

Curvature and critical Riemannian metric

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Let M be a compact orientable C^∞ manifold and g a C^∞ Riemannian metric on M satisfying

$$(0.1) \quad \int_M dV = 1$$

where dV is the volume element of M measured by g . We denote the set of all such metrics by $\mathcal{M}(M)$ or \mathcal{M} . When g is fixed we have a Riemannian manifold (M, g) .

Let us take a covering $\{U\}$ of M by coordinate neighborhoods and denote the local coordinates in U by x^h . In each neighborhood U we use the natural frame. Then the components of the curvature tensor of (M, g) in U is given by

$$K_{kji}{}^h = \partial_k \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} - \partial_j \left\{ \begin{matrix} h \\ ki \end{matrix} \right\} + \left\{ \begin{matrix} h \\ kp \end{matrix} \right\} \left\{ \begin{matrix} p \\ ji \end{matrix} \right\} - \left\{ \begin{matrix} h \\ jp \end{matrix} \right\} \left\{ \begin{matrix} p \\ ki \end{matrix} \right\}$$

where $\left\{ \begin{matrix} h \\ ji \end{matrix} \right\}$ are the Christoffel symbols derived from the components g_{ji} of g , Latin indices run the range $\{1, \dots, n\}$, and the summation convention is adopted. The Ricci tensor and the scalar curvature are given respectively by

$$K_{ji} = K_{pji}{}^p, \quad K = g^{ji} K_{ji}$$

where g^{ji} are defined by $g_{ik} g^{kh} = \delta_i^h$. Similarly all tensors will be expressed in terms of their components.

In a Riemannian manifold (M, g) indices can be raised and lowered by g^{ji} and g_{ji} so that for example $K^{kji}{}^h = K_{ac}{}^h g^{ak} g^{cj} g^{bi}$ are the contravariant components of the curvature tensor. Thus $K_{kji}{}^h K^{kji}{}^h$ is a scalar. Considering this at each point of M we get a scalar field.

Now let us consider the integral

$$J[g] = \int_M K_g dV_g,$$

where we write K_g and dV_g for K and dV respectively in order to emphasize that these depend on the metric g . $J[g]$ is the image of g by a map $J: \mathcal{M}(M) \rightarrow \mathbf{R}$. Critical points of this map are known as Einstein metrics.

M. Berger studied the second derivative of $J[g(t)]$ for curves $g(t)$ of $\mathcal{M}(M)$ and showed that it is not true that the index of $J[g]$ and also the index of $-J[g]$ are finite for every critical point [1]. Recently the present author proved that the index of $J[g]$ and the index of $-J[g]$ are both positive at each critical point [6]. This result diminishes our interest in J to a certain extent.

Let us consider the integral

$$I[g] = \int_M K_{kji\hbar} K^{kjih} dV_g.$$

Then I is also a mapping $I: \mathcal{M}(M) \rightarrow \mathbf{R}$. This integral has a remarkable property that $I[g]$ is non-negative and moreover that, if M does not admit a locally flat metric, then $I[g]$ is strictly positive.

If η is a diffeomorphism of M and η^* its pull back, we have $I[g] = I[\eta^*(g)]$. Hence we can deduce a mapping $\tilde{I}: \mathcal{M}/\mathcal{D} \rightarrow \mathbf{R}$ from the mapping $I: \mathcal{M} \rightarrow \mathbf{R}$ where \mathcal{D} is the diffeomorphism group of M and \mathcal{M}/\mathcal{D} is the space of orbits generated by \mathcal{D} of Riemannian metrics [2].

If \bar{g} is a critical point of I , then the orbit of \bar{g} by \mathcal{D} is a critical point of \tilde{I} and vice versa. In this case let us say that \bar{g} is a critical point of \tilde{I} for convenience sake. This convention is useful since there can exist no local minimum, in the strict sense, of the mapping I but \tilde{I} may possibly have a local minimum and the present paper concerns this.

Some years ago M. Berger obtained differential equations of g for the critical point, namely the critical Riemannian metric, of the mapping I [1]. In the present paper it is shown that a metric of constant curvature is a critical Riemannian metric. Moreover, the second derivative of $I[g(t)]$ at a critical point is calculated and the following Main Theorem is obtained.

THEOREM. *If M is diffeomorphic to S^n and \bar{g} is a metric of positive constant curvature on M , then the index of I and also of \tilde{I} at \bar{g} is zero and \tilde{I} has a local minimum at \bar{g} .*

§ 1. Critical Riemannian metric.

Here and in the sequel we write

$$(1.1) \quad I[g] = \int K_{kji\hbar} K^{kjih} dV.$$

We have dropped M and g in this formula. A C^∞ curve in \mathcal{M} will be represented locally by $g_{ji}(x^1, \dots, x^n; t)$ and we define a tensor field D_{ji} on $(M, g(t))$ by

$$(1.2) \quad D_{ji}(x, t) = \frac{\partial g_{ji}(x, t)}{\partial t}.$$

This symmetric $(0, 2)$ -tensor satisfies

$$(1.3) \quad \int D_p{}^p dV = 0$$

because of (0.1).

The curvature tensor $K_{kji}{}^h$ changes with g and we get [6]

$$(1.4) \quad \frac{\partial}{\partial t} K_{kji}{}^h = \nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h$$

where the tensor $D_{ji}{}^h$ is defined by

$$D_{ji}{}^h = \frac{\partial}{\partial t} \{ {}^h_{ji} \}$$

and satisfies

$$(1.5) \quad D_{ji}{}^h = -\frac{1}{2}(\nabla_j D_i{}^h + \nabla_i D_j{}^h - \nabla^h D_{ji})$$

and ∇ means covariant differentiation with respect to the metric tensor $g(t)$.

As we have

$$\frac{\partial}{\partial t} g^{ih} = -g^{ki} g^{jh} \frac{\partial}{\partial t} g_{kj} = -D^{ih},$$

we get

$$\begin{aligned} \frac{\partial}{\partial t} (K_{kjin} K^{kjih}) &= \frac{\partial}{\partial t} (K_{kji}{}^h K_{dcb}{}^a g^{kd} g^{jc} g^{ib} g_{ha}) \\ &= 2(\nabla_k D_{ji}{}^h - \nabla_j D_{ki}{}^h) K^{kji}{}_h \\ &\quad + K_{kji}{}^h K_{dcb}{}^a (-D^{kd} g^{jc} g^{ib} g_{ha} - g^{kd} D^{jc} g^{ib} g_{ha} \\ &\quad - g^{kd} g^{jc} D^{ib} g_{ha} + g^{kd} g^{jc} g^{ib} D_{ha}). \end{aligned}$$

Substituting (1.5) into this formula and taking some property of the curvature tensor into account, we can deduce

$$(1.6) \quad \frac{\partial}{\partial t} (K_{kjin} K^{kjih}) = 4K^{kji}{}_h \nabla_k \nabla_i D_j{}^h - 2K_{kji}{}^b K^{kji}{}_a D_{ba}.$$

Now, from (1.1) we get

$$\frac{d}{dt} I[g(t)] = \int \left[\frac{\partial}{\partial t} (K_{kjin} K^{kjih}) + \frac{1}{2} K_{kjin} K^{kjih} g^{qp} D_{qp} \right] dV.$$

Substituting (1.6) into this and applying Green's theorem, we get

$$\begin{aligned} \frac{d}{dt} I[g(t)] &= \int \left[4(\nabla_i \nabla_k K^{kji}{}_h) D_j{}^h \right. \\ &\quad \left. - 2K_{kji}{}^a K^{kji}{}_p D_{qp} + \frac{1}{2} K_{kjin} K^{kjih} D_p{}^p \right] dV. \end{aligned}$$

Then, applying Ricci's identity and Bianchi's identity, we get

$$(1.7) \quad \frac{d}{dt} I[g(t)] = \int \left[2\nabla^j \nabla^i K - 4\nabla_p \nabla^p K^{ji} + 4K^j_p K^{pi} - 4K^j_{qp} K^{qp} - 2K^{srqj} K_{srq}{}^i + \frac{1}{2} K_{dcba} K^{dcba} g^{ji} \right] D_{ji} dV.$$

A point \bar{g} is a critical point of I if and only if (1.7) vanishes at $t=0$ for all curves $g(t)$ such that $g(0)=\bar{g}$. Let us consider the integral which is obtained from the right-hand member of (1.7) by replacing $g(t)$ with \bar{g} . Then we can say that \bar{g} is a critical point of I if and only if this integral vanishes for all tensor fields D_{ji} satisfying (1.3) in which g is replaced with \bar{g} . Thus we see that \bar{g} is a critical point of I if and only if

$$(1.8) \quad 2\nabla^j \nabla^i K - 4\nabla_p \nabla^p K^{ji} + 4K^j_p K^{pi} - 4K^j_{qp} K^{qp} - 2K^{srqj} K_{srq}{}^i + \frac{1}{2} K_{dcba} K^{dcba} g^{ji} = c g^{ji}$$

is satisfied by $g=\bar{g}$ and some constant c . Precisely, c is obtained by transvecting with g_{ji} , hence

$$(1.9) \quad c = -\frac{2}{n} \nabla_p \nabla^p K + \left(\frac{1}{2} - \frac{2}{n} \right) K_{dcba} K^{dcba}.$$

Thus we get the following theorem [1].

THEOREM 1.1. *Let M be a compact orientable C^∞ manifold and \mathcal{M} be the space of Riemannian metrics on M satisfying (0.1). Then a necessary and sufficient condition for a Riemannian metric g to be a critical point of the functional (1.1) is that (1.8) is satisfied by this metric g where c is a constant in M . At that time c is given by (1.9).*

EXAMPLE. If M is diffeomorphic to S^n , M admits a Riemannian metric of positive constant curvature. Then

$$(1.10) \quad K_{kjin} = \frac{K}{n(n-1)} (g_{kh} g_{ji} - g_{jh} g_{ki})$$

satisfies (1.8) with

$$c = \left(\frac{1}{2} - \frac{2}{n} \right) \frac{2K^2}{n(n-1)}.$$

Hence this metric is a critical point of the functional (1.1) with the critical value

$$\frac{2K^2}{n(n-1)}.$$

Now, the purpose of the present paper is to prove the following theorem which is equivalent with the Main Theorem stated in the introduction.

THEOREM 1.2. *Let M be a C^∞ manifold diffeomorphic to S^n . Then the mapping $\tilde{I}: \mathcal{M}/\mathcal{D} \rightarrow \mathbf{R}$ given by the functional (1.1) has a local minimum at the Riemannian metric \bar{g} of positive constant curvature (1.10).*

§2. The second derivative of $I[g(t)]$.

In order to prove the main theorem let us first calculate the second derivative $I''[g(t)]$ for an arbitrary curve $g(t)$ of \mathcal{M} .

If we define W^{ji} by

$$(2.1) \quad W^{ji} = 2\nabla^j \nabla^i K - 4\nabla_p \nabla^p K^{ji} + 4K^j_p K^{pi} - 4K^j_{qp} K^{qp} \\ - 2K^{srqj} K_{srq}{}^i + \frac{1}{2} K_{dcba} K^{dcba} g^{ji},$$

we get

$$(2.2) \quad I''[g(t)] = \int \left[\frac{\partial W^{ji}}{\partial t} \frac{\partial g_{ji}}{\partial t} + W^{ji} \left(\frac{\partial^2 g_{ji}}{\partial t^2} + \frac{1}{2} \frac{\partial g_{ji}}{\partial t} g^{qp} \frac{\partial g_{qp}}{\partial t} \right) \right] dV$$

where

$$(2.3) \quad \frac{\partial W^{ji}}{\partial t} = 2 \frac{\partial}{\partial t} \nabla^j \nabla^i K - 4 \frac{\partial}{\partial t} \nabla_p \nabla^p K^{ji} \\ + 4 \frac{\partial}{\partial t} (K^j_p K^{pi}) - 4 \frac{\partial}{\partial t} (K^j_{qp} K^{qp}) - 2 \frac{\partial}{\partial t} (K^{srqj} K_{srq}{}^i) \\ + \frac{1}{2} \frac{\partial}{\partial t} (K_{dcba} K^{dcba} g^{ji}).$$

From the definition of $D_{ji}{}^h$ we have the following identity which is valid for any C^∞ tensor field depending on t differentiably, for example $T_i{}^h$,

$$(2.4) \quad \frac{\partial}{\partial t} \nabla_j T_i{}^h = \nabla_j \frac{\partial}{\partial t} T_i{}^h - D_{ji}{}^p T_p{}^h + D_{jp}{}^h T_i{}^p.$$

Applying this identity to the first and the second terms in the right-hand member of (2.3), we get

$$(2.5) \quad \frac{\partial}{\partial t} \nabla^j \nabla^i K = \frac{\partial}{\partial t} (g^{jq} g^{ip} \nabla_q \nabla_p K) \\ = -D^{jp} \nabla_p \nabla^i K - D^{ip} \nabla^j \nabla_p K \\ + \nabla^j \nabla^i \frac{\partial}{\partial t} K - D^{jip} \nabla_p K,$$

$$(2.6) \quad \frac{\partial}{\partial t} \nabla_p \nabla^p K^{ji} = \frac{\partial}{\partial t} (g^{qp} \nabla_q \nabla_p K^{ji}) \\ = -D^{qp} \nabla_q \nabla_p K^{ji} \\ + g^{qp} (-D_{qp}{}^r \nabla_r K^{ji} + D_{qr}{}^j \nabla_p K^{ri} + D_{qr}{}^i \nabla_p K^{jr}) \\ + g^{qp} \nabla_q \left(\nabla_p \frac{\partial}{\partial t} K^{ji} + D_{pr}{}^j K^{ri} + D_{pr}{}^i K^{jr} \right).$$

From (1.4) and (1.5) we get

$$(2.7) \quad \frac{\partial}{\partial t} K^{ji} = -D^{jp} K_p^i - D^{ip} K_p^j + \frac{1}{2} (\nabla_p \nabla^j D^{ip} + \nabla_p \nabla^i D^{jp} - \nabla_p \nabla^p D^{ji} - \nabla^j \nabla^i D_p^p),$$

$$(2.8) \quad \frac{\partial}{\partial t} K = -D^{qp} K_{qp} + \nabla_q \nabla_p D^{qp} - \nabla_q \nabla^q D_p^p,$$

$$(2.9) \quad \begin{aligned} \frac{\partial}{\partial t} (K^j_p K^{pi}) &= -D^{jq} K_{qp} K^{pi} - D^{iq} K_{qp} K^{pj} - K^{jq} D_{qp} K^{pi} \\ &+ \frac{1}{2} K^{qi} (\nabla_p \nabla^j D_q^p + \nabla_p \nabla_q D^{jp} - \nabla_p \nabla^p D_q^j - \nabla_q \nabla^j D_p^p) \\ &+ \frac{1}{2} K^{qj} (\nabla_p \nabla^i D_q^p + \nabla_p \nabla_q D^{ip} - \nabla_p \nabla^p D_q^i - \nabla_q \nabla^i D_p^p), \end{aligned}$$

$$(2.10) \quad \begin{aligned} \frac{\partial}{\partial t} (K^j_{qp} K^{qp}) &= -K_{rqp} K^{qp} D^{rj} - K^j_{qp} (D^{qr} K_r^p + D^{pr} K_r^q) \\ &+ \frac{1}{2} K^{qp} (\nabla^j \nabla_q D_p^i + \nabla^j \nabla_p D_q^i - \nabla^j \nabla^i D_{qp} \\ &\quad - \nabla_q \nabla^j D_p^i - \nabla_q \nabla_p D^{ji} + \nabla_q \nabla^i D_p^j) \\ &+ \frac{1}{2} K^j_{qp} (\nabla_r \nabla^q D^{pr} + \nabla_r \nabla^p D^{qr} - \nabla_r \nabla^r D^{qp} - \nabla^q \nabla^p D_r^r), \end{aligned}$$

$$(2.11) \quad \begin{aligned} \frac{\partial}{\partial t} (K^{srqj} K_{srq}^i) &= (\nabla_c \nabla_b D_a^j + \nabla_c \nabla_a D_b^j - \nabla_c \nabla^j D_{ba}) K^{cba i} \\ &+ (\nabla_c \nabla_b D_a^i + \nabla_c \nabla_a D_b^i - \nabla_c \nabla^i D_{ba}) K^{cba j} \\ &- K^{bsrj} K_{sr}^a D_{ba} - K^{sbrj} K_s^a D_{ba} \\ &- K^{srbj} K_{sr}^a D_{ba}, \end{aligned}$$

$$(2.12) \quad \begin{aligned} \frac{\partial}{\partial t} (K_{dcba} K^{dcba} g^{ji}) &= (4K^{dcba} \nabla_d \nabla_b D_{ca} - 2K_{dcba} \nabla^q D_{qp}) g^{ji} \\ &- K_{dcba} K^{dcba} D^{ji}. \end{aligned}$$

Applying (1.5), (2.7) and (2.8) we get from (2.5) and (2.6)¹⁾

$$(2.13) \quad \begin{aligned} \frac{\partial}{\partial t} \nabla^j \nabla^i K &= -D^{jq} \nabla_q \nabla^i K - D^{iq} \nabla_q \nabla^j K \\ &+ \nabla^j \nabla^i (-D^{qp} K_{qp} + \nabla_q \nabla_p D^{qp} - \nabla_q \nabla^q D_p^p) \\ &- \frac{1}{2} (\nabla^j D^{iq} + \nabla^i D^{jq} - \nabla^q D^{ji}) \nabla_q K, \end{aligned}$$

1) In this paper ∇AB always means $(\nabla A)B$. Thus we have $\nabla(AB) = \nabla AB + A\nabla B$.

$$\begin{aligned}
(2.14) \quad & \frac{\partial}{\partial t} \nabla_q \nabla^q K^{ji} \\
&= -D^{qp} \nabla_q \nabla_p K^{ji} - \nabla_q D^{qr} \nabla_r K^{ji} + \frac{1}{2} \nabla^r D_q{}^q \nabla_r K^{ji} \\
&\quad + (\nabla_q D_r{}^j + \nabla_r D_q{}^j - \nabla^j D_{qr}) \nabla^q K^{ri} \\
&\quad + (\nabla_q D_r{}^i + \nabla_r D_q{}^i - \nabla^i D_{qr}) \nabla^q K^{rj} \\
&\quad + \frac{1}{2} \nabla^p (\nabla_p D_r{}^j + \nabla_r D_p{}^j - \nabla^j D_{pr}) K^{ri} \\
&\quad + \frac{1}{2} \nabla^p (\nabla_p D_r{}^i + \nabla_r D_p{}^i - \nabla^i D_{pr}) K^{rj} \\
&\quad - \nabla_q \nabla^q \left(D^{js} K_s{}^i + D^{is} K_s{}^j - \frac{1}{2} \nabla_s \nabla^j D^{is} - \frac{1}{2} \nabla_s \nabla^i D^{js} \right. \\
&\quad \left. + \frac{1}{2} \nabla_s \nabla^s D^{ji} + \frac{1}{2} \nabla^j \nabla^i D_s{}^s \right).
\end{aligned}$$

Let us define F by

$$(2.15) \quad F = \frac{\partial W^{ji}}{\partial t} D_{ji}$$

and the notation \cong by $A \cong A + \text{divergence}$. Then we get

$$\begin{aligned}
(2.16) \quad & F \cong D_{ji} \left[-4D^{jq} \nabla_q \nabla^i K \right. \\
&\quad + 2\nabla^j \nabla^i (-D^{qp} K_{qp} + \nabla_q \nabla_p D^{qp} - \nabla_q \nabla^q D_p{}^p) \\
&\quad - 2\nabla^j D^{iq} \nabla_q K + \nabla^q D^{ji} \nabla_q K + 4D^{qp} \nabla_q \nabla_p K^{ji} \\
&\quad + 4\nabla_q D^{qp} \nabla_p K^{ji} - 2\nabla^q D_p{}^p \nabla_q K^{ji} \\
&\quad - 8(\nabla_q D_p{}^j + \nabla_p D_q{}^j - \nabla^j D_{qp}) \nabla^q K^{pi} \\
&\quad - 4\nabla^q (\nabla_q D_p{}^j + \nabla_p D_q{}^j - \nabla^j D_{qp}) K^{pi} \\
&\quad + 2\nabla_q \nabla^q (4D^{jp} K_p{}^i - 2\nabla_p \nabla^j D^{ip} + \nabla_p \nabla^p D^{ji} + \nabla^j \nabla^i D_p{}^p) \\
&\quad - 8D^{jq} K_{qp} K^{pi} - 4K^{jq} D_{qp} K^{pi} \\
&\quad + 4K^{jq} (\nabla_p \nabla^i D_q{}^p + \nabla_p \nabla_q D^{ip} - \nabla_p \nabla^p D_q{}^i - \nabla_q \nabla^i D_p{}^p) \\
&\quad + 4K_{rqp}{}^i K^{qp} D^{rj} + 4K_{qp}{}^j{}^i (D^{qr} K_r{}^p + D^{pr} K_r{}^q) \\
&\quad - 2K^{qp} (\nabla^j \nabla_q D_p{}^i + \nabla^j \nabla_p D_q{}^i - \nabla^j \nabla^i D_{qp} \\
&\quad \quad - \nabla_q \nabla^j D_p{}^i - \nabla_q \nabla_p D^{ji} + \nabla_q \nabla^i D_p{}^j) \\
&\quad - 2K_{qp}{}^j{}^i (\nabla_r \nabla^q D^{pr} + \nabla_r \nabla^p D^{qr} - \nabla_r \nabla^r D^{qp} - \nabla^q \nabla^p D_r{}^r) \\
&\quad - 4K^{cba}{}^j (\nabla_c \nabla_b D_a{}^i + \nabla_c \nabla_a D_b{}^i - \nabla_c \nabla^i D_{ba}) \\
&\quad + 4K^{bap}{}^j K_{qp}{}^a{}^i D_{ba} + 2K^{apbj} K_{qp}{}^a{}^i D_{ba} \\
&\quad + (2K^{dcba} \nabla_d \nabla_b D_{ca} - K_{dcb}{}^a K^{dcbp} D_{qp}) g^{ji} \\
&\quad \left. - \frac{1}{2} K_{dcba} K^{dcba} D^{ji} \right].
\end{aligned}$$

This formula is valid for any value of t .

We now want to evaluate the integral (2.2) at $t=0$, namely, at a critical point.

As we have $W^{ji} = cg^{ji}$ at $t=0$, we get

$$I''[g(0)] = \int \left[F + cg^{ji} \left(\frac{\partial^2 g_{ji}}{\partial t^2} + \frac{1}{2} \frac{\partial g_{ji}}{\partial t} g^{qp} \frac{\partial g_{qp}}{\partial t} \right) \right]_0 dV$$

where $[]_0$ means that we have put $t=0$.

From (0.1) we get

$$\int \left[g^{ji} \frac{\partial^2 g_{ji}}{\partial t^2} - D^{ji} D_{ji} + \frac{1}{2} (D_p^p)^2 \right] dV = 0.$$

Hence we can deduce

$$(2.17) \quad I''[g(0)] = \int G dV$$

where G is given by

$$(2.18) \quad G \cong \left[F + \left\{ -\frac{2}{n} \nabla_q \nabla^q K + \left(\frac{1}{2} - \frac{2}{n} \right) K_{acba} K^{acba} \right\} D_{ji} D^{ji} \right]_0,$$

for we have (1.9).

§ 3. The integrand of $I''[g(0)]$.

According to M. Berger and D. Ebin [1], [2] we can deduce properties of the Hessian of $I[g]$ at a critical point \bar{g} if we study $I''[g(0)]$ for all curves $g(t)$ such that $g(0) = \bar{g}$ and such that the tensor field $D_{ji} = [\partial g_{ji} / \partial t]_0$ satisfies $\nabla^j D_{ji} = 0$.

If D_{ji} satisfies

$$(3.1) \quad \nabla^j D_{ji} = 0,$$

we get

$$(3.2) \quad \nabla_q \nabla^j D^{qi} = K^j_q D^{qi} + K^j_p{}^i D^{qp}$$

and $D_{ji} \nabla^j T^i \cong 0$ for any vector field T^h on M . Applying such results we get after some straightforward calculation

$$(3.3) \quad \begin{aligned} G \cong & -4D_{ji} D^{jq} \nabla_q \nabla^i K - 2D_{ji} \nabla^j D^{iq} \nabla_q K \\ & + D_{ji} \nabla^q D^{ji} \nabla_q K + 4D_{ji} D^{qp} \nabla_q \nabla_p K^{ji} \\ & - 2\nabla^q D_p{}^p D_{ji} \nabla_q K^{ji} \\ & - 8D_{ji} (\nabla_q D_p{}^j + \nabla_p D_q{}^j - \nabla^j D_{qp}) \nabla^q K^{pi} \\ & - 4D_{ji} (\nabla^q \nabla_q D_p{}^j + K_p{}^q D_q{}^j - K^{jq} D_{qp}) K^{pi} \end{aligned}$$

$$\begin{aligned}
& +2\nabla_q \nabla^q D_{ji} (4D^{jp} K_p^i - 2K_r^j D^{rp} - 2K_p^j D^{ip} + \nabla^j \nabla^i D_p^p + \nabla_p \nabla^p D^{ji}) \\
& -8D_{ji} D^{jq} K_{qp} K^{pi} \\
& +4D_{ji} K^{jq} (K_{qp} D^{ip} - \nabla_p \nabla^p D_q^i - \nabla_q \nabla^i D_p^p) \\
& +4K_{rqp}^i K^{qp} D^{rj} D_{ji} \\
& -2K^{qp} (2D_{ji} \nabla^j \nabla_q D_p^i - D_{ji} \nabla^j \nabla^i D_{qp} - D_{ji} \nabla_q \nabla_p D^{ji}) \\
& -2D_{ji} K_{qp}^j \{2(K_r^q D^{ar} + K_r^q D^{pr}) - \nabla_r \nabla^r D^{qp} - \nabla^q \nabla^p D_r^r\} \\
& +2D_{ji} K^{cba} K_{cba}^p D_p^i \\
& -4D_{ji} K^{cba} (\nabla_c \nabla_a D_b^i - \nabla_c \nabla^i D_{ba}) \\
& +4K^{bqp} K_{qp}^a D_{ba} D_{ji} \\
& +2K^{dcb} \nabla_d \nabla_b D_{ca} D_p^p \\
& -K_{dcb}^q K^{dcb} D_{qp} D_r^r \\
& + \left(-\frac{2}{n} \nabla_p \nabla^p K - \frac{2}{n} K_{dcb} K^{dcb} \right) D_{ji} D^{ji}
\end{aligned}$$

at $t=0$, that is, at $\bar{g}=g(0)$.

§ 4. $I''[g(0)]$ when $g(0)$ is the metric of constant curvature.

If $g(0)$ is a metric of constant curvature, we have

$$\begin{aligned}
K_{kjh} &= \frac{K}{n(n-1)} (g_{kh} g_{ji} - g_{jh} g_{ki}), \\
K^{dcbj} K_{dcb}^i &= \frac{2K^2}{n^2(n-1)} g^{ji}, \\
K^{kqpj} K_{qp}^i &= \frac{K^2}{n^2(n-1)^2} \{ (n-2)g^{kj} g^{ih} + g^{ki} g^{jh} \}
\end{aligned}$$

and $\nabla_k K^{ji}=0$, $\nabla_k K=0$.

Hence we get from (3.3)

$$\begin{aligned}
G &\cong 2\nabla_q \nabla^q D_{ji} \left\{ \nabla^j \nabla^i D_p^p + \nabla_p \nabla^p D^{ji} + \frac{2K}{n(n-1)} (g^{ji} D_p^p - D^{ji}) \right\} \\
&\quad - \frac{2K}{n} D_{ji} \nabla_q \nabla^q D^{ji} \\
&\quad - \frac{2K}{n(n-1)} (D_q^q g_{ji} - D_{ji}) \left\{ \frac{2K}{n} D^{ji} - \frac{2K}{n(n-1)} (D_p^p g^{ji} - D^{ji}) \right. \\
&\quad \quad \left. - \nabla_p \nabla^p D^{ji} - \nabla^j \nabla^i D_p^p \right\} \\
&\quad + \frac{4K^2}{n^2(n-1)} D_{ji} D^{ji}
\end{aligned}$$

$$\begin{aligned}
 & -\frac{4K}{n(n-1)}D_{ji}(\nabla_q\nabla^jD^{iq}-\nabla_q\nabla^qD^{ji}) \\
 & +\frac{4K^2}{n^2(n-1)^2}\{(n-2)D_{ji}D^{ji}+(D_p^p)^2\} \\
 & -\frac{2K}{n(n-1)}D_q^q\nabla_r\nabla^rD_p^p-\frac{2K^2}{n^2(n-1)}(D_p^p)^2 \\
 & -\frac{4K^2}{n^2(n-1)}D_{ji}D^{ji}
 \end{aligned}$$

since such terms as $D_{ji}\nabla^j\nabla^iD_p^p$ are divergences because of (3.1).

We use Ricci's identities and identities such as (3.2) and get

$$\begin{aligned}
 2\nabla_q\nabla^qD_{ji}\nabla^j\nabla^iD_p^p & \cong -2\nabla^j\nabla_q\nabla^qD_{ji}\nabla^iD_p^p \\
 & \cong -\frac{4K}{n(n-1)}D_p^p\nabla_q\nabla^qD_r^r
 \end{aligned}$$

by virtue of (3.1). We also get

$$D_{ji}\nabla_p\nabla^jD^{ip} \cong \frac{K}{n(n-1)}\{D_{ji}D^{ji}-(D_p^p)^2\} + \frac{K}{n}D_{ji}D^{ji}$$

and finally

$$\begin{aligned}
 (4.1) \quad G & \cong 2\nabla_q\nabla^qD_{ji}\nabla_p\nabla^pD^{ji} \\
 & -\frac{2K}{n-1}\left(D_{ji}\nabla_p\nabla^pD^{ji}-\frac{1}{n}D_p^p\nabla_q\nabla^qD_r^r\right) \\
 & +\frac{4(n-2)}{n^2(n-1)^2}K^2D_{ji}D^{ji} \\
 & -\frac{2(n-3)}{n^2(n-1)^2}K^2(D_p^p)^2.
 \end{aligned}$$

§ 5. Proof of the main theorem.

If we put

$$(5.1) \quad D_{ji} = H_{ji} + \frac{H}{n}g_{ji}, \quad H = D_p^p,$$

we get $g^{ji}H_{ji} = 0$. Let us define G_1, G_2 by

$$(5.2) \quad G_1 = 2\left[\nabla_q\nabla^qH_{ji}\nabla_p\nabla^pH^{ji} - \frac{K}{n-1}H_{ji}\nabla_p\nabla^pH^{ji} + \frac{2(n-2)}{n^2(n-1)^2}K^2H_{ji}H^{ji}\right],$$

$$(5.3) \quad G_2 = \frac{2}{n}\left[\nabla_q\nabla^qH\nabla_p\nabla^pH - \frac{n-4}{n^2(n-1)}K^2H^2\right].$$

Then we have $G \cong G_1 + G_2$ and

$$(5.4) \quad I''[g(0)] = \int G_1 dV + \int G_2 dV.$$

Writing $\Delta f = -\nabla_p \nabla^p f$ we get

$$\begin{aligned} \int (\Delta f)^2 dV &= \int f \nabla_q \nabla^q \nabla^p \nabla_p f dV \\ &= \int f \nabla_q (\nabla^p \nabla^q \nabla_p f - K_p^q \nabla^p f) dV \\ &= \int (\nabla^q \nabla^p f) (\nabla_q \nabla_p f) dV + \frac{K}{n} \int f \Delta f dV. \end{aligned}$$

Since we have

$$\left(\nabla^q \nabla^p f + \frac{1}{n} g^{qp} \Delta f \right) \left(\nabla_q \nabla_p f + \frac{1}{n} g_{qp} \Delta f \right) = (\nabla^q \nabla^p f) (\nabla_q \nabla_p f) - \frac{1}{n} (\Delta f)^2,$$

we get

$$\begin{aligned} \int (\Delta f)^2 dV &= \int \left(\nabla^q \nabla^p f + \frac{1}{n} g^{qp} \Delta f \right) \left(\nabla_q \nabla_p f + \frac{1}{n} g_{qp} \Delta f \right) dV \\ &\quad + \frac{1}{n} \int (\Delta f)^2 dV + \frac{K}{n} \int f \Delta f dV, \end{aligned}$$

hence

$$\int (\Delta f)^2 dV \geq \frac{K}{n-1} \int \nabla_p f \nabla^p f dV.$$

It is well-known [3] that, if λ_1 is the first positive eigenvalue of the equation $\Delta \rho = \lambda \rho$, and if f satisfies $\int f dV = 0$, then

$$\int \nabla_p f \nabla^p f dV \geq \lambda_1 \int f^2 dV.$$

Moreover,

$$\lambda_1 = \frac{1}{n-1} K$$

for a space of positive constant curvature [5], [7]. Hence we have

$$\int (\Delta f)^2 dV \geq \frac{1}{(n-1)^2} K^2 \int f^2 dV$$

in our case.

If H does not vanish identically, we have

$$\int (\Delta H)^2 dV - \frac{n-4}{n^2(n-1)} K^2 \int H^2 dV \geq \left[\frac{1}{(n-1)^2} - \frac{n-4}{n^2(n-1)} \right] K^2 \int H^2 dV > 0,$$

hence $\int G_2 dV > 0$. Since we have $\int G_1 dV \geq 0$ always, we have proved $\int (G_1 + G_2) dV > 0$ in this case.

Let us now assume $H \equiv 0$. If moreover H_{ji} also vanishes everywhere on M , we have $D_{ji} \equiv 0$. We need not consider this case. If H_{ji} does not

identically vanish, we have $\int G_1 dV > 0$, hence $\int (G_1 + G_2) dV > 0$.

Thus we have proved the main theorem.

References

- [1] M. Berger, Quelques formules des variation pour une structure riemannienne, *Ann. sci. École Norm. Sup.*, 4^e série, 3 (1970), 285-294.
- [2] M. Berger and D. Ebin, Some decompositions of the space of symmetric tensors on a Riemannian manifold, *J. Differential Geometry*, 3 (1969), 379-392.
- [3] M. Berger, P. Gauduchon and E. Mazet, *Le spectre d'une variété riemannienne*, Springer, 1971.
- [4] D. Ebin, The manifolds of Riemannian metrics, *Proceedings of 1968 AMS Summer Institute on Global Analysis*.
- [5] A. Lichnerowicz, *Géométrie des groupes de transformations*, Dunod, 1958.
- [6] Y. Mutō, On Einstein metrics, *J. Differential Geometry*, to appear.
- [7] M. Obata, Certain conditions for a Riemannian manifold to be isometric with a sphere, *J. Math. Soc. Japan*, 14 (1962), 333-340.

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