1-\lambda_2} e_2^{\lambda_2-\lambda_3} \cdots e_{N-1}^{\lambda_{N-1}-\lambda_N} e_N^{\lambda_N}, \quad \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \geq 0 \tag{49}$$

as such basis. Therefore, (48) reduces to the statement that for any λ:

$$\lim_{\beta\to\infty} \mathbb{E}[e_{(\lambda)}(\mathbf{a} \boxtimes_\beta \mathbf{b})] = \lim_{\beta\to\infty} \prod_{i=1}^{N} \mathbb{E}\left[e_i^{\lambda_i-\lambda_{i+1}}(\mathbf{a} \boxtimes_\beta \mathbf{b})\right]. \tag{50}$$

We now prove (50). Fix λ and decompose $e_{(\lambda)}$ into a linear combination of Jack polynomials

$$e_{(\lambda)}(x_1, \ldots, x_N) = \sum_{\mu} T_\lambda^\mu(\beta/2) J_\mu(x_1, \ldots, x_N; \beta/2). \tag{51}$$

Note that the sum in (51) is finite, as $\mu = (\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N \geq 0)$ is required to satisfy $\sum_{i=1}^{N} \mu_i = \sum_{i=1}^{N} \lambda_i$. Further, (46) implies that

$$\lim_{\beta\to\infty} T_\lambda^\mu(\beta/2) = \begin{cases} 1, & \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases}$$

Thus, using Proposition 3.1, (46), and (39) we get

$$\begin{aligned}\lim_{\beta\to\infty}\mathbb{E}[e_{(\lambda)}(\mathbf{a}\boxtimes_\beta \mathbf{b})] &= \lim_{\beta\to\infty}\sum_\mu T^\mu_\lambda(\beta/2)\frac{J_\mu(\mathbf{a};\,\beta/2)J_\mu(\mathbf{b};\,\beta/2)}{J_\mu(1,\dots,1;\,\beta/2)}\\ &= \frac{\prod_{i=1}^N e_i^{\lambda_i-\lambda_{i+1}}(\mathbf{a})\prod_{i=1}^N e_i^{\lambda_i-\lambda_{i+1}}(\mathbf{b})}{\prod_{i=1}^N e_i^{\lambda_i-\lambda_{i+1}}\left(\underbrace{1,\dots,1}_{N}\right)}\\ &= \prod_{i=1}^N \mathbb{E}\left[e_i^{\lambda_i-\lambda_{i+1}}(\mathbf{a}\boxtimes_\beta \mathbf{b})\right] = \lim_{\beta\to\infty}\prod_{i=1}^N \mathbb{E}\left[e_i^{\lambda_i-\lambda_{i+1}}(\mathbf{a}\boxtimes_\beta \mathbf{b})\right],\end{aligned}$$

which proves (50).

Next, we investigate $\lim_{\beta\to\infty}\pi_\beta^{N\to k}(\mathbf{a})$. Again it suffices to prove that for every $\lambda_1\geq\cdots\geq\lambda_k\geq 0$,

$$\lim_{\beta\to\infty}\mathbb{E}\left[e_{(\lambda)}\left(\pi_\beta^{N\to k}(\mathbf{a})\right)\right] = \lim_{\beta\to\infty}\prod_{i=1}^k \mathbb{E}\left[e_i^{\lambda_i-\lambda_{i+1}}\left(\pi_\beta^{N\to k}(\mathbf{a})\right)\right]. \tag{52}$$

Using Proposition 3.2, (46), and (41) we have

$$\begin{aligned}\lim_{\beta\to\infty}\mathbb{E}\left[e_{(\lambda)}\left(\pi_\beta^{N\to k}(\mathbf{a})\right)\right] &= \lim_{\beta\to\infty}\sum_\mu T^\mu_\lambda(\beta/2)J_\mu(\mathbf{a};\,\beta/2)\frac{(k\beta/2)_{\mu;\beta/2}}{(N\beta/2)_{\mu;\beta/2}}\\ &= \prod_{i=1}^k e_i^{\lambda_i-\lambda_{i+1}}(\mathbf{a})\prod_{\substack{(i,j)\in\mathbb{Z}_{>0}\times\mathbb{Z}_{>0}\\ j\leq\lambda_i}}\frac{k-i+1}{N-i+1}\\ &= \prod_{i=1}^k e_i^{\lambda_i-\lambda_{i+1}}\left(\pi_\beta^{N\to k}(\mathbf{a})\right) = \lim_{\beta\to\infty}\prod_{i=1}^k e_i^{\lambda_i-\lambda_{i+1}}\left(\pi_\beta^{N\to k}(\mathbf{a})\right).\end{aligned}$$

Finally, we turn to $\lim_{\beta\to\infty}\mathbf{a}\boxplus_\beta\mathbf{b}$, and again prove that for every $\lambda_1\geq\cdots\geq\lambda_N\geq 0$

$$\lim_{\beta\to\infty}\mathbb{E}[e_{(\lambda)}(\mathbf{a}\boxplus_\beta \mathbf{b})] = \lim_{\beta\to\infty}\prod_{i=1}^N \mathbb{E}\left[e_i^{\lambda_i-\lambda_{i+1}}(\mathbf{a}\boxplus_\beta \mathbf{b})\right]. \tag{53}$$

Thus, using Proposition 3.3 and (46) we get

$$\lim_{\beta\to\infty} \mathbb{E}[e_{(\lambda)}(\mathbf{a} \boxplus_\beta \mathbf{b})]$$
$$= \sum_{\nu,\mu} \lim_{\beta\to\infty} \left[c^\lambda_{\mu,\nu}(J^{\mathrm{dual}};\, \beta/2) \cdot \frac{(N\beta/2)_{\lambda;\beta/2}}{(N\beta/2)_{\nu;\beta/2}(N\beta/2)_{\mu;\beta/2}} \right] \prod_{i=1}^{N} \left[e_i^{\mu_i-\mu_{i+1}}(\mathbf{a}) e_i^{\nu_i-\nu_{i+1}}(\mathbf{b}) \right].$$

By degree considerations, $c^\lambda_{\mu,\nu}(J^{\mathrm{dual}};\, \beta/2)$ is non-zero only if $\sum_i(\mu_i+\nu_i) = \sum_i \lambda_i$. In this case

$$\lim_{\beta\to\infty} \frac{(N\beta/2)_{\lambda;\beta/2}}{(N\beta/2)_{\nu;\beta/2}(N\beta/2)_{\mu;\beta/2}} = \frac{\prod\limits_{j\le\lambda_i} \left(N+1-i\right)}{\prod\limits_{j\le\lambda_i}(N+1-i)\cdot \prod\limits_{j\le\lambda_i}(N+1-i)} = \prod_{i=1}^{N} (N+1-i)^{\lambda_i-\mu_i-\nu_i}.$$

For the value of the constant $\lim_{\beta\to\infty} c^\lambda_{\mu,\nu}(J^{\mathrm{dual}};\, \beta/2)$ we use the automorphism ω_θ of the algebra of symmetric functions in infinitely many variables (cf. [35, Chapter VI, Section 10]) with $\theta = \beta/2$. It has the following action on Jack polynomials:

$$\omega_\theta\left(J^{\mathrm{dual}}_\lambda(\cdot;\, \theta)\right) = J_{\lambda'}(\cdot;\, \theta^{-1}),$$

where λ' is the transpose partition, defined through

$$\lambda'_j = \left|\{i \in \mathbb{Z}_{>0} : \lambda_i \ge j\}\right|, \quad j = 1, 2, \dots.$$

Then using (47) we conclude

$$\lim_{\beta\to\infty} c^\lambda_{\mu,\nu}(J^{\mathrm{dual}};\, \beta/2) = c^{\lambda'}_{\mu',\nu'}(m),$$

where m stands for the monomial symmetric functions; that is, the coefficients are defined from the decomposition

$$m_{\mu'} m_{\nu'} = \sum_{\lambda'} c^{\lambda'}_{\mu',\nu'}(m) m_{\lambda'}.$$

The monomial symmetric functions are easy to multiply directly: $c^{\lambda'}_{\mu',\nu'}(m)$ counts the number of ways to represent the vector $(\lambda'_i)_{i=1,2,\dots}$ as a sum of $(\mu'_i)_{i=1,2,\dots}$ and a permutation of the coordinates of the vector $(\nu'_i)_{i=1,2,\dots}$.

It remains to check that

$$\sum_{\nu,\mu} c^{\lambda'}_{\mu',\nu'}(m) \prod_{i=1}^{N} (N+1-i)^{\lambda_i-\mu_i-\nu_i} \prod_{i=1}^{N} \left[e_i^{\mu_i-\mu_{i+1}}(\mathbf{a}) e_i^{\nu_i-\nu_{i+1}}(\mathbf{b}) \right]$$
$$= \prod_{\ell=1}^{N} \left[\sum_{p=0}^{\ell} e_p(\mathbf{a}) e_{\ell-p}(\mathbf{b}) \frac{N(N-1)\cdots(N-\ell+1)}{N(N-1)\cdots(N-p+1)\cdot N(N-1)\cdots(N-\ell+p+1)} \right]^{\lambda_\ell-\lambda_{\ell+1}}, \quad (54)$$

which is straightforward given the above description of $c^{\lambda'}_{\mu',\nu'}(m)$.

3.4 Proof of Theorem 1.6

Let us start by computing the constant Z_N in Definition 1.3. For that we use the Dixon–Anderson integration formula, see [15], [21, Exercise 4.2, q. 2], which reads

$$\int_T \prod_{1\le i<j\le n} (t_i - t_j) \prod_{i=1}^{n} \prod_{j=1}^{n+1} \frac{|t_i - a_j|^{\alpha_j - 1}}{|b - t_i|^{\alpha_j}} \, \mathrm{d}t_1 \cdots \mathrm{d}t_n$$
$$= \frac{\prod_{j=1}^{n+1} \Gamma(\alpha_j)}{\Gamma\left(\sum_{j=1}^{n+1} \alpha_j\right)} \prod_{1\le i<j\le n+1} (a_i - a_j)^{\alpha_i+\alpha_j-1} \prod_{i=1}^{n+1} |b - a_i|^{\alpha_i - \sum_{j=1}^{n+1} \alpha_j}, \quad (55)$$

where the domain of integration T is given by

$$a_1 < t_1 < a_2 < t_2 \cdots < t_n < a_{n+1}.$$

Choosing $b = \infty$, $\alpha_i = \beta/2$, we verify (9) by induction in N.

The next step is to find $\tilde{x}_i^k$. We do this sequentially: first, for $k = N-1$ and $1 \le i \le N-1$, then for $k = N-2$, etc, until we reach $k = 1$. Using the definition of the β-corners process and the formula for Z_N we write down the conditional distribution of x_i^{k-1}, $1 \le i \le k-1$ given x_i^k, $1 \le i \le k$:

$$P\left(x_1^{k-1}, \ldots, x_{k-1}^{k-1} \,|\, x_1^k, \ldots, x_k^k\right) = \frac{\Gamma(k\beta/2)}{\Gamma(\beta/2)^k} \frac{\prod_{1\le i<j\le k-1} \left(x_j^{k-1} - x_i^{k-1}\right)}{\prod_{1\le i<j\le k} \left(x_j^k - x_i^k\right)^{\beta-1}} \times \left(\prod_{i=1}^{k} \prod_{j=1}^{k-1} \left| x_i^k - x_j^{k-1} \right| \right)^{\beta/2-1}. \quad (56)$$

As $\beta \to \infty$, the density (as a function of $x_1^{k-1}, \ldots, x_{k-1}^{k-1}$) concentrates near the point where the second line of (56) is maximized, so we need to solve the maximization problem:

$$\prod_{i=1}^{k} \prod_{j=1}^{k-1} \left| x_i^k - x_j^{k-1} \right| \to \max, \qquad \text{with fixed } x_1^k, \ldots, x_k^k. \tag{57}$$

Taking logarithmic derivatives of (57) in $x_1^{k-1}, \ldots, x_{k-1}^{k-1}$ we find that the optimal point $(\tilde{x}_1^{k-1}, \ldots, \tilde{x}_{k-1}^{k-1})$ should satisfy the $k-1$ equations

$$\sum_{j=1}^{k} \frac{1}{\tilde{x}_i^{k-1} - x_j^k} = 0, \quad i = 1, \ldots, k-1. \tag{58}$$

Observe that (58) is precisely the collection of equations that define the $k-1$ roots of the derivative of the polynomial $f(u) = \prod_{j=1}^{k}(u - x_j^k)$. This implies the first part of (12) of Theorem 1.6.

For the second part we Taylor-expand the density of the β-corners process near the point $\tilde{x}_i^k$, $1 \le i \le k \le N$. Set

$$x_i^k = \tilde{x}_i^k + \frac{\Delta x_i^k}{\sqrt{\beta}}$$

and rewrite (8) as

$$\begin{aligned}
&\frac{1}{Z_N} \prod_{k=1}^{N-1} \left[\prod_{1 \le i < j \le k} \left(\tilde{x}_j^k - \tilde{x}_i^k \right)^{2-\beta} \right] \cdot \left[\prod_{a=1}^{k} \prod_{b=1}^{k+1} \left| \tilde{x}_a^k - \tilde{x}_b^{k+1} \right|^{\beta/2-1} \right] \\
&\quad \times \prod_{k=1}^{N-1} \left[\prod_{1 \le i < j \le k} \left(1 + \frac{1}{\sqrt{\beta}} \cdot \frac{\Delta x_j^k - \Delta x_i^k}{\tilde{x}_j^k - \tilde{x}_i^k} \right)^{2-\beta} \right] \cdot \left[\prod_{a=1}^{k} \prod_{b=1}^{k+1} \left(1 + \frac{1}{\sqrt{\beta}} \cdot \frac{\Delta x_a^k - \Delta x_b^{k+1}}{\tilde{x}_a^k - \tilde{x}_b^{k+1}} \right)^{\beta/2-1} \right] \\
&= \frac{1}{Z_N} \prod_{k=1}^{N-1} \left[\prod_{1 \le i < j \le k} \left(\tilde{x}_j^k - \tilde{x}_i^k \right)^{2-\beta} \right] \cdot \left[\prod_{a=1}^{k} \prod_{b=1}^{k+1} \left| \tilde{x}_a^k - \tilde{x}_b^{k+1} \right|^{\beta/2-1} \right] \\
&\quad \times \prod_{k=1}^{N-1} \exp \left[-\sqrt{\beta} \sum_{1 \le i < j \le k} \frac{\Delta x_j^k - \Delta x_i^k}{\tilde{x}_j^k - \tilde{x}_i^k} + \frac{\sqrt{\beta}}{2} \sum_{a=1}^{k} \sum_{b=1}^{k+1} \frac{\Delta x_a^k - \Delta x_b^{k+1}}{\tilde{x}_a^k - \tilde{x}_b^{k+1}} \right] \\
&\quad \times \prod_{k=1}^{N-1} \exp \left[\sum_{1 \le i < j \le k} \frac{\left(\Delta x_j^k - \Delta x_i^k \right)^2}{2 \left(\tilde{x}_j^k - \tilde{x}_i^k \right)^2} - \sum_{a=1}^{k} \sum_{b=1}^{k+1} \frac{\left(\Delta x_a^k - \Delta x_b^{k+1} \right)^2}{4 \left(\tilde{x}_a^k - \tilde{x}_b^{k+1} \right)^2} + O\left(\beta^{-1/2} \right) \right].
\end{aligned} \tag{59}$$

As $\beta \to \infty$, the last line of (59) gives the desired density of ξ_i^k in the ∞-corners process, and it remains to show that the fourth line of (59) is identically equal to 1. Indeed, the coefficient of Δx_i^k in the expression under exponent is

$$-\sqrt{\beta} \sum_{\substack{1\le j\le k,\\ j\neq i}} \frac{1}{\tilde{x}_i^k - \tilde{x}_j^k} + \frac{\sqrt{\beta}}{2} \sum_{j=1}^{k-1} \frac{1}{\tilde{x}_i^k - \tilde{x}_j^{k-1}} + \frac{\sqrt{\beta}}{2} \sum_{j=1}^{k+1} \frac{1}{\tilde{x}_i^k - \tilde{x}_j^{k+1}}. \tag{60}$$

The last term in (60) is zero because of the equations (58). For the first two terms, recall that $\tilde{x}_i^k$ and $\tilde{x}_{k-1}^i$ are roots of a polynomial and its derivative, that is,

$$\sum_{a=1}^{k} \prod_{j\neq a} \left(u - \tilde{x}_j^k\right) = k \prod_{a=1}^{k-1} \left(u - \tilde{x}_a^{k-1}\right). \tag{61}$$

We then differentiate (61) in u and plug in $u = \tilde{x}_i^k$ to get

$$2 \sum_{a=1}^{k} \prod_{j\neq a,i} \left(\tilde{x}_i^k - \tilde{x}_j^k\right) = k \sum_{j=1}^{k-1} \prod_{a\neq j} \left(\tilde{x}_i^k - \tilde{x}_a^{k-1}\right) \tag{62}$$

and plug $u = \tilde{x}_i^k$ into (61) to get

$$\prod_{j\neq i} \left(\tilde{x}_i^k - \tilde{x}_j^k\right) = k \prod_{a=1}^{k-1} \left(\tilde{x}_i^k - \tilde{x}_a^{k-1}\right). \tag{63}$$

Dividing (62) by (63), we conclude that the first two terms in (60) cancel out. As a result, we see that the contribution of the fourth line of (59) must vanish as $\beta \to \infty$.

The last ingredient of the proof of Theorem 1.6 is to explain that the density (11) is integrable, that is, that the inverse covariance matrix arising in this density is indeed positive definite. This would have been immediate, if all the terms in the exponent had negative signs, yet the $i < j$ sum is positive, and therefore, an additional clarification is necessary. We prove that the integral of (11) is finite by integrating the variables ξ_i^k sequentially in k; that is, we first integrate over ξ_1^1, then over ξ_1^2, ξ_2^2, etc. Each step is an integration over $k-1$ variables $\xi_1^{k-1}, \dots, \xi_{k-1}^{k-1}$, and the integral is the limit (as $\beta \to \infty$) of the identity expressing the unit total mass of the conditional probability (56). Repeating the argument (59), this limit is an identity holding for any k reals $\zeta_1, \dots, \zeta_k$:

$$\int \cdots \int \exp\left(-\sum_{a=1}^{k} \sum_{b=1}^{k-1} \frac{(\zeta_a - \xi_b)^2}{\left(x_a^k - x_b^{k-1}\right)^2} \right) \mathrm{d}\xi_1 \dots \mathrm{d}\xi_{k-1} = Z \cdot \exp\left(-2 \sum_{1 \le a < b \le k} \frac{(\zeta_a - \zeta_b)^2}{\left(x_a^k - x_b^k\right)^2} \right), \quad (64)$$

where $Z > 0$ does not depend on $\zeta_1, \dots, \zeta_k$; it can be explicitly evaluated, but we do not need its value here. Note that in the integrand of (64) the expression in the exponent is clearly negative, and therefore the question of convergence of the integral does not arise. However, iteratively using (64) for $k = 2, \dots, N$, we compute the (finite) normalizing constant for the density (11).

4 Discrete Versions and Generalities

4.1 Expectation identities at general (q, t)

The main ingredient of our proofs, which is the expectation computations of Section 3.1, admits a generalization up to the hierarchy of symmetric functions to the level of Macdonald polynomials.

For Propositions 3.1, 3.3 the Macdonald version is as follows.

Proposition 4.1. Fix signatures $\lambda = (\lambda_1 \ge \dots \ge \lambda_N)$ and $\mu = (\mu_1 \ge \dots \ge \lambda_N)$, and let $\nu = (\nu_1 \ge \dots \nu_N)$ be a random signature with distribution given by the weight $\nu \mapsto c^{\nu}_{\lambda,\mu}(\hat{P}; q, t)$. For each positive signature $\rho = \rho_1 \ge \rho_2 \ge \dots \ge \rho_N \ge 0$ we have

$$\begin{aligned} &\mathbb{E} P_\rho\left(q^{\nu_1} t^{N-1}, q^{\nu_2} t^{N-2}, \dots, q^{\nu_N}; q, t\right) \\ &\quad = \frac{P_\rho\left(q^{\lambda_1} t^{N-1}, q^{\lambda_2} t^{N-2}, \dots, q^{\lambda_N}; q, t\right) \cdot P_\rho\left(q^{\mu_1} t^{N-1}, q^{\mu_2} t^{N-2}, \dots, q^{\mu_N}; q, t\right)}{P_\rho\left(t^{N-1}, t^{N-2}, \dots, 1; q, t\right)}. \end{aligned} \quad (65)$$

Proof. Using (17) and definition of $c^{\nu}_{\lambda,\mu}(\hat{P}; q, t)$ we have

$$\begin{aligned} \mathbb{E}\left[\frac{P_\rho(q^{\nu_1} t^{N-1}, \dots, q^{\nu_N}; q, t)}{P_\rho(t^{N-1}, \dots, 1; q, t)}\right] &= \mathbb{E}\left[\frac{P_\nu(q^{\rho_1} t^{N-1}, \dots, q^{\rho_N}; q, t)}{P_\nu(t^{N-1}, \dots, 1; q, t)}\right] \\ &= \sum_{\nu} \frac{P_\nu\left(q^{\rho_1} t^{N-1}, \dots, q^{\rho_N}; q, t\right)}{P_\nu\left(t^{N-1}, \dots, 1; q, t\right)} c^{\nu}_{\lambda,\mu}(\hat{P}; q, t) \\ &= \frac{P_\lambda\left(q^{\rho_1} t^{N-1}, \dots, q^{\rho_N}; q, t\right)}{P_\lambda\left(t^{N-1}, \dots, 1; q, t\right)} \cdot \frac{P_\mu\left(q^{\rho_1} t^{N-1}, \dots, q^{\rho_N}; q, t\right)}{P_\mu\left(t^{N-1}, \dots, 1; q, t\right)} \\ &= \frac{P_\rho\left(q^{\lambda_1} t^{N-1}, \dots, q^{\lambda_N}; q, t\right)}{P_\rho\left(t^{N-1}, \dots, 1; q, t\right)} \cdot \frac{P_\rho\left(q^{\lambda_1} t^{N-1}, \dots, q^{\lambda_N}; q, t\right)}{P_\rho\left(t^{N-1}, \dots, 1; q, t\right)}. \end{aligned}$$

■

For an analog of Proposition 3.2 we need a new definition generalizing $\pi^{\beta}_{N\to k}$.

Definition 4.2. Fix a $0 < k < N$, and a signature $\lambda = (\lambda_1 \geq y_2 \geq \cdots \geq \lambda_N)$. Define a random signature $\nu = (\nu_1 \geq \nu_k)$ with distribution $\pi^{q,t}_{N\to k}(\lambda)$ through the following decomposition

$$\frac{P_\lambda\left(t^{1-N}, t^{2-N}, \ldots, t^{-k}, z_1, \ldots, z_k; q, t\right)}{P_\lambda\left(t^{1-N}, \ldots, t^{-1}, 1; q, t\right)} = \sum_\nu \pi^{q,t}_{N\to k}(\lambda)[\nu] \cdot \frac{P_\nu\left(z_1, \ldots, z_k; q, t\right)}{P_\nu\left(t^{1-k}, \ldots, 1; q, t\right)}. \tag{66}$$

Plugging $z_i = t^{1-i}$ into (66) one proves that $\sum_\nu \pi^{q,t}_{N\to k}(\lambda)[\nu] = 1$. The non-negativity of weights follows from the branching rules for the Macdonald polynomials, see [35].

Let us emphasize, that we use negative powers of t in (66). On the other hand, the normalization of Macdonald polynomials $\hat{P}_\lambda$ entering the definition of $c^\nu_{\lambda,\mu}(\hat{P}; q, t)$ involved positive powers. For the latter this difference is not important due to homogeneity of P_λ and vanishing of $c^\nu_{\lambda,\mu}(\hat{P}; q, t)$ unless $\sum_i(\lambda_i + \mu_i - \nu_i) = 0$. However, for (66) this becomes important.

Proposition 4.3. Fix $k < N$, a signature λ of rank N, and let ν be $\pi^{q,t}_{N\to k}(\lambda)$–distributed. Then for any non-negative signature $\rho = (\rho_1 \geq \cdots \geq \rho_k \geq 0)$ of rank k we have

$$\mathbb{E} P_\rho\left(q^{\nu_1} t^{k-1}, \ldots, q^{\nu_k}; q, t\right) = \frac{P_\rho\left(t^{k-1}, \ldots, t, 1; q, t\right)}{P_\rho\left(t^{N-1}, \ldots, t, 1; q, t\right)} \cdot P_\rho\left(q^{\lambda_1} t^{N-1}, \ldots, q^{\lambda_N}; q, t\right) \tag{67}$$

where we also treated ρ as a signature of rank N by adding $N - k$ zero coordinates.

Proof. Using (17) and homogeneity of Macdonald polynomials we have

$$\begin{aligned}
&\mathbb{E}\left[\frac{P_\rho(q^{\nu_1} t^{k-1}, \ldots, q^{\nu_k}; q, t)}{P_\rho(t^{k-1}, \ldots, t, 1; q, t)}\right] \\
&= \mathbb{E}\left[\frac{P_\nu(q^{\rho_1} t^{k-1}, \ldots, q^{\rho_k}; q, t)}{P_\nu(t^{k-1}, \ldots, 1; q, t)}\right] = \sum_\nu \pi^{q,t}_{N\to k}(\lambda)[\nu] \frac{P_\nu\left(q^{\rho_1} t^{k-1}, \ldots, q^{\rho_k}; q, t\right)}{P_\nu\left(t^{k-1}, \ldots, t, 1; q, t\right)} \\
&= \sum_\nu \pi^{q,t}_{N\to k}(\lambda)[\nu] \frac{P_\nu\left(q^{\rho_1}, \ldots, q^{\rho_k} t^{1-k}; q, t\right)}{P_\nu\left(t^{1-k}, \ldots, t^{-1}, 1; q, t\right)} = \frac{P_\lambda\left(q^{\rho_1}, \ldots, q^{\rho_k} t^{1-k}, t^{-k}, \ldots, t^{1-N}; q, t\right)}{P_\lambda\left(t^{1-N}, \ldots, t^{-1}, 1; q, t\right)} \\
&= \frac{P_\lambda\left(q^{\rho_1} t^{N-1}, \ldots, q^{\rho_k} t^{N-k}, t^{N-k-1}, \ldots, 1; q, t\right)}{P_\lambda\left(t^{N-1}, \ldots, t, 1; q, t\right)} = \frac{P_\rho\left(q^{\lambda_1} t^{N-1}, \ldots, q^{\lambda_N}; q, t\right)}{P_\rho\left(t^{N-1}, \ldots, t, 1; q, t\right)}.
\end{aligned} \tag{68}$$

■

An analog of Proposition 4.3 at $q = t$ is implicitly used in [24], [48], [49] for the study of the extended Gelfand–Tsetlin graph.

As in Section 3.2, if we choose $\rho_1 = \cdots = \rho_\ell = 1$, $\rho_{\ell+1} = \rho_{\ell+2} = \cdots = 0$ in Propositions 4.1, 4.3, then the Macdonald polynomials would turn into elementary symmetric functions e_ℓ, and we get formulas for the expectations of e_ℓ. In particular, q does not enter into these formulas in any explicit form, which is a (q, t)–analog of the β-independence in Theorem 1.1.

4.2 Crystallization for general (q, t)

The Law of Large Numbers (crystallization) of Theorems 1.2, 1.6 is obtained from operations on Macdonald polynomials $P_\lambda(\cdot;\, q, t)$ by a triple limit transition:

$$q \to 1; \qquad t = q^\theta,\ \theta \to +\infty; \qquad \lambda_i = \varepsilon^{-1} r_i,\ \varepsilon \to 0. \tag{69}$$

In these theorems, we made the limit transitions in a particular order (first, $q \to 1$, $\lambda_i \to \infty$ to degenerate into random matrices, and only then $\theta \to \infty$), but different orders of taking limits are also possible and would lead to *another* set of answers. We do not address the full classification of the limiting behaviors here (it probably deserves a separate publication), but only mention two possible scenarios.

(1) If we start with $\theta \to \infty$ (so $t \to 0$), then Macdonald polynomials degenerate to q-Whittaker functions, as discussed in details in [7], [23]. Two different further $q \to 1$ limits were studied in the literature. The first one is parallel to the degeneration of q-Whittaker functions to Whittaker functions: the particles crystallize on a perfect lattice, while fluctuations are related to directed polymers in random media, see [7]. Another limit in [8] leads to more complicated Law of Large Numbers and Gaussian fluctuations.

(2) We can first degenerate Macdonald polynomials into Jacks and the latter into products of elementary symmetric functions, as in (46). After taking these limits, an analog of the β-corners process would involve weights given by products of Binomial coefficients, while the top-row (which was $y_1 < \cdots < y_N$ in Definition 1.3) is still discrete. Linearly rescaling the coordinates of the top row one finds yet another Law of Large Numbers. Using Stirling's formula and solving the associated maximization formula (as in the proof of Theorem 1.6) one can explicitly find the limit then. It

has the following description: kth particle of level $M-1$ splits the interval between kth and $(k+1)$st particles on level M in the proportion $k:(N-k)$.

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References

[1] Anderson, G. W. "A short proof of Selberg's generalized beta formula." *Forum Math*. 3 (1991): 415–7.

[2] Anker, J. P. "An introduction to Dunkl theory and its analytic aspects." Preprint arXiv:1611.08213.

[3] Baryshnikov, Y. "GUEs and queues." *Probab. Theory Relat. Fields.* 119, no. 2 (2001): 256–74.

[4] Borcea, J. and P. Brändén. "The Lee-Yang and Polya-Schur Programs. II. Theory of Stable Polynomials and Applications." *Comm. Pure Appl. Math.* 62, no. 12 (2009): 1595–631, arXiv:0809.3087.

[5] Borodin, A. "CLT for spectra of submatrices of Wigner random matrices." *Moscow Mat. J.* 14, no. 1 (2014): 29–38, arXiv:1010.0898.

[6] Borodin, A. "Stochastic higher spin six vertex model and Macdonald measures." Preprint arXiv:1608.01553.

[7] Borodin, A.and I. Corwin. "Macdonald processes." *Probab. Theory Relat. Fields.* 158 (2014): 225, arXiv:1111.4408.

[8] Borodin, A., I. Corwin, and P. L. Ferrari. "Anisotropic (2+1)d growth and Gaussian limits of q-Whittaker processes." Prepint arXiv:1612.00321.

[9] Borodin, A. and V. Gorin. "General beta Jacobi corners process and the Gaussian Free Field." *Commun. Pure Appl. Math.* 68, no. 10 (2015): 1774–844, arXiv:1305.3627.

[10] Brändén, P., J. Haglund, M. Visontai, and D. G. Wagner. "Proof of the monotone column permanent conjecture." In: *Notions of Positivity and the Geometry of Polynomials*, edited by P. Brändén, M. Passare, and M. Putinar. *Trends Math.* Basel: Springer, 2011.

[11] Bufetov, A. "Kerov's interlacing sequences and random matrices." *J. Math. Phys.* 54, no. 11(2013), arXiv:1211.1507.

[12] Bufetov, A. and V. Gorin. "Representations of classical Lie groups and quantized free convolution." *Geom. Funct. Anal.* 25 (2015): 763–814, arXiv:1311.5780.

[13] Bufetov, A. and V. Gorin. "Fluctuations of particle systems determined by Schur generating functions." Preprint arXiv:1604.01110.

[14] Che, Z. and B. Landon. "Local spectral statistics of the addition of random matrices." Preprint arXiv:1701.00513.

[15] Dixon, A. L. "Generalizations of Legendre's formula $KE' - (K - E)K' = \frac{1}{2}\pi$." *Proc. London Math. Soc.* 3 (1905): 206–24.

[16] Dumitriu, I. and A. Edelman. "Eigenvalues of Hermite and Laguerre ensembles: large beta asymptotics." *Annales de l'Institut Henri Poincare (B) Probab. Stat.* 41, no. 6 (2005): 1083–99, arXiv:math-ph/0403029.

[17] Dunkl, C. "Hankel transforms associated to finite reflection groups, Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991)." *Contemp. Math.* 138 (1992): 123–38.

[18] Edelman, A. "The random matrix technique of ghosts and shadows." *Markov Processes Relat. Fields.* 16, no. 4 (2010): 783–90.

[19] Erdős, L. and D. Schröder. "Fluctuations of rectangular Young diagrams of interlacing Wigner eigenvalues." (2016): preprint arXiv:1608.05163.

[20] Farmer, D. and R. C. Rhoades. "Differentiation evens out zero spacings." *Trans. Amer. Math. Soc.* 357 (2005): 3789–811, arXiv:math/0310252.

[21] Forrester, P. J. *Log-gases and Random Matrices*. Princeton: Princeton University Press, 2010.

[22] Fulton, W. "Eigenvalues, invariant factors, highest weights, and Schubert calculus." *Bull. Amer. Math. Soc.* 37 (2000): 209–49.

[23] Gerasimov, A. D. Lebedev, and S. Oblezin. "On a classical limit of q-deformed Whittaker functions." *Lett. Math. Phys.* 100 (2012): 279, arXiv:1101.4567.

[24] Gorin, V. and G. Olshanski. "A quantization of the harmonic analysis on the infinite-dimensional unitary group." *J. Funct. Anal.* 270 (2016): 375–418, arXiv:1504.06832.

[25] Gorin, V. and M. Shkolnikov. "Multilevel Dyson Brownian motions via Jack polynomials." *Probab. Theory Relat. Fields.* 163, no. 3 (2015): 413–63, arXiv:1401.5595.

[26] Gorin, V. and L. Zhang. "Interlacing adjacent levels of *&beta*;-Jacobi corners processes." Preprint arXiv:1612.02321.

[27] Guhr, T. and H. Kohler. "Recursive construction for a class of radial functions. I. Ordinary space." *J. Math. Phys.* 43, no. 5 (2002): 2707–40.

[28] Heckman, G. J. and E. M. Opdam. "Root systems and hypergeometric functions. I, II." *Compositio Mathematica.* 64, no. 3 (1987): 329–52 and 353–73.

[29] Heckman, G. J. and H. Schlichtkrull. *Harmonic Analysis and Special Functions on Symmetric Spaces*. Boston: Acad. Press, 1994.

[30] de Jeu, M. F. E. "The Dunkl transform." *Invent. Math.* 113, no. 1 (1993): 147–62.

[31] Kapovich, M. and J. J. Millson. "A path model for geodesics in Euclidean buildings and its applications to representation theory." *Groups, Geometry Dynamics.* 2 (2008): 405–80, arXiv: math.RT/0411182

[32] Kerov, S. "Asymptotics of the separation of roots of orthogonal polynomials." *St. Petersburg Math. J.* 5 (1994): 925–41.

[33] Koekoek, R. and R. F. Swarttouw. "The Askey-scheme of hypergeometric orthogonal polynomials and its q-analogue." Report 98–17, Faculty of Technical Mathematics and Informatics, Delft University of Technology, 1998. http://aw.twi.tudelft.nl/ koekoek/askey/.

[34] Leble, T. and S. Serfaty, "Large deviation principle for empirical fields of log and Riesz Gases." Preprint arXiv:1502.02970.

[35] Macdonald, I. G. *Symmetric Functions and Hall Polynomials*, 2nd ed. New York: The Clarendon Press, Oxford University Press, 1995.

[36] Marcus, A. W. "Polynomial convolutions and (finite) free probability." Preprint.

[37] Marcus, A. W. "Discrete unitary invariance." Preprint arXiv:1607.06679

[38] Marcus, A. W., D. A. Spielman, and N. Srivastava. "Finite free convolutions of polynomials." Preprint arXiv:1504.00350.

[39] Marcus, A. W., D. A. Spielman, and N. Srivastava. "Interlacing families I: bipartite Ramanujan graphs of all degrees." *Ann. of Math.* 182–1 (2015): 307–25, arXiv:1304.4132.

[40] Marcus, A. W., D. A. Spielman, and N. Srivastava. "Interlacing families IV: bipartite Ramanujan graphs of all sizes." *FOCS* (2015): 1358–77, arXiv:1505.08010.

[41] Metcalfe, A. "Universality properties of Gelfand-Tsetlin patterns." *Probab. Theory Relat. Fields.* 155, no. 303 (2013): arXiv:1105.1272.

[42] Mingo, J. A. and R. Speicher. "Second order freeness and fluctuations of random matrices: I. Gaussian and Wishart matrices and cyclic Fock spaces." *J. Funct. Anal.* 235, no. 1 (2006): 226 no. 1–270, arXiv:math/0405191.

[43] Mingo, J. A., P. Sniady, and R. Speicher. "Second order freeness and fluctuations of random matrices: II. Unitary Random Matrices." *Adv. Math.* 209, no. 1 (2007): 212–40, arXiv:math/0405258.

[44] Neretin, Yu. A. "Rayleigh triangles and non-matrix interpolation of matrix beta integrals." *Sbornik: Math.* 194, no. 4 (2003): 515–40.

[45] Nica, A. and R. Speicher. *Lectures on the Combinatorics of Free Probability*. Cambridge: Cambridge University Press, 2006.

[46] Okounkov, A. and G. Olshanski. "Shifted Jack polynomials, binomial formula, and applications." *Math. Res. Letters.* 4 (1997): 69–78, arXiv:q-alg/9608020.

[47] Olshanski, G. and A. Vershik. "Ergodic unitarily invariant measures on the space of infinite Hermitian matrices." In *Contemporary Mathematical Physics*. F. A. Berezin's memorial volume. *Amer. Math. Transl. Ser.* 2, vol. 175 (R. L. Dobrushin et al., eds), (1996): 137–75, arXiv:math/9601215.

[48] Olshanski, G. "Extended Gelfand-Tsetlin graph, its q-boundary, and q-B-splines." *Funct. Anal. Appl.* 50, no. 2 (2016): 107–30, arXiv:1607.04201.

[49] Olshanski, G. "An analogue of the Big q–Jacobi polynomials in the algebra of symmetric functions, in preparation."

[50] Opdam, E. M. "Root systems and hypergeometric functions. III, IV." *Compositio Mathematica.* 67, no. 1 (1988): 21–49; 67(2): 191–209.

[51] Opdam, E. M. "Dunkl operators, Bessel functions, and the discriminant of a finite Coxeter group." *Compos. Math.* 85 (1993): 333–73.

[52] Petrov, L. "Asymptotics of uniformly random lozenge tilings of polygons. Gaussian free field." *Ann. Probab.* 43, no. 1 (2014): 143, arXiv:1206.5123.

[53] Puder, D. and W. Sawin. Private communication.

[54] Ram, A. "Alcove walks, Hecke algebras, spherical functions, crystals and column strict tableaux." *Pure Appl. Math. Q.* 2, no. 4 (2006): 9631013.

[55] Rösler, M. "A positive radial product formula for the Dunkl kernel." *Trans. Amer. Math. Soc.* 355 (2003): 2413–38, arXiv:math/0210137.

[56] Sandier, E. and S. Serfaty. "1D log gases and the renormalized energy: crystallization at vanishing temperature." *Ann. Probab.* 43, no. 4 (2015): 2026–83, arXiv:1303.2968.

[57] Scwher, C. "Galleries, Hall-Littlewood polynomials, and structure constants of the spherical Hecke algebra." *Int. Math. Res. Not.* (2006): 1–31.

[58] Sheffield, S. "Gaussian free fields for mathematicians." *Probab. Theory Relat. Fields.* 139 (2007): 521–41, arXiv:math.PR/0312099.

[59] Schramm, O. and S. Sheffield. "Contour lines of the two-dimensional discrete Gaussian free field." *Acta Math.* 202, no. 1 (2009): 21 137, arXiv:math/0605337.

[60] Sodin, S. "Fluctuations of interlacing sequences." (2016): Preprint arXiv:1610.02690.

[61] Stanley, R. "Some combinatorial properties of Jack symmetric functions." *Adv. Math.* 77 (1989): 76–115.

[62] Sun, Y. "A new integral formula for Heckman-Opdam hypergeometric functions." *Adv. Math.* 289 (2016): 1157–204.

[63] Sun, Y. "Matrix models for multilevel Heckman-Opdam and multivariate Bessel measures." Preprint arXiv:1609.09096.

[64] Voiculescu, D. "Symmetries of some reduced free product $C*$-algebras." *Operator Algebras and Their Connections with Topology and Ergodic Theory* (Busteni, 1983), Lecture Notes in Mathematics, vol. 1132, Springer, 1985, 556–88.

[65] Voiculescu, D. "Addition of certain non-commuting random variables." *J. Funct. Anal.* 66 (1986): 323–46.

[66] Voiculescu, D. "Limit laws for random matrices and free products." *Invent. Math.* 104 (1991): 201–20.

[67] Voiculescu, D., K. Dykema, and A. Nica. "Free random variables." CRM Monograph Series, vol. 1, Providence, RI: American Mathematical Society, 1992.

[68] Walsh, J. L. "On the location of the roots of certain types of polynomials." *Trans. Am. Math. Soc.* 24, no. 3 (1922): 163–80.

[69] Werner, W. "Topics on the two-dimensional Gaussian Free Field." Preprint.