Currency Prices, the Nominal Exchange Rate, and Security Prices in a Two Country Dynamic Monetary Equilibrium *

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Abstract

This paper examines a continuous-time two country dynamic monetary equilibrium in which countries with possibly heterogeneous tastes and endowments hold their own money for the purpose of transaction services formulated via money in the utility function. Given a price system, no-arbitrage pricing results are provided for the price of each money and the nominal exchange rate. Characterizations are provided for equilibrium prices for general time-additive preferences and non-Markovian exogenous processes. Under a Markovian structure of model primitives, the currency prices are shown to solve a bivariate system of partial differential equations. Assuming that each country is endowed with heterogeneous separable power utility and the exogenous quantities all follow geometric Brownian motions, an equilibrium is shown to exist and additional characterization is provided. A further example of non-separable Cobb-Douglas preferences is investigated. The additional features over the customary environment of homogeneous logarithmic preferences are emphasized.

KEYWORDS: Asset Pricing, Currency Prices, Monetary Equilibrium, Nominal Exchange Rate

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1 Introduction

The majority of assets traded on world financial markets are denominated in nominal quantities with exposure to inflation and nominal exchange rate fluctuations. Although this exposure has been sufficient to prompt the introduction of inflation-indexed securities in many markets, most general equilibrium models have largely ignored the potential impact. The reason behind this avoidance has been one of tractability; past attempts at a general equilibrium synthesis between real, financial, and monetary markets have required rather harsh assumptions on primitives to obtain results. This problem is especially acute within the international asset pricing literature, as pointed out by Dumas (1994). In this paper, we partially address this problem by examining a continuous-time two country dynamic monetary equilibrium model in which agents have possibly heterogeneous tastes and endowments. The nominal quantities in the model are endogenously determined through a transaction service role for money.

Past theoretical work in international finance has evolved along several different modeling paradigms to examine a variety of issues. These approaches include (1) versions of the international CAPM [Solnik (1974), Adler and Dumas (1983)], (2) international general equilibrium models which examine real quantities [Dumas (1992), Nielsen and Saá-Requejo (1993), Uppal (1993), Serrat (1995), Zapatero (1995)], and (3) international general equilibrium models which incorporate nominal pricing by agents holding money to alleviate some market friction.

The model in this paper falls into the third category. Within this literature, several methods of incorporating money have been used including transaction cost technologies such as a cash-in-advance constraint [Lucas (1982), Svensson (1985)], overlapping generations formulations, and money in the utility function (MIUF) formulations. These methods have been well-debated [Kareken and Wallace (1980), McCallum (1983), LeRoy (1984), Feenstra (1986)]; in particular, Feenstra (1986) shows in partial equilibrium that a functional equivalence exists between a MIUF formulation and a transaction cost technology embedded in an agent's budget constraint. For tractability in a heterogeneous agent economy, we choose to employ a MIUF formulation. As will be seen, real money balances act like a durable good with no adjustment costs; hence, the equilibrium can be readily characterized using a representative agent with constant weights. This contrasts with modeling money using a transaction cost technology, where the complications in aggregation have forced cash-in-advance-based international models [Lucas (1982), Svensson(1985)] to rely on each agent having identical tastes and endowments. These assumptions lead to models that claim that domestic agents should act just like their foreign counterparts, a conclusion that seems difficult to swallow in the context of international financial markets.

The use of a MIUF formulation to characterize prices in an economy with money is not new to the nominal asset pricing literature. Stulz (1986a, 1986b) and Foresi (1990) extend the Cox, Ingersoll, and Ross (1985) model, taking a money supply process as exogenous and provide CCAPM results in the presence of real money holdings. However, they do not explicitly solve for the price of money and so do not close the model. Shi (1994), using a transaction cost technology, circumvents the problem of solving for the price of money by viewing the price of money as another exogenously specified "production" technology, while endogenously determining the money supply. Recognizing the complications of a production model, Bakshi and Chen (1996), in a single country model, provide a full characterization, including the price of money, by appealing to the first order conditions of a sequence of discrete-time, single agent, pure-exchange economies. They arrive at continuous-time characterizations by limiting arguments. The Bakshi and Chen (1996) model can be viewed as a special case of the model we present here.

In an international context, previous work examining nominal asset pricing has been restricted to agents with logarithmic preferences. Stulz (1984, 1987) presents a two country MIUF production economy based on Cox, Ingersoll, and Ross (1985). Stulz does not fully characterize all nominal quantities, but provides joint restrictions on the endogenous nominal interest rates, the prices of money, and the nominal exchange rate.[1] Collin Dufresne and Shi (1996), also in a two country Cox, Ingersoll, and Ross (1985) framework, are able to characterize all quantities of interest by treating the price of each currency as an exogenously specified "production" technology (money being held to alleviate Cobb-Douglas exchange costs). In the context of developing a model to price foreign exchange contingent claims, Bakshi and Chen (1997) build a continuous-time counterpart to the two country monetary model of Lucas (1982). Due to the difficulties in imposing a cash-in-advance constraint in continuous-time, they impose a "cash constraint" on identical logarithmic agents and determine the price of money as the ratio of the exogenous output to the exogenous money supply. These works, along with the two country cash-in-advance models of Lucas (1982) and Svensson (1985), still leave open a potentially interesting question: how are nominal equilibrium prices affected when agents are heterogeneous in tastes and endowments?[2]

Our objective in this paper is to provide a two country dynamic monetary equilibrium which fully characterizes all endogenous quantities and allows for heterogeneity in tastes and endowments. Each country issues money to its inhabitants for the purpose of transaction services. Agents consume by holding claims to a single stock shared by both countries which pays an exogenously specified stochastic dividend.[3] For risk sharing purposes, agents can also trade in an adequate number of securities such that the market is dynamically complete at all times.

The supply of a country's currency is controlled entirely by that country's government, allowing

[1]Stulz (1984) begins by examining an agent endowed with a variation of a CES utility; however, he only obtains partial equilibrium results for this more general utility function.

[2]International cash-in-advance models, with a variety of objectives differing from ours (e.g. Bekaert (1994, 1996)), still maintain the perfect-pooling construction to characterize the equilibrium. International general equilibrium models that examine real quantities have also placed restrictive assumptions on preferences [Dumas (1992), Uppal (1993) both with power preferences; Nielsen and Saá-Requejo (1993), Zapatero (1995) both with logarithmic preferences]. For a description of the voluminous international macroeconomic literature exploring issues in addition to asset pricing, see Obstfeld and Rogoff (1996).

[3]In contrast to most international equilibrium models, we simplify our economy by assuming that there exists a single consumption good consumed in both countries. By doing so, we avoid price index complications that can further complicate characterizing the nominal quantities in the model.

the currency to in some sense define the notion of a nation. To make this notion even sharper, we assume that each country only holds its own money for the purpose of transaction services. To motivate this assumption, the following table (for the last quarter of 1994) shows that although foreign currency holding is sizable, a unit of foreign currency supports more units of imports than a unit of domestic currency supports units of GDP, implying a home bias in currency.[4] We make the extreme assumption of a complete bias, but our methods easily extend to the case of countries holding both currencies.

Country	$\nu_{FOR} \equiv \frac{\text{Foreign Currency Holding}}{\text{Imports}}$	$\nu_{DOM} \equiv \frac{\text{Money Supply}}{\text{GDP}}$	$\frac{\nu_{DOM}}{\nu_{FOR}}$
Canada	0.07	0.17	2.36
Denmark	0.08	0.33	4.22
Finland	0.17	1.22	7.24
Japan	0.06	0.33	5.71
Norway	0.20	1.62	8.29
Sweden	0.08	0.54	7.13
UK	0.29	0.98	3.43

Within the context of our model, we completely characterize the equilibrium. Specifically, we provide no-arbitrage pricing results for the price of each money and the nominal exchange rate independent of preferences and the method of incorporating money in the economy. Each currency price is equal to the present value of future implicit real "dividends" given by the product of the future currency price and the associated nominal interest rate. This pricing expression provides an example of a backward stochastic differential equation. Accordingly, the nominal exchange rate is equal to the ratio of the present values of these currency dividends. Each country's consumption-money-portfolio problem is fully characterized using martingale techniques [Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987)] including the incorporation of monetary transfers from each country's government. Aggregating with a representative agent construction, we provide equilibrium characterizations of all endogenous quantities, including a currency price characterization suitable for numerical analysis. Specializing the aggregate consumption and the two money supplies to be jointly Markovian, the currency prices are characterized as the solution to a bivariate system of nonlinear partial differential equations. Equity prices are shown to satisfy a modified CCAPM in which a security's risk premium is driven not only by the instantaneous covariance of its return with the aggregate consumption, but also with aggregate real money balances in each currency. Our analysis complements and extends studies of intertemporal complete markets with heterogeneous tastes [Dumas (1989), Duffie and Zame (1989), Karatzas, Lehoczky, and Shreve

[4]These data are taken from *International Financial Statistics* published by the IMF. Foreign currency holdings are a nominal measure of demand deposits of domestic citizens held in foreign banks. The money supply measure is a narrow measure of money including demand deposits and currency outside banks. Imports (goods and services) and GDP are measured in 1990 units of the home currency. The ratios $\nu_{FOR} \equiv$ Foreign Currency Holding/Imports and $\nu_{DOM} \equiv$ Money Supply/GDP loosely capture the amount of money needed to support a unit of consumption. The final ratio, ν_{DOM}/ν_{FOR}, captures the relative importance to domestic investors of holding their own versus foreign currency in servicing consumption.

(1990), Wang (1996) (one good); Lakner (1989) (multiple goods)]; however, our second good, real money balances, is endogenous due to its dependence on the price of money which substantially complicates equilibrium determination.

As an application, we present two examples and discuss the resulting characterization. First, to isolate the effect of heterogeneity on nominal prices, countries are assumed to derive power separable utility from consumption and real money balances, but may differ in their use of money through different exponents on real money balances in power preferences. We derive closed-form expressions for all quantities of interest and full comparative statics. Additional features not seen with homogeneous logarithmic preferences include an influence of the endowment and taste differences on currency prices, and hence the nominal exchange rate. Our second example, whose focus is the interaction between real and nominal quantities, assumes non-separable Cobb-Douglas preferences over consumption and money holdings. In this case, the inclusion of money impacts real economic quantities, and cross-currency effects arise where one country's money supply impacts the other country's currency price.

Our economic setup is described in Section 2, including no-arbitrage pricing results and each country's consumption-money-portfolio problem. Section 3 defines equilibrium and provides characterization of all equilibrium prices. Section 4 presents additional results when each country has separable power preferences and all exogenous processes follow geometric Brownian motions. Section 5 provides the Cobb-Douglas example. All proofs are in the Appendix.

2 General Formulation

We present a continuous-time two country variation on the Lucas (1978) pure-exchange economy where each country, in addition to having preferences over consumption, also desires to hold its own money for transaction services. The economy has a finite horizon $[0, T]$, with uncertainty represented by a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathsf{P})$, on which is defined a three-dimensional Brownian motion $W = (W_1, W_2, W_3)^\top$. The common information is given by the augmented filtration $\mathbf{F} = \{\mathcal{F}_t : t \in [0, T]\}$ generated by W under the probability measure P. All stochastic processes are assumed progressively measurable with respect to $\mathbf{F}$, all stated equalities involving random variables hold P-a.s., and all stochastic differential equations are assumed to have solutions. Quantities associated with the domestic and foreign country are indexed by $i = \{D, F\}$.

2.1 The Commodity and Money Supplies, Prices of Money, and the Nominal Exchange Rate

A single infinitely-divisible commodity (the numeraire) and two monies exist in this economy. We model the supply of the perishable commodity δ as an Itô process of the form

$$d\delta(t) = \delta(t) \left[\mu_\delta(t)dt + \sigma_\delta(t)^\top dW(t)\right], \quad \delta(0) = \delta_0. \tag{2.1}$$

Each money supply process M_i expressed in units of money i (nominal) also follows an Itô process

$$dM_i(t) = M_i(t)\left[\mu_{M_i}(t)dt + \sigma_{M_i}(t)^\top dW(t)\right], \quad M_i(0) = M_{i0}, \quad i = \{D, F\}. \tag{2.2}$$

Here μ_δ, μ_{M_i} and σ_δ, σ_{M_i} are exogenously specified one- and three-dimensional bounded (possibly non-Markovian) processes. The money supply management by each monetary authority is taken as exogenous. The fashion by which money enters the economy is presented in Section 2.3.

It will be shown in equilibrium that the relative price q_i of each money in units of the commodity follows an Itô process

$$dq_i(t) = q_i(t)\left[\mu_{q_i}(t)dt + \sigma_{q_i}(t)^\top dW(t)\right], \quad i = \{D, F\}, \tag{2.3}$$

where μ_{q_i} and σ_{q_i} are one- and three-dimensional bounded processes to be determined in equilibrium. We assume $q_i(T) = 0$, since no agent receives additional gains from holding money beyond the horizon.[5] To be consistent with past literature, expected inflation in each currency is defined as $-\mu_{q_i}$, the negative of the instantaneous expected rate of change of the currency price.

The nominal exchange rate e is defined as the number of units of foreign money per unit of the domestic money, $\frac{1}{q_F}/\frac{1}{q_D}$. By an application of Itô's Lemma, the dynamics of the nominal exchange rate e are given by

$$de(t) = e(t)\left[\mu_e(t)dt + \sigma_e(t)^\top dW(t)\right], \tag{2.4}$$

where $\mu_e(t) \equiv \mu_{q_D}(t) - \mu_{q_F}(t) + \sigma_{q_F}(t)^\top(\sigma_{q_F}(t) - \sigma_{q_D}(t))$ and $\sigma_e(t) \equiv \sigma_{q_D}(t) - \sigma_{q_F}(t)$.

2.2 Security Markets

We assume that domestic and foreign agents can frictionlessly trade in four securities: a real riskless bond, two nominal riskless bonds denominated in the two currencies, and a stock. The three bonds are assumed to be in zero net supply, while the stock, which pays out the commodity at the rate δ, is assumed to be in a positive net supply of one share.

Each security price is modeled as an Itô process. The real riskless bond is locally riskless in real terms with an instantaneous real rate of return of $r(t)dt$. The two nominal riskless bonds are locally riskless in each of their respective currencies with an instantaneous nominal rate of return of $R_i(t)dt$. The prices of the real riskless bond B_0, the nominal bonds B_D and B_F, and the stock S have real dynamics given by

$$dB_0(t) = B_0(t)r(t)dt, \tag{2.5}$$

$$dB_i(t) = B_i(t)\left[\left(\mu_{q_i}(t) + R_i(t)\right)dt + \sigma_{q_i}(t)^\top dW(t)\right], \quad i \in \{D, F\}, \tag{2.6}$$

$$dS(t) + \delta(t)dt = S(t)\left[\mu_S(t)dt + \sigma_S(t)^\top dW(t)\right], \quad S(T) = 0. \tag{2.7}$$

We use $\mathcal{P}$ to denote the endogenous price system $(q_D, q_F, B_0, B_D, B_F, S)$. Comparing (2.6) with (2.3), we see that R_i represents the additional compensation the nominal bond gives over currency

[5] This assumption will be further discussed in Remark 3.1.

i for not providing transaction services to the agent.[6] We can view R_i as a measure of the transaction services provided by currency i.

The real short rate r, the three-dimensional vector of drifts $\mu \equiv (\mu_{q_D} + R_D, \mu_{q_F} + R_F, \mu_S)^\top$, and the volatility matrix $\sigma \equiv [\sigma_{q_D}, \sigma_{q_F}, \sigma_S]^\top$ are bounded processes, and determined endogenously in equilibrium, with S verified to follow the posited Itô process (2.7). Assuming $\sigma(t)$ is invertible for all t, the market is dynamically complete since the number of risky securities equals the number of dimensions of uncertainty.[7] In particular, future money transfers as well as dividends are fully hedgeable. Accordingly, we define the *real state price density process* ξ as a process with the dynamics

$$d\xi(t) = -\xi(t)\left[r(t)dt + \theta(t)^\top dW(t)\right], \tag{2.8}$$

where θ is the market price of risk process, $\theta(t) \equiv \sigma(t)^{-1}[\mu(t) - r(t)\mathbf{1}]$, where $\mathbf{1} \equiv (1,1,1)^\top$. $\xi(t,\omega)$ is interpreted as the Arrow-Debreu price per unit of probability P of one unit of consumption good in state $\omega \in \Omega$ at time t. In addition, we assume $E\left[\int_0^T \xi(s)^2 S(s)^2 ds\right] < \infty$, $E\left[\int_0^T \xi(s)^2 q_i(s)^2 ds\right] < \infty$, and $E\left[\int_0^T \xi(s)^2 q_i(s)^2 M_i(s)^2 ds\right] < \infty$ to develop the pricing expressions given in Lemma 2.1.

Remark 2.1 (Extended Fisher Equation and Uncovered Interest Rate Parity) The market price of risk process yields a no-arbitrage relationship for the instantaneous nominal short rate of return in each currency $R_i(t)$,

$$R_i(t) = r(t) - \mu_{q_i}(t) + \sigma_{q_i}(t)^\top \theta(t), \quad i \in \{D, F\}. \tag{2.9}$$

The nominal short rate for currency i is equal to the real short rate plus expected inflation plus the instantaneous covariance between changes in the price of money and changes in the real state price density, collapsing to the standard Fisher equation only if changes in the money price and state prices are uncorrelated. Stulz (1986a), Foresi (1990), and Bakshi and Chen (1996) specialize (2.9) and present it as an implication of the consumption CAPM with money. Using (2.4) and (2.9), we arrive at the uncovered interest rate parity relation linking the two nominal short-rates:

$$R_F(t) - \mu_e(t) + \sigma_e(t)^\top \sigma_e(t) = R_D(t) + \sigma_e(t)^\top (\sigma_{q_D}(t) - \theta(t)). \tag{2.10}$$

The left hand side of (2.10) is the domestic nominal expected rate of return of a foreign currency deposit, which equals the domestic nominal short-rate plus an instantaneous covariance between changes in the exchange rate and changes in the difference between the price of money and the real state price density. If the nominal exchange rate is locally riskless, the difference between the two nominal short rates is just the rate of growth of the nominal exchange rate.

From the existence of the real state price deflator, the following "no-arbitrage" asset pricing equations can be established for the stock and the price of each money.

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[6] This assumes that $R_i(t) > 0$ which will be shown to hold in equilibrium.

[7] Since r, μ and σ are endogenous, the posited restrictions should be verified to hold in equilibrium. The invertibility of σ could be especially difficult to verify. Alternatively, in addition to the stock, we could assume the existence of zero net-supply securities as in Karatzas, Lehoczky, and Shreve (1990) to guarantee market completeness.

Lemma 2.1 *Given a price system $\mathcal{P}$, the price of the stock is given by*

$$S(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s)\delta(s)ds|\mathcal{F}_t\right]. \tag{2.11}$$

The price of each money is given by

$$q_i(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s)q_i(s)R_i(s)ds|\mathcal{F}_t\right], \quad i \in \{D, F\}, \tag{2.12}$$

and hence the nominal exchange rate by

$$e(t) = \frac{E\left[\int_t^T \xi(s)q_D(s)R_D(s)ds|\mathcal{F}_t\right]}{E\left[\int_t^T \xi(s)q_F(s)R_F(s)ds|\mathcal{F}_t\right]}. \tag{2.13}$$

Equation (2.11) yields the familiar asset pricing equation: the stock price equals the expected present value of future dividends. Equation (2.12) may be interpreted analogously, that the real price of a unit of money i equals the expected present value of its implicit future "dividends" $q_i R_i$. Hence, $q_i R_i$ must represent the real value (and R_i the nominal value) of the transaction services provided by holding that money. The exchange rate is then the ratio of the present values of future dividends of the domestic to the foreign currency. Equation (2.12) is an example of a backward stochastic differential equation. [See El Karoui, Peng, and Quenez (1997) for a recent survey on the use of backward stochastic differential equations in finance.]

This no-arbitrage approach allows for convenient representation of each money price, the nominal exchange rate, and the stock price. The strength of appealing to no-arbitrage pricing is that no preference or money-modeling assumptions are required. Equation (2.12) is equally valid in a production economy or one where money is held due to agent-specific transaction costs. Furthermore, the no-arbitrage approach naturally combines with the martingale techniques used in solving each country's optimization (Section 2.3) and in constructing equilibrium (Section 3).[8]

2.3 Countries' Endowments, Preferences, and Optimization

The domestic country is endowed at time zero with ϵ_D shares of the stock, and the foreign country $\epsilon_F = 1 - \epsilon_D$. The monies enter the economy by endowing each country with a claim to its own currency. The initial wealth of country i is given by $x_{i0} = \epsilon_i S(0) + q_i(0)M_i(0)$.

Each country i chooses a nonnegative consumption process c_i, a nonnegative nominal money holding process m_i in currency i, and a portfolio process π_i. The consumption process and the

[8] In contrast to our emphasis on nominal quantities, Nielsen and Saá-Requejo (1993), Serrat (1995), and Zapatero (1995) have employed martingale techniques in international equilibrium models to examine only real quantities. To explore the relationship between empirical purchasing power parity tests and theoretical characterizations of nominal exchange rates, Apte, Sercu, and Uppal (1997) use a no-arbitrage-based approach to characterize the nominal exchange rate in terms of observables in a multicountry, multigood economy.

money process are assumed to satisfy $E\left[\int_0^T c_i(t)^2\right] < \infty$ and $E\left[\int_0^T m_i(t)^2\right] < \infty$. To preclude arbitrage strategies, we also assume that $E\left[\int_0^T \xi(t)^2 ||\pi_i(t)||^2 dt\right] < \infty$ and $E\left[\int_0^T \xi(t)^2 q_i(t)^2 m_i(t)^2 dt\right] < \infty$. Here $\pi_i \equiv (\pi_{iD}, \pi_{iF}, \pi_{iS})^\top$ denotes the vector of amounts (in units of the commodity) invested in the domestic nominal bond, the foreign nominal bond, and the stock respectively. Given an endowment x_{i0}, an *admissible policy* (c_i, m_i, π_i) is defined as one for which the associated wealth process, $X_i(t) \equiv \pi_{i0}(t) + \pi_i(t)^\top \mathbf{1} + q_i(t)m_i(t)$, satisfies the dynamic budget constraint

$$
\begin{aligned}
(2.14) \quad dX_i(t) \quad = \quad & X_i(t)\, r(t)\, dt - c_i(t)\, dt + \pi_i(t)^\top \left[(\mu(t) - r(t)\mathbf{1})dt + \sigma(t)\, dW(t)\right] \\
& + q_i(t)\, m_i(t) \left[(\mu_{q_i}(t) - r(t))\, dt + \sigma_{q_i}(t)^\top\, dW(t)\right] \\
& + q_i(t)\, M_i(t) \left[(\mu_{M_i}(t) + \sigma_{q_i}(t)^\top \sigma_{M_i}(t))\, dt + \sigma_{M_i}(t)^\top\, dW(t)\right], \quad t \in [0, T],
\end{aligned}
$$

and obeys the no bankruptcy condition $X_i(T) \geq 0$. Equation (2.14) is the standard wealth evolution equation modified to incorporate real money holdings by accounting for the capital gains or losses over the riskless short rate from holding money (second line) and the change in wealth due to real money transfers (third line).[9]

Preferences for country i over consumption and real money balances in currency i are represented by a time-additive utility function $E\left[\int_0^T u_i(c_i(t), q_i(t)m_i(t))dt\right]$, where $u_i : [0,\infty) \times [0,\infty) \to \Re \cup \{-\infty\}$ is increasing, strictly concave, and three times continuously differentiable in both arguments. We define the gradient as $Du_i(c_i, q_i m_i) \equiv \left(\frac{\partial u_i(c, q_i m_i, t)}{\partial c_i}, \frac{\partial u_i(c_i, q_i m_i)}{\partial q_i m_i}\right)$, which has an inverse function $J_i(\cdot,\cdot) : (0,\infty)^2 \to (0,\infty)^2$. The standard Inada conditions are imposed on $J_i(h_1, h_2)$: for all h_1, h_2, $\lim_{\lambda\uparrow\infty} ||J_i(\lambda h_1, \lambda h_2)|| = 0$ and $\lim_{\lambda\downarrow 0} ||J_i(\lambda h_1, \lambda h_2)|| = \infty$. Country i's dynamic optimization problem is to maximize $E\left[\int_0^T u_i(c_i(t), q_i(t)m_i(t))dt\right]$ over all admissible policies (c_i, m_i, π_i) for which the expected integral is well-defined. Each country only has preferences over its own currency.[10]

Each country's dynamic consumption-portfolio optimization problem can be converted into a static variational problem [Cox and Huang (1989), Karatzas, Lehoczky, and Shreve (1987)], which in our context is summarized by Proposition 2.1, incorporating money holdings and transfers.

Proposition 2.1 *The dynamic optimization problem for each country $i \in \{D, F\}$ is equivalent to the static variational problem*

$$
(2.15) \qquad \max_{(c_i, m_i)} E\left[\int_0^T u_i(c_i(t), q_i(t)\, m_i(t))dt\right]
$$

subject to

$$
(2.16) \quad E\left[\int_0^T \xi(t)\, (c_i(t) + R_i(t) q_i(t) m_i(t))\, dt\right] \leq \xi(0)\epsilon_i S(0) + E\left[\int_0^T \xi(t) R_i(t) q_i(t) M_i(t) dt\right].
$$

[9] Capital gains or losses from holding money are $m_i(t)dq_i(t)$. Changes in wealth due to real money transfers are $q_i(t)dM_i(t) + d[q_i, M_i](t)$. The latter covariation term results from money being endowed to each agent in nominal terms, which is then deflated by $q_i(t)$.

[10] Our analysis readily extends to countries deriving utility from money balances in both currencies, $u_i(c_i, q_D m_{iD}, q_F m_{iF})$, $i \in \{D, F\}$, where m_{ij} denotes country i's holding of currency $j \in \{D, F\}$. Only the obvious modifications to market clearing, the definition of equilibrium, and the representative agent must be made.

The static budget constraint (2.16) states that each country's expected present value of lifetime consumption and real money holdings must be less than or equal to the present value of that country's lifetime endowments, the initial stock endowment plus money transfers. Proposition 2.2 characterizes the optimal solution $(\hat{c}_i, \hat{m}_i, \hat{\pi}_i)$.

Proposition 2.2 *The necessary and sufficient conditions for optimality are*

$$u_{ic}(\hat{c}_i(t),\, q_i(t)\, \hat{m}_i(t)) = y_i\, \xi(t), \tag{2.17}$$

$$u_{im}(\hat{c}_i(t),\, q_i(t)\, \hat{m}_i(t)) = y_i\, \xi(t)\, R_i(t), \tag{2.18}$$

where $u_{ic}(c, qm, t) \equiv \frac{\partial u_i(c,qm,t)}{\partial c}$, $u_{im}(c, qm, t) \equiv \frac{\partial u_i(c,qm,t)}{\partial qm}$, *and* y_i *is the unique nonnegative number solving*

$$E\left[\int_0^T \xi(t)\,(1, R_i(t))\, J_i(y_i\xi(t), y_i\xi(t)R_i(t))dt\right] = \xi(0)\epsilon_i S(0) + E\left[\int_0^T \xi(t)R_i(t)q_i(t)M_i(t)dt\right]. \tag{2.19}$$

Moreover, the optimal financial wealth $\hat{X}_i$ *is given by*

$$\hat{X}_i(t) = \frac{1}{\xi(t)}E\left[\int_t^T \xi(s)\hat{c}_i(s)ds|\mathcal{F}_t\right] + \frac{1}{\xi(t)}E\left[\int_t^T \xi(s)R_i(s)q_i(s)(\hat{m}_i(s) - M_i(s))ds|\mathcal{F}_t\right]. \tag{2.20}$$

Country i's marginal utilities of consumption and real money balances are proportional to the cost of consumption ξ and to the real cost of holding currency i, $\xi\, R_i$. Country i's optimal time t wealth equals the expected present value of future consumption plus net real money holdings. Moreover, (2.17) – (2.18) imply that the nominal short-rate is

$$R_i(t) = \frac{u_{im}(\hat{c}_i(t),\, q_i(t)\, \hat{m}_i(t))}{u_{ic}(\hat{c}_i(t),\, q_i(t)\, \hat{m}_i(t))}. \tag{2.21}$$

R_i, in addition to its interpretation as a measure of the liquidity services provided by money, can be interpreted as the relative price between real money balances and the commodity.

3 Equilibrium and Characterization

We characterize financial security prices and the price of each money by appealing to general equilibrium restrictions. Definition 3.1 enforces clearing in the commodity market, the money markets, the nominal bond markets, the stock market, and the real bond market, respectively.

Definition 3.1 *Given preferences and endowments* $(u_D(\cdot), u_F(\cdot), \epsilon_D, \epsilon_F)$*, an equilibrium is an allocation* $((\hat{c}_D, \hat{m}_D, \hat{\pi}_D), (\hat{c}_F, \hat{m}_F, \hat{\pi}_F))$ *and a price system* $\mathcal{P}$ *such that* $(\hat{c}_i, \hat{m}_i, \hat{\pi}_i)$ *is an optimal solution to country i's optimization problem for* $i \in \{D, F\}$ *and markets clear for* $t \in [0, T]$*:*

$$\hat{c}_D(t) + \hat{c}_F(t) = \delta(t), \tag{3.1}$$

$$\hat{m}_i(t) = M_i(t), \quad i \in \{D, F\}, \tag{3.2}$$

$$\hat{\pi}_{Di}(t) + \hat{\pi}_{Fi}(t) = 0, \quad i \in \{D, F\}, \tag{3.3}$$

$$\hat{\pi}_{DS}(t) + \hat{\pi}_{FS}(t) = S(t), \tag{3.4}$$

$$\hat{X}_D(t) + \hat{X}_F(t) = S(t) + q_D(t)\, M_D(t) + q_F(t)\, M_F(t). \tag{3.5}$$

For analytical convenience, we introduce a representative agent (e.g., Huang (1987)), whose preferences over consumption and both real money balances are given by

$$U(c, q_D M_D, q_F M_F; \Lambda) \equiv \max_{\substack{c_D, c_F \\ m_D, m_F}} \lambda_D u_D(c_D, q_D m_D) + \lambda_F u_F(c_F, q_F m_F) \tag{3.6}$$

subject to $c_D + c_F = c$, $m_D = M_D$, $m_F = M_F$, and $c_i \geq 0$, $m_i \geq 0$ $i \in \{D, F\}$ where $\Lambda \equiv (\lambda_D, \lambda_F) \in (0, \infty)^2$. We define the gradient as

$$DU(c, q_D M_D, q_F M_F; \Lambda) \equiv \begin{pmatrix} \frac{\partial U(c, q_D M_D, q_F M_F; \Lambda)}{\partial c} \\ \frac{\partial U(c, q_D M_D, q_F M_F; \Lambda)}{\partial q_D M_D} \\ \frac{\partial U(c, q_D M_D, q_F M_F; \Lambda)}{\partial q_F M_F} \end{pmatrix}^{\top}.$$

$U(\cdot, \cdot, \cdot; \Lambda)$ inherits all the properties of an individual's utility function [Lakner (1989)], and the inverse of $DU(\cdot, \cdot, \cdot; \Lambda)$ is

$$J(h_1, h_2, h_3; \Lambda) \equiv \begin{pmatrix} J_D^1(h_1/\lambda_D, h_2/\lambda_D) + J_F^1(h_1/\lambda_F, h_3/\lambda_F) \\ J_D^2(h_1/\lambda_D, h_2/\lambda_D) \\ J_F^2(h_1/\lambda_F, h_3/\lambda_F) \end{pmatrix}$$

where J_i^j is the jth element of the mapping J_i.

From the MIUF formulation and complete markets assumption, the representative agent's weights are identified as $\lambda_D = 1/y_D$ and $\lambda_F = 1/y_F$ to match each country's equilibrium allocation. Countries' individual weights remain constant even in the presence of money because the ratio of the weights equals the ratio of countries' marginal utilities with respect to any good, which by (2.17) is a constant. This yields the equilibrium conditions in Proposition 3.1.

Proposition 3.1 *If an equilibrium exists, the equilibrium state price density ξ, the nominal short-rates R_i, and the currency prices q_i satisfy*

$$\xi(t) = U_c(t; \Lambda), \tag{3.7}$$

$$R_i(t) = \frac{U_{M_i}(t; \Lambda)}{U_c(t; \Lambda)}, \quad i \in \{D, F\}, \tag{3.8}$$

$$q_i(t) = \frac{1}{U_c(t; \Lambda)} E\left[\int_t^T q_i(s)\, U_{M_i}(s; \Lambda) ds | \mathcal{F}_t\right], \quad i \in \{D, F\}, \tag{3.9}$$

where $\Lambda = (1/y_D, 1/y_F)$ satisfies

$$\begin{aligned} & E\left[\int_0^T (U_c(t, \Lambda), U_{M_i}(t, \Lambda))\, J_i(\frac{U_c(t, \Lambda)}{\lambda_i}, \frac{U_{M_i}(t, \Lambda)}{\lambda_i}) dt\right] \\ &= \epsilon_i E\left[\int_0^T U_c(t, \Lambda) \delta(t) dt\right] + E\left[\int_0^T U_{M_i}(t, \Lambda)\, q_i(t, \Lambda) M_i(t) dt\right], \quad i \in \{D, F\}, \end{aligned} \tag{3.10}$$

and the derivatives of $U(t;\Lambda)$ *are shorthand for the derivatives of* $U(\delta(t), q_D(t)M_D(t), q_F(t)M_F(t);\Lambda)$*. Conversely, if there exists* ξ*,* R_i*,* q_i*, and* Λ *satisfying (3.7)-(3.10), then the equilibrium conditions are satisfied by the associated optimal policies. Consequently, the equilibrium nominal exchange rate is given by*

$$e(t) = \frac{E\left[\int_t^T q_D(s)\, U_{M_D}(s;\Lambda)ds|\mathcal{F}_t\right]}{E\left[\int_t^T q_F(s)\, U_{M_F}(s;\Lambda)ds|\mathcal{F}_t\right]}. \tag{3.11}$$

The existence of equilibrium is established by showing that there exists a vector of equilibrium weights Λ such that (3.10) is satisfied and that all equilibrium quantities satisfy the earlier assumptions. Compared with a standard heterogeneous-agent single or multiple consumption good model [Karatzas, Lehoczky, and Shreve (1990), Wang (1996), Lakner (1989)], computation of the equilibrium weights additionally requires computation of each currency price in terms of the weights.

Via the "world" representative agent construction, Proposition 3.1 provides a general characterization for the quantities of interest even in the presence of countries with heterogeneous preferences (and endowments). The existing literature [Stulz (1984, 1987), Collin Dufresne and Shi (1996), Bakshi and Chen (1997)] did not take advantage of a representative agent and so was restricted to logarithmic preferences. The other main respect in which Proposition 3.1 differs from the existing literature is in the price of money representation (3.9). Other authors either failed to recognize (3.9) [Stulz] and so could not close the model, or pinned down the price of money essentially exogenously [Bakshi and Chen via a binding cash constraint, and Collin Dufresne and Shi by pure assumption]. One paper that does employ an analogue of (3.9) is Bakshi and Chen (1996) in a single-country, single-agent world; our model reduces to theirs when $e_F = 0$.

Proposition 3.2 characterizes the real quantities in the economy.

Proposition 3.2 *Assuming an equilibrium exists, the market price of risk, the real short-rate, and the stock price are given by*

$$\theta(t) = A_c(t)\delta(t)\sigma_\delta(t) + \sum_{i\in\{D,F\}} A_{M_i}(t)q_i(t)M_i(t)\left(\sigma_{M_i}(t) + \sigma_{q_i}(t)\right), \tag{3.12}$$

$$\begin{aligned} r(t) = {} & A_c(t)\delta(t)\mu_\delta(t) + \frac{1}{2}B_{cc}(t)\delta(t)^2\sigma_\delta(t)^\top\sigma_\delta(t) \\ & + \sum_{i\in\{D,F\}} A_{M_i}(t)q_i(t)M_i(t)\left(\mu_{q_i}(t) + \sigma_{q_i}(t)^\top\sigma_{M_i}(t) + \mu_{M_i}(t)\right) \\ & + \sum_{i\in\{D,F\}} B_{cM_i}(t)\delta(t)q_i(t)M_i(t)\sigma_\delta(t)^\top\left(\sigma_{q_i}(t) + \sigma_{M_i}(t)\right) \\ & + \sum_{i\in\{D,F\}}\sum_{j\in\{D,F\}} B_{M_iM_j}(t)q_i(t)M_i(t)q_j(t)M_j(t)\left\{\frac{1}{2}\sigma_{q_i}(t)^\top\sigma_{q_j}(t)\right. \\ & \left. + \sigma_{q_i}(t)^\top\sigma_{M_j}(t) + \frac{1}{2}\sigma_{M_i}(t)^\top\sigma_{M_j}(t)\right\}, \end{aligned} \tag{3.13}$$

$$(3.14) \qquad S(t) \quad = \quad \frac{1}{U_c(t;\Lambda)} E\left[\int_t^T U_c(s;\Lambda)\delta(s)ds|\mathcal{F}_t\right],$$

where $A_k(t) \equiv -U_{ck}(t;\Lambda)/U_c(t;\Lambda)$ *and* $B_{kl}(t) \equiv -U_{ckl}(t;\Lambda)/U_c(t;\Lambda)$.

By rearranging (3.12), we arrive at a version of the CCAPM for this economy,

$$\mu(t) - r(t)\mathbf{1} = A_c(t)\text{cov}\left(\frac{dP(t)}{P(t)}, d\delta(t)\right)$$
$$+A_{M_D}(t)\text{cov}\left(\frac{dP(t)}{P(t)}, dq_D(t)M_D(t)\right) + A_{M_F}(t)\text{cov}\left(\frac{dP(t)}{P(t)}, dq_F(t)M_F(t)\right)$$

where P is the vector of risky security prices $P \equiv (B_D, B_F, S)$. Hence, in addition to a security's risk premium being explained by the instantaneous covariance of its return with the aggregate commodity endowment [Breeden (1979), Duffie and Zame (1989)], it is also explained by the instantaneous covariance of its return with real money balances in each currency. Since the signs of $A_{M_D}(t)$ and $A_{M_F}(t)$ are not restricted, it might be possible to construct examples where an asset's risk premium is negative while its instantaneous covariance with the aggregate commodity is positive, causing the typical implication of the CCAPM to fail.

Equation (3.13) retains the standard interpretation that the real short-rate equals the negative of the instantaneous expected rate of change in the representative agent's marginal utility of consumption [Breeden (1979), Cox, Ingersoll, and Ross (1985)]. However, the expected marginal utility of consumption may now depend also on real money balances, yielding the last three terms in (3.13). These additional terms have an ambiguous effect on the size of the real short-rate. The stock price (3.14) retains the form of a benchmark economy with no money since the MIUF formulation preserves the equilibrium condition of state prices being proportional to the representative agent's marginal utility of consumption. In Stulz (1984, 1987) and Collin Dufresne and Shi (1996), a CCAPM follows directly from an agent's dynamic programming problem, but does not contain the additional terms since a non-separable MIUF formulation is not analyzed.

The expressions for the real economic quantities in Proposition 3.2 are not complete characterizations since each quantity is still a function of the endogenous currency prices. To determine the role of money in asset pricing requires a solution of (3.9) in terms of the primitives of the economy. In general, explicit calculation of the price of money q_i is rather difficult. By specializing the commodity process δ and the money supply processes M_i to be jointly Markovian; however, the price of each money can be characterized as the solution to a system of nonlinear partial differential equations (PDEs) given by the following new result.

Proposition 3.3 *Assume the stochastic processes* δ, M_D, *and* M_F *are jointly Markovian. Let* $Q_D(\delta, M_D, M_F, t) \equiv q_D(t)$, $Q_F(\delta, M_D, M_F, t) \equiv q_F(t)$ *be given by (3.9) and assume* Q_D, Q_F *are continuously differentiable with respect to* t *and twice continuously differentiable with respect to* δ,

M_D, M_F. Q_D and Q_F then solve the bivariate system of partial differential equations

$$\left(\mathcal{L}+\frac{\partial}{\partial t}\right) Q_D U_c(\delta, Q_D M_D, Q_F M_F) + Q_D U_{M_D}(\delta, Q_D M_D, Q_F M_F) = 0 \tag{3.15}$$

$$\left(\mathcal{L}+\frac{\partial}{\partial t}\right) Q_F U_c(\delta, Q_D M_D, Q_F M_F) + Q_F U_{M_F}(\delta, Q_D M_D, Q_F M_F) = 0 \tag{3.16}$$

subject to the boundary conditions $Q_D(\delta, M_D, M_F, T) = 0$ and $Q_F(\delta, M_D, M_F, T) = 0$, where $\mathcal{L}$ denotes the differential generator of (δ, M_D, M_F).

Proposition 3.3 will be useful for deriving each currency price in the subsequent examples.

Remark 3.1 (Price of Money Indeterminacy) Sections 2 and 3 have relied on the assumption of zero currency prices at time T, $q_i(T) = 0$. In fact each terminal currency price is indeterminate since no-arbitrage requires $q_i(T) \geq 0$ and the market clearing conditions provide no additional restrictions thereon. This no-arbitrage restriction contrasts with the results of Ōhashi (1991) which show that when the common information structure in the economy is continuous, the terminal value of a dividend paying stock must be zero to preclude arbitrage. In our model, the terminal value of money can be positive since agents will not short-sell "overpriced" money, due to the transaction services provided. We have chosen to focus on $q_i(T) = 0$ for consistency with the economic notion that money should have no value once its use as a transaction facilitator is complete. This restriction is similar to employing a transversality condition on the price of money in an infinite horizon model [Bakshi and Chen (1996)]. If we do not impose this restriction, equations (2.12) and (3.9) are modified to

$$q_i(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s) q_i(s) R_i(s) ds | \mathcal{F}_t\right] + \frac{1}{\xi(t)} E\left[\xi(T) q_i(T) | \mathcal{F}_t\right] \tag{3.17}$$

$$= \frac{1}{U_c(t;\Lambda)} E\left[\int_t^T q_i(s)\, U_{M_i}(s;\Lambda) ds | \mathcal{F}_t\right] + \frac{1}{U_c(t;\Lambda)} E\left[U_c(T;\Lambda) q_i(T) | \mathcal{F}_t\right]. \tag{3.18}$$

Without further restrictions on $q_i(T)$, a family of solutions exists for (3.17) and (3.18).

4 The Case of Separable Power Preferences

The two assumptions of the economy of this section are formally stated below. Assumption 4.1 plays a critical role in obtaining an explicit representation of the currency prices, while Assumption 4.2 represents a generalization from separable logarithmic preferences.

Assumption 4.1 *The commodity δ and each nominal money supply M_i are geometric Brownian motions. Hence, the drift and diffusion coefficients given in (2.1) and (2.2) are constants.*

Assumption 4.2 *Each country's agent has preferences of the form*

$$u_i(c_i, q_i m_i) = \beta_i \frac{{c_i}^{\gamma}}{\gamma} + (1-\beta_i)\frac{(q_i m_i)^{\alpha_i}}{\alpha_i}, \quad i \in \{D, F\}, \tag{4.1}$$

where $\gamma < 1$, $\alpha_i < 1$, *and* $\beta_i \in (0,1)$.

The utility function (4.1) is not homothetic and has a variable elasticity of substitution between consumption and real money balances. Due to the separability, we do not expect the money supply processes to affect real quantities. In this way we may focus on the nominal quantities which are new to this model. We allow for heterogeneity in tastes through each country tilting differently between the commodity and real money balances ($\beta_D \neq \beta_F$) as well as agents having differing transaction services provided by real money balances ($\alpha_D \neq \alpha_F$). In our analysis and comparative statics, we emphasize the additional features beyond those for the customary logarithmic preferences.

Proposition 4.1 presents the solution to each country's static budget constraint under equilibrium (3.10), explicitly showing the existence of an equilibrium. Equation (4.2) reveals that a country's weight is purely driven by its stock endowment and attitudes toward commodity risk normalized by the consumption preference parameter β_i.

Proposition 4.1 *An equilibrium exists with the equilibrium weights given by*

$$\lambda_i = \frac{\epsilon_i^{1-\gamma}}{\beta_i}, \quad i \in \{D, F\}. \tag{4.2}$$

Proposition 4.2 summarizes the price of each money by appealing to the PDEs of Proposition 3.3. The consequent nominal short-rates and exchange rate are also reported.

Proposition 4.2 *The price of each money* $i \in \{D, F\}$ *is*

$$q_i(t) = \left(\frac{(1-\beta_i) g(b_i, \alpha_i, t)}{\beta_i}\right)^{\frac{1}{1-\alpha_i}} \frac{(\epsilon_i \delta(t))^{\frac{1-\gamma}{1-\alpha_i}}}{M_i(t)}, \tag{4.3}$$

where

$$g(b_i, \alpha_i, t) \equiv \left(\frac{\exp[b_i(1-\alpha_i)(T-t)] - 1}{b_i}\right), \tag{4.4}$$

$$b_i \equiv -\left(\mu_{M_i} - {\sigma_{M_i}}^{\top} \sigma_{M_i}\right) + \alpha_i \left(\frac{1-\gamma}{1-\alpha_i}\right)\left[\mu_\delta + \frac{1}{2}\left(\frac{1-(2-\gamma)\alpha_i}{1-\alpha_i}\right) - \sigma_\delta^{\top} \sigma_{M_i}\right]. \tag{4.5}$$

The nominal short-rate for each currency $i \in \{D, F\}$ *is given by*

$$R_i(t) = \frac{b_i}{\exp\left[b_i(1-\alpha_i)(T-t)\right] - 1}, \tag{4.6}$$

and the nominal exchange rate by

$$e(t) = \frac{\left(\frac{(1-\beta_D) g(b_D, \alpha_D, t)}{\beta_D}\right)^{\frac{1}{1-\alpha_D}} \epsilon_D^{\frac{1-\gamma}{1-\alpha_D}} M_F(t)}{\left(\frac{(1-\beta_F) g(b_F, \alpha_F, t)}{\beta_F}\right)^{\frac{1}{1-\alpha_F}} \epsilon_F^{\frac{1-\gamma}{1-\alpha_F}} M_D(t)} \delta(t)^{\frac{(1-\gamma)(\alpha_D - \alpha_F)}{(1-\alpha_D)(1-\alpha_F)}}. \tag{4.7}$$

Consequently, comparative statics for q_D, R_D, *and* e *are as follows* (q_F, R_F *are symmetric*)*:*

	$q_D(t)$	$R_D(t)$	$e(t)$
$\partial/\partial\mu_{M_D}$	$-ve$	$+ve$	$-ve$
$\partial/\partial\mu_{M_F}$	0	0	$+ve$
$\partial/\partial\sigma_{M_D}{}^\top\sigma_{M_D}$	$+ve$	$-ve$	$+ve$
$\partial/\partial\sigma_{M_F}{}^\top\sigma_{M_F}$	0	0	$-ve$
$\partial/\partial\mu_\delta$	*-ve iff* $\alpha_i < 0$ *0 iff* $\alpha_i = 0$	*+ve iff* $\alpha_i < 0$ *0 iff* $\alpha_i = 0$	*0 if* $\alpha_D = \alpha_F = 0$ *Ambiguous otherwise*
$\partial/\partial\sigma_\delta{}^\top\sigma_\delta$	*+ve iff* $\alpha_i < 0$ *0 iff* $\alpha_i = 0$	*+ve iff* $\alpha_i \in \left(0, \frac{1}{2-\gamma}\right)$	*0 if* $\alpha_D = \alpha_F = 0$ *Ambiguous otherwise*
$\partial/\partial\sigma_\delta{}^\top\sigma_{M_D}$	*+ve iff* $\alpha_D < 0$ *0 iff* $\alpha_D = 0$	*-ve iff* $\alpha_D < 0$ *0 iff* $\alpha_D = 0$	*+ve iff* $\alpha_D < 0$ *0 iff* $\alpha_D = 0$
$\partial/\partial\sigma_\delta{}^\top\sigma_{M_F}$	0	0	*-ve iff* $\alpha_D < 0$ *0 iff* $\alpha_F = 0$

Equation (4.3) presents the currency prices as a function of exogenous variables. As is familiar, each currency price, q_i, is increasing in the current level of the commodity supply and decreasing in the current money supply i. By allowing differing exponents in the power preferences over commodity versus money balances, the currency price may exhibit a differing sensitivity to the commodity supply versus the money supply. This induces a stochastic velocity of money, $\nu_i \equiv \delta/(q_i M_i)$, as in Svensson (1985):

$$\nu_i(t) = \left(\frac{(1-\beta_i)g(b_i,\alpha_i,t)}{\beta_i}\right)^{\frac{1}{1-\alpha_i}} \frac{\delta(t)^{\frac{\gamma-\alpha_i}{1-\alpha_i}}}{\epsilon_i^{\frac{1-\gamma}{1-\alpha_i}}}, \quad i \in \{D, F\}.$$

Due to the separability of the representative agent's preferences over the two currencies, the other country's currency supply or dynamics do not influence the currency i price. Equation (4.3) shows the price level to be unambiguously increasing in agent i's endowment. As agent i becomes wealthier, he desires to hold more real money balances, but since the level of the money supply has not changed, the currency price must increase for the money market to clear. As country i tilts preferences away from holding real balances (as β_i increases), the equilibrium price of currency i must fall to prevent a reduction in i's money holdings. When preferences over real money balances deviate from logarithmic ($\alpha_i \neq 0$), the currency price is also driven by the commodity supply dynamics. The assumed separable preferences and the segmentation of the currency markets lead the currency i price to be unaffected by the other country's endowments or tastes, and lead to these unambiguous dependencies on country i's characteristics.

From (4.6), the nominal short-rate for each currency is driven by the currency dynamics, and when preferences over real balances deviate from logarithmic, by the commodity growth dynamics. Examining the extended Fisher equation (2.9), the latter influence depends on the relative effect of the commodity dynamics on the real interest rate, expected inflation, and the risk premium on nominal bonds. For example, an increase in μ_δ increases the real short-rate and decreases expected inflation. When real money balance preferences are logarithmic, these two opposing effects exactly offset each other. When $\alpha_i < 0$, the increase in the real short-rate dominates, leading to an increase in R_i.

From (4.7), the exchange rate is decreasing in the domestic money supply (which drives the domestic currency price down) and increasing in the foreign money supply. If both countries have the same exponent in their preferences for money balances ($\alpha_D = \alpha_F$), a change in the commodity supply affects both currency prices proportionately and so has no effect on the exchange rate. This breaks down when the two countries differ in their money balance preferences. If $\alpha_D < \alpha_F$, an increase in the commodity supply causes a smaller increase in the domestic currency and hence the exchange rate drops. As a consequence of the currency price dependencies, the exchange rate is increasing in the domestic endowment and decreasing as the domestic country tilts preferences away from holding real balances (as β_D increases). The dependencies on the foreign endowment and preferences are the reverse. When both countries have logarithmic preferences over real money balances, the exchange rate shows no dependence on the dynamics of the commodity supply; any deviation from logarithmic preferences breaks this result down.

Corollary 4.1 *The mean growth and volatility of the currency and exchange rate are*

$$\begin{aligned}
(4.8)\quad \mu_{q_i}(t) &= \frac{b_i \exp[b_i(1-\alpha_i)(T-t)]}{1-\exp[b_i(1-\alpha_i)(T-t)]} - \mu_{M_i} + \sigma_{M_i}{}^\top \sigma_{M_i} \\
&\quad + \frac{1-\gamma}{1-\alpha_i}\left(\mu_\delta - \sigma_\delta{}^\top \sigma_{M_i}\right) + \frac{1}{2}\left(\frac{1-\gamma}{1-\alpha_i}\right)\left(\frac{\alpha_i-\gamma}{1-\alpha_i}\right)\sigma_\delta{}^\top \sigma_\delta, \\
(4.9)\quad \sigma_{q_i}(t) &= \frac{1-\gamma}{1-\alpha_i}\sigma_\delta - \sigma_{M_i}, \\
(4.10)\quad \mu_e(t) &= R_F(t)\exp[b_F(1-\alpha_F)(T-t)] - R_D(t)\exp[b_D(1-\alpha_D)(T-t)] \\
&\quad + \mu_{M_F} - \mu_{M_D} + \sigma_{M_D}{}^\top \sigma_{M_D} - \sigma_{M_D}{}^\top \sigma_{M_F} \\
&\quad + (1-\gamma)\left(\frac{\alpha_D-\alpha_F}{(1-\alpha_D)(1-\alpha_F)}\right)\left(\mu_\delta + \sigma_\delta{}^\top(\sigma_{M_F} - \sigma_{M_D})\right) \\
&\quad + \frac{1}{2}(1-\gamma)\left(\frac{\alpha_D-\alpha_F}{(1-\alpha_D)(1-\alpha_F)}\right)\left((1-\gamma)\left(\frac{\alpha_D-\alpha_F}{(1-\alpha_D)(1-\alpha_F)}\right)-1\right)\sigma_\delta{}^\top\sigma_\delta, \\
(4.11)\quad \sigma_e(t) &= (1-\gamma)\left(\frac{\alpha_D-\alpha_F}{(1-\alpha_D)(1-\alpha_F)}\right)\sigma_\delta - \sigma_{M_D} + \sigma_{M_F},
\end{aligned}$$

where b_i and R_i are as in Proposition 4.2.

The currency i price risk $\sigma_{q_i}{}^\top \sigma_{q_i}$ is increasing in the variability of both the money supply and the aggregate commodity supply, but may be more or less sensitive to these variabilities unless $\gamma = \alpha_i$.

Equation (4.9) also implies that the currency price risk is decreasing in ${\sigma_\delta}^\top \sigma_{M_i}$; if the money and the commodity supply covary strongly, the currency price remains smoother. The expected inflation rate, $-\mu_{q_i}$, is decreasing in the expected rate of growth of the commodity supply and increasing in the expected rate of growth of the money supply. An increase in the instantaneous covariance between the commodity and currency i (${\sigma_\delta}^\top \sigma_{M_i}$) implies a higher future expected growth rate of currency i relative to the commodity, leading to an increase in the expected inflation rate. Recalling that the expected growth rate in the nominal exchange rate is $\mu_e = \mu_{q_D} - \mu_{q_F} + {\sigma_{q_F}}^\top \sigma_{q_F} - {\sigma_{q_D}}^\top \sigma_{q_F}$, the dynamics of each money supply affect μ_e through their effects on expected inflation in each currency, the foreign currency price risk, and the covariance between the two currencies. For example, an increase in μ_{M_F} leads to in increase in foreign inflation causing μ_e to increase. Finally, equation (4.11) implies that the exchange rate risk $\sigma_e^\top \sigma_e$ is increasing in the variability of both money supplies, but decreasing in the covariance between them. Unless both countries have the same exponent in their preferences for money balances, the exchange rate risk is increasing in the commodity supply risk.

Proposition 4.3 summarizes the real quantities in the economy.

Proposition 4.3 *The equilibrium real state price density is given by*

$$\xi(t) = \delta(t)^{\gamma - 1}. \tag{4.12}$$

The market price of risk and the real short-rate are constants given by

$$\theta(t) = (1-\gamma)\sigma_\delta, \tag{4.13}$$

$$r(t) = (1-\gamma)\mu_\delta - \frac{1}{2}(1-\gamma)(2-\gamma){\sigma_\delta}^\top \sigma_\delta. \tag{4.14}$$

Consequently, the stock price and its dynamics are

$$S(t) = \delta(t)\frac{\exp(\gamma(\mu_\delta - 1/2(1-\gamma){\sigma_\delta}^\top \sigma_\delta)(T-t)) - 1}{\gamma(\mu_\delta - 1/2(1-\gamma){\sigma_\delta}^\top \sigma_\delta)}, \tag{4.15}$$

$$\mu_S = \mu_\delta - \frac{\gamma(\mu_\delta - 1/2(1-\gamma){\sigma_\delta}^\top \sigma_\delta)\exp(\gamma(\mu_\delta - 1/2(1-\gamma){\sigma_\delta}^\top \sigma_\delta)(T-t))}{\exp(\gamma(\mu_\delta - 1/2(1-\gamma){\sigma_\delta}^\top \sigma_\delta)(T-t)) - 1}, \tag{4.16}$$

$$\sigma_S = \sigma_\delta. \tag{4.17}$$

The real quantities in the economy are identical to a benchmark economy with no money since the representative agent's marginal utility of consumption is not dependent on real money balances.

5 The Case of Non-Separable Cobb-Douglas Preferences

For this section, we retain Assumption 4.1, but replace Assumption 4.2 by Assumption 5.1. The Cobb-Douglas utility employed is non-separable and so allows for an impact of the money supplies on the real economic quantities. Even with homogeneous utility functions across countries, money introduces considerable complexity to the problem. We have not derived fully analytical results and need to employ numerical techniques to investigate some of the equilibrium behavior. Adding additional heterogeneity to the example would require a fully numerical analysis because complicated wealth effects in the representative agent utility function would prevent its explicit derivation.

Assumption 5.1 *Each country's agent has preferences of the form*

$$u_i(c_i, q_i m_i) = \frac{1}{\gamma}\left[c_i^{\beta}(q_i m_i)^{1-\beta}\right]^{\gamma}, \quad i \in \{D, F\}, \tag{5.1}$$

where $\gamma < 1$, $\beta \in (0,1)$.

The parameter $1-\gamma$ represents the relative risk aversion over the composite good $c_i^{\beta}(q_i m_i)^{1-\beta}$, with $\gamma = 0$ corresponding to (separable) logarithmic preferences.

An analogue of Proposition 4.1, and hence existence of equilibrium has not been established. Proposition 5.1 presents a quasi-analytical solution of the equilibrium currency prices and other nominal quantities. The stochastic part of the currency prices is simpler than (4.3), but they are also driven by a deterministic time-dependent term $h_i(t)$ requiring numerical solution.

Proposition 5.1 *The price of each money* $i \in \{D, F\}$ *is given by*

$$q_i(t) = \frac{1-\beta}{\beta} h_i(t) \frac{\delta(t)}{M_i(t)}, \tag{5.2}$$

where the deterministic functions $h_i(t) \equiv h_i(t; b_i, \lambda_D, \lambda_F, \gamma, \beta)$ *solve the following bivariate system of ordinary differential equations:*

$$(b_D h_D + h_D')H(h_D, h_F) + (1-\beta)\gamma h_D \mathcal{H}(h_D, h_F, h_D', h_F') + \lambda^{\frac{1}{1-\beta\gamma}} h_D^{\gamma(\beta-1)} = 0, \tag{5.3}$$

$$(b_F h_F + h_F')H(h_D, h_F) + (1-\beta)\gamma h_F \mathcal{H}(h_D, h_F, h_D', h_F') + \lambda^{\frac{1}{1-\beta\gamma}} h_F^{\gamma(\beta-1)} = 0, \tag{5.4}$$

subject to the boundary conditions $h_D(T) = 0$, $h_F(T) = 0$, *and where*

$$H(h_D, h_F) \equiv \lambda_D^{\frac{1}{1-\beta\gamma}} h_D^{\frac{(1-\beta)\gamma}{1-\beta\gamma}} + \lambda_F^{\frac{1}{1-\beta\gamma}} h_F^{\frac{(1-\beta)\gamma}{1-\beta\gamma}}, \tag{5.5}$$

$$\mathcal{H}(h_D, h_F, h_D', h_F') \equiv \lambda_D^{\frac{1}{1-\beta\gamma}} h_D^{\frac{1-\gamma}{\beta\gamma-1}} h_D' + \lambda_F^{\frac{1}{1-\beta\gamma}} h_F^{\frac{1-\beta}{\beta\gamma-1}} h_F', \tag{5.6}$$

$$b_i \equiv -\left(\mu_{M_i} - \sigma_{M_i}{}^{\top}\sigma_{M_i}\right) + \gamma\left(\mu_\delta - \frac{1}{2}(1-\gamma)\sigma_\delta^{\top}\sigma_\delta - \sigma_\delta^{\top}\sigma_{M_i}\right), \tag{5.7}$$

and (λ_D, λ_F) *are determined up to a multiplicative constant from one of the country's static budget constraints:*

$$\epsilon_D = \frac{\lambda_D^{\frac{1}{1-\beta\gamma}} \int_0^T \exp\left\{\gamma(\mu_\delta - \frac{1-\gamma}{2}\sigma_\delta^{\top}\sigma_\delta)\right\} h_D(t)^{\frac{(1-\gamma)\gamma}{1-\beta\gamma}} H(h_D(t), h_F(t))^{-\beta\gamma} dt}{\int_0^T \exp\left\{\gamma(\mu_\delta - \frac{1-\gamma}{2}\sigma_\delta^{\top}\sigma_\delta)\right\} H(h_D(t), h_F(t))^{1-\beta\gamma} dt}. \tag{5.8}$$

The nominal short-rate for each currency $i \in \{D, F\}$ is given by

$$R_i(t) = \frac{\lambda_i^{\frac{1}{1-\beta\gamma}} h_i(t)^{\frac{1-\gamma}{\beta\gamma-1}}}{H(h_D(t), h_F(t))}, \tag{5.9}$$

and the nominal exchange rate by

$$e(t) = \frac{h_D(t)}{h_F(t)} \frac{M_F(t)}{M_D(t)}. \tag{5.10}$$

As in Section 4, each currency price i is increasing in the current level of the commodity supply and decreasing in the current money supply i. With Cobb-Douglas utility, the currency exhibits equal sensitivity to the consumption and the money supplies; in this respect it more closely resembles the benchmark case of homogeneous separable logarithmic preferences (where the currency price is also given by the dividend-to-money supply ratio weighted by a deterministic quantity). For Cobb-Douglas preferences, the velocity of money is nonstochastic, as $\nu_i(t) = \beta/((1-\beta)h_i(t))$, $i \in \{D, F\}$.

However, unlike the separable case of Section 4, each currency price is now dependent on the other country's money supply dynamics. This dependence is captured in the quantity h_i (itself depending on the quantities b_D and b_F, which depend on μ_{M_D}, μ_{M_F}, σ_{M_D}, and σ_{M_F}). We refer to these effects as "cross-monetary effects," which also impact the home nominal short-rate. They arise because the non-separability of the utility function allows for either country's pricing of money and consumption to interact. So, while the two countries do not directly interact in the currency markets, there is an indirect interaction via the fully integrated commodity market. This utility also deviates from the logarithmic case by again exhibiting an impact of the consumption supply dynamics, μ_δ and σ_δ, on currency prices, nominal short-rates, and the exchange rate.

The nominal exchange rate (5.10) somewhat resembles the case of separable logarithmic preferences in that it exhibits no dependence on the current level of the consumption supply. However, unlike with logarithmic preferences, the exchange rate does exhibit a dependence on the growth and volatility of the consumption supply and on the covariance of the monetary and consumption supply growths. In the presence of non-myopic countries, currency prices are impacted by the future consumption supply.

To further investigate the details of the dependencies, either new to this case or also apparent in Section 4, we require numerical solution of the h_i terms and hence currency prices, nominal short-rates, and exchange rate. The system of ODEs in Proposition 5.1 was numerically integrated using the IMSL subroutine DIVPAG which employs an Adams-Moulton method. Some assumptions made to narrow down the parameter space were: symmetric endowment across countries ($\epsilon_i = 0.5$);[11] and negativity of the b_i terms (to match the empirical data).[12] The ranges or choice of other parameters

[11]This assumption was more simply incorporated by taking $\lambda_D = \lambda_F = 1$, which in all numerical trials indeed generated $\epsilon_i = 0.5$.

[12]Using annual U.S. Citibase data from 1952 to 1996 for consumption (nondurables plus services) and the money supply (M1) led to the following parameter estimates: $\mu_\delta = 0.0206$, $\sigma_\delta^\top \sigma_\delta = 0.000222$, $\mu_{M_i} = 0.0489$, and $\sigma_{M_i}^\top \sigma_{M_i} =$

were $\beta = 0.5$; $\gamma = -1.5, -1.0, -0.5, 0.25, 0.5, 0.75$; $T = 50$; $t = 0, 10, 20, 30, 40$. For brevity, the following table reports only the dependencies which were persistent over all trials. The quantities q_F and R_F behave symmetrically to q_D and R_D.

	$q_D(t)$	$R_D(t)$	$e(t)$
$\partial/\partial\mu_{M_D}$	$-ve$	$+ve$	*-ve for* $\gamma > 0$
$\partial/\partial\mu_{M_F}$	*+ve for* $\gamma > 0$ *-ve for* $\gamma < 0$	*+ve for* $\gamma > 0$	*+ve for* $\gamma > 0$
$\partial/\partial{\sigma_{M_D}}^\top\sigma_{M_D}$	$+ve$	$-ve$	*+ve for* $\gamma > 0$
$\partial/\partial{\sigma_{M_F}}^\top\sigma_{M_F}$	*-ve for* $\gamma > 0$ *+ve for* $\gamma < 0$	*-ve for* $\gamma > 0$	*-ve for* $\gamma > 0$
$\partial/\partial{\sigma_\delta}^\top\sigma_{M_D}$	*-ve for* $\gamma > 0$ *+ve for* $\gamma < 0$	*+ve for* $\gamma > 0$ *-ve for* $\gamma < 0$	*-ve for* $\gamma > 0$
$\partial/\partial{\sigma_\delta}^\top\sigma_{M_F}$	$+ve$	*+ve for* $\gamma > 0$	*+ve for* $\gamma > 0$

The dependencies of the currency prices and nominal short-rates on their own money supply dynamics are as in the separable case of Section 4. The cross-monetary effects on the domestic country arise via the $\left(\mu_{M_F} - {\sigma_{M_F}}^\top\sigma_{M_F}\right)$ term in b_F (5.7), so the mean growth and volatility of the foreign money supply will always act in opposition. This can be considered as an impact of the "risk-adjusted" (or "certainty equivalent") foreign money supply growth. The direction of the cross-monetary effects is driven by whether countries are more or less risk averse than logarithmic agents. If the countries are more risk averse than log ($\gamma < 0$), their consumption is relatively insensitive to price changes. In this case, a high (certainty equivalent) growth in the foreign money supply depresses the current value of money in both the foreign and domestic currencies. If the countries are less risk averse than log ($\gamma > 0$), on the other hand, a high growth in the foreign money supply increases both the domestic currency price and the domestic short rate. In all trials, currency prices are increasing in the covariance between the other country's money supply and the consumption supply. We can only make unambiguous statements about the dependency of the exchange rate when the foreign and domestic currency prices have opposing dependency; this is the case when $\gamma > 0$.

0.00125. Perturbations of these parameters as well as preference parameters over reasonable quantities implied $b_i \in (-0.1, 0)$.

Corollary 5.1 *The expected nominal inflation rates, the currency risk, and the exchange rate mean growth and volatility are given by*

$$\mu_{q_i}(t) = \mu_\delta - \mu_{M_i} + \sigma_{M_i}{}^\top \sigma_{M_i} + \frac{h_i'(t)}{h_i(t)}, \quad i \in \{D, F\}, \tag{5.11}$$

$$\sigma_{q_i}(t) = \sigma_\delta - \sigma_{M_i}, \quad i \in \{D, F\}, \tag{5.12}$$

$$\mu_e(t) = \left(\mu_{M_F} - \sigma_{M_F}{}^\top \sigma_{M_F}\right) - \left(\mu_{M_D} - \sigma_{M_D}{}^\top \sigma_{M_D}\right) + \left(\frac{h_D'(t)}{h_D(t)} - \frac{h_F'(t)}{h_F(t)}\right), \tag{5.13}$$

$$\sigma_e(t) = \sigma_{M_F} - \sigma_{M_D}, \tag{5.14}$$

where h_i is as defined in Proposition 5.1.

The behavior of the currency and exchange rate risks are identical to the case of separable (including logarithmic) preferences explained in Section 4. It is the mean growths in the currency prices and exchange rate that are impacted by the non-separability of preferences, via the deterministic quantities h_i. Cross-monetary effects appear in these growth terms.

Proposition 5.2 summarizes the real quantities in the economy.

Proposition 5.2 *The equilibrium real state price density is given by*

$$\xi(t) = \beta \left(\frac{1-\beta}{\beta}\right)^{(1-\beta)\gamma} H(t)\delta(t)^{\gamma-1}. \tag{5.15}$$

The market price of risk and the real short-rate are given by

$$\theta(t) = (1-\gamma)\sigma_\delta, \tag{5.16}$$

$$r(t) = (1-\gamma)\mu_\delta - \frac{1}{2}(1-\gamma)(2-\gamma)\sigma_\delta{}^\top \sigma_\delta - (1-\beta)\gamma \frac{H(t)}{\mathcal{H}(t)}. \tag{5.17}$$

Consequently, the stock price dynamics are given by

$$\mu_S(t) = (1-\gamma)\mu_\delta + \frac{1}{2}\gamma(1-\gamma)\sigma_\delta{}^\top \sigma_\delta - (1-\beta)\gamma \frac{H(t)}{\mathcal{H}(t)} \tag{5.18}$$

$$\sigma_S(t) = \sigma_\delta, \tag{5.19}$$

where $H(t) \equiv H(h_D(t), h_F(t))$ and $\mathcal{H}(t) \equiv \mathcal{H}(h_D(t), h_F(t), h_D'(t), h_F'(t))$ are as in Proposition 5.1.

A clear effect of the non-separability of preferences over Section 4 is that the inclusion of money now impacts the real quantities in the economy. The pricing of the real and nominal quantities now interact. As with the currency prices, it is the levels and mean growths of the real state price density and the stock price which are impacted (via the function h_i) by both money supplies, while the volatilities are not impacted. The monetary impact on real quantities is driven by whether countries are more or less risk averse than logarithmic preferences. According to the numerical trials, when countries are less risk averse than log ($\gamma > 0$) the additional term $(1-\beta)\gamma H/\mathcal{H}$ is negative; hence, the real interest rate is increased above its value in a benchmark non-monetary model. As a result, so is the mean rate of return of the stock, since the market price of risk is unaffected. When countries are more risk averse than log ($\gamma < 0$), the interest rate and mean stock returns are depressed relative to a non-monetary model.

6 Summary and Conclusion

We have examined a continuous-time two country dynamic monetary equilibrium. With mild assumptions on all exogenous distributions, we provide no-arbitrage pricing results for the price of each money and the nominal exchange rate, independent of preferences and the method of incorporating money in the economy. Assuming that all agents have time-additive preferences, we examine the consumption-money-portfolio problem of each agent and provide an equilibrium characterization of all endogenous quantities, including a representation of the price of money suitable for numerical analysis. We further investigate the effect of incorporating money into a standard international equilibrium model through examples which assume that all exogenous processes follow geometric Brownian motions. The main contribution of this paper is in providing a benchmark economy which incorporates market-determined prices of money and a nominal exchange rate without requiring strong assumptions on endowments and preferences. Appropriate modifications could extend to an arbitrary number of countries and to non-segmented money markets. While our model is limited to an exchange economy with exogenous money supplies, much of our analysis is applicable to a Cox, Ingersoll, and Ross (1985)-type production economy with linear constant-return-to-scale production technologies. Interesting extensions of our model include endogenizing the money supplies by taking government monetary policies as given and incorporating production as in Kollmann (1998).

Appendix: Proofs

Proof of Lemma 2.1: The stock price representation is well-established (e.g., Karatzas, Lehoczky, and Shreve (1990), Theorem 8.2). The price of money representation uses a similar argument. Using (2.3), (2.8), and (2.9), the deflated price of money i, $\xi(t)\, q_i(t)$, has dynamics

$$\xi(t)\, q_i(t) = \xi(0)\, q_i(0) - \int_0^t \xi(s)\, q_i(s)\, R_i(s)\, ds + \int_0^t \xi(s)\, q_i(s)\, \left(\sigma_{q_i}(s)^\top - \theta(s)^\top\right) dW(s).$$

Defining $G_i(t) \equiv \xi(t)\, q_i(t) + \int_0^t \xi(s)\, q_i(s)\, R_i(s)\, ds$, we arrive at

$$G_i(t) = \xi(0)\, q_i(0) + \int_0^t \xi(s)\, q_i(s)\, \left(\sigma_{q_i}(s)^\top - \theta(s)^\top\right) dW(s),$$

which is a martingale from the assumed boundedness and integrability conditions. Assuming $q_i(T) = 0$, yields (2.12). By the same method, a similar expression can be derived for the real value of money,

$$q_i(t)M_i(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s)q_i(s)M_i(s)\eta_i(s)ds|\mathcal{F}_t\right], \tag{A.1}$$

where $\eta_i(s) \equiv R_i(s) - \mu_{M_i} + \sigma_{M_i}(s)^\top(\theta(s) - \sigma_{q_i}(s))$. *Q.E.D.*

Proof of Proposition 2.1: Since the objective functions of the dynamic and the static optimization are equivalent, it suffices to show that both problems have the same budget sets.

Let (c_i, m_i, π_i) be an admissible policy. From (2.9) and (2.14) the discounted wealth process $\xi(t)\, X_i(t)$ follows

$$\begin{aligned}\xi(t)\, X_i(t) - \xi(0)\, X_i(0) &= -\int_0^t \xi(s)\, c_i(s)ds + \int_0^t \xi(t)\, \pi_i(s)^\top \sigma(s)\, dW(s) \\ &- \int_0^t \xi(s)\, m_i(s)\, q_i(s) R_i(s)\, ds + \int_0^t \xi(s)\, M_i(s)\, q_i(s)\left(\mu_{M_i}(s) + \sigma_{q_i}(s)^\top \sigma_{M_i}(s) - \sigma_{M_i}(s)^\top \theta(s)\right) ds \\ &+ \int_0^t \xi(s)\, q_i(s)\left\{m_i(s)\, \sigma_{q_i}(s)^\top + M_i(s)\sigma_{M_i}(s)^\top\right\} dW(s) - \int_0^t \xi(s)\, X(s)\, \theta(s)^\top dW(s)\end{aligned}$$

by an application of Itô's lemma. Evaluating at $t = T$ and rearranging gives

$$\begin{aligned}&\int_0^T \xi(s)\,(c_i(s) + q_i(s)\, R_i(s)\, m_i(s))\, ds + \xi(T)\, X_i(T) = \xi(0)\, X_i(0) \\ &\quad + \int_0^T \xi(s)\, M_i(s)\, q_i(s)\left(\mu_{M_i}(s) + \sigma_{q_i}(s)^\top \sigma_{M_i}(s) - \sigma_{M_i}(s)^\top \theta(s)\right) ds + \int_0^T \xi(s)\, \pi_i(s)^\top \sigma(s)\, dW(s) \\ &\quad + \int_0^T \xi(s)\, q_i(s)\left\{m_i(s)\, \sigma_{q_i}(s)^\top + M_i(s)\sigma_{M_i}(s)^\top\right\} dW(s) - \int_0^T \xi(s)\, X(s)\, \theta(s)^\top dW(s).\end{aligned}$$

Taking expectations, using the definition of admissible, recognizing that the stochastic integrals on the right-hand side are martingales, and substituting (A.1), we arrive at (2.16).

To show the converse, suppose that (2.16) holds with equality for (c_i, m_i). Define the martingale $L(t) = E\left[\int_0^T \xi(t)\,(c_i(t) + q_i(t)R_i(t)\,(m_i(t) - M_i(t)))\,|\mathcal{F}_t\right]$. By the martingale representation theorem, there exists a process ψ satisfying $E\left[\int_0^T ||\psi(t)||^2 dt\right] < \infty$ such that

$$L(t) = X_i(0) + \int_0^t \psi(s)^\top dW(s).$$

Define a proposed deflated gains process as $\xi(t)\,X_i(t) \equiv L(t) - J(t)$. By matching coefficients with (A.2), we arrive at a candidate trading strategy

$$\pi_i(t) = \frac{1}{\xi(t)}(\sigma(t)^\top)^{-1}\left[\psi(t) - q_i(t)\xi(t)\left(m_i(t)\sigma_{q_i}(t)^\top - M_i(t)\sigma_{M_i}(t)^\top\right) + (L(t) - J(t))\,\theta(t)\right]$$

which can be shown to finance (c_i, m_i) and satisfy the integrability conditions.[13] *Q.E.D.*

Proof of Proposition 2.2: Since the Slater condition is satisfied for the optimization problem (2.15) – (2.16), we can use Lagrangian theory to characterize its optimal solution. The first order conditions of the Lagrangian of (2.15) – (2.16) yield (2.17) – (2.19). The optimal wealth equation is obtained using standard arguments (e.g., Karatzas, Lehoczky, and Shreve (1987)). *Q.E.D.*

Proof of Proposition 3.1: Assume that the equilibrium conditions (3.1) – (3.5) are satisfied for ξ, R_i, and q_i where $i \in \{D, F\}$. From the equilibrium conditions,

$$\begin{pmatrix} \delta(t) \\ q_D(t)M_D(t) \\ q_F(t)M_F(t) \end{pmatrix} = \begin{pmatrix} \hat{c}_D(t) + \hat{c}_F(t) \\ q_D(t)\hat{m}_D(t) \\ q_F(t)\hat{m}_F(t) \end{pmatrix} = \begin{pmatrix} J_D^1(y_D\xi(t), y_D\xi(t)R_D(t)) + J_F^1(y_F\xi(t), y_F\xi(t)R_F(t)) \\ J_D^2(y_D\xi(t), y_D\xi(t)R_D(t)) \\ J_F^2(y_F\xi(t), y_F\xi(t)R_F(t)) \end{pmatrix}.$$

The right-most term equals $J(\xi(t), \xi(t)R_D(t), \xi(t)R_F(t); \Lambda)$, so inverting yields

$$\xi(t) = U_c(t;\Lambda), \quad R_D(t)\xi(t) = U_{M_D}(t;\Lambda), \quad R_F(t)\xi(t) = U_{M_F}(t;\Lambda).$$

Equations (3.7)-(3.9) follow. Equation (3.10) follows by substituting each country's optimal policy into the static budget constraint.

To prove the converse, assume that there exists ξ, R_i, and q_i where $i \in \{D, F\}$ satisfying (3.7) and (3.10). Clearing in the consumption good market (3.1) and the money markets (3.2) follows trivially from the solutions to the agents' maximization problems.

To show that the real bond market clears (3.5), sum the optimal wealth (2.20) across both agents and substitute (3.1) – (3.2), giving

$$\hat{X}_D(t) + \hat{X}_F(t) = \frac{1}{\xi(t)}E\left[\int_t^T \xi(s)\delta(s)ds|\mathcal{F}_t\right]$$
$$+\frac{1}{\xi(t)}E\left[\int_t^T \xi(s)\,q_D(s)\,M_D(s)\,\eta_D(s)ds|\mathcal{F}_t\right] + \frac{1}{\xi(t)}E\left[\int_t^T \xi(s)\,q_F(s)\,M_F(s)\,\eta_F(s)ds|\mathcal{F}_t\right],$$

where $\eta_D(s)$ and $\eta_F(s)$ are as in the proof of Lemma 2.1. From Lemma 2.1, the right-hand side equals $S(t) + q_D(t)\,M_D(t) + q_F(t)\,M_F(t)$, yielding (3.5).

To show market clearing in the stock market and the nominal bond market, consider summing across all agents the deflated wealth process given by (A.2) and substituting the market clearing conditions (3.1), (3.2), and (3.5), obtaining

$$d\,\xi(t)\;(S(t) + q_D(t)\,M_D(t) + q_F(t)\,M_F(t)) = \text{dt terms } + \xi(t)\,(\pi_D(t) + \pi_F(t))^\top\,\sigma(t)\,dW(t)$$
$$+\xi(t)\,q_D(t)\,M_D(t)\,(\sigma_{q_D}(t) + \sigma_{M_D}(t) - \theta(t))^\top\,dW(t) + \xi(t)\,q_F(t)\,M_F(t)\,(\sigma_{q_F}(t) + \sigma_{M_F}(t) - \theta(t))^\top\,dW(t)$$
$$-\xi(t)\,S(t)\theta(t)^\top\,dW(t).$$

[13] If money transfers were not proportional to the money supply, the proof can be modified. The term $E\left[\int_0^T \xi(t)R_i(t)q_i(t)M_i(t)\right]$ on the right-hand side of (2.16) is replaced by

$$\xi(0)\zeta(0)q_i(0)M_i(0) + E\left[\int_0^T \xi(t)\zeta(t)q_i(t)M_i(t)\left(\mu_{M_i}(t) + \sigma_{M_i}(t)^\top\sigma_{q_i}(t) - \sigma_{M_i}(t)^\top\theta(t)\right)dt\right]$$

where ζ is the proportion of the money transfer endowed to an agent.

From (2.2), (2.7), and (2.8), we compute

$$d\,\xi(t)\ \left(S(t)+q_D(t)\,M_D(t)+q_F(t)\,M_F(t)\right)=\text{dt terms }\ +\xi(t)\,S(t)\left(\sigma_S(t)^\top-\theta(t)^\top\right)\,dW(t)$$
$$+\xi(t)\,q_D(t)\,M_D(t)\left(\sigma_{q_D}(t)+\sigma_{M_D}(t)-\theta(t)\right)^\top dW(t)+\xi(t)\,q_F(t)\,M_F(t)\left(\sigma_{q_F}(t)+\sigma_{M_F}(t)-\theta(t)\right)^\top dW(t).$$

Matching diffusion coefficients in the above two representations, we arrive at $S(t)\sigma_S(t)^\top=(\pi_D(t)+\pi_F(t))^\top\,\sigma(t)$, a.s. Since markets are complete, $\sigma(t)$ is invertible. Hence, the unique solution to the above system of equations is given by (3.3) and (3.4). *Q.E.D.*

Proof of Proposition 3.2: Applying Itô's Lemma to (3.7) and matching the deterministic and diffusion terms with (2.8), yields (3.12) – (3.13). Substituting (3.7) into (2.11) yields (3.14). *Q.E.D.*

Proof of Proposition 3.3: The differential generator for $H(\delta,M_D,M_F,t)$, a continuously differentiable function with respect to t and a twice continuously differentiable function with respect to δ, M_D, and M_F, is

$$\mathcal{L}H(\delta,M_D,M_F,t)=\mu_\delta(t)\frac{\partial H}{\partial\delta}+\sum_{i\in\{D,F\}}\left(\mu_{M_i}(t)\frac{\partial H}{\partial M_i}+\sigma_\delta(t)^\top\sigma_{M_i}\frac{\partial^2 H}{\partial\delta\partial M_i}\right)$$
$$+\frac{1}{2}\sigma_\delta(t)^\top\sigma_\delta(t)\frac{\partial^2 H}{\partial\delta^2}+\frac{1}{2}\sum_{i\in\{D,F\}}\sum_{j\in\{D,F\}}\sigma_{M_i}{}^\top\sigma_{M_j}\frac{\partial^2 H}{\partial M_i\partial M_j}.$$

By rearranging (3.9), we obtain for $i\in\{D,F\}$

$$M_i(t)\equiv Q_i(t)U_c(t;\Lambda)+\int_0^t Q_i(s)U_{M_i}(s;\Lambda)ds=E\left[\int_0^T Q_i(s)U_{M_i}(s;\Lambda)ds|\mathcal{F}_t\right].$$

Under appropriate regularity conditions, M_i is a martingale under P; hence, its drift must be 0, resulting in the two partial differential equations given in the proposition. *Q.E.D.*

Proof of Proposition 4.1: To show that an equilibrium exists, we must solve (3.10) and verify that the resulting equilibrium quantities all satisfy our earlier assumptions.

Before examining the system of equations given by (3.10), we will solve the price of each money q_i as a function of the exogenous processes and the equilibrium weights. Substituting

$$DU(c,q_DM_D,q_FM_F;\Lambda)=\begin{pmatrix}\left((\beta_D\lambda_D)^{\frac{1}{1-\gamma}}+(\beta_F\lambda_F)^{\frac{1}{1-\gamma}}\right)^{1-\gamma}c^{\gamma-1}\\(1-\beta_D)\lambda_D\,(q_DM_D)^{\alpha_D-1}\\(1-\beta_F)\lambda_F\,(q_FM_F)^{\alpha_F-1}\end{pmatrix}$$

into Proposition 3.1 yields:

$$\text{(A.2)}\qquad \xi(t)=\delta(t)^{\gamma-1},$$
$$\text{(A.3)}\qquad R_i(t)=(1-\beta_i)\lambda_i(q_i(t)M_i(t))^{\alpha_i-1}\delta(t)^{1-\gamma},\quad i\in\{D,F\},$$
$$\text{(A.4)}\qquad q_i(t)=\delta(t)^{1-\gamma}E\left[\int_t^T q_i(s)(1-\beta_i)\lambda_i(q_i(s)M_i(s))^{\alpha_i-1}ds|\mathcal{F}_t\right],\quad i\in\{D,F\},$$

where we have normalized the weights by letting $(\beta_D\lambda_D)^{\frac{1}{1-\gamma}}+(\beta_F\lambda_F)^{\frac{1}{1-\gamma}}=1$ without loss of generality since the solution to (3.10) is unique up to a positive constant. Examining (A.4), we conjecture that the price of each money is of the form $q_i(t) = h_i(t)\frac{\delta(t)^{\frac{1-\gamma}{1-\alpha_i}}}{M_i(t)}$ where $h_i(t)$ is a deterministic function of time. Substituting into (3.15) – (3.16) yields (4.4), (4.5), and

$$q_i(t) = ((1-\beta_i)\lambda_i g(b_i, \alpha_i, t))^{\frac{1}{1-\alpha_i}} \frac{\delta(t)^{\frac{1-\gamma}{1-\alpha_i}}}{M_i(t)}, \quad i \in \{D, F\}.$$

Substituting the gradient of the representative agent, the optimal consumption and money holding policies, and the price of each money into (3.10) yields (4.2). *Q.E.D.*

Proof of Proposition 4.2: Equations (4.3), (4.6), and (4.7) follow by substituting (4.2) into (A.3) – (A.4). *Q.E.D.*

Proof of Corollary 4.1: Applying Itô's lemma to 4.3), (4.6), (4.7) yields the result. *Q.E.D.*

Proof of Proposition 4.3: Equations (4.13) and (4.14) result from computing the dynamics of (A.2). The price of the stock follows by directly computing (2.11). *Q.E.D.*

Proof of Proposition 5.1: Substituting

$$DU(c, q_D M_D, q_F M_F; \Lambda) = \begin{pmatrix} \beta c^{\beta\gamma-1}\left(\lambda_D (q_D M_D)^{\frac{(\beta-1)\gamma}{\beta\gamma-1}} + \lambda_F (q_F M_F)^{\frac{(\beta-1)\gamma}{\beta\gamma-1}}\right)^{1-\beta\gamma} \\ (1-\beta)\lambda_D^{\frac{1}{1-\beta\gamma}} c^{\beta\gamma} (q_D M_D)^{\frac{1-\gamma}{\beta\gamma-1}} \left(\lambda_D (q_D M_D)^{\frac{(\beta-1)\gamma}{\beta\gamma-1}} + \lambda_F (q_F M_F)^{\frac{(\beta-1)\gamma}{\beta\gamma-1}}\right)^{-\beta\gamma} \\ (1-\beta)\lambda_F^{\frac{1}{1-\beta\gamma}} c^{\beta\gamma} (q_F M_F)^{\frac{1-\gamma}{\beta\gamma-1}} \left(\lambda_D (q_D M_D)^{\frac{(\beta-1)\gamma}{\beta\gamma-1}} + \lambda_F (q_F M_F)^{\frac{(\beta-1)\gamma}{\beta\gamma-1}}\right)^{-\beta\gamma} \end{pmatrix}$$

into Proposition 3.3, and conjecturing (5.2) for the form of the currency prices, yields (5.3) – (5.4). Equations (5.8) – (5.10) follow by substituting (5.2) into (3.7) – (3.11). *Q.E.D.*

Proof of Corollary 5.1: Applying Itô's lemma to (5.2), (5.9), (5.10) yields the result. *Q.E.D.*

Proof of Proposition 5.2: Applying Itô's lemma to the marginal utility of consumption of the representative agent given in the proof of Proposition 5.1 yields (5.16) and (5.17). *Q.E.D.*

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