# Cubic surfaces with a Galois invariant double-six

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**Abstract**

We present a method to construct non-singular cubic surfaces over $\mathbb{Q}$ with a Galois invariant double-six. We start with cubic surfaces in the hexahedral form of L. Cremona and Th. Reye. For these, we develop an explicit version of Galois descent.

# 1 Introduction

**1.1.** —— The configuration of the 27 lines upon a smooth cubic surface is highly symmetric. The group of all permutations preserving the canonical class and respecting the intersection pairing is isomorphic to the Weyl group $W(E_6)$ of order 51 840.

When $S$ is a cubic surface over $\mathbb{Q}$ then the absolute Galois group $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ operates on the 27 lines. This yields a subgroup $G \subseteq W(E_6)$.

**1.2.** —— There are 350 conjugacy classes of subgroups of $W(E_6)$. Only for a few of them, explicit examples of cubic surfaces are known up to now.

General cubic surfaces [Ek, EJ1] lead to the full $W(E_6)$. In [EJ2], we constructed examples for the index two subgroup which is the simple group of order 25 920.

Other examples may be constructed by fixing a $\mathbb{Q}$-rational line or tritangent plane. Generically, this yields the maximal subgroups in $W(E_6)$ of indices 27 and 45, respectively. It is not yet clear which smaller groups arise by further specialization.

On the other hand, there are a number of rather small subgroups in $W(E_6)$ for which examples may be obtained easily. Blowing up six points in $\mathbf{P}^2_\mathbb{Q}$ forming a Galois invariant set leads to a cubic surface with a Galois invariant sixer.

*Key words and phrases.* Cubic surface, Hexahedral form, Double-six, Explicit Galois descent

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This means [Do] that there are six lines which are mutually skew and, as a set, invariant under the operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. It is clear that examples for all the 56 corresponding conjugacy classes of subgroups may be constructed in this way.

There are a few more easy cases, e.g. diagonal surfaces, but all in all not more than 70 of the 350 conjugacy classes of subgroups may be realized by such elementary methods.

**1.3.** —— In this article, we present a method to construct cubic surfaces over $\mathbb{Q}$ with a Galois invariant double-six. A *double-six* on a cubic surface is the configuration of twelve lines described in [Ha, Remark V.4.9.1]. Double-sixes were intensively studied by the classical algebraic geometers [Do].

In particular, we give explicit examples of cubic surfaces such that the 27 lines decompose, under the operation of Galois, into orbits of sizes twelve and fifteen. I.e., such that the *orbit structure* is $[12, 15]$. Our method is based on the hexahedral form for cubic surfaces due to L. Cremona and Th. Reye. For these, we develop an explicit version of Galois descent.

A short calculation in `GAP` shows that there are 102 conjugacy classes of subgroups of $W(E_6)$ which fix a double-six but no sixer. We have explicit examples of cubic surfaces for each of them. Some of the most interesting ones are presented in the final section.

# 2 The Segre cubic

**2.1. Definition.** —— The *Segre cubic* $S$ is the threefold defined in $\mathbf{P}^5$ by the equations

$$X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 = 0\,,$$
$$X_0 + X_1 + X_2 + X_3 + X_4 + X_5 = 0\,.$$

**2.2.** —— The followings facts are elementary and easily checked.

**Facts.** i) On the Segre cubic, there are exactly ten singular points. These are $(-1 : -1 : -1 : 1 : 1 : 1)$ and permutations of coordinates.

ii) $S$ contains the 15 planes given by

$$X_{i_0} + X_{i_1} = X_{i_2} + X_{i_3} = X_{i_4} + X_{i_5} = 0$$

for $\{i_0, \ldots, i_5\} = \{0, \ldots, 5\}$.

iii) Every plane passes through four of the singular points. Every singular point is met by six planes.

**2.3. Fact.** —— *On $S$, we have*
i) $(X_0+X_1)(X_0+X_2)(X_1+X_2) = -(X_3+X_4)(X_3+X_5)(X_4+X_5)$,
ii) $(X_0+X_2)(X_0+X_3)(X_0+X_4)(X_0+X_5) = (X_1+X_2)(X_1+X_3)(X_1+X_4)(X_1+X_5)$.
*Consequently, the form*

$$(X_0+X_1)(X_0+X_2)(X_0+X_3)(X_0+X_4)(X_0+X_5) \in \Gamma(S, \mathscr{O}(5))$$

*is invariant under permutation of coordinates.*

**Proof.** i) Raise the relation $X_0+X_1+X_2 = -(X_3+X_4+X_5)$ to the third power and use $X_0^3+X_1^3+X_2^3 = -(X_3^3+X_4^3+X_5^3)$.
ii) Using i), we see

$$\begin{aligned} &(X_0+X_2)(X_0+X_3)(X_0+X_4)(X_0+X_5) \\ =\ &(X_0+X_2)(X_0+X_3)(X_0+X_4)(X_0+X_5)(X_4+X_5)/(X_4+X_5) \\ =\ &-(X_0+X_2)(X_0+X_3)(X_1+X_2)(X_1+X_3)(X_2+X_3)/(X_4+X_5) \end{aligned}$$

and the final term is symmetric in $X_0$ and $X_1$. □

**2.4. Remark.** —— The divisor of $(X_0+X_1)(X_0+X_2)(X_0+X_3)(X_0+X_4)(X_0+X_5)$ is the sum over the 15 planes in the threefold $S$.

# 3 Cubic surfaces in hexahedral form

**3.1. Notation.** —— One way to write down a cubic surface explicitly is the so-called *hexahedral form*. Denote by $S^{(a_0,\ldots,a_5)}$ the cubic surface given in $\mathbf{P}^5$ by the system of equations

$$\begin{aligned} X_0^3 + X_1^3 + X_2^3 + X_3^3 + X_4^3 + X_5^3 &= 0\,, \\ X_0 + X_1 + X_2 + X_3 + X_4 + X_5 &= 0\,, \\ a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 &= 0\,. \end{aligned}$$

**3.2.** —— It is well-known that a non-singular cubic surface contains 27 lines and that these form a total of 45 tritangent planes. Suppose that the cubic surface $S^{(a_0,\ldots,a_5)}$ in hexahedral form is non-singular.

Then, we have 15 of the 45 tritangent planes on $S^{(a_0,\ldots,a_5)}$ explicitly given by the 15 planes on the Segre cubic. I.e., by the formulas

$$X_i + X_j = 0$$

for $i \neq j$. We will call them the *obvious* tritangent planes.

Correspondingly, 15 of the 27 lines are given by

$$X_{i_0} + X_{i_1} = X_{i_2} + X_{i_3} = X_{i_4} + X_{i_5} = 0$$

for $\{i_0, \ldots, i_5\} = \{0, \ldots, 5\}$. These lines will be called the *obvious* lines on $S^{(a_0,\ldots,a_5)}$.

**3.3. Remark.** —— The configuration of the 15 obvious lines is the same as that of the fifteen lines $F_{ij}$ in the blown-up model of a cubic surface [Ha, Theorem V.4.9]. The twelve non-obvious lines form a double-six.

**3.4. Remark.** —— The hexahedral form is due to L. Cremona [Cr] based on previous investigations by Th. Reye [Re]. A general cubic surface over an algebraically closed field may be brought into hexahedral form over that field. Cubic surfaces in hexahedral form with rational coefficients are, however, very special; see Remark 5.6.

# 4 The discriminantal locus

**4.1. Definition.** —— Let $\sigma_i$ denote the $i$-th elementary symmetric function in $a_0, \ldots, a_5$. Then, the form

$$d_4 := \sigma_2^2 - 4\sigma_4 + \sigma_1(2\sigma_3 - \tfrac{3}{2}\sigma_1\sigma_2 + \tfrac{5}{16}\sigma_1^3)$$

is called the *Coble quartic* [Co].

**4.2. Remark.** —— The Coble quartic is a homogeneous form of degree four. A calculation which is conveniently done in `maple` shows that $d_4$ is invariant under shift. I.e., $d_4(a_0, \ldots, a_5) = d_4(a_0 + c, \ldots, a_5 + c)$. Further, the Coble quartic is absolutely irreducible.

**4.3. Proposition.** —— *The cubic surface $S^{(a_0,\ldots,a_5)}$ is singular if and only if*

i) $\pm a_0 \pm a_1 \pm a_2 \pm a_3 \pm a_4 \pm a_5 = 0$ *for a combination of three plus and three minus signs, or*

ii) $d_4(a_0, \ldots, a_5) = 0$.

**Proof.** There are two ways the intersection of the Segre cubic $S$ with the hyperplane given by $a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5 = 0$ may become singular. On one hand, the hyperplane might meet a singular point of $S$. This is equivalent to statement i).

On the other hand, the hyperplane could be tangent to $S$ in a certain point $(x_0 : \ldots : x_5) \in S$. This means that $a_0X_0 + a_1X_1 + a_2X_2 + a_3X_3 + a_4X_4 + a_5X_5$ is a linear combination of

$$3x_0^2X_0 + 3x_1^2X_1 + 3x_2^2X_2 + 3x_3^2X_3 + 3x_4^2X_4 + 3x_5^2X_5$$

and $X_0 + X_1 + X_2 + X_3 + X_4 + X_5$.

Equivalently, $(a_0 : \ldots : a_5)$ is in the closure of the image of the rational map

$$\pi\colon \mathbf{P} := \mathbf{P}_S(\mathscr{O} \oplus \mathscr{O}(-2)) \dashrightarrow \mathbf{P}^5\,,$$
$$(s, r; x_0 : \ldots : x_5) \mapsto ((rx_0^2 + s) : \ldots : (rx_5^2 + s))\,.$$

Here, on the $\mathbf{P}^1$-bundle $\mathrm{pr}\colon \mathbf{P} \to S$, the section $r \in \Gamma(\mathbf{P}, \mathscr{O}(1))$ corresponds to the section $(1,0) \in \Gamma(S, \mathscr{O} \oplus \mathscr{O}(-2))$ while the rational section $s'$ corresponds to the rational section $(0, x_0^{-2})$. In particular, the section $s := s'x_0^2 \in \Gamma(\mathbf{P}, \mathscr{O}(1) \otimes \mathrm{pr}^* \mathscr{O}_S(2))$ together with $r$ generates $\mathscr{O}(1)$ in every fiber of pr.

We claim that $\pi$ is generically finite. For this, it suffices to find a single fiber which is finite. Start with the point $(-5 : -1 : -1 : -1 : 4 : 4) \in S(\mathbb{Q})$. For $s = 0$, we find the image $P := (25 : 1 : 1 : 1 : 16 : 16) \in \mathbf{P}^5$.

We assert that the fiber $\pi^{-1}(P)$ is finite. For this, we may clearly assume that $r \neq 0$. I.e., we may normalize to $r = 1$. Then, we obtain the condition

$$\big(\pm\sqrt{25-s} : \pm\sqrt{1-s} : \pm\sqrt{1-s} : \pm\sqrt{1-s} : \pm\sqrt{16-s} : \pm\sqrt{16-s}\big) \in S\,.$$

In particular, the sum of the coordinates is required to be zero. This leads to an algebraic equation of degree 32 in $s$ which is not the zero equation. Indeed, for $s = 1$, we see that $\pm\sqrt{24} \pm \sqrt{15} \pm \sqrt{15}$ does not vanish whatever combination of signs we choose. Hence, the fiber over $P$ is finite.

Consequently, the image of $\pi$ is four-dimensional and irreducible in $\mathbf{P}^5$. It must be given by a single equation. It is easy to check that the equations $X_0 + \cdots + X_5 = 0$ and $X_0^3 + \cdots + X_5^3 = 0$ indeed imply $d_4(X_0^2, \ldots, X_5^2) = 0$. $\square$

**4.4. Remark.** — The actual discriminant is a polynomial of degree 24 factoring into $d_4$ and the squares of the ten linear polynomials $(a_0 \pm a_1 \pm a_2 \pm a_3 \pm a_4 \pm a_5)$ as above. The necessity of taking the squares is motivated by [EJ2, Theorem 2.12] and Proposition 5.3, below.

# 5 The 30 non-obvious tritangent planes

**5.1. Notation.** — i) Put $a_i' := a_i - \frac{\sigma_1}{6}$ for $i = 0, \ldots, 5$. This essentially means to normalize the sum of the coefficients $a_0, \ldots, a_5$ to zero.
ii) Using this shortcut, we will write

$$\tau_2^{i,j} := \sigma_2(a_0', \ldots, a_5') + 2(a_i'^2 + a_i'a_j' + a_j'^2)\,.$$

**5.2. Fact.** — *For $\{i_0, \ldots, i_5\} = \{0, \ldots, 5\}$, one has the equality*

$$d_4 = -(\tau_2^{i_0,i_1}\tau_2^{i_2,i_3} + \tau_2^{i_0,i_1}\tau_2^{i_4,i_5} + \tau_2^{i_2,i_3}\tau_2^{i_4,i_5})\,.$$

**Proof.** Due to the symmetry of $d_4$, it suffices to verify the equality $d_4 = -(\tau_2^{0,1}\tau_2^{2,3} + \tau_2^{0,1}\tau_2^{4,5} + \tau_2^{2,3}\tau_2^{4,5})$. This is a direct calculation. $\square$

**5.3. Proposition** (Coble). — *Let $a_0, \ldots, a_5$ be such that $S^{(a_0,\ldots,a_5)}$ is non-singular and $\{i_0, \ldots, i_5\} = \{0, \ldots, 5\}$. Then, there are five tritangent planes containing the line given by $X_{i_0} + X_{i_1} = X_{i_2} + X_{i_3} = X_{i_4} + X_{i_5} = 0$.*

*Among them, there are the three obvious ones given by* $X_{i_0}+X_{i_1}=0$, $X_{i_2}+X_{i_3}=0$, *and* $X_{i_0}+X_{i_1}+X_{i_2}+X_{i_3}$ $(=-X_{i_4}-X_{i_5})=0$.
*The two others may be written down explicitly in the form*

$$(\tau_2^{i_0,i_1} \pm \sqrt{d_4})(X_{i_0}+X_{i_1}) - (\tau_2^{i_2,i_3} \mp \sqrt{d_4})(X_{i_2}+X_{i_3}) = 0\,.$$

**Proof.** Write $\ell$ for the line prescribed. We describe a plane through $\ell$ by the equation

$$X_{i_0}+X_{i_1} = \lambda(X_{i_2}+X_{i_3})$$

where $\lambda$ is an unknown. The intersection of $S^{(a_0,\ldots,a_5)}$ with this plane is a cubic curve decomposing into a line and a conic. The discriminant of the conic turns out to be

$$\lambda(1+\lambda)(C_0+C_1\lambda+C_2\lambda^2)$$

for

$$C_0 := (a_{i_0}-a_{i_1})^2-(a_{i_2}+a_{i_3}-a_{i_4}-a_{i_5})^2\,,$$
$$C_1 := (a_{i_0}-a_{i_1})^2+(a_{i_2}-a_{i_3})^2-2(a_{i_0}+a_{i_1}-a_{i_4}-a_{i_5})(a_{i_2}+a_{i_3}-a_{i_4}-a_{i_5})-(a_{i_4}-a_{i_5})^2\,,$$
$$C_2 := (a_{i_2}-a_{i_3})^2-(a_{i_0}+a_{i_1}-a_{i_4}-a_{i_5})^2\,.$$

Observe that $C_2$ (and $C_0$) do not vanish unless $S^{(a_0,\ldots,a_5)}$ is singular. The zeroes at $\lambda=0$ and $\lambda=-1$ (as well as that at $\lambda=\infty$) correspond to the three obvious tritangent planes containing $\ell$.

In the notation introduced above, the formulas for $C_0$, $C_1$, and $C_2$ become by far simpler. Indeed, direct calculations show $C_0=-2(\tau_2^{i_2,i_3}+\tau_2^{i_4,i_5})$, $C_1=-4\tau_2^{i_4,i_5}$, and $C_2=-2(\tau_2^{i_0,i_1}+\tau_2^{i_4,i_5})$. The solutions of the quadratic equation $C_0+C_1\lambda+C_2\lambda^2=0$ are

$$\lambda = \frac{1}{\tau_2^{i_0,i_1}+\tau_2^{i_4,i_5}}\left[-\tau_2^{i_4,i_5} \pm \sqrt{(\tau_2^{i_4,i_5})^2-(\tau_2^{i_0,i_1}+\tau_2^{i_4,i_5})(\tau_2^{i_2,i_3}+\tau_2^{i_4,i_5})}\right].$$

Fact 5.2 implies that the radicand is equal to $d_4$. Another application of the same fact yields $\lambda=(\tau_2^{i_2,i_3}\pm\sqrt{d_4})/(\tau_2^{i_0,i_1}\mp\sqrt{d_4})$, which is the assertion. $\square$

**5.4. Remarks.** —— i) Observe that

$$(\tau_2^{i_0,i_1}+\sqrt{d_4})(\tau_2^{i_0,i_1}-\sqrt{d_4}) = (\tau_2^{i_0,i_1}+\tau_2^{i_2,i_3})(\tau_2^{i_0,i_1}+\tau_2^{i_4,i_5}) \neq 0\,.$$

Thus, the coefficients given are different from zero.
ii) The equation $(\tau_2^{i_0,i_1}+\sqrt{d_4})(X_{i_0}+X_{i_1})=(\tau_2^{i_2,i_3}-\sqrt{d_4})(X_{i_2}+X_{i_3})$ is equivalent to

$$(\tau_2^{i_0,i_1}-\sqrt{d_4})(X_{i_0}+X_{i_1}) = (\tau_2^{i_4,i_5}+\sqrt{d_4})(X_{i_4}+X_{i_5}) \qquad \text{or}$$
$$(\tau_2^{i_2,i_3}+\sqrt{d_4})(X_{i_2}+X_{i_3}) = (\tau_2^{i_4,i_5}-\sqrt{d_4})(X_{i_4}+X_{i_5})\,.$$

**5.5.** —— The combinatorial structure behind these formulas is as follows. While the obvious tritangent planes contain three obvious lines each, the non-obvious tritangent planes contain one obvious line and two non-obvious ones. Therefore, to give one of the non-obvious tritangent planes depends on fixing one of the *obvious* lines. For this, equivalently, one of the 15 decompositions

$$\{0, \ldots, 5\} = \{i_0, i_1\} \cup \{i_2, i_3\} \cup \{i_4, i_5\}$$

has to be chosen. For every such decomposition, there are two non-obvious planes. We have three mutually equivalent equations for each of them.

**5.6. Remark.** —— For a cubic surface in hexahedral form, the explicit description of the lines and tritangent planes is relatively simple. This makes these surfaces valuable form a computational perspective. Observe, if $a_0, \ldots, a_5 \in K$ for a field $K$ then the 27 lines on $S^{(a_0,\ldots,a_5)}$ are defined over a quadratic extension of $K$ and 15 of them are defined over $K$, already. Nevertheless, it seems that Proposition 5.3 is due to A. Coble [Co] and was unknown before the year 1915.

# 6 Explicit Galois descent

**6.1.** —— Let $A$ be a commutative semisimple algebra of finite dimension over $\mathbb{Q}$. Then, there are exactly $\dim A$ algebra homomorphisms $A \to \overline{\mathbb{Q}}$. We will denote them by $\tau_0, \ldots, \tau_{\dim A - 1}$. For an element $a \in A$, we have its trace $\mathrm{tr}(a) := \mathrm{tr}_{\mathbb{Q}}(\cdot a \colon A \to A) \in \mathbb{Q}$. In terms of the algebra homomorphisms, we may write $\mathrm{tr}(a) = \tau_0(a) + \ldots + \tau_{\dim A - 1}(a)$.

We extend the concept of a trace to polynomials with coefficients in $A$ by applying the trace coefficient-wise. I.e., we put

$$\mathrm{Tr}\left(\sum c_{i_0,\ldots,i_n} x_0^{i_0} \cdot \ldots \cdot x_n^{i_n}\right) := \sum \mathrm{tr}(c_{i_0,\ldots,i_n}) x_0^{i_0} \cdot \ldots \cdot x_n^{i_n}\,.$$

**6.2. Definition** (The trace construction)**.** —— Let $A$ be a commutative semisimple $\mathbb{Q}$-algebra of dimension six and $l = c_0x_0 + \ldots + c_3x_3$ be a non-zero linear form with coefficients in $A$. Suppose that $\mathrm{Tr}(l) = 0$.

Then, we say that the cubic form $\mathrm{Tr}(l^3)$ is obtained from $l$ by the *trace construction*. Correspondingly for the cubic surface $S(l)$ over $\mathbb{Q}$ given by $\mathrm{Tr}(l^3) = 0$.

**6.3. Proposition.** —— *Suppose that we are given a commutative semisimple $\mathbb{Q}$-algebra $A$ of dimension six. Further, let $l$ be a linear form in four variables $x_0, \ldots, x_3$ with coefficients in $A$ satisfying* $\mathrm{Tr}(l) = 0$.

*Denote by $d$ the dimension of the $\mathbb{Q}$-vector space $\langle l^{\tau_0}, \ldots, l^{\tau_5}\rangle \subseteq \Gamma(\mathbf{P}^3_K, \mathscr{O}(1))$. Finally, fix a field $K$ such that $K \supseteq \mathrm{im}\,\tau_i$ for $i = 0, \ldots, 5$.*

*Then,*

i) $l^{\tau_0}, \ldots, l^{\tau_5}$ *define a rational map*

$$\underline{\iota}\colon S(l) \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K \dashrightarrow \mathbf{P}^5_K \,.$$

*The image of* $\underline{\iota}$ *is contained in a linear subspace of dimension* $d-1$*.*

ii) *If* $d = 4$ *then* $S(l)$ *is a cubic surface over* $\mathbb{Q}$ *such that* $S(l) \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K$ *has hexahedral form.*

*More precisely, if* $a_0 l^{\tau_0} + \ldots + a_5 l^{\tau_5} = 0$ *is a relation, linearly independent of the relation* $l^{\tau_0} + \ldots + l^{\tau_5} = 0$*, then* $\underline{\iota}$ *induces an isomorphism*

$$\iota\colon S(l) \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K \stackrel{\cong}{\longrightarrow} S^{(a_0,\ldots,a_5)} \,.$$

iii) *If* $d \leq 3$ *then* $S(l)$ *is the cone over a (possibly degenerate) cubic curve.*

**Proof.** i) is standard.

ii) In this case, the forms $l^{\tau_0}, \ldots, l^{\tau_5}$ generate the $K$-vector space $\Gamma(\mathbf{P}^3_K, \mathscr{O}(1))$ of all linear forms. Therefore, they define a closed immersion of $\mathbf{P}^3_K$ into $\mathbf{P}^5_K$. In particular, $\iota$ is a closed immersion.

We have $l^{\tau_0} + \ldots + l^{\tau_5} = 0$ and the other linear relation $a_0 l^{\tau_0} + \ldots + a_5 l^{\tau_5} = 0$. The cubic surface $S(l) \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K \subset \mathbf{P}^3_K$ is given by $(l^{\tau_0})^3 + \ldots + (l^{\tau_5})^3 = 0$. Consequently, $\underline{\iota}$ maps $S(l) \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec} K$ to the cubic surface in hexahedral form $S^{(a_0,\ldots,a_5)} \subset \mathbf{P}^5_K$.

iii) is clear. $\square$

**6.4.** — As an application of the trace construction, we have an explicit version of Galois descent. For this, some notation has to be fixed.

**6.5. Notation.** — For $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, denote by $t_\sigma\colon \operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\overline{\mathbb{Q}}$ the morphism of schemes induced by $\sigma^{-1}\colon \overline{\mathbb{Q}} \leftarrow \overline{\mathbb{Q}}$. This yields a morphism

$$t_\sigma^{\mathbf{P}^5}\colon \mathbf{P}^5_{\overline{\mathbb{Q}}} \longrightarrow \mathbf{P}^5_{\overline{\mathbb{Q}}}$$

of $\overline{\mathbb{Q}}$-schemes which is *twisted by* $\sigma$. I.e., compatible with $t_\sigma\colon \operatorname{Spec}\overline{\mathbb{Q}} \to \operatorname{Spec}\overline{\mathbb{Q}}$. Observe that, on $\overline{\mathbb{Q}}$-rational points,

$$t_\sigma^{\mathbf{P}^5} : (x_0 : \ldots : x_5) \mapsto \big(\sigma(x_0) : \ldots : \sigma(x_5)\big) \,.$$

We will usually write $t_\sigma$ instead of $t_\sigma^{\mathbf{P}^5}$. The morphism $t_\sigma$ maps the cubic surface $S^{(a_0,\ldots,a_5)}$ to $S^{(\sigma(a_0),\ldots,\sigma(a_5))}$.

Assume that $a_0, \ldots, a_5$ are pairwise different from each other and the set $\{a_0, \ldots, a_5\}$ is $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-invariant. Then, every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ uniquely determines a permutation $\pi_\sigma \in S_6$ such that $\sigma(a_i) = a_{\pi_\sigma(i)}$. This yields a group homomorphism $\Pi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to S_6$. We will denote the automorphism of $\mathbf{P}^5$, given by the permutation $\pi_\sigma$ on coordinates, by $\pi_\sigma$, too.

Putting everything together, we see that

$$\pi_\sigma \circ t_\sigma \colon S^{(a_0,\ldots,a_5)} \longrightarrow S^{(a_0,\ldots,a_5)}$$

is an automorphism twisted by $\sigma$. These automorphisms form an operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S^{(a_0,\ldots,a_5)}$ from the left.

**6.6. Theorem** (Explicit Galois descent). — *Let $a_0, \ldots, a_5 \in \overline{\mathbb{Q}}$ be the distict zeroes of a (possibly reducible) degree six polynomial $f \in \mathbb{Q}[T]$.*

i) *Then, there exist a cubic surface $S = S_{(a_0,\ldots,a_5)}$ over $\mathbb{Q}$ and an isomorphism*

$$\iota\colon S \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}} \stackrel{\cong}{\longrightarrow} S^{(a_0,\ldots,a_5)}$$

*such that, for every $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, the diagram*

$$\begin{array}{ccc} S \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}} & \xrightarrow{\ \iota\ } & S^{(a_0,\ldots,a_5)} \\ \uparrow{\scriptstyle \mathrm{id}\times t_\sigma} & & \uparrow{\scriptstyle \pi_\sigma \circ t_\sigma} \\ S \times_{\mathrm{Spec}\,\mathbb{Q}} \mathrm{Spec}\,\overline{\mathbb{Q}} & \xrightarrow{\ \iota\ } & S^{(a_0,\ldots,a_5)} \end{array}$$

*commutes.*

ii) *The properties given determine the $\mathbb{Q}$-scheme $S$ up to unique isomorphism.*

iii) *Explicitly, $S$ may be obtained by the trace construction as follows.*
*Consider the commutative semisimple algebra $A := \mathbb{Q}[T]/(f)$. Then,*

$$S := S(l)$$

*for $l = c_0x_0 + \ldots + c_3x_3$ any linear form such that $\mathrm{Tr}(l) = 0$, $\mathrm{Tr}(Tl) = 0$, and $c_0, \ldots, c_3 \in A$ are linearly independent over $\mathbb{Q}$.*

**Proof.** i) and ii) These assertions are particular cases of standard results from the theory of Galois descent [Se, Chapitre V, §4, n° 20, or J, Proposition 2.5]. In fact, the scheme $S^{(a_0,\ldots,a_5)}$ is (quasi-)projective over $\overline{\mathbb{Q}}$. Thus, everything which is needed are "descent data", an operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on $S^{(a_0,\ldots,a_5)}$ such that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts by a morphism of $\overline{\mathbb{Q}}$-schemes which is twisted by $\sigma$.

iii) The $\mathbb{Q}$-linear system of equations

$$\begin{aligned} \mathrm{tr}(c) &= 0\,, \\ \mathrm{tr}(Tc) &= 0 \end{aligned}$$

has a four-dimensional space $\mathbb{L}$ of solutions. Indeed, the bilinear form $(x, y) \mapsto \mathrm{tr}(xy)$ is non-degenerate [Bou, §8, Proposition 1]. Hence, the first two conditions on $l$ express that $c_0, \ldots, c_3 \in \mathbb{L}$ while the last one is equivalent to saying that $\langle c_0, \ldots, c_3 \rangle$ is a basis of that space.

To exclude the possibility that $S$ degenerates to a cone and to obtain the isomorphism $\iota$, we will use Proposition 6.3.ii). This requires to show that the linear forms $l^{\tau_i} = c_0^{\tau_i} x_0 + \ldots + c_3^{\tau_i} x_3$ for $0 \leq i \leq 5$ form a generating system of the vector space of all linear forms. Equivalently, we claim that the $6 \times 4$-matrix

$$(c_j^{\tau_i})_{0 \leq i \leq 5, 0 \leq j \leq 3}$$

has rank 4.

To prove this, we extend $\{c_0, \ldots, c_3\}$ to a $\mathbb{Q}$-basis $\{c_0, \ldots, c_5\}$ of $A$. It is enough to verify that the $6 \times 6$-matrix $(c_j^{\tau_i})_{0 \leq i,j \leq 5}$ is of full rank. This assertion is actually independent of the particular choice of a basis. We may do the calculations as well with $\{1, T, \ldots, T^5\}$. We find the Vandermonde matrix

$$\begin{pmatrix} 1 & T^{\tau_0} & \cdots & (T^{\tau_0})^5 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & T^{\tau_5} & \cdots & (T^{\tau_1})^5 \end{pmatrix}$$

of determinant equal to

$$\prod_{i<j} (T^{\tau_i} - T^{\tau_j}) = \prod_{i<j} (a_i - a_j) \neq 0 \, .$$

Observe that the six algebra homomorphisms $\tau_i \colon A \to \overline{\mathbb{Q}}$ are given by $T \mapsto a_i$.

Consequently, the linear forms $l^{\tau_i}$ yield the desired isomorphism

$$\iota \colon S \times_{\operatorname{Spec} \mathbb{Q}} \operatorname{Spec} K \stackrel{\cong}{\longrightarrow} S^{(a_0, \ldots, a_5)} \, .$$

Indeed, we have the equations $\mathrm{tr}(Tc_i) = 0$. Explicitly, they express that, for each $i \in \{0, \ldots, 3\}$,

$$0 = (Tc_i)^{\tau_0} + \ldots + (Tc_i)^{\tau_5} = a_0 c_i^{\tau_0} + \ldots + a_5 c_i^{\tau_5} \, .$$

This means $a_0 l^{\tau_0} + \ldots + a_5 l^{\tau_5} = 0$.

It remains to verify the commutativity of the diagram. For this, we cover $S^{(a_0, \ldots, a_5)}$ by the affine open subsets given by $X_j \neq 0$ for $j = 0, \ldots, 5$. Observe that the morphisms to be compared are both morphisms of $\overline{\mathbb{Q}}$-schemes twisted by $\sigma$. Hence, we may compare the pull-back maps between the algebras of regular functions by testing their generators.

For arbitrary $i \neq j$, consider the rational function $X_i / X_j$. Its pull-back under $\iota$ is $l^{\tau_i} / l^{\tau_j}$. Therefore, the pull-back of $X_i / X_j$ along the upper left corner is

$$l^{\sigma^{-1} \circ \tau_i} / l^{\sigma^{-1} \circ \tau_j} = (c_0^{\sigma^{-1} \circ \tau_i} x_0 + \ldots + c_3^{\sigma^{-1} \circ \tau_i} x_3) / (c_0^{\sigma^{-1} \circ \tau_j} x_0 + \ldots + c_3^{\sigma^{-1} \circ \tau_j} x_3) \, .$$

On the other hand, the pull-back of $X_i/X_j$ under $\pi_\sigma \circ t_\sigma$ is $X_{\pi_{\sigma^{-1}}(i)}/X_{\pi_{\sigma^{-1}}(j)}$. Consequently, for the pull-back along the lower right corner, we find

$$\begin{aligned} l^{\tau_{\pi_{\sigma^{-1}}(i)}}/l^{\tau_{\pi_{\sigma^{-1}}(j)}} &= (c_0^{\tau_{\pi_{\sigma^{-1}}(i)}} x_0 + \ldots + c_3^{\tau_{\pi_{\sigma^{-1}}(i)}} x_3)/(c_0^{\tau_{\pi_{\sigma^{-1}}(j)}} x_0 + \ldots + c_3^{\tau_{\pi_{\sigma^{-1}}(j)}} x_3)\,, \\ &= (c_0^{\sigma^{-1}\circ\tau_i} x_0 + \ldots + c_3^{\sigma^{-1}\circ\tau_i} x_3)/(c_0^{\sigma^{-1}\circ\tau_j} x_0 + \ldots + c_3^{\sigma^{-1}\circ\tau_j} x_3)\,. \end{aligned}$$

Indeed, the embeddings $\tau_{\pi_{\sigma^{-1}}(i)}, \sigma^{-1}\circ\tau_i \colon A \to \overline{\mathbb{Q}}$ are the same as one may check on the generator $T$,

$$\tau_{\pi_{\sigma^{-1}}(i)}(T) = a_{\pi_{\sigma^{-1}}(i)} = \sigma^{-1}(a_i) = \sigma^{-1}(\tau_i(T)) = (\sigma^{-1}\circ\tau_i)(T)\,.$$

This completes the proof. □

Let $S^{(a_0,\ldots,a_5)}$ be a cubic surface over $\overline{\mathbb{Q}}$ satisfying the assumptions of Theorem 6.6. Its Galois descent may be computed as follows.

**6.7. Algorithm** (Computation of the Galois descent)**.** —— Given a separable polynomial $f \in \mathbb{Q}[T]$ of degree six, this algorithm computes the Galois descent $S_{(a_0,\ldots,a_5)}$ to $\operatorname{Spec}\mathbb{Q}$ of the cubic surface $S^{(a_0,\ldots,a_5)}$ for $a_0,\ldots,a_5$ the zeroes of $f$.

i) Compute, according to the definition, the traces $t_i := \operatorname{tr} T^i$ for $i = 0,\ldots,5$. Use these values to compute $t_6 := \operatorname{tr} T^6$.

ii) Determine the kernel of the $2\times 6$-matrix

$$\begin{pmatrix} t_0 & t_1 & t_2 & t_3 & t_4 & t_5 \\ t_1 & t_2 & t_3 & t_4 & t_5 & t_6 \end{pmatrix}.$$

Choose linearly independent kernel vectors $(c_i^0,\ldots,c_i^5) \in \mathbb{Q}^6$ for $i = 0,\ldots,3$.

iii) Compute the term

$$\left[\sum_{j=0}^{5} (c_0^j x_0 + \ldots + c_3^j x_3) T^j\right]^3$$

modulo $f(T)$. This is a cubic form in $x_0,\ldots,x_3$ with coefficients in $\mathbb{Q}[T]/(f)$.

iv) Finally, apply the trace coefficient-wise and output the resulting cubic form in $x_0,\ldots,x_3$ with 20 rational coefficients.

**6.8. Remark.** —— Observe that the computations in steps i), iii), and iv) are executed in the algebra $\mathbb{Q}[T]/(f)$ of dimension six. In order to perform Algorithm 6.7, it is not necessary to realize the Galois hull or any other large algebra on the machine.

# 7 The Galois operation on the descent variety

**7.1. Proposition.** — *Let* $a_0, \ldots, a_5 \in \overline{\mathbb{Q}}$ *be as in Theorem 6.6. Further, suppose that* $S^{(a_0,\ldots,a_5)}$ *is non-singular.*

a) *Then,* $d_4(a_0, \ldots, a_5) \in \mathbb{Q}^*$.

b) *Further, on the descent variety* $S_{(a_0,\ldots,a_5)}$ *over* $\mathbb{Q}$*, there are*

i) *15 obvious tritangent planes given by*

$$E_{i,j}\colon \iota^* X_i + \iota^* X_j = 0$$

*for* $0 \leq i < j \leq 5$,

ii) *30 non-obvious tritangent planes given by*

$$E_\pi^{\pm\sqrt{d_4}} = E_{i_0,\ldots,i_5}^{\pm\sqrt{d_4}}\colon (\tau_2^{i_0,i_1} \pm \sqrt{d_4})(\iota^* X_{i_0} + \iota^* X_{i_1}) - (\tau_2^{i_2,i_3} \mp \sqrt{d_4})(\iota^* X_{i_2} + \iota^* X_{i_3}) = 0$$

*for* $\left(\begin{smallmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ i_0 & i_1 & i_2 & i_3 & i_4 & i_5 \end{smallmatrix}\right) \in S_6$. *Here, for two permutations* $\pi, \pi' \in S_6$*, one has*

$$E_\pi^{\sqrt{d_4}} = E_{\pi'}^{s\sqrt{d_4}}$$

*if and only if* $\pi' = \pi\rho$ *for* $\rho$ *in the stabilizer of the decomposition* $\{0,1\} \cup \{2,3\} \cup \{4,5\}$ *and* $s = \operatorname{sn}\rho$*. Here,* $\operatorname{sn}\rho$ *denotes the signature of the projection of* $\rho$ *to* $S_3$*. Note that the stabilizer of* $\{0,1\} \cup \{2,3\} \cup \{4,5\}$ *is a group of order 48, isomorphic to* $S_3 \ltimes (\mathbb{Z}/2\mathbb{Z})^3$.

c) *An element* $\sigma \in \operatorname{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *acts on the tritangent planes according to the rules*

$$\sigma(E_{i,j}) = E_{\pi_\sigma(i),\pi_\sigma(j)}, \qquad \sigma\big(E_{i_0,\ldots,i_5}^{\sqrt{d_4}}\big) = E_{\pi_\sigma(i_0),\ldots,\pi_\sigma(i_5)}^{\sigma(\sqrt{d_4})}.$$

*The latter rule is equivalent to* $\sigma\big(E_\pi^{\sqrt{d_4}}\big) = E_{\pi_\sigma \circ \pi}^{\sigma(\sqrt{d_4})}$.

**Proof.** a) $d_4$ is given by a symmetric polynomial in $a_0, \ldots, a_5$. Therefore, we have $d_4(a_0, \ldots, a_5) \in \mathbb{Q}$. According to Proposition 4.3.ii), smoothness implies $d_4(a_0, \ldots, a_5) \neq 0$.

b) The isomorphism

$$\iota\colon S_{(a_0,\ldots,a_5)} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}} \longrightarrow S^{(a_0,\ldots,a_5)}$$

is provided by Theorem 6.6. We therefore obtain all tritangent planes by pull-back from $S^{(a_0,\ldots,a_5)}$. The formulas for them are given in 3.2 and Proposition 5.3. The discussion clarifying which formulas lead to the same plane was carried out in Remark 5.4.ii).

c) From the commutative diagram given in Theorem 6.6.i), we see that the operation of $\sigma$ on $S_{(a_0,\ldots,a_5)} \times_{\operatorname{Spec}\mathbb{Q}} \operatorname{Spec}\overline{\mathbb{Q}}$ goes over into the automorphism

$$\pi_\sigma \circ t_\sigma\colon S^{(a_0,\ldots,a_5)} \longrightarrow S^{(a_0,\ldots,a_5)}.$$

$\pi_\sigma$ permutes the coordinates while $t_\sigma$ is the operation of $\sigma$ on the coefficients.

The first assertion immediately follows from this. For the second one, note that $d_4 \in \mathbb{Q}$. Furthermore, according to 5.1.ii),

$$\begin{aligned}\sigma(\tau_2^{i,j}) &= \sigma\big(\sigma_2(a'_0,\dots,a'_5) + 2(a_i'^2 + a'_i a'_j + a_j'^2)\big)\\ &= \sigma_2(a'_0,\dots,a'_5) + 2[(a'_{\pi_\sigma(i)})^2 + a'_{\pi_\sigma(i)} a'_{\pi_\sigma(j)} + (a'_{\pi_\sigma(j)})^2] = \tau_2^{\pi_\sigma(i),\pi_\sigma(j)} . \qquad \square\end{aligned}$$

**7.2. Remark.** — From ii), we immediately see that the Galois operation on the obvious lines

$$L_{\{i_0,i_1\},\{i_2,i_3\},\{i_4,i_5\}} \colon \iota^* X_{i_0} + \iota^* X_{i_1} = \iota^* X_{i_2} + \iota^* X_{i_3} = \iota^* X_{i_4} + \iota^* X_{i_5} = 0$$

is given by

$$\sigma(L_{\{i_0,i_1\},\{i_2,i_3\},\{i_4,i_5\}}) = L_{\{\pi_\sigma(i_0),\pi_\sigma(i_1)\},\{\pi_\sigma(i_2),\pi_\sigma(i_3)\},\{\pi_\sigma(i_4),\pi_\sigma(i_5)\}} .$$

**7.3.** — Concerning the operation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ on the twelve non-obvious lines, there is the particular case that it does not interchange the two sixers the twelve lines consist of. This case is of minor interest from the point of view of Galois descent. In fact, such examples may be constructed more easily by blowing up six points of $\mathbf{P}^2$ which form one or several Galois orbits. The next result asserts that we run into this case only for very particular choices of the starting polynomial $f$.

**7.4. Proposition.** — *Let $a_0, \dots, a_5 \in \overline{\mathbb{Q}}$ be as in Theorem 6.6. Further, assume that $S^{(a_0,\dots,a_5)}$ is non-singular.*
*Let $K \subset \overline{\mathbb{Q}}$ be any subfield. Then, $\mathrm{Gal}(\overline{\mathbb{Q}}/K)$ stabilizes the two sixers formed by the twelve non-obvious lines on $S_{(a_0,\dots,a_5)}$ if and only if*

$$\sqrt{d_4(a_0,\dots,a_5) \cdot \Delta(a_0,\dots,a_5)} \in K$$

*for* $\Delta(a_0,\dots,a_5) := \prod\limits_{i<j} (a_i - a_j)^2$.

**Proof.** We will show this result in several steps.

*First step.* Assume that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ stabilizes the sixers. Then, $\sigma$ operates on the 30 non-obvious tritangent planes by an even permutation if $\pi_\sigma \in A_6$ and by an odd permutation if $\pi_\sigma \in S_6 \backslash A_6$.

Here, we use R. Hartshorne's notation (which is in principle due to L. Schläfli [Sch1, p. 116]) for the 27 lines. Since the sixers are assumed stable, the operation of $\sigma$ is given by

$$E_i \mapsto E_{\pi(i)}, \quad G_i \mapsto G_{\pi(i)}, \quad F_{ij} \mapsto F_{\pi(i),\pi(j)}$$

for a certain permutation $\pi \in S_6$. Clearly, $\pi$ is even if and only if $\pi_\sigma$ is. The operation on the 30 non-obvious tritangent planes is therefore described by

$$[F_{ij}, E_i, G_j] \mapsto [F_{\pi(i),\pi(j)}, E_{\pi(i)}, G_{\pi(j)}] .$$

The group $S_6$ is generated by all 2-cycles. Without restriction, let us check the action of (01). In this case, exactly the twelve tritangent planes $[F_{ij}, E_i, G_j]$ for $i, j \in \{2, 3, 4, 5\}$ are fixed. On the others, the operation is a product of nine 2-cycles. This is an odd permutation.

*Second step.* Assume that $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ flips the two sixers. Then, $\sigma$ operates on the 30 non-obvious tritangent planes by an even permutation if $\pi_\sigma \in S_6 \backslash A_6$ and by an odd permutation if $\pi_\sigma \in A_6$.

Here, the mapping is given by

$$[F_{ij}, E_i, G_j] \mapsto [F_{\pi(i),\pi(j)}, E_{\pi(j)}, G_{\pi(i)}] \, .$$

This means, we have the map from above followed by $[F_{ij}, E_i, G_j] \mapsto [F_{ij}, E_j, G_i]$. This consists of 15 2-cycles and is, therefore, odd.

*Third step.* The provision

$$E^{\sqrt{d_4}}_{i_0,\ldots,i_5} \mapsto E^{\sqrt{d_4}}_{\pi_\sigma(i_0),\ldots,\pi_\sigma(i_5)} \, , \qquad E^{-\sqrt{d_4}}_{i_0,\ldots,i_5} \mapsto E^{-\sqrt{d_4}}_{\pi_\sigma(i_0),\ldots,\pi_\sigma(i_5)}$$

always yields an even permutation.

Again, it suffices to check the action of the 2-cycle (01). That stabilizes exactly three of the obvious lines, namely $L_{\{0,1\},\{2,3\},\{4,5\}}$, $L_{\{0,1\},\{2,4\},\{3,5\}}$, and $L_{\{0,1\},\{2,5\},\{3,4\}}$. The corresponding six non-obvious tritangent planes remain in place, too. The others form twelve 2-cycles. This is an even permutation.

*Fourth step.* Conclusion.

We see that $\sigma$ stabilizes the sixers if and only if $\sigma(\sqrt{d_4}) = \mathrm{sgn}\, \pi_\sigma \cdot \sqrt{d_4}$. This is exactly the characterizing property of the square root of the discriminant. $\square$

**7.5. Remark.** — The two notations used for the lines and tritangent planes correspond to the hexahedral and blown-up models of a cubic surface. The interchange between the two leads to an instance of the exceptional automorphism of $S_6$.

In fact, in the hexahedral model, the 15 obvious lines correspond to the partitions of $\{0, \ldots, 5\}$ into three doubletons. In the blown-up model, they are given by the subsets of size two. The exceptional automorphism may be constructed by taking exactly this as the starting point [Gl].

# 8 The real picture

**8.1.** — Corresponding to the trivial group and the four conjugacy classes of elements of order two in $W(E_6)$, there are five types of non-singular cubic surfaces over the real field. They may be characterized in terms of the reality of the 27 lines as follows.

I. There are 27 real lines.

II. There are 15 real lines.

III. There are seven real lines.

IV. There are three real lines. The others form six orbits of two lines which are skew and six orbits of two lines with a point in common.

V. There are three real lines. The others form twelve orbits each consisting of two lines with a point in common.

**8.2.** — These five types have been known very early in the history of Algebraic Geometry. They appear in the article [Sch1, pp. 114f.] of L. Schläfli from 1858 where several more details are given. Considerably later, in 1872, Schläfli [Sch2] showed that the associated space of real points is connected in types I through IV. In the fifth type, the smooth manifold $S(\mathbb{R})$ has two connected components.

**8.3. Proposition.** — *Let $S$ be a non-singular cubic surface over $\mathbb{Q}$ with a Galois invariant double-six. Then, $S(\mathbb{R})$ decomposes into two components if and only if the complex conjugation flips the sixers and does not fix any of the six blow-up points.*

**Proof.** Assume that $S$ is of type V. If the complex conjugation $\sigma$ would not flip the sixers then each 2-cycle $(ij)$ of blow-up points led to an orbit $\{E_i, E_j\}$ consisting of two skew lines. Further, if the operation on the blow-up points were trivial then we had 27 real lines. Both possibilities are contradictory. Hence, $\sigma$ flips the two sixers. Finally, if we had a blow-up point fixed by $\sigma$ then this would cause an orbit $\{E_i, G_i\}$ of two skew lines.

On the other hand, assume that $\sigma$ flips the sixers and the operation on the six blow-up points is given by the permutation $(01)(23)(45)$. Then, the three lines $F_{01}$, $F_{23}$, and $F_{45}$ are $\sigma$-invariant, i.e. real. It is easy to check that every line $\ell$ different from these three is mapped to a line having exactly one point in common with $\ell$. □

**8.4. Corollary.** — *Let $a_0, \ldots, a_5 \in \overline{\mathbb{Q}}$ be as in Theorem 6.6. Further, assume that $S^{(a_0,\ldots,a_5)}$ is non-singular.*

*Then, $S_{(a_0,\ldots,a_5)}(\mathbb{R})$ has two connected components if and only if exactly four of the $a_0, \ldots, a_5$ are real and $d_4(a_0, \ldots, a_5) > 0$.*

**Proof.** The exceptional automorphism of $S_6$ maps a permutation of type $(01)(23)(45)$ to a 2-cycle. Hence, the requirement that $\sigma$ does not fix any blow-up point is equivalent to saying that $\pi_\sigma$ is a 2-cycle. This means exactly that four of the $a_0, \ldots, a_5$ are real and the other two are complex conjugate to each other.

In this case, the discriminant $\Delta(a_0, \ldots, a_5)$ is automatically negative. Further, according to Proposition 7.4, complex conjugation flips the two sixers on $S_{(a_0,\ldots,a_5)}$ if and only if $\Delta(a_0, \ldots, a_5) \cdot d_4(a_0, \ldots, a_5) < 0$. The assertion follows. □

**8.5. Remark.** —— The calculations for the other cases are not difficult. We summarize the results in the table below.

Table 1: Real types of cubic surfaces generated by descent

| $\#a_i$ real | #blow-up points fixed | #real lines and type for $d_4{\cdot}\Delta < 0$ | for $d_4{\cdot}\Delta > 0$ |
|---|---|---|---|
| 0 | 4 | $7_{\mathrm{III}}$ | $15_{\mathrm{II}}$ |
| 2 | 2 | $3_{\mathrm{IV}}$ | $7_{\mathrm{III}}$ |
| 4 | 0 | $3_{\mathrm{V}}$ | $3_{\mathrm{IV}}$ |
| 6 | 6 | $15_{\mathrm{II}}$ | $27_{\mathrm{I}}$ |

# 9 Examples

**9.1.** —— Using Algorithm 6.7, we generated a series of examples of smooth cubic surfaces over $\mathbb{Q}$. Our list of examples realizes each of the 102 conjugacy classes of subgroups of $W(E_6)$ which fix a double-six but no sixer. It is available at the web page `http:/www.uni-math.gwdg.de/jahnel` of the second author. In this section, we present a few cubic surfaces from our list which are, as we think, of particular interest.

**9.2. Example.** —— The polynomial

$$f := T^6 - 30T^4 + 20T^3 - 90T^2 - 1344T + 3970 \in \mathbb{Q}[T]$$

has Galois group $S_6$ and discriminant $2^{11}{\cdot}3^{12}{\cdot}5^5{\cdot}13^2{\cdot}31^2{\cdot}317^2$. Coble's radicand is equal to $1260 = 2^2{\cdot}3^2{\cdot}5{\cdot}7$. Algorithm 6.7 yields the non-singular cubic surface $S$ given by the equation

$$\begin{aligned}-4x^3 + 14x^2y - x^2z - 7x^2w + 14xy^2 + 2xyz + 8xyw - 4xzw \\ - 8xw^2 - 9y^3 - 10y^2z + 8y^2w + yz^2 - 4yzw - 3yw^2 - 3zw^2 - 9w^3 = 0\,.\end{aligned}$$

In this case, the Galois group operating on the 27 lines is of order 1440, isomorphic to $S_6 \times \mathbb{Z}/2\mathbb{Z}$. This is the maximal possible group stabilizing a double-six. We have orbit structure $[12, 15]$. The quadratic field splitting the double-six is $\mathbb{Q}(\sqrt{14})$. As $f$ has exactly two real roots, $S$ is of Schläfli's type III. This means, there are seven real lines on $S$. The manifold $S(\mathbb{R})$ is connected.

**9.3. Example.** —— Consider the polynomial

$$f := T^6 - 390T^4 - 10\,180T^3 + 10\,800T^2 + 2\,164\,296T + 13\,361\,180 \in \mathbb{Q}[T]\,.$$

Then, we have Galois group $S_6$ and discriminant $2^{17} \cdot 3^6 \cdot 5^5 \cdot 761^2 \cdot 44\,010\,848\,671^2$. Coble's radicand is equal to $108\,900 = 2^2 \cdot 3^2 \cdot 5^2 \cdot 11^2$. Algorithm 6.7 yields the non-singular cubic surface $S$ given by the equation

$$\begin{aligned} -x^2z - x^2w - 3xy^2 + xz^2 + 14xzw + 8xw^2 - 2y^3 - 11y^2z \\ + y^2w + 4yz^2 + 4yzw + 10yw^2 + 4z^3 - 11z^2w + 9zw^2 - 6w^3 = 0\,. \end{aligned}$$

Here, as Coble's radicand is a perfect square, only $S_6$ operates on the 27 lines. The orbit structure is still $[12, 15]$. The quadratic field $\mathbb{Q}(\sqrt{10})$ splits the double-six. Again, $f$ has exactly two real roots. Hence, $S$ is of Schläfli's type III. There are seven real lines on $S$ and the manifold $S(\mathbb{R})$ is connected.

**9.4. Example.** —— The polynomial

$$f := T^6 + 60T^4 - 40T^3 - 900T^2 + 15\,072T - 27\,860 \in \mathbb{Q}[T]$$

has Galois group $A_6$ and discriminant $2^{16} \cdot 3^{14} \cdot 5^6 \cdot 23^2 \cdot 59^2 \cdot 1831^2$. Coble's radicand is equal to $7200 = 2^5 \cdot 3^2 \cdot 5^2$. Algorithm 6.7 yields the non-singular cubic surface $S$ given by the equation

$$\begin{aligned} 5x^3 - 9x^2y + x^2z + 6x^2w + 3xy^2 + xyz + 6xyw - 2xz^2 \\ - 4xzw - y^3 - 3y^2z + 2yz^2 + 2yzw + 4z^3 + 2z^2w + 2zw^2 = 0\,. \end{aligned}$$

Here, the Galois group operating on the 27 lines is $A_6 \times \mathbb{Z}/2\mathbb{Z}$ with orbit structure $[12, 15]$. The double-six is split by the quadratic field $\mathbb{Q}(\sqrt{2})$. The starting polynomial $f$ has exactly two real roots. Thus, $S$ is of Schläfli's type III. There are seven real lines on $S$ and the manifold $S(\mathbb{R})$ is connected.

**9.5. Example.** —— Consider the polynomial $f := T(T^5 - 5T - 2) \in \mathbb{Q}[T]$. Its discriminant is $-3\,000\,000 = -2^6 \cdot 3 \cdot 5^6$. Coble's radicand is equal to $20 = 2^2 \cdot 5$. The Galois group of the second factor is $S_5$. Algorithm 6.7 yields the non-singular cubic surface $S$ given by the equation

$$\begin{aligned} 2x^3 + x^2y - 4x^2z - x^2w + 2xy^2 + 2xyz + 2xyw - 2xz^2 - 4xzw \\ - 2xw^2 + 2y^2z - y^2w + yz^2 + 2yzw - 5yw^2 - 3z^2w + 6zw^2 + 9w^3 = 0\,. \end{aligned}$$

The Galois group operating on the 27 lines is isomorphic to $S_5 \times \mathbb{Z}/2\mathbb{Z}$. The orbit structure is nevertheless $[12, 15]$. $\mathbb{Q}(\sqrt{-15})$ is the field splitting the double-six. Here, $f$ has exactly four real roots and $S$ is of Schläfli's type V. There are only three real lines on $S$. The manifold $S(\mathbb{R})$ has two connected components.

**9.6. Example.** —— For the polynomial

$$f := T(T^5 - 60T^3 - 90T^2 + 675T + 810) \in \mathbb{Q}[T]\,,$$

the second factor has Galois group $S_5$. The discriminant of $f$ is equal to $-2^{12} \cdot 3^{21} \cdot 5^8 \cdot 13^2$ and Coble's radicand is $900 = 2^2 \cdot 3^2 \cdot 5^2$. Algorithm 6.7 yields the non-singular cubic surface $S$ given by the equation

$$3x^3 + 2x^2z + xy^2 - 2xyz - 2xyw - xzw + 2xw^2 - yzw - yw^2 - z^3 + z^2w = 0\,.$$

Since Coble's radicand is a perfect square, the Galois group operating on the 27 lines is isomorphic to $S_5$. The orbit structure is still $[12, 15]$. $\mathbb{Q}(\sqrt{-3})$ splits the double-six. The polynomial $f$ has exactly four real roots and $S$ is of Schläfli's type V. There are only three real lines on $S$ and the manifold $S(\mathbb{R})$ has two connected components.

**9.7. Example.** —— Consider the polynomial $f := T(T^5 + 20T + 16) \in \mathbb{Q}[T]$. Then, we have a discriminant of $2^{24} \cdot 5^6$. Coble's radicand is equal to $-80 = -2^4 \cdot 5$. The Galois group of the second factor is $A_5$. In this example, Algorithm 6.7 yields the non-singular cubic surface $S$ given by the equation

$$\begin{aligned} -3x^3 - 7x^2y - 4x^2z + 5x^2w + 4xy^2 + 10xyz - 4xyw - 2xz^2 \\ + 2xzw + xw^2 - 4y^2z + yz^2 - 4yzw - 16yw^2 + z^2w - 5zw^2 = 0\,. \end{aligned}$$

The Galois group operating on the 27 lines is isomorphic to $A_5 \times \mathbb{Z}/2\mathbb{Z}$. The orbit structure is $[12, 15]$. The double-six is split by the quadratic field $\mathbb{Q}(\sqrt{-5})$. The polynomial $f$ has exactly two real roots and $S$ is of Schläfli's type IV. There are only three real lines on $S$ but the manifold $S(\mathbb{R})$ is connected.

**9.8. Example.** —— The polynomial

$$f := T^6 - 456T^4 - 904T^3 + 102\,609T^2 + 1\,041\,060T + 2\,935\,300 \in \mathbb{Q}[T]$$

is irreducible and has the Galois group $(S_3 \times S_3) \ltimes \mathbb{Z}/2\mathbb{Z}$ of order 72. This group has three subgroups of index two. Correspondingly, the splitting field of $f$ contains the quadratic number fields $\mathbb{Q}(\sqrt{-2})$, $\mathbb{Q}(\sqrt{-1})$, and $\mathbb{Q}(\sqrt{2})$. The discriminant of $f$ is equal to $-2^{21} \cdot 3^{24} \cdot 5^2 \cdot 3049^2 \cdot 6823^2$. Coble's radicand is $-202\,500 = -2^2 \cdot 3^4 \cdot 5^4$. Algorithm 6.7 yields the non-singular cubic surface $S$ given by the equation

$$\begin{aligned} -2x^3 + 3x^2z + 9x^2w - 4xy^2 - 8xyz - 10xzw + 4xw^2 - 4y^3 - 3y^2z \\ - 4y^2w - 2yz^2 - 2yzw + 8yw^2 - z^3 + z^2w - 6zw^2 - 2w^3 = 0\,. \end{aligned}$$

Here, the quadratic field splitting the double-six is $\mathbb{Q}(\sqrt{2})$. The Galois group operating on the 27 lines is only of order 72, isomorphic to $(S_3 \times S_3) \ltimes \mathbb{Z}/2\mathbb{Z}$.

The orbit structure is $[6, 6, 6, 9]$. There are actually three Galois invariant double-sixes but there is no Galois invariant sixer. The orbits of size six do not consist of skew lines. The polynomial $f$ has no real roots. Hence, $S$ is of Schläfli's type II. There are fifteen real lines on $S$ and the manifold $S(\mathbb{R})$ is connected.

**9.9. Remark.** —— To find a suitable starting polynomial $f$ for a given conjugacy class of subgroups, our strategy was as follows.

i) Choose a commutative semisimple algebra in explicit form $A = \mathbb{Q}[X]/(F)$. Make sure that $\deg F = 6$ and that $F$ has the Galois group required. For this, a list of polynomials such as that in [MM] is helpful.

ii) Fix a square class $d(\mathbb{Q}^*)^2$ for Coble's radicand which corresponds to the group desired.

iii) A general element $t \in A$ generates $A$ as a $\mathbb{Q}$-algebra. I.e., it provides an isomorphism $\mathbb{Q}[T]/(f_t) \to A$ for $f_t$ a suitable polynomial of degree six. The condition that the Coble radicand of $f_t$ lies in the given square class leads to a double cover of $\mathbf{P}^5$. This is a (possibly singular) Fano fivefold. We need a $\mathbb{Q}$-rational point on a Zariski open subset of that fivefold.

In practice, a naive search up to height 20 was always sufficient. In a few cases where no rational point was found, we replaced the algebra $A$ by a different one.

**9.10. Remark.** —— The cubic surface provided by Algorithm 6.7 is determined by the starting polynomial $f$. The actual equation, however, heavily depends on the vector space basis chosen in step ii).

Computer algebra systems such as `magma` typically output an echelonized basis. It turned out that echelonized bases lead to huge coefficients and, hence, are highly unsuitable for our purposes. In the examples, we presented equations coming from reduced bases. A general method to reduce the coefficients of a given cubic surface is described in [E].

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