Research Article

Cubic Spline Method for a Generalized Black-Scholes Equation

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We develop a numerical method based on cubic polynomial spline approximations to solve a a generalized Black-Scholes equation. We apply the implicit Euler method for the time discretization and a cubic polynomial spline method for the spatial discretization. We show that the matrix associated with the discrete operator is an M-matrix, which ensures that the scheme is maximum-norm stable. It is proved that the scheme is second-order convergent with respect to the spatial variable. Numerical examples demonstrate the stability, convergence, and robustness of the scheme.

1. Introduction

An option is a tradable financial contract whose value depends on the value of an underlying asset. The buyer of the contract obtains the right, but not the obligation, to buy or to sell an asset at a specified price on or before a maturity date. A call option provides the right to buy the underlying asset for a certain price, whereas a put option confers the right to sell the underlying asset for a certain price. A European option can only be exercised at the maturity date, while an American option can be exercised at any time prior to its maturity date. Black and Scholes [1] showed that the value of a European option is governed by a second-order parabolic differential equation with respect to the underlying asset price and time, which is known as the Black-Scholes equation. The value of an American option is determined by a linear complementarity problem or as a free boundary value problem involving the Black-Scholes differential operator [2]. It is often necessary to use numerical methods to solve these partial differential equations, as analytic solutions are not generally available.

There are several numerical methods for the valuation of European and American options. The first numerical method for option pricing was the lattice method proposed in Cox et al. [3] and was improved in Hull and White [4], which is equivalent to an explicit time-stepping scheme. Since the Black-Scholes equation with constant or space-independent parameters can be transformed into a diffusion equation, the finite difference methods applied to constant-coefficient heat equations have also been developed (see, e.g., Schwartz

[5], Courtadon [6], Wilmott et al. [2], and Rogers and Talay [7]) for pricing options. Vázquez [8] presented an upwind numerical approach for the Black-Scholes equation. Cen and Le [9–11] presented stable finite difference schemes on a piecewise uniform mesh for pricing European and American options. Wang [12] and Angermann and Wang [13] proposed a fitted finite volume method for the discretization of the Black-Scholes equation. Rambeerich et al. [14] applied the exponential time integration scheme to price options. Other methods, such as meshless approach [15, 16], element-free kp-Ritz method [17, 18], and element-free Galerkin method [19], also can be used to solve the generalized Black-Scholes equation.

The spline collocation methods are useful methods for solving partial differential equations. Spline solutions have their own advantages. For example, once the solution has been computed, the information between mesh points is available. Numerical methods based on spline collocation methods also have been used to solve option pricing problems. Christara et al. [20] proposed a quadratic spline collocation method to the American option pricing problems. Holtz and Kunoth [21] developed a B-spline-based monotone multigrid method for the valuation of American options. Khabir and Patidar [22] applied a B-spline collocation method to solve the heat equation which is obtained from the Black-Scholes equation by an Euler transformation. Kadalbajoo et al. [23, 24] used cubic B-spline collocation methods for the Black-Scholes equation.

In this paper, we present a numerical method based on cubic polynomial spline to solve a generalized Black-Scholes equation. We combine the implicit Euler method for discretizing the time variable with the cubic polynomial spline scheme for discretizing the spatial variable. The matrix associated with the discrete operator is an M-matrix, which ensures that the scheme is maximum-norm stable. We will show that the scheme is second-order convergent with respect to the spatial variable. Numerical results support the theoretical results.

The rest of the paper is organized as follows. In the next section, a generalized Black-Scholes equation is introduced. The temporal semidiscretization is described in Section 3. The spatial discretization is constructed in Section 4. The fully discrete scheme is presented in Section 5. Finally, numerical experiments are provided to support these theoretical results in Section 6.

2. The Continuous Problem

We consider the following generalized Black-Scholes equation:

$$-\frac{\partial v}{\partial t} - \frac{1}{2}\widehat{\sigma}^2\left(S,t\right)S^2\frac{\partial^2 v}{\partial S^2} - r\left(t\right)S\frac{\partial v}{\partial S} + r\left(t\right)v = 0, \tag{1}$$
$$\left(S,t\right) \in \mathbb{R}^+ \times \left(0,T\right),$$

equipped with the terminal and boundary conditions

$$v\left(S,T\right) = \max\left(S - E, 0\right), \quad x \in \mathbb{R}^+, $$
$$v\left(0,t\right) = 0, \quad t \in [0,T]. \tag{2}$$

Here, $v(S,t)$ is the value of European call option at asset price S and at time t, E is the exercise price, T is the maturity, $r(t)$ is the risk-free interest rate, and $\widehat{\sigma}(S,t)$ is the volatility function of underlying asset. When $\widehat{\sigma}$ and r are constant functions, it becomes the classical Black-Scholes model. The existence and uniqueness of a classical solution of (1)-(2) are well known (see [25, 26]).

Note that (1) degenerates when S goes to zero. We transform the Black-Scholes equations (1)-(2) into a nondegenerate partial differential equation by using a log transformation $x = \ln S$

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2\left(x,t\right)\frac{\partial^2 u}{\partial x^2} - \left(r\left(t\right) - \frac{1}{2}\sigma^2\left(x,t\right)\right)\frac{\partial u}{\partial x}$$
$$+ r\left(t\right)u = 0, \quad \left(x,t\right) \in \mathbb{R} \times \left(0,T\right),$$
$$u\left(x,T\right) = \max\left(e^x - E, 0\right), \quad x \in \mathbb{R}, \tag{3}$$
$$u\left(x,t\right) = 0, \quad \text{as } x \longrightarrow -\infty, \ t \in [0,T],$$

where $u(x,t) = v(S,t) = v(e^x,t)$ and $\sigma(x,t) = \widehat{\sigma}(e^x,t)$.

For applying the numerical method, we truncate the infinite domain $\mathbb{R} \times (0,T)$ into a truncated domain $\Omega = (x_{\min}, x_{\max}) \times (0,T)$, where $x_{\min}$ and $x_{\max}$ are chosen properly so that, for practical purposes, they do not affect the option price. Based on Willmott et al.'s estimate [2] that the upper bound of the asset price is typically three or four times the strike price, it is reasonable for us to set $x_{\max} = \ln(4E)$,

while, for the lower bound of the asset price, since $-\ln(4E)$ is negative enough, we take $x_{\min} = -\ln(4E)$ for convenience in numerical experiments. Therefore, in the remaining of this paper, we will consider the following problem:

$$-\frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2\left(x,t\right)\frac{\partial^2 u}{\partial x^2} - \left(r\left(t\right) - \frac{1}{2}\sigma^2\left(x,t\right)\right)\frac{\partial u}{\partial x}$$
$$+ r\left(t\right)u = 0, \quad \left(x,t\right) \in \Omega,$$
$$u\left(x,T\right) = \max\left(e^x - E, 0\right), \quad x \in \left(x_{\min}, x_{\max}\right), \tag{4}$$
$$u\left(x_{\min}, t\right) = 0, \quad t \in [0,T],$$
$$u\left(x_{\max}, t\right) = e^{x_{\max}} - Ee^{-\int_t^T r(s)ds}, \quad t \in [0,T].$$

Here, the right boundary condition is chosen according to Vázquez [8]. Normally, this truncation of the domain leads to a negligible error in the value of the option [25].

3. The Temporal Semidiscretization

To approximate the solution (4), first, we apply the implicit Euler method to discretize the temporal variable. This scheme, on a uniform mesh

$$\overline{\Omega}^K = \left\{t_j = j\Delta t,\ 0 \le j \le K,\ \Delta t = \frac{T}{K}\right\}, \tag{5}$$

reads

$$u^K = u\left(x,T\right) = \max\left(e^x - E, 0\right),$$
$$\left(I + \Delta t L_x\right)u^j\left(x\right) = u^{j+1}\left(x\right),$$
$$u^j\left(x_{\min}\right) = 0, \tag{6}$$
$$u^j\left(x_{\max}\right) = e^{x_{\max}} - Ee^{-\int_{t_j}^T r(s)ds},$$
$$\text{for } j = K - 1, \dots, 1, 0,$$

where

$$L_x u^j\left(x\right) = -\frac{1}{2}\left(\sigma^j\left(x\right)\right)^2\frac{d^2 u^j}{dx^2}$$
$$- \left(r^j - \frac{1}{2}\left(\sigma^j\left(x\right)\right)^2\right)\frac{du^j}{dx} \tag{7}$$
$$+ r^j u^j\left(x\right),$$

and $u^j(x)$ denotes the approximation of the exact solution $u(x,t)$ at the time level t_j.

Similarly to Kellogg and Tsan [27], we can prove that the differential operator $(I + \Delta t L_x)$ satisfies a maximum principle, and, consequently,

$$\left\|\left(I + \Delta t L_x\right)^{-1}\right\|_\infty \le \frac{1}{1 + r\Delta t}. \tag{8}$$

Hence, we can obtain the following result.

Lemma 1. *The temporal semidiscretization scheme* (6) *is unconditionally stable.*

Estimates for the global error are deduced from appropriate bounds of the local error, where the auxiliary problem

$$\left(I + \Delta t L_x\right) \widehat{u}^j = u\left(x, t_{j+1}\right),$$
$$\widehat{u}^j \left(x_{\min}\right) = 0,$$
$$\widehat{u}^j \left(x_{\max}\right) = e^{x_{\max}} - E e^{-\int_{t_j}^{T} r(s) \mathrm{d}s}, \quad (9)$$

is introduced to define the local error.

Lemma 2 (local error estimate). *The local error associated with the method* (9), *defined by* $e^j = u(x, t_j) - \widehat{u}^j(x)$, *satisfies*

$$\left\| e^j \right\|_\infty = O\left((\Delta t)^2\right). \quad (10)$$

Proof. Using Taylor expansion, we have

$$-\frac{u\left(x, t_{j+1}\right) - u\left(x, t_j\right)}{\Delta t} = -u_t\left(x, t_j\right) + O\left(\Delta t\right)$$
$$= -L_x u\left(x, t_j\right) + O\left(\Delta t\right), \quad (11)$$

for $1 \le j < K$. From (9) and (11), it is straightforward to show that the local error is the solution of the problem

$$\left(I + \Delta t L_x\right) e^j = O\left((\Delta t)^2\right),$$
$$e^j\left(x_{\min}\right) = e^j\left(x_{\max}\right) = 0, \quad (12)$$

and, therefore, the result follows from the maximum principle for the operator $(I + \Delta t L_x)$. □

Lemma 3 (global error estimate). *The global error associated with the implicit Euler method* (6), *given by* $E^j = u(x, t_j) - u^j(x)$, *satisfies*

$$E = \sup_{j \le K} \left\| E^j \right\|_\infty = O\left(\Delta t\right), \quad (13)$$

and, therefore, the temporal semidiscretization scheme is a first-order convergent scheme.

Proof. The global truncation error at time t_j can be decomposed as

$$E^j = u\left(x, t_j\right) - u^j\left(x\right)$$
$$= \left(u\left(x, t_j\right) - \widehat{u}^j\left(x\right)\right) + \left(\widehat{u}^j\left(x\right) - u^j\left(x\right)\right). \quad (14)$$

By relations (6) and (9), we have

$$\left(I + \Delta t L_x\right)\left(\widehat{u}^j\left(x\right) - u^j\left(x\right)\right) = u\left(x, t_{j+1}\right) - u^{j+1}\left(x\right). \quad (15)$$

Applying the maximum principle for the operator $(I + \Delta t L_x)$, we can obtain

$$\left\| \widehat{u}^j\left(x\right) - u^j\left(x\right) \right\|_\infty \le \frac{1}{1 + r\Delta t} \left\| u(x, t_{j+1}) - u^{j+1}(x) \right\|_\infty. \quad (16)$$

Thus, from (14)–(16), we have

$$\left\| E^j \right\|_\infty \le C \left(\left\| e^j \right\|_\infty + \left\| e^{j+1} \right\|_\infty + \cdots + \left\| e^K \right\|_\infty \right)$$
$$\le C\Delta t, \quad (17)$$

for $0 \le j \le K$, where C is a positive constant independent from Δt. □

4. The Spatial Discretization

For the approximate solution of the semidiscretization problem (6), the spatial discretization is performed on a uniform mesh

$$\overline{\Omega}^N = \left\{ x_i = ih, \ 0 \le i \le N, \ h = \frac{(x_{\max} - x_{\min})}{N} \right\}, \quad (18)$$

for the computational domain $[x_{\min}, x_{\max}]$. Thus, at each time point t_j, we apply a cubic spline scheme on the above uniform mesh Ω^N to approximate problem (6).

Let $S_\Delta^j(x)$ be the approximate solution of the exact solution $u^j(x)$ of the boundary value problem (6) at the jth time level. At each subinterval $[x_i, x_{i+1}]$, the cubic spline function $S_\Delta^j(x)$ has the following form:

$$S_\Delta^j(x) = a_i^j + b_i^j\left(x - x_i\right) + c_i^j\left(x - x_i\right)^2$$
$$+ d_i^j(x - x_i)^3, \quad i = 0, 1, \ldots, N - 1, \quad (19)$$

where a_i^j, b_i^j, c_i^j, and d_i^j are constants. Using the notation U_i^j for approximation of $u^j(x)$ at mesh points x_i and $S_\Delta^j(x_i) = U_i^j$, $S_\Delta^j(x_{i+1}) = U_{i+1}^j$ as interpolatory constraints. From algebraic manipulation, we can obtain

$$a_i^j = U_i^j,$$
$$b_i^j = \frac{U_{i+1}^j - U_i^j}{h} - \frac{h\left(M_{i+1}^j - M_i^j\right)}{6},$$
$$c_i^j = \frac{M_i^j}{2}, \quad (20)$$
$$d_i^j = \frac{M_{i+1}^j - M_i^j}{6h},$$

where $M_i^j = \left(S_\Delta^j\right)''(x_i)$. Using the continuity of the first derivative at mesh point x_i, we get the following equation:

$$M_{i+1}^j + 4M_i^j + M_{i-1}^j = \frac{6}{h^2}\left(U_{i+1}^j - 2U_i^j + U_{i-1}^j\right). \quad (21)$$

Substituting

$$M_i^j = \frac{-2u_i^{j+1} - \Delta t\left(2r^j - \left(\sigma_i^j\right)^2\right)u_{x,i}^j + 2\left(1 + \Delta t r^j\right)u_i^j}{\Delta t\left(\sigma_i^j\right)^2} \quad (22)$$

in (21) and using the following approximations for first-order derivative of u^j

$$u_{x,i}^j \simeq \frac{U_{i+1}^j - U_{i-1}^j}{2h},$$
$$u_{x,i+1}^j \simeq \frac{3U_{i+1}^j - 4U_i^j + U_{i-1}^j}{2h}, \quad (23)$$
$$u_{x,i-1}^j \simeq \frac{-U_{i+1}^j + 4U_i^j - 3U_{i-1}^j}{2h},$$

we get the following spline difference scheme:

$$L^N U_i^j = \frac{2h^2}{\left(\sigma_{i-1}^j\right)^2} U_{i-1}^{j+1} + \frac{8h^2}{\left(\sigma_i^j\right)^2} U_i^{j+1}$$

$$+ \frac{2h^2}{\left(\sigma_{i+1}^j\right)^2} U_{i+1}^{j+1}, \quad i = 1, 2, \ldots, N-1, \qquad (24)$$

$$U_0^j = 0, \qquad U_N^j = e^{x_{\max}} - Ee^{-\int_{t_j}^T r(s)ds},$$

where

$$L^N U_i^j = \left[-6\Delta t - \frac{1}{2}h\Delta t p_{i+1}^j + 2h\Delta t p_i^j \right.$$

$$\left. + \frac{3}{2}h\Delta t p_{i-1}^j + \frac{2h^2\left(1 + \Delta t r^j\right)}{\left(\sigma_{i-1}^j\right)^2} \right] U_{i-1}^j$$

$$+ \left[12\Delta t + 2h\Delta t p_{i+1}^j - 2h\Delta t p_{i-1}^j \right.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (25)$$

$$\left. + \frac{8h^2\left(1 + \Delta t r^j\right)}{\left(\sigma_i^j\right)^2} \right] U_i^j$$

$$+ \left[-6\Delta t - \frac{3}{2}h\Delta t p_{i+1}^j - 2h\Delta t p_i^j \right.$$

$$\left. + \frac{1}{2}h\Delta t p_{i-1}^j + \frac{2h^2\left(1 + \Delta t r^j\right)}{\left(\sigma_{i+1}^j\right)^2} \right] U_{i+1}^j$$

and $p_i^j = 2r^j/(\sigma_i^j)^2 - 1$.

It is easy to see that the matrix associated with the discrete operator L^N is an M-matrix for sufficiently small h. Hence, the following discrete maximum principle holds true.

Lemma 4 (discrete maximum principle). *For sufficiently small h, the operator L^N defined by* (25) *on the uniform mesh Ω^N satisfies a discrete maximum principle; that is, if w is a mesh function that satisfies $w_0 \geq 0$, $w_N \geq 0$, and $L^N w_i \geq 0$ $(1 \leq i < N)$, then $w_i \geq 0$, for all i.*

From the above lemma, we can conclude that the spatial discretization scheme (24) is maximum-norm stable.

To prove the convergence of the spline difference scheme, we discretize the auxiliary problem (9) and obtain

$$L^N \widehat{U}_i^j = \frac{2h^2}{\left(\sigma_{i-1}^j\right)^2} u\left(x_{i-1}, t_{j+1}\right) + \frac{8h^2}{\left(\sigma_i^j\right)^2} u\left(x_i, t_{j+1}\right)$$

$$+ \frac{2h^2}{\left(\sigma_{i+1}^j\right)^2} u\left(x_{i+1}, t_{j+1}\right), \quad i = 1, 2, \ldots, N-1,$$

$$\widehat{U}_0^j = 0, \qquad \widehat{U}_N^j = e^{x_{\max}} - Ee^{-\int_{t_j}^T r(s)ds}.$$

$$\qquad\qquad\qquad\qquad\qquad\qquad (26)$$

Lemma 5. *Let $\widehat{u}^j(x)$ be the solution of* (9) *and $\{\widehat{U}_i^j\}$ be the solution of* (26). *Then, we have the following error estimate:*

$$\left|\widehat{u}^j\left(x_i\right) - \widehat{U}_i^j\right| \leq Ch^2\Delta t, \quad 0 \leq i \leq N. \qquad (27)$$

Proof. We use a Taylor expansion at $x = x_i$ to obtain the following local truncation error estimate:

$$\left|L^N\left(\widehat{u}_i^j - \widehat{U}_i^j\right)\right| = \left| L^N \widehat{u}_i^j - \frac{2h^2}{\left(\sigma_{i-1}^j\right)^2} u\left(x_{i-1}, t_{j+1}\right) \right.$$

$$- \frac{8h^2}{\left(\sigma_i^j\right)^2} u\left(x_i, t_{j+1}\right)$$

$$\left. - \frac{2h^2}{\left(\sigma_{i+1}^j\right)^2} u\left(x_{i+1}, t_{j+1}\right) \right|$$

$$= \left| L^N \widehat{u}_i^j - \frac{2h^2}{\left(\sigma_{i-1}^j\right)^2}\left(I + \Delta t L_x\right)\widehat{u}_{i-1}^j \right. \qquad (28)$$

$$- \frac{8h^2}{\left(\sigma_i^j\right)^2}\left(I + \Delta t L_x\right)\widehat{u}_i^j$$

$$\left. - \frac{2h^2}{\left(\sigma_{i+1}^j\right)^2}\left(I + \Delta t L_x\right)\widehat{u}_{i+1}^j \right|$$

$$\leq \frac{1}{3}h^4\Delta t\left(\left|p_{i-1}^j\right| + 2\left|p_i^j\right| + \left|p_{i+1}^j\right|\right)$$

$$\times \left|\frac{d^3\widehat{u}^j}{dx^3}\left(\xi_i\right)\right|$$

$$\leq Ch^4\Delta t.$$

Hence, using the discrete maximum principle (Lemma 4) for the discrete operator L^N, we have

$$\left|\widehat{u}^j\left(x_i\right) - \widehat{U}_i^j\right| \leq Ch^2\Delta t, \quad 0 \leq i \leq N, \qquad (29)$$

which completes the proof. ∎

5. The Fully Discrete Scheme

Combining the time semidiscretization scheme (6) with the spatial discretization scheme (24), we can obtain the following fully discretization scheme:

$$U_i^K = \max\left(e^x - E, 0\right),$$

$$L^N U_i^j = \frac{2h^2}{\left(\sigma_{i-1}^j\right)^2} U_{i-1}^{j+1} + \frac{8h^2}{\left(\sigma_i^j\right)^2} U_i^{j+1}$$

$$+ \frac{2h^2}{\left(\sigma_{i+1}^j\right)^2} U_{i+1}^{j+1}, \quad 1 \leq i < N, \qquad (30)$$

$$U_0^j = 0, \qquad U_N^j = e^{x_{\max}} - Ee^{-\int_{t_j}^T r(s)ds},$$

$$\text{for } j = K-1, \ldots, 1, 0,$$

where the discrete operators L^N are described in Section 4 and U_i^j is the fully discrete approximation to the exact solution of (4) at the mesh point (x_i, t_j).

Now, we can get the main result for our difference scheme.

Theorem 6. *Let $u(x,t)$ be the exact solution of* (4) *and let U be the discrete solution of the fully discrete scheme* (30). *Then, the global error of our difference scheme satisfies*

$$\left| u\left(x_i, t_j\right) - U_i^j \right| \leq C\left(h^2 + \Delta t\right), \quad 0 \leq i \leq N, \ 0 \leq j \leq K, \tag{31}$$

where C is a positive constant independent of h and Δt.

Proof. The global error at the time t_j can be decomposed in the form

$$\begin{aligned} \left| u\left(x_i, t_j\right) - U_i^j \right| &\leq \left| u\left(x_i, t_j\right) - \widehat{u}^j\left(x_i\right) \right| \\ &\quad + \left| \widehat{u}^j\left(x_i\right) - \widehat{U}_i^j \right| + \left| \widehat{U}_i^j - U_i^j \right|. \end{aligned} \tag{32}$$

From Lemmas 2 and 5, we can obtain

$$\left| u\left(x_i, t_j\right) - U_i^j \right| \leq C\Delta t\left(h^2 + \Delta t\right) + \left| \widehat{U}_i^j - U_i^j \right|. \tag{33}$$

Further, it is easy to see that $\widehat{U}^j - U^j$ can be written as the solution of one step of (30) with zero boundary conditions and $u(x, t_{j+1}) - U^{j+1}$ as the final value. Applying the stability of the discrete scheme (Lemma 4), we have

$$\left| \widehat{U}_i^j - U_i^j \right| \leq C\left\| u\left(x, t_{j+1}\right) - U^{j+1} \right\|_{\infty}. \tag{34}$$

Then, from (33) and (34), a recurrence relation for the global errors follows, and, from it, the result of Theorem 6 can be obtained immediately. □

6. Numerical Experiments

In this section, we present some numerical results to examine the performance and convergence of the cubic spline method. Errors and convergence rates for the numerical scheme are presented for three examples.

Example 1. European call option with parameters: $\sigma = 0.4$, $r = 0.08$, $T = 1$, $E = 1$, $x_{\min} = -\ln(4E)$, and $x_{\max} = \ln(4E)$: in this case, the analytical solution is

$$v(S, \tau) = SN\left(d_1\right) - Ee^{-r(T-\tau)}N\left(d_2\right), \tag{35}$$

where

$$\begin{aligned} N(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} \mathrm{d}y, \\ d_1(S, \tau) &= \frac{\ln(S/E) + \left(r + (1/2)\sigma^2\right)(T-\tau)}{\sigma\sqrt{T-\tau}}, \\ d_2(S, \tau) &= d_1(S, \tau) - \sigma\sqrt{T-\tau}. \end{aligned} \tag{36}$$

The maximum error is given in Table 1. The analytical and numerical solution profiles are given in Figure 1.

TABLE 1: Numerical results for Example 1.

N	K	Error $e^{N,K}$	Rate $r^{N,K}$
8	4	$1.2469e - 02$	—
16	16	$2.9318e - 03$	2.089
32	64	$7.2583e - 04$	2.014
64	256	$1.8143e - 04$	2.000
128	1024	$4.5346e - 05$	2.000

TABLE 2: Numerical results for Example 2.

N	K	Error $e^{N,K}$	Rate $r^{N,K}$
8	4	$1.8366e - 02$	—
16	16	$6.7729e - 03$	1.439
32	64	$1.4290e - 03$	2.245
64	256	$3.6112e - 04$	1.985
128	1024	$8.9871e - 05$	2.007

TABLE 3: Numerical results for Example 3.

N	K	Error $e^{N,K}$	Rate $r^{N,K}$
8	4	$1.8364e - 02$	—
16	16	$6.7715e - 03$	1.439
32	64	$1.4277e - 03$	2.246
64	256	$3.5967e - 04$	1.989
128	1024	$8.8418e - 05$	2.024

Example 2. European call option with parameters: $\sigma = 0.1$, $r = 0.06$, $T = 1$, $E = 1$, $x_{\min} = -\ln(4E)$ and $x_{\max} = \ln(4E)$: as in Example 1, in this case, the analytical solution is known.

The maximum error is given in Table 1. The analytical and numerical solution profiles are given in Figure 2.

Example 3. European call option with parameters: $\sigma(S, \tau) = 0.15(0.5 + 2\tau)((S/100 - 1.2)^2/((S/100)^2 + 1.44))$, $r = 0.06$, $T = 1$, $E = 1$, $x_{\min} = -\ln(4E)$, and $x_{\max} = \ln(4E)$: here, the volatility function $\sigma(S, \tau)$ is the same as the one given in Toivanen [28] and Kadalbajoo et al. [23, 24].

In this case, the exact solution is not known. We use the approximated solution of $N = 2048$ and $K = 4096$ as the exact solution. We present the error estimates for different N and K. Let $U^{2048,4096}$ denote "the exact solution." We measure the accuracy in the discrete maximum norm
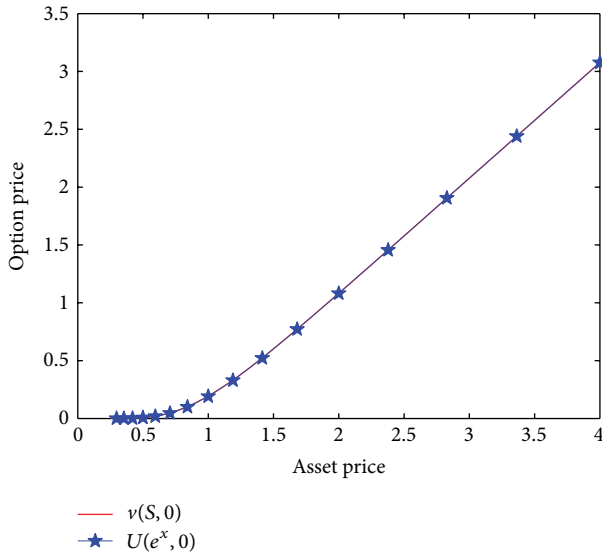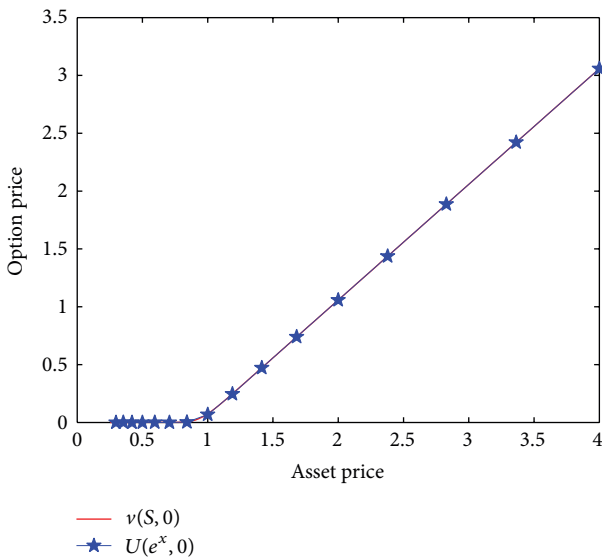
$$e^{N,K} = \max_{i,j} \left| U_{ij}^{N,K} - U_{i,j}^{2048,4096} \right| \tag{37}$$

and the convergence rate

$$R^{N,K} = \log_2\left(\frac{e^{N,K}}{e^{2N,4K}} \right). \tag{38}$$

The error estimates and convergence rates are listed in Table 3. The analytical and numerical solution profiles are given in Figure 3.

From Figures (1)–(3), it is seen that the numerical solutions by our method are nonoscillatory. From Tables 1,

Figure 1: Option value at $t = 0$ for Example 1.

Figure 2: Option value at $t = 0$ for Example 2.
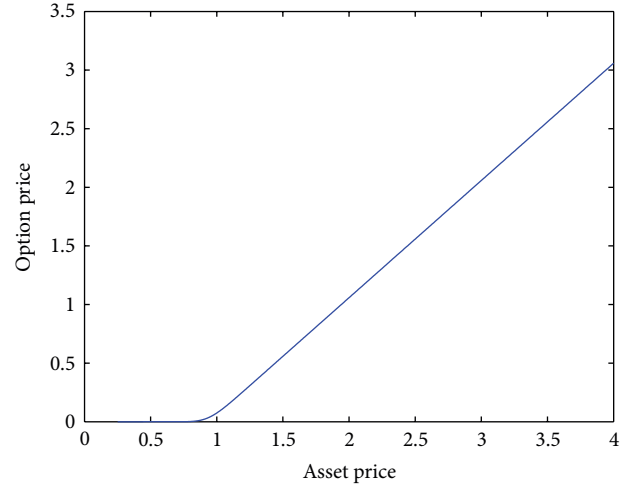

Figure 3: Option value at $t = 0$ for Example 3.

2, and 3, we see that $e^{N,K}/e^{2N,4K}$ is close to 4, which supports the convergence estimate of Theorem 6. They indicate that the theoretical results are fairly sharp.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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