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CUBIC SPLINE COALESCENCE FRACTAL INTERPOLATION THROUGH MOMENTS

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Abstract

This paper generalizes the classical cubic spline with the construction of the cubic spline coalescence hidden variable fractal interpolation function (CHFIF) through its *moments*, i.e. its second derivative at the mesh points. The second derivative of a cubic spline CHFIF is a typical fractal function that is self-affine or non-self-affine depending on the parameters of the generalized iterated function system. The convergence results and effects of hidden variables are discussed for cubic spline CHFIFs.

Keywords: Iterated Function Systems; Fractal Interpolation Functions; Coalescence; Cubic Spline; Moments; Self-Affine; Non-Self-Affine; Convergence.

1. INTRODUCTION

Fractals represent powerful techniques to approximate natural objects such as trees, clouds, landscapes, glaciers, and waves that cannot be described by using classical geometry. With the introduction of the term *fractals* by Mandelbrot,[1] the fractal geometry has been successfully used in various domains such as economics,[2] physics,[3] graphics,[4] life sciences,[5] signal processing,[6] image processing,[7] etc. Fractal interpolation function (FIF) is

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introduced by Barnsley[8,9] through the iterated function system (IFS) to model a large class of self-affine or self-similar objects. The construction of a FIF is based on the fixed point of the Read-Bajraktarević operator that preserves self-affinity or self-similarity.

The term "hidden variable" has been introduced by Barnsley *et al.*[10] and Massopust.[11] The hidden variable FIF (HFIF) is more diverse, appealing and irregular than FIF for the same set of interpolation data since the values of hidden variable FIF continuously depend on all parameters which define it. Since the HFIF is the projection of a vector valued function, it is usually non-self-affine in nature. However, in practical applications of FIFs, the interpolation data might be generated simultaneously from self-affine or non-self-affine functions.[12] To approximate self-affine or non-self-affine functions simultaneously, the coalescence hidden variable FIF (CHFIF) is introduced (see for instance Refs. 13 and 14).

The existence of a differentiable FIF or spline FIF (SFIF) is introduced by Barnsley and Harrington.[15] However, the construction of SFIFs with only a fixed type of boundary conditions is allowed in their construction. The construction of SFIFs with any type of boundary conditions is given in Refs. 14, 16 to 18. The derivative of a SFIF is a typical fractal that is self-affine in nature. CHFIFs can be integrated successively in order to get more diverse and appealing spline CHFIFs, where hidden variables play significant role in their shapes. Such type of splines are useful in practical applications since their derivatives can be either of self-affine or non-self-affine fractal functions. The construction and convergence analysis of cubic spline CHFIFs have practical importance in view of significant applications of cubic splines in science and engineering problems.[19,20]

In the present paper, the construction of cubic spline CHFIF $f_1(x)$ on a mesh Δ is developed through *moments* $M_n^* = f_1{''}(x_n), n = 0, 1, 2, \ldots, N$, with any type of boundary conditions as in the classical cubic spline. The advantage of such a construction is that, for a prescribed data and boundary conditions, one can have infinite number of cubic spline CHFIFs that are self-affine or non-self-affine, depending on choice of hidden variables and boundary conditions of the associated cubic spline fractal functions. The convergence results of cubic spline CHFIFs on two classes of sequence of uniform or non-uniform meshes are proved for the data generating function $\Phi(x)$, where $\Phi \in C^r[x_0, x_N]$, for $r = 2, 3$, or 4.

The organization of this paper is as follows. In Sec. 2, we discuss the construction of cubic spline CHFIFs through its moments. Our construction admits all types of boundary conditions as in classical cubic splines. The convergence results of cubic spline CHFIFs are described on two classes of sequence of meshes in Sec. 3. Finally, effects of hidden variables on the cubic spline CHFIFs are illustrated through suitably chosen examples in Sec. 4.

2. CUBIC SPLINE HFIF

First, we discuss in Sec. 2.1 the basics of CHFIFs. The construction of cubic spline CHFIFs is described in Sec. 2.2.

2.1. Basics of CHFIFs

Let $x_0 < x_1 < \cdots < x_N$ be a partition of an interval $I = [x_0, x_N] \subset \mathbb{R}$ and $\{(x_n, y_n) \in I \times \mathbb{R} : n = 0, 1, 2, \ldots, N\}$ be a set of data points. This data set is extended to a generalized set of data $\{(x_n, y_n, z_n) \in \mathbb{R}^3 : n = 0, 1, 2, \ldots, N\}$ with real parameters z_n, $n = 0, 1, 2, \ldots, N$. Set, $\tilde{g}_1 = \text{Min}_n\, y_n$, $\tilde{g}_2 = \text{Max}_n\, y_n$, $\tilde{h}_1 = \text{Min}_n\, z_n$, $\tilde{h}_2 = \text{Max}_n\, z_n$, and $K = I \times D$, where $D = J_1 \times J_2$, J_1 and J_2 are suitable compact sets in $\mathbb{R}$ such that $[\tilde{g}_1, \tilde{g}_2] \subseteq J_1$, $[\tilde{h}_1, \tilde{h}_2] \subseteq J_2$. Let $L_n : I \to I_n = [x_{n-1}, x_n]$ be a contraction map satisfying

$$L_n(x_0) = x_{n-1}, \quad L_n(x_N) = x_n. \tag{2.1}$$

Let $F_n : K \to D$ be a vector valued function satisfying

$$\left.\begin{aligned} &F_n(x_0, y_0, z_0) = (y_{n-1}, z_{n-1}),\\ &F_n(x_N, y_N, z_N) = (y_n, z_n),\\ &d(F_n(x, y, z), F_n(x^*, y^*, z^*))\\ &\quad \le s d_E((y, z), (y^*, z^*)), \end{aligned}\right\} \tag{2.2}$$

for $n = 1, 2, \ldots, N$, where (x, y, z), $(x^*, y^*, z^*) \in K$, $0 \le s < 1$, d is the sup. metric on K, and d_E is the Euclidean metric on $\mathbb{R}^2$. In order to define the CHFIF, functions L_n and F_n are chosen such that $L_n(x) = a_n x + b_n$ and

$$\begin{aligned} F_n(x, y, z) &= A_n(y, z)^T + (p_n(x), q_n(x))^T\\ &= (F_n^1(x, y, z), F_n^2(x, z))^T, \end{aligned} \tag{2.3}$$

where A_n is an upper triangular matrix $\begin{pmatrix} \alpha_n & \beta_n \\ 0 & \gamma_n \end{pmatrix}$ and $p_n(x)$, $q_n(x)$ are continuous functions having two free parameters. These parameters can be determined by using Eq. (2.2). We choose α_n as free variable with $|\alpha_n| < 1$ and β_n as constrained free variable with respect to γ_n such that $|\beta_n|+|\gamma_n| < 1$. The generalized IFS that is needed for construction of a CHFIF corresponding to the data $\{(x_n, y_n, z_n) : n = 0, 1, \ldots, N\}$ is now defined as

$$\{\mathbb{R}^3; \omega_n(x, y, z) = (L_n(x), F_n(x, y, z)), \quad n = 1, 2, \ldots, N\}. \tag{2.4}$$

It is known[10] that the IFS defined in Eq. (2.4) associated with the data $\{(x_n, y_n, z_n) : n = 0, 1, \ldots, N\}$ is hyperbolic with respect to a metric τ that is equivalent to the Euclidean metric on $\mathbb{R}^3$. Hence, there exists a unique non-empty compact set $G \subseteq \mathbb{R}^3$, called as attractor of the IFS (2.4), such that $G = \bigcup_{i=1}^{N} \omega_i(G)$. This attractor G provides the existence of a unique vector valued interpolant f in the following proposition.

Proposition 2.1.[10] *The attractor G of the IFS defined by Eq.* (2.4) *is the graph of the continuous vector valued function $f : I \to D$ such that $f(x_n) = (y_n, z_n)$ for all $n = 1, 2, \ldots, N$, i.e., $G = \{(x, y, z) : x \in I$ and $f(x) = (y(x), z(x))\}$.*

Proposition 2.1 gives that the graph of the vector valued function $f(x) = (f_1(x), f_2(x))$ is the attractor of the IFS $\{\mathbb{R}^3;\ \omega_n(x, y, z), n = 1, 2, \ldots, N\}$ if and only if the fixed point f of Read-Bajraktarević operator T on the space of continuous vector valued functions from I to D satisfies

$$Tf(x) = f(x) = F_n(L_n^{-1}(x), f(L_n^{-1}(x))), \quad x \in I_n, \quad n = 1, 2, \ldots, N. \tag{2.5}$$

The image Tf of the vector valued function f can be written component wise as $(T_1 f_1, T_2 f_2)$, where T_1 and T_2 are component wise Read-Bajraktarević operators from I to $\mathbb{R}$. The function $f_1(x)$ in the projection $\{(x, f_1(x)) : x \in I\}$ of the attractor G on $\mathbb{R}^2$, is called *coalescence FIF* or *coalescence hidden variable FIF* (CHFIF) for the given data $\{(x_n, y_n) : n = 0, 1, \ldots, N\}$. It is easily seen that CHFIFs satisfy the following functional equation for $x \in I$.

$$\begin{aligned} T_1 f_1(L_n(x)) &= f_1(L_n(x)) \\ &= F_n^1(x, f_1(x), f_2(x)) \\ &= \alpha_n f_1(x) + \beta_n f_2(x) + p_n(x). \end{aligned} \tag{2.6}$$

Similarly, the function $f_2(x)$ in the projection $\{(x, f_2(x)) : x \in I\}$ of the attractor G is a self-affine fractal function that interpolates the data $\{(x_n, z_n) : n = 0, 1, \ldots, N\}$ and satisfies the following functional equation.

$$\begin{aligned} T_2 f_2(L_n(x)) &= f_2(L_n(x)) \\ &= F_n^2(x, f_2(x)) \\ &= \gamma_n f_2(x) + q_n(x), \quad x \in I. \end{aligned} \tag{2.7}$$

Since the projection of the attractor is not always union of affine transformations of itself, HFIFs are generally non-self-affine by nature. By choosing $y_n = z_n$ and $\alpha_n + \beta_n = \gamma_n$, CHFIF $f_1(x)$ obtained as the projection on $\mathbb{R}^2$ of the attractor of the IFS (2.4) coincides with a self-affine fractal function $f_2(x)$ for the same interpolation data. Hence, the CHFIF is self-affine or self-similar in this case. This type of CHFIFs can be used to approximate the random steps of Gaussian, increments of the fractional Brownian function and wave-height functions.[3]

2.2. Construction of Cubic Spline CHFIFs

A function $S(x)$ is said to be a cubic spline on a grid $x_0 < x_1 < \cdots < x_N$ if it satisfies (i) $S(x)$ is a polynomial of degree 3 on each subinterval $[x_{n-1}, x_n]$ and (ii) $S^r(x)$ is continuous on $[x_0, x_N]$ for $r = 0, 1, 2$. The following proposition provides the existence of a SFIF.

Proposition 2.2.[16] *Let $\{(x_n, y_n) : n = 0, 1, 2, \ldots, N\}$ be the interpolation data with $x_0 < x_1 < x_2 < \ldots < x_N$. Let $L_n(x) = a_n x + b_n$ that satisfies Eq.* (2.1) *and $F_n(x, y) = \alpha_n y + q_n(x)$ for $n = 1, 2, \ldots, N$. Suppose for some integer $r \geq 0$, $|\alpha_n| < a_n^r$, and $q_n \in C^r[x_0, x_N]; n = 1, 2, \ldots, N$. Let*

$$F_{n,k}(x, y) = \frac{\alpha_n y + q_n^{(k)}}{a_n^k},$$

$$x_{0,k} = \frac{q_1^{(k)}(x_0)}{a_1^k - \alpha_1},$$

$$x_{N,k} = \frac{q_N^{(k)}(x_N)}{a_N^k - \alpha_N}, \quad k = 1, 2, \ldots, r.$$

If $F_{n-1,k}(x, y)(x_N, y_{N,k}) = F_{n,k}(x_0, y_{o,k})$ for $n = 2, 3, \ldots, N$ and $k = 1, 2, \ldots, r$, then $\{(L_n(x), F_n(x, y))\}_{n=1}^N$ determines a FIF $f \in C^r[x_0, x_N]$ and $f^{(k)}$, $k = 1, 2, \ldots, r$, is the FIF determined by $\{(L_n(x), F_{n,k}(x, y))\}_{n=1}^N$.

Based on the Proposition 2.2, we define the cubic spline CHFIF f_1 through *moments*, $M_n^* = f_1^{''}(x_n)$, $n = 0, 1, 2, \ldots, N$, as follows.

Definition 2.1. A function $f_1(x)$ (or $f_1(Y;x)$) is called a cubic spline CHFIF interpolating to the data set $\{(x_n, y_n) : n = 0, 1, \ldots, N\}$ with mesh $\Delta : x_0 < x_1 < x_2 < \cdots < x_N$, if (i) $f_1 \in C^2[x_0, x_N]$, (ii)f_1 satisfies the interpolation conditions $f_1(x_n) = y_n, n = 0, 1, \ldots, N$, and (iii) the graph of f_1 is the projection of the attractor of a IFS, $\{\mathbb{R}^3; \omega_n(x, y, z), n = 1, 2, \ldots, N\}$ to $\mathbb{R}^2$, where for $n = 1, 2, \ldots, N$, $\omega_n(x, y) = (L_n(x), F_n(x, y, z))$, $L_n(x)$ is defined as in Eq. (2.1), $F_n(x, y, z) = (a_n^2(\alpha_n y + \beta_n z + p_n(x)), a_n^2(\gamma_n z + q_n(x)))$, $0 < |\alpha_n| < 1$, $0 < |\beta_n| + |\gamma_n| < 1$, and $p_n(x)$, $q_n(x)$ are suitable cubic polynomials.

Denote moments of self-affine fractal function as $M_n = f_2''(x_n)$ for $n = 0, 1, 2, \ldots, N$. The moments M_n^* and M_n; $n = 0, 1, 2, \ldots, N$, are used to determine the polynomials $p_n(x)$ and $q_n(x)$ (see Appendix). Thus, the desired IFS for the construction of the cubic spline CHFIF f_1 is given by

$$\begin{aligned}\{\mathbb{R}^3; \omega_n(x, y, z) &= (L_n(x), F_n(x, y, z)\\ &= (F_n^1(x, y, z), F_n^2(x, z))),\\ &\qquad n = 1, 2, \ldots, N\},\end{aligned} \tag{2.8}$$

where $L_n(x) = a_n x + b_n$,

$$\begin{aligned}F_n^1(x, y, z) = a_n^2\Bigg\{&\alpha_n y + \beta_n z + \frac{(M_n^* - \alpha_n M_N^* - \beta_n M_N)(x - x_0)^3}{6(x_N - x_0)} + (M_{n-1}^* - \alpha_n M_0^* - \beta_n M_0)\\ &\times \frac{(x_N - x)^3}{6(x_N - x_0)} - \frac{(M_{n-1}^* - \alpha_n M_0^* - \beta_n M_0)(x_N - x_0)(x_N - x)}{6} - (M_n^* - \alpha_n M_N^* - \beta_n M_N)\\ &\times \frac{(x_N - x_0)(x - x_0)}{6} + \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0 - \beta_n z_0\right)\frac{x_N - x}{x_N - x_0} + \left(\frac{y_n}{a_n^2} - \alpha_n y_N - \beta_n z_N\right)\frac{x - x_0}{x_N - x_0}\Bigg\},\end{aligned} \tag{2.9}$$

and

$$\begin{aligned}F_n^2(x, z) = a_n^2\Bigg\{&\gamma_n z + \frac{(M_n - \gamma_n M_N)(x - x_0)^3}{6(x_N - x_0)} + \frac{(M_{n-1} - \gamma_n M_0)(x_N - x)^3}{6(x_N - x_0)}\\ &- \frac{(M_{n-1} - \gamma_n M_0)(x_N - x_0)(x_N - x)}{6} - \frac{(M_n - \gamma_n M_N)(x_N - x_0)(x - x_0)}{6}\\ &+ \left(\frac{z_{n-1}}{a_n^2} - \gamma_n z_0\right)\frac{x_N - x}{x_N - x_0} + \left(\frac{z_n}{a_n^2} - \gamma_n z_N\right)\frac{x - x_0}{x_N - x_0}\Bigg\}.\end{aligned} \tag{2.10}$$

The projection of the attractor G of the IFS given by Eq. (2.8), i.e. $\{(x, f_1(x)) \mid x \in I\}$ is the graph of the required cubic spline CHFIF that may be self-affine or non-self-affine depending on the hidden variables. Suppose that the data $\{(x_n, y_n) : n = 0, 1, 2, \ldots, N\}$ is generated by a continuous function Φ that is approximated by the cubic spline CHFIF f_1. Then f_1 is called (i) the complete cubic spline CHFIF if it has boundary conditions of Type-I, i.e. $f_1'(x_0) = \Phi'(x_0)$, $f_1'(x_N) = \Phi'(x_N)$; (ii) the natural cubic spline CHFIF with $M_0^* = M_N^* = 0$ if it has boundary conditions of Type-II, i.e. $f_1''(x_0) = \Phi''(x_0) = M_0^*$, $f_1''(x_N) = \Phi''(x_N) = M_N^*$; and (iii) the periodic cubic spline CHFIF if it has boundary conditions of Type-III, i.e. $f_1(x_0) = f_1(x_N)$, $f_1'(x_0) = f_1'(x_N)$, $f_1''(x_0) = f_1''(x_N)$.

Remark 2.1. (1) If free variables $\alpha_n = 0$ and constrained free variables $\beta_n = 0$; $n = 1, 2, \ldots, N$, $F_n^1(x, y, z)$ reduces to a cubic polynomial in each sub-interval of I. Hence, the IFS (2.8) generates the classical cubic spline $S(x)$ as a special case of the cubic spline CHFIF.

(2) In general, a cubic spline CHFIF is not self-affine as it is the projection of attractor of a non-diagonal IFS. But, if $y_n = z_n$, $n = 0, 1, 2, \ldots, N$, $\alpha_n + \beta_n = \gamma_n$ for $n = 1, 2, \ldots, N$, and f_1 , f_2 have the same boundary conditions, the cubic spline CHFIF is self-affine in nature, i.e. second derivative of the cubic spline CHFIF is a typical self-affine fractal function.

(3) If f_1 is periodic, the necessary condition for the existence of the periodic cubic spline CHFIF for prescribed moments M_n is given by

$$\begin{aligned}\sum_{n=1}^{N}[(h_n + h_{n+1})M_n^* - 2\alpha_n h_n M_N^* - 2a_n\beta_n({f_2}'(x_N)\\ - {f_2}'(x_0)) - \beta_n h_n(M_0 + M_N)] = 0.\end{aligned} \tag{2.11}$$

With $\alpha_n = 0$ and $\beta_n = 0$ for $n = 1, 2, \ldots, N$, Eq. (2.11) reduces to the necessary condition for the existence of the periodic classical cubic spline associated with M_n.[21] Also, for $\beta_n = 0$ for $n = 1, 2, \ldots, N$, Eq. (2.11) reduces to the necessary condition for the existence of the periodic SFIF associated with M_n^*.[16]

3. CONVERGENCE OF CUBIC SPLINE CHFIFs

Let $\mathcal{G}^* = \{f \in C^2(I, \mathbb{R}^2) \;:\; f(x_n) = (y_n, z_n),\; n = 0, 1, 2, \ldots, N\}$, where $I = [x_0, x_N]$. Let f_1 and f_2 be the components of the vector valued function f such that $f = (f_1, f_2)$. From Eqs. (2.6) and (2.9), the cubic spline CHFIF satisfies the implicit relation

$$f_1(L_n(x)) = a_n^2(\alpha_n f_1(x) + \beta_n f_2(x) + p_n(x)), \quad x \in I \tag{3.1}$$

and from Eqs. (2.7) and (2.10), the self-affine cubic spline fractal function satisfies the functional relation

$$f_2(L_n(x)) = a_n^2(\gamma_n f_2(x) + q_n(x)), x \in I, \tag{3.2}$$

where $p_n(x)$ and $q_n(x)$ are cubic polynomials for $n = 1, 2, \ldots, N$. In this section, we assume that for $n = 1, 2, \ldots, N$; $|\beta_n| + |\gamma_n| \leq s < 1$ and $|\alpha_n| \leq s^* < 1$, where s and s^* are some fixed real numbers. In view of Eqs. (2.9) and (2.10), denote $p_n(\alpha_n, \beta_n, x) \equiv p_n(x)$ and $q_n(\gamma_n, x) \equiv q_n(x)$ for $n = 1, 2, \ldots, N$. Further, let for $x \in I_n$, $n = 1, 2, \ldots, N$, $q_n(x)$ satisfies

$$\left|\frac{\partial^{1+r} q_n(\tau_n, x)}{\partial \gamma_n \partial x^r}\right| \leq K_r, \tag{3.3}$$

where $|\tau_n| \in (0, s a_n^r)$ and K_r is a positive constant. We need the following lemma to prove our main convergence Theorems 3.1 to 3.3.

Lemma 3.1. *Let $f_1(x)$ be the cubic spline CHFIF through generalized interpolation data and $S(x)$ be the classical cubic spline with respect to the mesh $\Delta : x_0 < x_1 < \ldots < x_N$, interpolating $\{y_0, y_1, \ldots, y_N\}$ at the mesh points with same type of boundary conditions. Suppose, there exist positive constants K_r^*, K_r^{**}, $r = 0, 1, 2$ such that*

$$\left.\begin{aligned}\left|\frac{\partial^{1+r} p_n(\xi_n, \beta_n, x)}{\partial \alpha_n \partial x^r}\right| &\leq K_r^*, \\ \left|\frac{\partial^{1+r} p_n(\alpha_n, \eta_n, x)}{\partial \beta_n \partial x^r}\right| &\leq K_r^{**}\end{aligned}\right\} \tag{3.4}$$

for $|\xi_n| \in (0, s^* a_n^r)$, $|\eta_n| \in (0, s a_n^r)$, $x \in I_n$, $n = 1, 2, \ldots, N$, and $r = 0, 1, 2$. Then,

$$\|f_1{}^{(r)} - S^{(r)}\|_\infty \leq \|\Delta\|^{2-r} \frac{\max\limits_{1\leq n\leq N} |\alpha_n|(\|S^{(r)}\|_\infty + K_r^*) + \max\limits_{1\leq n\leq N} |\beta_n|(\|f_2{}^{(r)}\|_\infty + K_r^{**})}{|I|^{2-r} - \|\Delta\|^{2-r} \max\limits_{1\leq n\leq N} |\alpha_n|}, \quad r = 0, 1, 2, \tag{3.5}$$

where $|I|$ is the length of the interval I.

Proof. Denote $\mathcal{B}_r^* = (\bigotimes_{n=1}^N [-s^* a_n^r, s^* a_n^r]; \bigotimes_{n=1}^N (-s a_n^r, s a_n^r))$ for $r = 0, 1, 2$. Since the proof is analogous for any value of r, we prove it for $r = 0$ as follows. Let $(\alpha, \beta) = (\alpha_1, \alpha_2, \ldots, \alpha_N; \beta_1, \beta_2, \ldots, \beta_N) \in \mathcal{B}_0^*$. For a given self-affine cubic spline fractal function f_2 and given boundary conditions, cubic spline CHFIF f_1 is unique for an element (α, β) in $\mathcal{B}_0^*$. So, using Eq. (3.1), the component wise Read-Bajraktarević operator $T_1{}^*_{(\alpha^*, \beta^*)}$ of $T^* : \mathcal{G}^* \to \mathcal{G}^*$, can be written as for $x \in I_n$ and $n = 1, 2, \ldots, N$,

$$\begin{aligned}T_1{}^*_{(\alpha,\beta)} f_1{}^*(x) = a_n^2[&\alpha_n f_1{}^*(L_n^{-1}(x)) \\ &+ \beta_n f_2{}^*(L_n^{-1}(x)) \\ &+ p_n(\alpha_n, \beta_n, L_n^{-1}(x))].\end{aligned} \tag{3.6}$$

Suppose f_1 is the fixed point of $T_1{}^*_{(\alpha,\beta)}$, where $(\alpha; \beta) \in \mathcal{B}_0^*$ such that $\alpha_n \neq 0$ and $\beta_m \neq 0$ for some n and m. Also, if $(\alpha^*, \beta^*) = (0, 0, \ldots, 0;\ 0, 0, \ldots, 0) \in \mathcal{B}_0^*$, then the classical cubic spline S is the fixed point of $T_1{}^*_{(\alpha^*, \beta^*)}$ with the prescribed boundary conditions. Hence, by Eq. (3.6), for $x \in I_n$, $n = 1, 2, \ldots, N$,

$$\begin{aligned}|T_1{}^*_{(\alpha,\beta)} f_1(x) - T_1{}^*_{(\alpha,\beta)} S(x)| &= |a_n^2[\alpha_n f_1(L_n^{-1}(x)) + \beta_n f_2(L_n^{-1}(x)) \\ &\quad + p_n(\alpha_n, \beta_n, L_n^{-1}(x))] \\ &\quad - a_n^2[\alpha_n S(L_n^{-1}(x)) + \beta_n f_2(L_n^{-1}(x) \\ &\quad + p_n(\alpha_n, \beta_n, L_n^{-1}(x))]| \\ &= a_n^2|\alpha_n||f_1(L_n^{-1}(x)) - S(L_n^{-1}(x))| \\ &\leq \frac{\|\Delta\|^2}{|I|^2} \max_{1\leq n\leq N} |\alpha_n|\; \|f_1 - S\|_\infty.\end{aligned}$$

The above inequality gives

$$\|T_1{}^*_{(\alpha,\beta)} f_1 - T_1{}^*_{(\alpha,\beta)} S\|_\infty \leq \frac{\|\Delta\|^2}{|I|^2} \max_{1\leq n\leq N} |\alpha_n|\; \|f_1 - S\|_\infty. \tag{3.7}$$

Now, using the mean value theorem and Eq. (3.5), for $x \in I_n$, $n = 1, 2, \cdots, N$,

$$\begin{aligned}
&|T1^*_{(\alpha,\beta)}S(x) - T1^*_{(\alpha^*,\beta^*)}S(x)| \\
&\quad = |a_n^2[\alpha_n S(L_n^{-1}(x)) + \beta_n f_2(L_n^{-1}(x)) + p_n(\alpha_n, \beta_n, L_n^{-1}(x)) - p_n(0,0,L_n^{-1}(x))]| \\
&\quad \le a_n^2 \left\{ |\alpha_n| \|S\|_\infty + |\beta_n| \|f_2\|_\infty + |\alpha_n| \left| \frac{\partial p_n(\xi_n, \beta_n, L_n^{-1}(x))}{\partial \alpha_n} \right| + |\beta_n| \left| \frac{\partial p_n(0, \eta_n, L_n^{-1}(x))}{\partial \beta_n} \right| \right\} \\
&\quad \le \frac{\|\Delta\|^2}{|I|^2} \left\{ \max_{1\le n\le N} |\alpha_n|(\|S\|_\infty + K_0^*) + \max_{1\le n\le N} |\beta_n|(\|f_2\|_\infty + K_0^{**}) \right\}.
\end{aligned}$$

It follows that

$$\|T1^*_{(\alpha,\beta)}S - T1^*_{(\alpha^*,\beta^*)}S\|_\infty \le \frac{\|\Delta\|^2}{|I|^2} \left\{ \max_{1\le n\le N} |\alpha_n|(\|S\|_\infty + K_0^*) + \max_{1\le n\le N} |\beta_n|(\|f_2\|_\infty + K_0^{**}) \right\}. \tag{3.8}$$

Using inequalities (3.7) and (3.8) in

$$\begin{aligned}
\|f_1 - S\|_\infty &= \|T1^*_{(\alpha,\beta)}f_1 - T1^*_{(\alpha^*,\beta^*)}S\|_\infty \\
&\le \|T1^*_{(\alpha,\beta)}f_1 - T1^*_{(\alpha,\beta)}S\|_\infty + \|T1^*_{(\alpha,\beta)}S - T1^*_{(\alpha^*,\beta^*)}S\|_\infty
\end{aligned}$$

resulted into

$$\|f_1 - S\|_\infty \le \|\Delta\|^2 \frac{\max_{1\le n\le N} |\alpha_n|(\|S\|_\infty + K_0^*) + \max_{1\le n\le N} |\beta_n|(\|f_2\|_\infty + K_0^{**})}{|I|^2 - \|\Delta\|^2 \max_{1\le n\le N} |\alpha_n|}.$$

Hence, Lemma 3.1 is proved for $r = 0$. The proof is similar to $r = 1, 2$, and thus omitted. □

Remark 3.1. By assuming $\beta_n = 0$ and replacing α_n by γ_n and inequality (3.4) by inequality (3.3) in Lemma 3.1, the self-affine cubic spline fractal function f_2 satisfies the following estimate:

$$\|f_2{}^{(r)} - S^{(r)}\|_\infty \le \frac{\|\Delta\|^{2-r}s}{|I|^{2-r} - \|\Delta\|^{2-r}s}(\|S^{(r)}\|_\infty + K_r), \tag{3.9}$$

when f_2 and S are constructed with the same boundary conditions.

Let Δ_k be a sequences of meshes on $[x_0, x_N]$ as

$$\Delta_k : x_0 = x_{k,0} < x_{k,1} < \cdots < x_{k,N_k} = x_N.$$

Denote, $h_{k,n_k} = x_{k,n_k} - x_{k,n_k-1}$ and $\|\Delta_k\| = \max_{1\le n_k\le N_k} h_{k,n_k}$. We prove that the sequence of cubic spline CHFIFs $\{f_{1\Delta_k}(x)\}$ with boundary conditions of Type-I, Type-II, or Type-III converge to the data generating function $\Phi(x)$ on a sequences of meshes $\{\Delta_k\}$ at the rate of $\|\Delta_k\|^2$, when the corresponding self-affine cubic spline fractal function $f_{2\Delta_k}(x)$ is constructed with any one of the boundary conditions of Type-I, Type-II, or Type-III, where $\Phi \in C^r(I)$, $r = 2, 3$, or 4. In view of Lemma 3.1, we define two types of sequences of meshes $\{\Delta_k\}$ on $[x_0, x_N]$ for study of the convergence of cubic spline CHFIFs to a data generating function, depending upon free variables α_{k,n_k} and constrained free variables β_{k,n_k}.

Class A. $\left\{ \{\Delta_k\} : \max_{1\le n_k\le N_k} \{|\alpha_{k,n_k}|, |\beta_{k,n_k}|\} \le \|\Delta_k\| < 1 \text{ for each } k \right\}$.

Class B. $\{\{\Delta_k\} : |\alpha_{k,i}| > \|\Delta_k\| \text{ or } |\beta_{k,j}| > \|\Delta_k\| \text{ for some } i, j, 1 \le i, j \le N_k \text{ and for each } k\}$.

For the function Φ in $C^2[x_0, x_N]$ generating the interpolation data, the convergence result of cubic spline CHFIFs are given in the following theorem if cubic spline fractal functions $f_{2\Delta_k}$ satisfy any boundary conditions of Type-I, Type-II, or Type-III on a sequence of meshes.

Theorem 3.1. *Let $\Phi \in C^2[x_0, x_N]$ and cubic spline CHFIFs $f_{1\Delta_k}(x)$ satisfy any boundary conditions of Type-I, Type-II or Type-III on a sequence of meshes $\{\Delta_k\}$ on $[x_0, x_N]$ with $\lim_{k\to\infty} \|\Delta_k\| = 0$. If $\{\Delta_k\}$ is in Class A, then*

$$\|\Phi^{(r)} - f_{1\Delta_k}^{(r)}\|_\infty = \circ(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2, \tag{3.10}$$

and if $\{\Delta_k\}$ is in Class B, then

$$\|\Phi^{(r)} - f_{1\Delta_k}^{(r)}\|_\infty = \bigcirc(\|\Delta_k\|^{2-r}), \quad r = 0, 1, 2. \tag{3.11}$$

Proof. By Lemma 3.1, with the same boundary conditions, $f_{1\Delta_k}$ and S_{Δ_k} satisfies the following relation on each element of the sequence $\{\Delta_k\}$ for $r = 0, 1, 2$.

$$\|f_{1\Delta_k}^{(r)} - S_{\Delta_k}^{(r)}\|_\infty \le \|\Delta_k\|^{2-r} \frac{\max\limits_{1\le n_k\le N_k} |\alpha_{k,n_k}|(\|S_{\Delta_k}^{(r)}\|_\infty + K_r^*) + \max\limits_{1\le n_k\le N_k} |\beta_{k,n_k}|(\|f_{2\Delta_k}^{(r)}\|_\infty + K_r^{**})}{|I|^{2-r} - \|\Delta_k\|^{2-r} \max\limits_{1\le n_k\le N_k} |\alpha_{k,n_k}|}. \tag{3.12}$$

From Ref. 21, pp. 28, classical cubic splines with boundary conditions of Type-I, Type-II or Type-III satisfy,

$$\|\Phi^{(r)} - S_{\Delta_k}^{(r)}\|_\infty \le 5\|\Delta_k\|^{2-r}\omega(\Phi^{(r)}; \|\Delta_k\|), \quad r = 0, 1, 2, \tag{3.13}$$

where $\omega(\Phi; x)$ is the modulus of continuity of $\Phi(x)$. Inequality (3.13) gives

$$\|S_{\Delta_k}^{(r)}\|_\infty \le \|\Phi^{(r)}\|_\infty + 5\|\Delta_k\|^{2-r}\omega(\Phi^{(r)}; \|\Delta_k\|). \tag{3.14}$$

From inequality (3.9), with the same boundary conditions, $f_{2\Delta_k}$ and S_{Δ_k} satisfy

$$\|f_{2\Delta_k}^{(r)} - S_{\Delta_k}^{(r)}\|_\infty \le \frac{\|\Delta_k\|^{2-r}s}{|I|^{2-r} - \|\Delta_k\|^{2-r}s} \times (\|S_{\Delta_k}^{(r)}\|_\infty + K_r).$$

Using inequality (3.14), $\|f_{2\Delta_k}^{(r)}\|_\infty$ exists as $k \to \infty$ for $r = 0, 1, 2$. Inequalities (3.12) to (3.14) together with inequality (3.9) gives

$$\|\Phi^{(r)} - f_{1\Delta_k}^{(r)}\|_\infty \le \|\Delta_k\|^{2-r} \left\{ 5\omega(\Phi^{(r)}; \|\Delta_k\|) + \frac{\max\limits_{1\le n_k\le N_k} \{|\alpha_{k,n_k}|, |\beta_{k,n_k}|\}(\|S_{\Delta_k}^{(r)}\|_\infty + \|f_{2\Delta_k}^{(r)}\|_\infty + K_r^* + K_r^{**})}{|I|^{2-r} - \|\Delta_k\|^{2-r} \max\limits_{1\le n_k\le N_k} |\alpha_{k,n_k}|} \right\}. \tag{3.15}$$

By the assumptions $\Phi \in C^2(I)$ and $\max_{1\le n_k\le N_k}\{|\alpha_{k,n_k}|, |\beta_{k,n_k}|\} \le \|\Delta_k\| < 1$, the right hand side of inequality (3.15) tends to zero as $k \to \infty$. Hence, the convergence result (3.10) for the Class A follows from inequality (3.15). Since $\max_{1\le n_k\le N_k} |\alpha_{n_k}| \le s^* < 1$ and $\max_{1\le n_k\le N_k} |\beta_{n_k}| < s$, inequality (3.15) reduces to

$$\|\Phi^{(r)} - f_{1\Delta_k}^{(r)}\|_\infty \le \|\Delta_k\|^{2-r} \left\{ 5\omega(\Phi^{(r)}; \|\Delta_k\|) + \frac{\max\{s^*, s\}(\|S_{\Delta_k}^{(r)}\|_\infty + \|f_{2\Delta_k}^{(r)}\|_\infty + K_r^* + K_r^{**})}{|I|^{2-r} - \|\Delta_k\|^{2-r}s^*} \right\}$$

Finally, the convergence result (3.11) for Class B follows from the above inequality. □

The convergence results of cubic spline CHFIFs to the function Φ in $C^3[x_0, x_N]$ are given in the following if cubic spline fractal functions $f_{2\Delta_k}$ satisfy any boundary conditions of Type-I, Type-II, or Type-III on a sequence of meshes.

Theorem 3.2. *Let $\Phi \in C^3[x_0, x_N]$ and cubic spline CHFIFs $f_{1\Delta_k}(x)$ satisfy boundary conditions of Type-I, Type-II or Type-III on a sequence of meshes $\{\Delta_k\}$ on $[x_0, x_N]$ with*

$$\lim_{k\to\infty} \|\Delta_k\| = 0 \quad \textit{and} \quad \frac{\|\Delta_k\|}{\min_{1\le n_k\le N_k} h_{k,n_k}} \le \theta < \infty.$$

If $\{\Delta_k\}$ is in Class A or Class B, then respectively for $r = 0, 1, 2$, $\|\Phi^{(r)} - f_{1\Delta_k}^{(r)}\|_\infty = \circ(\|\Delta_k\|^{2-r})$ or $\|\Phi^{(r)} - f_{1\Delta_k}^{(r)}\|_\infty = \bigcirc(\|\Delta_k\|^{2-r})$.

Proof. It is known (Ref. 21, pp. 32) that classical cubic splines with boundary conditions of Type-I, Type-II or Type-III, satisfy

$$\|\Phi^{(r)} - S_{\Delta_k}^{(r)}\|_\infty \le \frac{5}{3}\|\Delta_k\|^{3-r}(3 + \bar{K})\omega(\Phi^{(3)}; \|\Delta_k\|), \quad r = 0, 1, 2, \tag{3.16}$$

where $\bar{K} = 8\theta^2(1 + 2\theta)(1 + 3\theta)$. From inequalities (3.9) and (3.16), it is clear that $\|S_{\Delta_k}^{(r)}\|_\infty$ and $\|f_{2\Delta_k}^{(r)}\|_\infty$ are bounded. Thus, the following error estimate holds for cubic spline CHFIFs with boundary conditions of Type-I, Type-II or Type-III:

$$\|\Phi^{(r)} - f_{1\Delta_k}^{(r)}\|_\infty \le \|\Delta_k\|^{2-r} \left\{ \frac{5}{3}\|\Delta_k\|(3 + \bar{K})\omega(\Phi^{(3)}; \|\Delta_k\|) + \frac{\max\limits_{1\le n_k\le N_k} \{|\alpha_{k,n_k}|, |\beta_{k,n_k}|\}(\|S_{\Delta_k}^{(r)}\|_\infty + \|f_{2\Delta_k}^{(r)}\|_\infty + K_r^* + K_r^{**})}{|I|^{2-r} - \|\Delta_k\|^{2-r} \max\limits_{1\le n_k\le N_k} |\alpha_{k,n_k}|} \right\}.$$

The above estimate gives the convergence results of Theorem 3.2 depending on the sequence mesh of Class A or Class B. □

The convergence results of cubic spline CHFIFs to the function Φ in $C^4[x_0, x_N]$ are given in the following if cubic spline fractal functions $f_{2_{\Delta_k}}$ satisfy any boundary conditions of Type-I or Type-II on a sequence of meshes.

Theorem 3.3. *Let $\Phi \in C^4[x_0, x_N]$ and cubic spline CHFIFs $f_{1_{\Delta_k}}(x)$ satisfy boundary conditions of Type-I or Type-II on a sequence of meshes $\{\Delta_k\}$ on $[x_0, x_N]$ with $\lim_{k\to\infty} \|\Delta_k\| = 0$ and $\frac{\|\Delta_k\|}{\min_{1\le n_k\le N_k} h_{k,n_k}} \le \eta < \infty$. If $\{\Delta_k\}$ is in Class A or Class B, then respectively for $r = 0, 1, 2$, $\|\Phi^{(r)} - f_{1}{}^{(r)}_{\Delta_k}\|_\infty = \circ(\|\Delta_k\|^{2-r})$ or $\|\Phi^{(r)} - f_{1}{}^{(r)}_{\Delta_k}\|_\infty = \bigcirc(\|\Delta_k\|^{2-r})$.*

Proof. It is known[22] that, classical cubic splines with boundary conditions of Type-I or Type-II, satisfy

$$\|\Phi^{(r)} - S^{(r)}_{\Delta_k}\|_\infty \le L_r \|\Phi^{(4)}\|_\infty \|\Delta_k\|^{4-r}, \quad r = 0, 1, 2, 3, \quad (3.17)$$

where $L_0 = 5/384$, $L_1 = 1/24$, $L_2 = 3/8$, and $L_3 = (\eta + \eta^{-1})/2$. From inequalities (3.9) and (3.17) it follows that $\|S^{(r)}_{\Delta_k}\|_\infty$ and $\|f_{2}{}^{(r)}_{\Delta_k}\|_\infty$ are bounded. Hence, the error estimate in this case for cubic spline CHFIFs with boundary conditions of Type-I or Type-II, is given by

$$\|\Phi^{(r)} - f_{1}{}^{(r)}_{\Delta_k}\|_\infty \le \|\Delta_k\|^{2-r} \left\{ L_r \|\Phi^{(4)}\|_\infty \|\Delta_k\|^2 + \frac{\max_{1\le n_k\le N_k}\{|\alpha_{k,n_k}|, |\beta_{k,n_k}|\}(\|S^{(r)}_{\Delta_k}\|_\infty + \|f_{2}{}^{(r)}_{\Delta_k}\|_\infty + K^*_r + K^{**}_r)}{|I|^{2-r} - \|\Delta_k\|^{2-r} \max_{1\le n_k\le N_k} |\alpha_{k,n_k}|} \right\}, \quad r = 0, 1, 2.$$

Thus, the above inequality gives the convergence results of Theorem 3.3. □

Remark 3.2. The spline CHFIF f_1 uniformly converge in the C^1 norm to the data generating function Φ if $\{\Delta_k\}$ is in Class B. If there exists a positive number u such that $\max_{1\le n_k\le N_k}\{|\alpha_{k,n_k}|, |\beta_{k,n_k}|\} \le \|\Delta_k\|^u$ for all $k = 0, 1, 2, \ldots$, then f_1 converge uniformly to Φ in the C^2 norm on I.

4. EXAMPLES OF CUBIC SPLINE CHFIFs

In this section, we construct examples of cubic spline CHFIFs as the fixed point of the IFS given by Eq. (2.8). Suppose that $\{(0, 0), (\frac{2}{5}, 1), (\frac{3}{4}, -1), (1, 2)\}$ is the given interpolation data for cubic spline CHFIFs. Chose free variables $\alpha_n = 0.8$, $n = 1, 2, 3$; hidden variables $z_0 = 3$, $z_1 = 2$, $z_2 = 8$, $z_3 = 5$, $\gamma_1 = 0.3$, $\gamma_2 = 0.35$, $\gamma_3 = 0.4$; and constrained free variables $\beta_1 = 0.4$, $\beta_2 = 0.6$, $\beta_3 = 0.5$. Using Eq. (2.1), each IFS has $L_1(x) = \frac{2}{5}x$, $L_2(x) = \frac{7}{20}x + \frac{2}{5}$, and $L_3(x) = \frac{1}{4}x + \frac{3}{4}$. For the first two examples, we compute $F^2_n(x, z)$ for the self-affine cubic spline fractal function f_2 with a boundary condition $f_2'(x_0) = 10$ and $f_2'(x_3) = 1$. The moments are evaluated (Table 1) by using the system of equations (see Appendix). These moments are used in Eq. (2.10) for the construction of $F^2_n(x, z)$ (Table 2). For constructing an example of the cubic spline CHFIF with boundary conditions of Type-I, we choose $f_1'(x_0) = 2$ and $f_1'(x_3) = 5$. Equations (A.3) are solved with these choices to get

Table 1 Derivatives Used for IFSs of CHFIFs.

Fig.	$f_2'(x_0)$	M_0	M_1	M_2	M_3	$f_2'(x_3)$	$f_1'(x_0)$	M^*_0	M^*_1	M^*_2	M^*_3	$f_1'(x_3)$
1	10	−209.49	236.76	−235.69	388.58	1	2	−0.66	−90.65	156.63	−131.82	5
2	10	−209.49	236.76	−235.69	388.58	1	2	201.38	−123.87	113.56	−50.32	5
3	10	−418.8	923.6	−836.1	1933.1	1	2	921.8	1420.3	1614.3	1646.2	5
4	−13.43	10	122.47	−171.90	1	−21.64	2	−95.98	−289.22	−29.01	−383.10	5
5	10	−146.01	125.51	−197.81	202.07	1	2	−67.79	−235.24	21.97	−316.34	5
6	2	−77.87	−331.38	−59.68	−462.54	5	2	−77.87	−331.38	−59.68	−462.54	5

Table 2 $F_n(x, y, z) = (F_n^1(x, y, z),\ F_n^2(x, z))$ **for Cubic Spline CHFIFs.**

Fig. 1	$F_1(x,y,z) = (0.128y + 0.064z - 1.7189x^3 + 5.3755x^2 - 3.0406x - 0.192,\ \ 0.048z + 7.1155x^3 - 11.7315x^2 + 3.52x + 2.856)$
	$F_2(x,y,z) = (0.05y + 0.0313z - 5.0268x^3 + 8.1848x^2 - 0.3205x - 1.0938,\ \ 0.0429z - 13.9198x^3 + 18.9927x^2 + 0.8413x + 1.8714)$
	$F_3(x,y,z) = (0.05y + 0.0313z - 5.0268x^3 + 8.1848x^2 - 0.3205x - 1.0938,\ \ 0.025z + 4.0109x^3 - 4.7468x^2 - 2.3141x + 7.925)$
Fig. 2	$F_1(x,y,z) = (0.128y - 0.096z + 6.2655x^3 - 6.8335x^2 + 1.504x + 0.288,\ \ 0.048z + 7.1155x^3 - 11.7315x^2 + 3.52x + 2.856)$
	$F_2(x,y,z) = (0.098y - 0.0735z + 16.2854x^3 - 25.1537x^2 + 6.8193x + 1.2205,\ \ 0.0429z - 13.9198x^3 + 18.9927x^2 + 0.8413x + 1.8714)$
	$F_3(x,y,z) = (0.05y + 0.0125z - 0.8557x^3 - 0.1762x^2 + 3.9068x - 1.0375,\ \ .025z + 4.0109x^3 - 4.7468x^2 - 2.3141x + 7.925)$
Fig. 3	$F_1(x,y,z) = (0.128y + 0.064z - 27.2463x^3 + 28.1503x^2 - 0.096x + 0.448,\ \ 0.048z + 16.9824x^3 - 23.4544x^2 + 3.52x - 6.664)$
	$F_2(x,y,z) = (0.098y + 0.0735z - 36.6824x^3 + 57.2174x^2 - 22.6575x + 1.5145,\ \ 0.0429z - 52.7327x^3 + 65.5487x^2 + 6.2269x - 9.6999)$
	$F_3(x,y,z) = (0.05y + 0.0313z - 17.9536x^3 + 33.9448x^2 - 13.0599x - 0.7813,\ \ 0.025z + 19.0457x^3 - 20.8914x^2 - 15.1293x + 9.175)$
Fig. 4	$F_1(x,y,z) = 0.128y + 0.064z + 1.0682x^3 - 1.8558x^2 + 1.4035x - 0.192,\ \ 0.048z + 3.6899x^3 - 0.7521x^2 - 4.0339x + 2.856)$
	$F_2(x,y,z) = (0.098y + 0.0735z + 10.1125x^3 - 13.3792x^2 + 0.9237x + 0.7795,\ \ 0.0429z - 5.9516x^3 + 7.7897x^2 + 4.0761x + 1.8714)$
	$F_3(x,y,z) = (0.05y + 0.0313z - 1.2489x^3 + 1.3367x^2 + 2.7497x - 1.0938,\ \ 0.025z + 1.5994x^3 - 5.2041x^2 + 0.5547x + 7.925)$
Fig. 5	$F_1(x,y,z) = (0.128y + 0.064z - 2.8759x^3 + 3.5879x^2 - 0.096x - 0.192,\ \ -0.08z + 11.882x^3 - 17.522x^2 + 4.8x + 3.24)$
	$F_2(x,y,z) = (0.098y + 0.0735z + 5.0471x^3 - 5.7207x^2 - 1.6694x + 0.7795,\ \ 0.0367z - 8.7332x^3 + 10.3705x^2 + 4.2892x + 1.8898)$
	$F_3(x,y,z) = (0.05y + 0.0313z - 3.2659x^3 + 4.663x^2 + 1.4404x - 1.0938,\ \ -0.025z + 5.6159x^3 - 8.0068x^2 - 0.5591x + 8.075)$
Fig. 6	$F_1(x,y,z) = (0.08y + 0.048z + 1.446x^3 - 1.246x^2 + 0.544x,\ \ 0.128z + 1.446x^3 - 1.246x^2 + 0.544x)$
	$F_2(x,y,z) = (0.049y + 0.049z + 11.83x^3 - 16.4813x^2 + 2.4552x + 1,\ \ 0.098z + 11.83x^3 - 16.4813x^2 + 2.4552x + 1)$
	$F_3(x,y,z) = (0.0187y + 0.0313z - 0.9909x^3 + 0.0817x^2 + 3.8091x - 1,\ \ 0.05z - 0.9909x^3 + 0.0817x^2 + 3.8091x - 1)$

the values of moments $M_0^*, M_1^*, M_2^*, M_3^*$ (Table 1). These moments are now used in Eq. (2.9) for the construction of $F_n^1(x, y, z)$ (Table 2). Iterations of the IFS code (2.8) generates the desired cubic spline CHFIF (Fig. 1). To illustrate the effect of constrained free variables on the shape of the cubic spline CHFIF in comparison with Fig. 1, we take $\beta_1 = \beta_2 = -0.6$, and $\beta_3 = 0.2$. Using the computed values of moments $M_0^*, M_1^*, M_2^*, M_3^*$ (Table 1), we evaluate $F_n^1(x, y, z)$, $n = 1, 2, 3$ (Table 2). Iterations of the IFS code (2.8) generates the desired cubic spline CHFIF (Fig. 2). Similarly, perturbations in the free variables α_n would affect the shape of the cubic spline CHFIF.

The effect of change in hidden variables, i.e. parameter z_n, boundary conditions of fractal function f_2 and free variables γ_n, on the shape of the cubic spline CHFIF are illustrated in Figs. 3, 4 and 5, respectively by comparing these with Fig. 1. In Fig. 3, we only modify free parameters as $z_0 = -7$, $z_1 = -10, z_2 = 9, z_3 = -8$. In Fig. 4, we choose boundary conditions for self-affine fractal function f_2 as $M_0 = 10$, $M_3 = 1$ instead of $f_2'(x_0) = 10$, $f_2'(x_3) = 1$. In Fig. 5, we change only free variables $\gamma_1 = -0.5$, $\gamma_2 = 0.3$ and $\gamma_3 = -0.4$. Depending on changes in hidden variables, moments of the self-affine cubic spline fractal function and cubic spline CHFIF are calculated (Table 1). These are used to determine $F_n(x, y, z)$ (Table 2) for the IFS code (2.8). Finally, we assume $y_n = z_n$ for $n = 0, 1, 2, 3$, $\alpha_n + \beta_n = \gamma_n$, i.e. $\alpha_1 = \beta_3 = 0.5$, $\alpha_2 = \beta_2 = 0.4$ and $\alpha_3 = \beta_1 = 0.5$

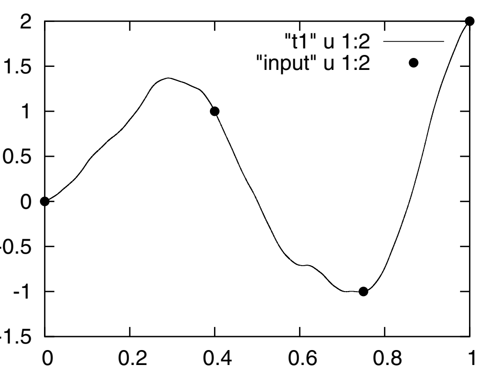

Fig. 1 Cubic spline CHFIF with boundary conditions of Type-I.

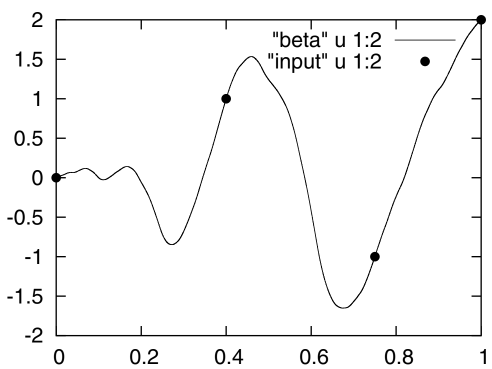

Fig. 2 Cubic spline CHFIF with boundary conditions of Type-I and different β_n.

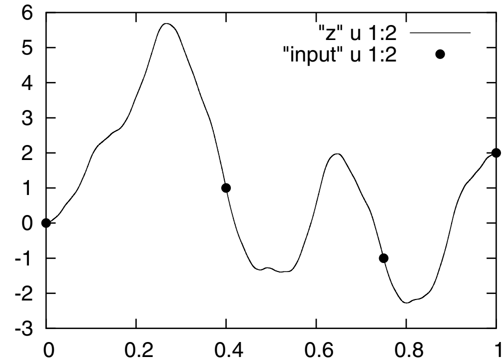

Fig. 3 Cubic spline CHFIF with boundary conditions of Type-I and different z_n.

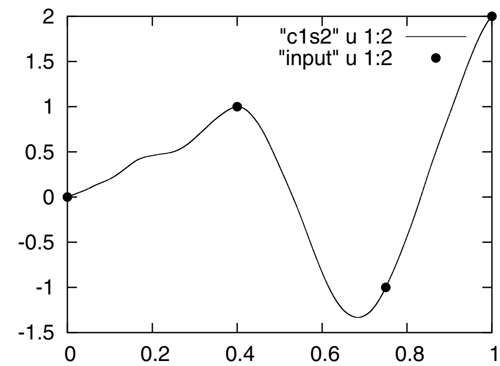

Fig. 4 Cubic spline CHFIF with boundary conditions of Type-I and different boundary conditions for self-affine cubic spline fractal function.

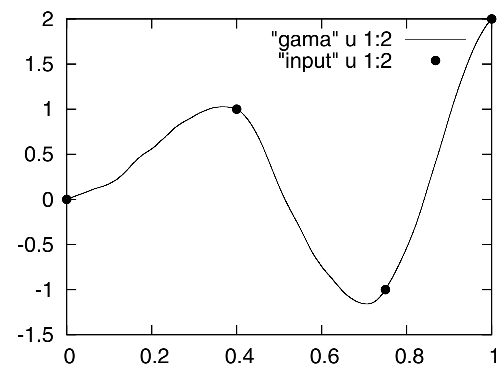

Fig. 5 Cubic spline CHFIF with boundary conditions of Type-I and different γ_n.

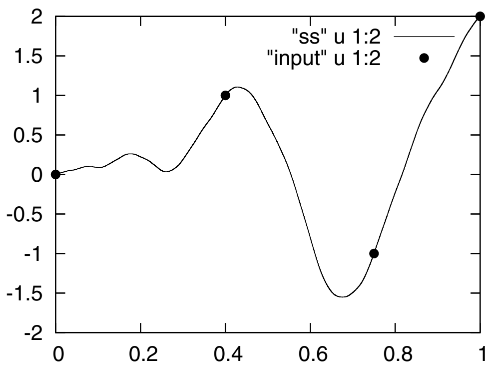

Fig. 6 Self-affine cubic spline CHFIF.

with the same boundary conditions. In this case, the projection of the attractor generates the self-affine cubic spline CHFIF (Fig. 6). Hence, our approach offers more flexibility and diversity in the choice of self-affine or non-self-affine cubic spline CHFIFs to an experimenter depending on the need of the problem.

5. CONCLUSION

The construction of the cubic spline CHFIF through moments is initiated for the first time to approximate non-self-affine smooth objects. This construction allows admissibility of any kind of boundary conditions and generalizes results of the classical cubic spline.

For a data generating function $\Phi \in C^r[x_0, x_n]$, $r = 2, 3$, or 4, it is proved that, cubic spline CHFIFs converge to Φ with arbitrary degree of accuracy when the step size approaches zero on two different classes of mesh. These upper bounds on error in approximation of Φ and its derivatives by the cubic spline CHFIF and its derivatives respectively with different boundary conditions are also obtained. If the data generating function $\Phi(x)$ satisfies $\omega(\Phi^{(2)}, t) = \bigcirc(|t|^\mu(\log|t|)^n)$, $n = 0, 1$, or 2, $0 < \mu < 1$, Φ can be approximated satisfactorily by a cubic spline CHFIF $f_1(x)$ by choosing only free variables α_n and γ_n suitably, since β_n and z_n do not affect the smoothness of the CHFIF $f_1''(x)$.[14]

Hidden variables, free variables and constrained variables play an important role in determining the shape of the cubic spline CHFIF. For prescribed boundary conditions, an infinite number of cubic spline CHFIFs can be constructed interpolating the same data by changing free variables α_n, constrained free variables β_n, hidden variables γ_n, free parameter z_n or boundary conditions of self-affine cubic spline fractal function. Thus, for simulating objects having non-self-affine or self-affine smooth shapes, the cubic spline CHFIF offers more flexibility. It is felt that spline FIF should find rich applications since classical splines have vast applications in CAM/CAD and other mathematical, engineering applications.[19,20] The self-affine and non-self-affine nature of smooth objects in various scientific applications can also be effectively captured with the use of cubic spline CHFIFs.

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REFERENCES

1. B. B. Mandelbrot, *The Fractal Geometry of Nature* (W. H. Freeman, San Francisco, 1983).
2. B. B. Mandelbrot, *Fractals and Scaling in Finance* (Springer-Verlag, New York, 1997).
3. J. Feder, *Fractals* (Plenum Press, New York 1988).
4. M. Frame and B. B. Mandelbrot, *Fractals, Graphics, and Mathematics Education* (The Mathematical Association of America, Washington, 2000).
5. B. J. West, *Physiology, Promiscuity and Prophecy at the Millennium: A Tale of Tails*, in Studies of Nonlinear Phenomena in the Life Sciences, Vol. 9 (World Scientific, Singapore, 1998).
6. J. L. Véhel, K. Daoudi and E. Lutton, Fractal modelling of speech signals, *Fractals* **2**(3) (1994) 379–382.
7. Y. Fisher, *Fractal Image Compression* (Springer-Verlag, New York, 1996).
8. M. F. Barnsley, Fractal functions and interpolations, *Constr. Approx.* **2** (1986) 303–329.
9. M. F. Barnsley, *Fractals Everywhere* (Academic Press, Orlando, Florida, 1988).
10. M. F. Barnsley, J. Elton, D. P. Hardin and P. R. Massopust, Hidden variable fractal interpolation functions, *SIAM J. Math. Anal.* **20**(5) (1989) 1218–1242.
11. P. R. Massopust, Fractal functions, in *Fractal Surfaces and Wavelets* (Academic Press, 1994).
12. J. Santamaria, M. E. Gómez, J. L. Vicent, K. M. Krishnan and I. K. Schuller, Scaling of the interface roughness in Fe-Cr superlattices: self-affine versus non-self-affine, *Phys. Rev. Lett.* **19** (2002) 190601.
13. A. K. B. Chand and G. P. Kapoor, Hidden variable bivariate fractal interpolation surfaces, *Fractals* **11**(3) (2003) 277–288.
14. A. K. B. Chand, A study on coalescence and spline fractal interpolation functions, PhD thesis, IIT Kanpur, India (2005).

15. M. F. Barnsley and A. N. Harrington, The calculus of fractal interpolation functions, *J. Approx. Theory* **57** (1989) 14–34.
16. A. K. B. Chand and G. P. Kapoor, Generalized cubic spline fractal interpolation functions, *SIAM J. Numer. Anal.* **44**(2) (2006) 655–676.
17. M. A. Navascués and M. V. Sebastián, Some results on convergence of cubic spline fractal interpolation functions, *Fractals* **11**(1) (2003) 1–7.
18. M. A. Navascués and M. V. Sebastián, Smooth fractal interpolation, *J. Inequalities Appl.*, Article ID 78734 (2006) 1–20.
19. G. Farin, *Curves and Surfaces for Computer Aided Geometric Design*: *A Practical Guide* (Academic Press, San Diego, 1990).
20. G. D. Knott, *Interpolating Cubic Splines* (Birkhäuser, Boston, 2000).
21. J. Ahlberg, E. Nilson and J. Walsh, *The Theory of Splines and Their Applications* (Academic Press, 1967).
22. C. A. Hall and W. W. Meyer, Optimal error bounds for cubic spline interpolation, *J. Approx. Theory* **16** (1976) 105–122.

APPENDIX

In this section, the details of employing moments M_n^* and M_n; $n = 0, 1, 2, \ldots, N$ in the construction of cubic spline CHFIFs in Sec. 2 are given. Since $f_1^{''}$ is affine,

$$f_1^{''}(L_n(x)) = \alpha_n f_1^{''}(x) + \beta_n f_2^{''}(x) + \frac{k_n(x - x_0)}{x_N - x_0} + l_n, \quad n = 1, 2, \ldots, N. \tag{A.1}$$

Using Eq. (2.1) and Eq. (A.1), $k_n = M_n^* - M_{n-1}^* - \alpha_n(M_N^* - M_0^*) - \beta_n(M_N - M_0)$ and $l_n = M_{n-1}^* - \alpha_n M_0^* - \beta_n M_0$. So, Eq. (A.1) reduces to

$$f_1^{''}(L_n(x)) = \alpha_n f_1^{''}(x) + \beta_n f_2^{''}(x) + \frac{(M_n^* - \alpha_n M_N^* - \beta_n M_N)(x - x_0)}{x_N - x_0} + \frac{(M_{n-1}^* - \alpha_n M_0^* - \beta_n M_0)(x_N - x)}{x_N - x_0}.$$

Integrating the above equations twice and using Eq. (2.1), the cubic spline CHFIF satisfies

$$\begin{aligned} f_1(L_n(x)) = a_n^2 \Bigg\{ & \alpha_n f_1(x) + \beta_n f_2(x) + \frac{(M_n^* - \alpha_n M_N^* - \beta_n M_N)(x - x_0)^3}{6(x_N - x_0)} \\ & + \frac{(M_{n-1}^* - \alpha_n M_0^* - \beta_n M_0)(x_N - x)^3}{6(x_N - x_0)} - \frac{(M_{n-1}^* - \alpha_n M_0^* - \beta_n M_0)(x_N - x_0)(x_N - x)}{6} \\ & - \frac{(M_n^* - \alpha_n M_N^* - \beta_n M_N)(x_N - x_0)(x - x_0)}{6} + \left(\frac{y_{n-1}}{a_n^2} - \alpha_n y_0 - \beta_n z_0\right)\frac{x_N - x}{x_N - x_0} \\ & + \left(\frac{y_n}{a_n^2} - \alpha_n y_N - \beta_n z_N\right)\frac{x - x_0}{x_N - x_0}\Bigg\}, \quad n = 1, 2, \ldots, N. \end{aligned} \tag{A.2}$$

Introduce the following notations:

$$h_n = x_n - x_{n-1}, \quad C_n^* = \frac{-6a_{n+1}\alpha_{n+1}}{h_n + h_{n+1}}, \quad C_n = \frac{-(\alpha_n h_n + 2\alpha_{n+1}h_{n+1})}{h_n + h_{n+1}}, \quad \lambda_n = \frac{h_{n+1}}{h_n + h_{n+1}},$$

$$\begin{aligned} d_n^* = & \frac{6[(y_{n+1} - y_n)/h_{n+1} - (y_n - y_{n-1})/h_n]}{h_n + h_{n+1}} - \frac{6(a_{n+1}\alpha_{n+1} - a_n\alpha_n)}{h_n + h_{n+1}}\frac{y_N - y_0}{x_N - x_0} \\ & - \frac{6(a_{n+1}\beta_{n+1} - a_n\beta_n)}{h_n + h_{n+1}}\frac{z_N - z_0}{x_N - x_0} + \frac{6(a_{n+1}\beta_{n+1}f_2'(x_0) - a_n\beta_n f_2'(x_N))}{h_n + h_{n+1}} \\ & + \frac{\beta_n h_n + 2\beta_{n+1}h_{n+1}}{h_n + h_{n+1}}M_0 + \frac{2\beta_n h_n + \beta_{n+1}h_{n+1}}{h_n + h_{n+1}}M_N \end{aligned}$$

$$\mu_n = 1 - \lambda_n, \quad D_n = \frac{-(2\alpha_n h_n + \alpha_{n+1}h_{n+1})}{h_n + h_{n+1}}, \quad D_n^* = \frac{6a_n\alpha_n}{h_n + h_{n+1}}, \quad n = 1, 2, \ldots, N - 1,$$

$$d_0^* = 6/h_1[y_1 - y_0 - \alpha_1 a_1^2(y_N - y_0) - \beta_1 a_1^2(z_N - z_0)] + \beta_1[6a_1 f_2{}'(x_0) + 2h_1 M_0 + h_1 M_N],$$

$$C_0^* = 6(1 - a_1\alpha_1), \quad C_0 = 2(1-\alpha_1)h_1, \quad \lambda_0 = h_1, \quad D_0 = -\alpha_1 h_1, \quad C_N = -\alpha_N h_N, \quad \mu_N = h_N,$$
$$D_N = 2(1-\alpha_N)h_N, \quad D_N^* = -6(1 - a_N\alpha_N), \quad d_N^* = -6/h_N[y_N - y_{N-1} - \alpha_N a_N^2(y_N - y_0)$$
$$- \beta_N a_N^2(z_N - z_0)] + \beta_N[h_N M_0 + 2h_N M_N - 6a_N f_2{}'(x_N)].$$

Using Eq. (A.2), we can write the functional relation for $f_1'(x_0)$, continuity relations at $f_1'(x_n)$ for $n = 1, 2, \cdots, N-1$, and the functional relation for $f_1'(x_N)$ in the matrix form as

$$\begin{bmatrix} C_0^* & C_0 & \lambda_0 & 0 & 0 & \dots & 0 & 0 & 0 & D_0 & 0 \\ C_1^* & C_1+\mu_1 & 2 & \lambda_1 & 0 & \dots & 0 & 0 & 0 & D_1 & D_1^* \\ C_2^* & C_2 & \mu_2 & 2 & \lambda_2 & \dots & 0 & 0 & 0 & D_2 & D_2^* \\ C_3^* & C_3 & 0 & \mu_3 & 2 & \dots & 0 & 0 & 0 & D_3 & D_3^* \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots \\ C_{N-3}^* & C_{N-3} & 0 & 0 & 0 & \dots & 2 & \lambda_{N-3} & 0 & D_{N-3} & D_{N-3}^* \\ C_{N-2}^* & C_{N-2} & 0 & 0 & 0 & \dots & \mu_{N-2} & 2 & \lambda_{N-2} & D_{N-2} & D_{N-2}^* \\ C_{N-1}^* & C_{N-1} & 0 & 0 & 0 & \dots & 0 & \mu_{N-1} & 2 & \lambda_{N-1}+D_{N-1} & D_{N-1}^* \\ 0 & C_N & 0 & 0 & 0 & \dots & 0 & 0 & \mu_N & D_N & D_N^* \end{bmatrix} \begin{bmatrix} f_1{}'(x_0) \\ M_0^* \\ M_1^* \\ M_2^* \\ \vdots \\ M_{N-2}^* \\ M_{N-1}^* \\ M_N^* \\ f_1{}'(x_N) \end{bmatrix} = \begin{bmatrix} d_0^* \\ d_1^* \\ d_2^* \\ d_3^* \\ \vdots \\ d_{N-3}^* \\ d_{N-2}^* \\ d_{N-1}^* \\ d_N^* \end{bmatrix}. \tag{A.3}$$

The system of Eqs. (A.3) consisting of $(N+1)\times(N+3)$ coefficient matrix has unknowns $f_1{}'(x_0), M_0^*, M_1^*, \dots, M_N^*, f_1{}'(x_N)$.

First an analogue of Eq. (A.3), can be constructed for the self-affine fractal function f_2 by taking $\beta_n = 0$ and $\alpha_n = \gamma_n$. The solution of the corresponding system of equations with suitable boundary conditions determines $f_2{}'(x_0)$ and $f_2{}'(x_N)$ and moments M_n, $n = 0, 1, 2, \dots, N$ of f_2.

Next, using values of $f_2{}'(x_0)$, $f_2{}'(x_N)$, M_0, and M_N, with suitable boundary conditions, the system of Eqs. (A.3) is solved and $f_1{}'(x_0), M_0^*, M_1^*, \dots, M_N^*$, and $f_1{}'(x_N)$ are determined. These values of M_n and M_n^*; $n = 0, 1, 2, \dots, N$ are finally used in the construction of a cubic spline CHFIF from the IFS that is given by Eq. (2.8).