Tôhoku Math. Journ.
25 (1973), 557-567.

CURVATURE AND SPECTRUM OF RIEMANNIAN MANIFOLD

Dedicated to Professor Shigeo Sasaki on his 60th birthday

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(Received April 15, 1973)

1. Introduction. Let M be an n-dimensional compact Riemannian manifold without boundary, and Δ the Laplacian acting on smooth p-forms on M, $0 \leqq p \leqq n$. Being elliptic, the operator Δ has an infinite sequence

$$0 \leqq \lambda_{p,1} \leqq \lambda_{p,2} \leqq \cdots \leqq \lambda_{p,k} \leqq \cdots \uparrow \infty$$

of eigenvalues, each of which is repeated as many times as its multiplicity indicates, and the corresponding sequence $\{\phi_{p,k}\}_{k=1}^{\infty}$ of eigenforms forms a complete orthonormal set in the space of p-forms with Riemannian inner product. The series

$$\sum_{k=1}^{\infty} \exp(-\lambda_{p,k}t)\phi_{p,k}(x) \otimes \phi_{p,k}(y)$$

converges uniformly on compact subsets of $(0, \infty) \times M \times M$ to the fundamental solution $e^p(t, x, y)$ of the heat operator $\partial/\partial t + \Delta$ acting on p-forms, and we have,

$$(1) \qquad \sum_{k=1}^{\infty} \exp(-\lambda_{p,k}t) = \int_M \operatorname{Tr} e^p(t, x, x)\,, \qquad t > 0\,,$$

Tr denoting the trace. The geometric and probabilistic interpretation of $e^p(t, x, y)$ may be seen in Itô [2]. Moreover, we have the following Minakshisundaram's asymptotic expansion for $\operatorname{Tr} e^p(t, x, x)$:

$$(2) \qquad \operatorname{Tr} e^p(t, x, x) \underset{t\to 0+}{\sim} (4\pi t)^{-n/2}\{u_{p,0}(x) + tu_{p,1}(x) + \cdots + t^k u_{p,k}(x) + \cdots\}\,,$$

where the coefficients $u_{p,k}(x)$ are local Riemannian invariants. Let $a_{p,k} = \int_M u_{p,k}$. Then by (1) and (2) we have,

$$(3) \qquad \sum_{k=1}^{\infty} \exp(-\lambda_{p,k}t) \underset{t\to 0+}{\sim} (4\pi t)^{-n/2}(a_{p,0} + ta_{p,1} + \cdots + t^k a_{p,k} + \cdots)\,.$$

We call the sequence $\operatorname{Spec}^p(M) = \{\lambda_{p,1}, \lambda_{p,2}, \cdots, \lambda_{p,k}, \cdots\}$ of eigenvalues the spectrum of M for p-forms. Many of the relations between spectra and geometric quantities of Riemannian manifolds have been obtained by calculating the coefficients of the Minakshisundaram's expansion in terms of curvatures of M. In the present paper we shall calculate the coefficient $u_{1,3}$ in (2) for the Einsteinian case to obtain the following:

THEOREM. *Let M be a compact locally symmetric Einstein space and M' a compact Riemannian manifold. Suppose that M and M' have the same spectra for functions and for 1-forms respectively, i.e.,* $\mathrm{Spec}^0(M) = \mathrm{Spec}^0(M')$ *and* $\mathrm{Spec}^1(M) = \mathrm{Spec}^1(M')$. *Then M' is also locally symmetric.*

I should like to express my hearty thanks to Professor T. Sakai who has kindly read through this manuscript to point out several errors.

2. Preliminaries. Let M be a Riemannian manifold with Riemannian metric $\langle\ ,\ \rangle$ and Levi-Civita connection ∇. Let $R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z$ and $\rho(X, Y) = \mathrm{Tr}\,(Z \to R(X, Z)Y)$ be curvature tensor and Ricci tensor respectively. We denote ; the covariant differentiation. Throughout this paper we use the Einstein's summation convention.

Let us fix a point m of M and choose a normal coordinate system (V, x^i) origined at the point m. We shall omit (m) for the quantities at the point m, for example,

$$R_{ijkl} = R_{ijkl}(m) = \left\langle R\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\right\rangle(m)\ ,$$

$$\rho_{ij} = \rho_{ij}(m) = \rho\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)(m)\ .$$

Let $x \in V$ be a point with the coordinates (x^i). Then $g_{ij}(x) = \langle \partial/\partial x^i, \partial/\partial x^j\rangle(x)$ is represented in terms of curvature tensor and its successive covariant derivatives at the point m, see Sakai [5].

$$\begin{aligned} g_{ij}(x) = \delta_{ij} &+ \frac{1}{3} R_{kijl}x^k x^l + \frac{1}{3!} R_{kijl;p}x^k x^l x^p + \frac{1}{5!}\Big(6R_{kijl;pq} \\ &+ \frac{16}{3} R_{kilu}R_{pjqu}\Big)x^k x^l x^p x^q + \frac{1}{6!}(8R_{kijl;pqr} + 16R_{kilu}R_{pjqu;r} \\ &+ 16R_{kjlu}R_{piqu;r})\,x^k x^l x^p x^q x^r + \frac{1}{7!}(10R_{kijl;pqrs} \\ &+ 34R_{kilu;pq}R_{rjsu} + 34R_{kjlu;pq}R_{risu} + 55R_{kilu;p}R_{qjru;s} \\ &- 16R_{kilu}R_{pjqv}R_{rusv})x^k x^l x^p x^q x^r x^s + o(|x|^6)\ . \end{aligned} \tag{4}$$

From $g_{ij}(x)g^{jk}(x) = \delta_i^k$ and (4) we have,

$$\begin{aligned} g^{ij}(x) = \delta^{ij} &+ A^{ij}_{kl}x^k x^l + A^{ij}_{klp}x^k x^l x^p + A^{ij}_{klpq}x^k x^l x^p x^q \\ &+ A^{ij}_{klpqr}x^k x^l x^p x^q x^r + o(|x|^5)\ , \end{aligned} \tag{5}$$

where

$$A^{ij}_{kl} = -\frac{1}{3} R_{kijl} ,$$

$$A^{ij}_{klp} = -\frac{1}{3!} R_{kijl;p} ,$$

$$A^{ij}_{klpq} = \frac{1}{5!}(-6R_{kijl;pq} + 8R_{kilu}R_{pjqu}) ,$$

$$A^{ij}_{klpqr} = \frac{1}{6!}(-8R_{kijl;pqr} + 24R_{kilu}R_{pjqu;r} + 24R_{kjlu}R_{piqu;r}) .$$

Then from

$$\Gamma^c_{ab}(x) = \frac{1}{2} g^{ci}(x)\left(\frac{\partial g_{ia}}{\partial x^b} + \frac{\partial g_{ib}}{\partial x^a} - \frac{\partial g_{ab}}{\partial x^i}\right)(x) ,$$

(4), (5), and Bianchi's identity we have,

$$(6) \qquad \begin{aligned} \Gamma^c_{ab}(x) = {} & B^{cab}_k x^k + B^{cab}_{kl} x^k x^l + B^{cab}_{klp} x^k x^l x^p + B^{cab}_{klpq} x^k x^l x^p x^q \\ & + B^{cab}_{klpqr} x^k x^l x^p x^q x^r + o(|x|^5) , \end{aligned}$$

where

$$B^{cab}_k = -\frac{1}{3}(R_{kabc} + R_{kbac}) ,$$

$$B^{cab}_{kl} = -\frac{1}{3!}\left\{(R_{kabc;l} + R_{kbac;l}) + \frac{1}{2}(R_{kalc;b} + R_{kblc;a} - R_{kalb.c})\right\} ,$$

$$\begin{aligned} B^{cab}_{klp} = -\frac{1}{5!}\Big\{ & 6(R_{kabc;lp} + R_{kbac:lp}) + 3(R_{kalc.bp} + R_{kalc:pb}) \\ & + 3(R_{kblc;ap} + R_{kblc;pa}) - 3(R_{kalb;cp} + R_{kalb;pc}) \\ & + \frac{32}{3} R_{kclu}(R_{pabu} + R_{pbau}) - 8(R_{kalu}R_{bcpu} + R_{kblu}R_{acpu})\Big\} , \end{aligned}$$

$$\begin{aligned} B^{cab}_{klpq} = -\frac{1}{6!}\{ & 8(R_{kabc;lpq} + R_{kbac;lpq}) + 4(R_{kalc;bpq} + R_{kalc;pbq} + R_{kalc;pqb}) \\ & + 4(R_{kblc;apq} + R_{kblc;paq} + R_{kblc;pqa}) - 4(R_{kalb;cpq} + R_{kalb;pcq} + R_{kalb;pqc}) \\ & + 44R_{kclu}(R_{pabu;q} + R_{pbau;q}) + 4R_{kclu}R_{paqb;u} \\ & - 32(R_{kalu}R_{bcpu;q} + R_{kblu}R_{acpu;q}) \\ & - 24(R_{bcku}R_{lapu;q} + R_{acku}R_{lbpu;q}) + 32(R_{kabu} + R_{kbau})R_{lcpu;q}\} , \end{aligned}$$

$$\begin{aligned} B^{cab}_{klpqr} = -\frac{1}{7!}\Big[& 10(R_{kabc;lpqr} + R_{kbac;lpqr}) \\ & + 5(R_{kalc;bpqr} + R_{kalc;pbqr} + R_{kalc;pqbr} + R_{kalc;pqrb}) \end{aligned}$$

$$
\begin{aligned}
&+ 5(R_{kblc;apqr} + R_{kblc;paqr} + R_{kblc;pqar} + R_{kblc;pqra}) \\
&- 5(R_{kalb;cpqr} + R_{kalb;pcqr} + R_{kalb;pqcr} + R_{kalb;pqrc}) \\
&- 51(R_{acku;lp}R_{qbru} + R_{bcku;lp}R_{qaru} + R_{kalu;pq}R_{bcru} + R_{kblu;pq}R_{acru}) \\
&- 17R_{kalu}(R_{bcpu;qr} + R_{pcqu;rb} - R_{pbqu;rc}) \\
&- 17R_{kblu}(R_{acpu;qr} + R_{pcqu;ra} - R_{paqu;rc}) + 67(R_{kabu} + R_{kbau})R_{lcpu;qr} \\
&+ R_{kclu}\{67(R_{pabu;qr} + R_{pbau;qr}) + 25(R_{paqu;br} + R_{paqu;rb}) \\
&+ 25(R_{pbqu;ar} + R_{pbqu;ra}) - 42(R_{paqb;ur} + R_{paqb;ru})\} \\
&- 110(R_{kalu;p}R_{bcqu;r} + R_{kblu;p}R_{acqu;r}) + 155R_{kclu;p}(R_{qabu;r} + R_{qbau;r}) \\
&- 15R_{kclu;p}R_{qabr;u} - 8R_{kalu}R_{pbqv}(R_{curv} + R_{cvru}) \\
&+ 24R_{kulv}(R_{acpu}R_{qbrv} + R_{bcpu}R_{qarv}) \\
&+ R_{kclu}\Big\{-104(R_{paqv}R_{burv} + R_{pbqv}R_{aurv}) + 8(R_{paqv}R_{bvru} + R_{pbqv}R_{avru}) \\
&+ \frac{248}{3}R_{puqv}(R_{rabv} + R_{rbav})\Big\}\Big] .
\end{aligned}
$$

Let us define $g(x) = \det(g_{ij}(x))$ on V. Then we have,

(7) $$
\begin{aligned}
g^{-1/4}(x) = 1 &+ G_{kl}x^k x^l + G_{klp}x^k x^l x^p + G_{klpq}x^k x^l x^p x^q \\
&+ G_{klpqr}x^k x^l x^p x^q x^r + G_{klpqrs}x^k x^l x^p x^q x^r x^s + o(|x|^6) ,
\end{aligned}
$$

where

$$G_{kl} = \frac{1}{12}\rho_{kl} ,$$

$$G_{klp} = \frac{1}{24}\rho_{kl;p} ,$$

$$G_{klpq} = \frac{1}{4!}\Big(\frac{3}{10}\rho_{kl;pq} + \frac{1}{12}\rho_{kl}\rho_{pq} + \frac{1}{15}R_{kulv}R_{puqv}\Big) ,$$

$$G_{klpqr} = \frac{1}{5!}\Big(\frac{1}{3}\rho_{kl;pqr} + \frac{5}{12}\rho_{kl}\rho_{pq;r} + \frac{1}{3}R_{kulv}R_{puqv;r}\Big) ,$$

$$
\begin{aligned}
G_{klpqrs} = \frac{1}{6!}\Big(&\frac{5}{14}\rho_{kl;pqrs} + \frac{3}{4}\rho_{kl}\rho_{pq;rs} + \frac{4}{7}R_{kulv}R_{puqv;rs} + \frac{5}{8}\rho_{kl;p}\rho_{qr;s} \\
&+ \frac{15}{28}R_{kulv;p}R_{qurv;s} + \frac{5}{72}\rho_{kl}\rho_{pq}\rho_{rs} \\
&+ \frac{1}{6}\rho_{kl}R_{puqv}R_{rusv} + \frac{8}{63}R_{kulv}R_{pvqw}R_{rwsu}\Big) .
\end{aligned}
$$

For the proof, see Sakai [5].

3. Coefficients $u_{1,k}$. Let $\sigma_x\colon [0, 1] \to M$ be the minimal geodesic joining the points m and x. By the parallel translation the curve σ_x defines an isomorphism $P_{x,t}$ of the cotangent space $T^*_{\sigma_x(t)}(M)$ at $\sigma_x(t)$ onto the cotangent space $T^*_x(M)$ at x. For a cotangent vector $\alpha = \alpha(m) \in T^*_m(M)$, let us define

$$U_0(\alpha, x) = g^{-1/4}(x)P_{x,0}(\alpha) \,. \tag{8}$$

Then $U_0(\alpha, x)$ is a smooth 1-form in the neighborhood V, and we define $U_k(\alpha, x)$, $k \geqq 1$, inductively as follows:

$$U_k(\alpha, x) = -g^{-1/4}(x)\int_0^1 g^{1/4}(\sigma_x(t))P_{x,t}(\varDelta U_{k-1}(\alpha, \sigma_x(t)))t^{k-1}dt \,. \tag{9}$$

See Berger [1] and Patodi [4]. The map $\alpha \rightarrow U_k(\alpha, m)$ is a linear endomorphism on $T^*_m(M)$ and the coefficients $u_{1,k}(m)$ is given by $\mathrm{Tr}\,(\alpha \rightarrow U_k(\alpha, m))$.

LEMMA 1. *For a cotangent vector* $\alpha = \alpha(m) = \alpha_i dx^i(m) \in T^*_m(M)$, $P_{x,0}(\alpha) = \alpha_i(x)dx^i$ *is represented as follows*:

$$\begin{aligned} \alpha_i(x) = \alpha_i &+ \frac{1}{3!}\alpha_j R_{kijl}x^k x^l + \frac{2}{4!}\alpha_j R_{kijl;p}x^k x^l x^p \\ &+ \frac{1}{5!}\alpha_j(3R_{kijl;pq} + R_{kilu}R_{pjqu})x^k x^l x^p x^q \\ &+ \frac{2}{6!}\alpha_j(2R_{kijl;pqr} + R_{kilu;p}R_{qjru} + 2R_{kilu}R_{pjqu;r})x^k x^l x^p x^q x^r \\ &+ \frac{1}{7!}\alpha_j(5R_{kijl;pqrs} + 10R_{kilu}R_{pjqu;rs} + 3R_{kjlu}R_{piqu;rs} \\ &+ 10R_{kilu;p}R_{qjru;s} - R_{kilu}R_{pjqv}R_{rusv})x^k x^l x^p x^q x^r x^s + o(|x|^6) \,. \end{aligned} \tag{10}$$

PROOF. Since $P_{x,0}(\alpha)$ is parallel along σ_x, we have

$$\frac{d\alpha_i(\sigma_x(t))}{dt} - \alpha_k(\sigma_x(t))\Gamma^k_{ij}(\sigma_x(t))x^j = 0 \,. \tag{11}$$

On the other hand, by the Taylor's expansion,

$$\alpha_i(x) = \alpha_i + \left.\frac{d\alpha_i(\sigma_x(t))}{dt}\right|_{t=0} + \frac{1}{2!}\left.\frac{d^2\alpha_i(\sigma_x(t))}{dt^2}\right|_{t=0} + \cdots \,. \tag{12}$$

The lemma follows from (11), (12), and (6).

Then from (7), (8), and (10), $U_0(\alpha, x) = (U_0(\alpha, x))_i dx^i$ is represented as follows:

$$\begin{aligned} (U_0(\alpha, x))_i = E^i &+ E^i_{kl}x^k x^l + E^i_{klp}x^k x^l x^p + E^i_{klpq}x^k x^l x^p x^q \\ &+ E^i_{klpqr}x^k x^l x^p x^q x^r + E^i_{klpqrs}x^k x^l x^p x^q x^r x^s + o(|x|^6) \,, \end{aligned} \tag{13}$$

where

$$E^i = \alpha_i ,$$

$$E^i_{kl} = \frac{1}{3!}\left(\frac{1}{2}\alpha_i\rho_{kl} + \alpha_j R_{kijl}\right),$$

$$E^i_{klp} = \frac{1}{4!}(\alpha_i\rho_{kl;p} + 2\alpha_j R_{kijl;p}),$$

$$E^i_{klpq} = \frac{1}{5!}\left\{\alpha_i\left(\frac{3}{2}\rho_{kl;pq} + \frac{5}{12}\rho_{kl}\rho_{pq} + \frac{1}{3}R_{kulv}R_{puqv}\right) + \alpha_j\left(3R_{kijl;pq} + \frac{5}{3}\rho_{kl}R_{pijq} + R_{kilu}R_{pjqu}\right)\right\},$$

$$E^i_{klpqr} = \frac{1}{6!}\left\{\alpha_i\left(2\rho_{kl;pqr} + \frac{5}{2}\rho_{kl}\rho_{pq;r} + 2R_{kulv}R_{puqv;r}\right) + \alpha_j(4R_{kijl;pqr} + 4R_{kilu}R_{pjqu;r} + 5\rho_{kl}R_{pijq;r} + 5\rho_{kl;p}R_{qijr} + 2R_{kilu;p}R_{qjru})\right\},$$

$$\begin{aligned}E^i_{klpqrs} = \frac{1}{7!}\Big\{&\alpha_i\Big(\frac{5}{2}\rho_{kl;pqrs} + \frac{21}{4}\rho_{kl}\rho_{pq;rs} + 4R_{kulv}R_{puqv;rs} + \frac{35}{8}\rho_{kl;p}\rho_{qr;s} \\ &+ \frac{15}{4}R_{kulv,p}R_{qurv;s} + \frac{35}{72}\rho_{kl}\rho_{pq}\rho_{rs} \\ &+ \frac{7}{6}\rho_{kl}R_{puqv}R_{rusv} + \frac{8}{9}R_{kulv}R_{pvqw}R_{rwsu}\Big) \\ &+ \alpha_j\Big(5R_{kijl;pqrs} + \frac{21}{2}\rho_{kl;pq}R_{rijs} + \frac{21}{2}\rho_{kl}R_{pijq;rs} \\ &+ 10R_{kilu}R_{pjqu;rs} + 3R_{kjlu}R_{piqu;rs} \\ &+ \frac{35}{2}\rho_{kl;p}R_{qijr;s} + 10R_{kilu;p}R_{qjru;s} + \frac{35}{12}\rho_{kl}\rho_{pq}R_{rijs} \\ &+ \frac{7}{2}\rho_{kl}R_{piqu}R_{rjsu} + \frac{7}{3}R_{kulv}R_{puqv}R_{rijs} - R_{kilu}R_{pjqv}R_{rusv}\Big)\Big\}.\end{aligned}$$

4. Calculation. From now on we assume that M is a compact Einstein space, i.e., $\rho(X, Y) = c\langle X, Y\rangle$, $c \in R$. Then the Laplacian Δ acting on 1-forms is locally represented as follows:

$$\begin{aligned}(14)\qquad (\Delta\omega)_h(x) &= -g^{ij}(x)\omega_{h;ij}(x) + c\omega_h(x) \\ &= -g^{ij}(x)\Big\{\frac{\partial^2\omega_h(x)}{\partial x^i\partial x^j} - \omega_u(x)\frac{\Gamma^u_{hj}(x)}{\partial x^i} \\ &\quad - \frac{\partial\omega_u(x)}{\partial x^i}\Gamma^u_{hj}(x) - \frac{\partial\omega_u(x)}{\partial x^j}\Gamma^u_{hi}(x) - \frac{\partial\omega_h(x)}{\partial x^u}\Gamma^u_{ij}(x) \\ &\quad + \omega_u(x)(\Gamma^u_{vj}(x)\Gamma^v_{hi}(x) + \Gamma^u_{vh}(x)\Gamma^v_{ij}(x))\Big\} + c\omega_h(x),\end{aligned}$$

for a 1-form $\omega(x) = \omega_h(x)dx^h$. Therefore, from (5), (6), (13), and (14), $\Delta U_0(\alpha, x) = (\Delta U_0(\alpha, x))_h dx^h$ is given as follows:

(15) $$(\Delta U_0(\alpha, x))_h = S^h + S^h_{kl}x^kx^l + S^h_{klp}x^kx^lx^p + S^h_{klpq}x^kx^lx^px^q + o(|x|^4)\ ,$$

where

$$\begin{aligned}
S^h &= H^{hii} + cE^h\ ,\\
S^h_{kl} &= H^{hii}_{kl} + A^{ij}_{kl}H^{hij} + cE^h_{kl}\ ,\\
S^h_{klpq} &= H^{hii}_{klpq} + A^{ij}_{kl}H^{hij}_{pq} + A^{ij}_{klp}H^{hij}_{q} + A^{ij}_{klpq}H^{hij} + cE^h_{klpq}\ ,\\
H^{hij} &= E^uB^{uhj}_i - (E^h_{ij} + E^h_{ji})\ ,\\
H^{hij}_k &= E^u(B^{uhj}_{ik} + B^{uhj}_{ki}) - Q^h_{ijk}\ ,\\
H^{hij}_{kl} &= E^u(B^{uhj}_{ikl} + B^{uhj}_{kil} + B^{uhj}_{kli} - B^{uvj}_kB^{vhi}_l - B^{uvh}_kB^{vij}_l) + (E^u_{ik} + E^u_{ki})B^{uhj}_l\\
&\quad + (E^u_{jk} + E^u_{kj})B^{uhi}_l + (E^h_{uk} + E^h_{ku})B^{uij}_l + E^u_{kl}B^{uhj}_i - Q^h_{ijkl}\ ,\\
H^{hij}_{klpq} &= E^u(B^{uhj}_{iklpq} + B^{uhj}_{kilpq} + B^{uhj}_{klipq} + B^{uhj}_{klpiq} + B^{uhj}_{klpqi} - B^{uvj}_kB^{vhi}_{lpq}\\
&\quad - B^{uvj}_{kl}B^{vhi}_{pq} - B^{uvj}_{klp}B^{vhi}_q - B^{uvh}_kB^{vij}_{lpq} - B^{uvh}_{kl}B^{vij}_{pq} - B^{uvh}_{klp}B^{vij}_q)\\
&\quad + E^u_{kl}(B^{uhj}_{ipq} + B^{uhj}_{piq} + B^{uhj}_{pqi} - B^{uvj}_pB^{vhi}_q - B^{uvh}_pB^{vij}_q)\\
&\quad + (E^u_{ik} + E^u_{ki})B^{uhj}_{lpq} + (E^u_{jk} + E^u_{kj})B^{uhi}_{lpq} + (E^h_{uk} + E^h_{ku})B^{uij}_{lpq}\\
&\quad + (E^u_{ikl} + E^u_{kil} + E^u_{kli})B^{uhj}_{pq} + (E^u_{jkl} + E^u_{kjl} + E^u_{klj})B^{uhi}_{pq}\\
&\quad + (E^h_{ukl} + E^h_{kul} + E^h_{klu})B^{uij}_{pq} + E^u_{klp}(B^{uhj}_{iq} + B^{uhj}_{qi})\\
&\quad + (E^u_{iklp} + E^u_{kilp} + E^u_{klip} + E^u_{klpi})B^{uhj}_q\\
&\quad + (E^u_{jklp} + E^u_{kjlp} + E^u_{kljp} + E^u_{klpj})B^{uhi}_q\\
&\quad + (E^h_{uklp} + E^h_{kulp} + E^h_{klup} + E^h_{klpu})B^{uij}_q + E^u_{klpq}B^{uhj}_i - Q^h_{ijklpq}\ ,\\
Q^h_{ijk} &= E^h_{ijk} + E^h_{jik} + E^h_{jki} + E^h_{ikj} + E^h_{kij} + E^h_{kji}\ ,\\
Q^h_{ijkl} &= E^h_{ijkl} + E^h_{jikl} + E^h_{jkil} + E^h_{jkli} + E^h_{ikjl} + E^h_{kijl}\\
&\quad + E^h_{kjil} + E^h_{kjli} + E^h_{iklj} + E^h_{kilj} + E^h_{klij} + E^h_{klji}\ ,\\
Q^h_{ijklpq} &= E^h_{ijklpq} + E^h_{jiklpq} + E^h_{jkilpq} + E^h_{jklipq} + E^h_{jklpiq} + E^h_{jklpqi}\\
&\quad + E^h_{ikjlpq} + E^h_{kijlpq} + E^h_{kjilpq} + E^h_{kjlipq} + E^h_{kjlpiq} + E^h_{kjlpqi}\\
&\quad + E^h_{ikljpq} + E^h_{kiljpq} + E^h_{klijpq} + E^h_{kljipq} + E^h_{kljpiq} + E^h_{kljpqi}\\
&\quad + E^h_{iklpjq} + E^h_{kilpjq} + E^h_{klipjq} + E^h_{klpijq} + E^h_{klpjiq} + E^h_{klpjqi}\\
&\quad + E^h_{iklpqj} + E^h_{kilpqj} + E^h_{klipqj} + E^h_{klpiqj} + E^h_{klpqij} + E^h_{klpqji}\ .
\end{aligned}$$

By the parallel translation along σ_x we have,

$$(P_{x,t}(\Delta U_0(\alpha, \sigma_x(t))))_h = S^h + \left(\frac{1-t^2}{2}S^uB^{uhk}_l + t^2S^h_{kl}\right)x^kx^l$$

$$
\text{(16)}\quad \begin{aligned}
&+\left(\frac{1-t^3}{3}S^uB_{lp}^{uhk}+t^3S_{klp}^h\right)x^kx^lx^p\\
&+\left\{\frac{(1-t^2)^2}{8}S^uB_l^{uvk}B_q^{vhp}+\frac{1-t^4}{4}S^uB_{lpq}^{uhk}\right.\\
&\left.+\frac{t^2(1-t^2)}{2}S_{kl}^uB_q^{uhp}+t^4S_{klpq}^h\right\}x^kx^lx^px^q+o(|x|^4)\,.
\end{aligned}
$$

Therefore, by (9), (7), and (16), $U_1(\alpha, x) = (U_1(\alpha, x))_h dx^h$ is represented as follows:

$$
\text{(17)}\qquad (U_1(\alpha, x))_h = T^h + T_{kl}^h x^k x^l + T_{klp}^h x^k x^l x^p + T_{klpq}^h x^k x^l x^p x^q + o(|x|^4)\,,
$$

where

$$
\begin{aligned}
T^h &= -S^h\,,\\
T_{kl}^h &= -\frac{1}{3}(S^uB_l^{uhk}+S_{kl}^h+2S^hG_{kl})\,,\\
T_{klpq}^h &= -\frac{1}{15}(S^uB_l^{uvk}B_q^{vhp}+3S^uB_{lpq}^{uhk}+S_{kl}^uB_q^{uhp}+3S_{klpq}^h\\
&\quad +4S^uB_l^{uhk}G_{pq}+2S_{kl}^hG_{pq}-2S^hG_{kl}G_{pq}+12S^hG_{klpq})\,.
\end{aligned}
$$

Similarly $U_2(\alpha, x) = (U_2(\alpha, x))_h dx^h$ is represented as follows:

$$
\text{(18)}\qquad (U_2(\alpha, x))_h = F^h + F_k^h x^k + F_{kl}^h x^k x^l + o(|x|^2)\,,
$$

where

$$
\begin{aligned}
F^h &= -\frac{1}{2}(T^uB_i^{uhi}-2T_{ii}^h+cT^h)\,,\\
F_{kl}^u &= -\frac{1}{4}\Big\{T^u(B_{ikl}^{uhi}+B_{kil}^{uhi}+B_{kli}^{uhi}-B_k^{uvi}B_l^{vhi}-B_k^{uvh}B_l^{vii})\\
&\quad +2(T_{ik}^u+T_{ki}^u)B_l^{uhi}+(T_{uk}^h+T_{ku}^h)B_l^{uii}+T_{kl}^uB_i^{uhi}\\
&\quad -2(T_{iikl}^h+T_{ikil}^h+T_{ikli}^h+T_{kiil}^h+T_{kili}^h+T_{klii}^h)\\
&\quad +T^uA_{kl}^{ij}B_i^{uhj}-(T_{ij}^h+T_{ji}^h)A_{kl}^{ij}+cT_{kl}^h\\
&\quad +(T^uB_i^{uhi}-2T_{ii}^h+cT^h)G_{kl}+\frac{1}{2}(T^uB_i^{uvi}-2T_{ii}^v+cT^v)B_l^{vhk}\Big\}\,.
\end{aligned}
$$

Finally $U_3(\alpha, m) = (U_3(\alpha, m))_h dx^h(m)$ is represented as follows:

$$
\text{(19)}\quad \begin{aligned}
(U_3(\alpha, m))_h &= -\frac{1}{6}\Big\{T^u(B_{ijj}^{uhi}+B_{jij}^{uhi}+B_{jji}^{uhi}-B_j^{uvi}B_j^{vhi}-B_j^{uvh}B_j^{vii})\\
&\quad +2(T_{ij}^u+T_{ji}^u)B_j^{uhi}+(T_{uj}^h+T_{ju}^h)B_j^{uii}+T_{jj}^uB_i^{uhi}\\
&\quad -4(T_{iijj}^h+T_{ijij}^h+T_{ijji}^h)+T^uA_{kk}^{ij}B_j^{uhi}
\end{aligned}
$$

$$- 2T^h_{ij}A^{ij}_{kk} + cT^h_{ii} - \frac{1}{2}(T^u B^{uvi}_i - 2T^v_{ii} + cT^v)B^{vhj}_j$$

$$+ (T^u B^{uhi}_i - 2T^h_{ii} + cT^h)G_{jj} - c(T^u B^{uhi}_i - 2T^h_{ii} + cT^h)\Big\} .$$

For the calculation of $u_{1,3}(m) = \operatorname{Tr}(\alpha \to U_3(\alpha, m))$ we use the formulas (2.1) through (2.11) in Sakai [5] and the following:

(20) $$R_{abcd}R_{aucv}R_{budv} = \frac{1}{2}c|R|^2 - \frac{1}{4}R_{abcd}R_{abuv}R_{cduv} + \frac{1}{4}|\nabla R|^2 - \frac{1}{8}\Delta|R|^2 ,$$

(21) $$R_{abcd}R_{aucv}R_{bvdu} = \frac{1}{2}c|R|^2 - \frac{1}{2}R_{abcd}R_{abuv}R_{cduv} + \frac{1}{4}|\nabla R|^2 - \frac{1}{8}\Delta|R|^2 ,$$

which are easily derived from the Ricci's identity in the Einsteinian case. Thus we have,

(22) $$\begin{aligned} u_{1,3}(m) = \frac{1}{6!}\Big\{&\Big(\frac{5}{18}n^4 - \frac{76}{9}n^3 + \frac{572}{9}n^2 - \frac{1172}{9}n\Big)c^3 \\ &+\Big(\frac{1}{9}n^2 - \frac{29}{3}n + \frac{632}{9}\Big)c|R|^2 - 5R_{abcd}R_{abuv}R_{cduv} \\ &+ \frac{5}{3}|\nabla R|^2 - \frac{5}{6}\Delta|R|^2\Big\} + \frac{2}{3}\operatorname{Tr}(\alpha_h \to T^h_{iijj} + T^h_{ijij} + T^h_{ijji}) ,\end{aligned}$$

(23) $$\begin{aligned} &\operatorname{Tr}(\alpha_h \to T^h_{iijj} + T^h_{ijij} + T^h_{ijji}) \\ &= \frac{1}{6!}\Big\{\Big(\frac{1}{3}n^4 - \frac{38}{15}n^3 + \frac{8}{15}n^2 + \frac{176}{15}n\Big)c^3 \\ &\quad + \Big(\frac{8}{15}n^2 - \frac{16}{5}n - \frac{146}{15}\Big)c|R|^2 + 3R_{abcd}R_{abuv}R_{cduv} - |\nabla R|^2 \\ &\quad + \frac{1}{2}\Delta|R|^2\Big\} - \frac{1}{5}\operatorname{Tr}(\alpha_h \to S^h_{iijj} + S^h_{ijij} + S^h_{ijji}) ,\end{aligned}$$

(24) $$\begin{aligned} &\operatorname{Tr}(\alpha_h \to S^h_{iijj} + S^h_{ijij} + S^h_{ijji}) \\ &= \frac{1}{6!}\Big\{\Big(-\frac{5}{6}n^3 + 13n^2 - \frac{2}{3}n\Big)c^3 + (27n + 69)c|R|^2 \\ &\quad + 90R_{abcd}R_{abuv}R_{cduv}\Big\} + \operatorname{Tr}(\alpha_h \to H^{huu}_{iijj} + H^{huu}_{ijij} + H^{huu}_{ijji}) ,\end{aligned}$$

(25) $$\begin{aligned} \operatorname{Tr}(\alpha_h \to H^{huu}_{iijj}) = \frac{1}{6!}\Big\{&\Big(-\frac{5}{12}n^4 + \frac{26}{9}n^3 + \frac{442}{63}n^2 - \frac{76}{3}n\Big)c^3 \\ &+ \Big(-n^2 + \frac{10}{21}n + \frac{121}{7}\Big)c|R|^2 + \Big(-\frac{4}{7}n - \frac{405}{14}\Big)R_{abcd}R_{abuv}R_{cduv}\end{aligned}$$

$$+\left(\frac{30}{7}n+\frac{20}{7}\right)|\nabla R|^2+\left(-\frac{15}{7}n-\frac{10}{7}\right)\Delta|R|^2\Big\}\,,$$

$$\operatorname{Tr}(\alpha_h\to H^{huu}_{ijij})=\frac{1}{6!}\Big\{\left(-\frac{5}{12}n^3-\frac{46}{3}n^2-\frac{26}{3}n\right)c^3$$

$$(26)\qquad +\left(-\frac{1}{3}n^2-8n+\frac{437}{21}\right)c|R|^2+\left(-\frac{8}{7}n+20\right)R_{abcd}R_{abuv}R_{cduv}$$

$$+\left(-\frac{95}{84}n-\frac{529}{84}\right)|\nabla R|^2+\left(-\frac{50}{21}n+\frac{5149}{168}\right)\Delta|R|^2\Big\}\,,$$

$$\operatorname{Tr}(\alpha_h\to H^{huu}_{ijji})=\frac{1}{6!}\Big\{\left(\frac{85}{36}n^3-\frac{31}{9}n^2+\frac{50}{3}n\right)c^3$$

$$(27)\qquad +\left(-\frac{1}{6}n^2-\frac{85}{42}n-\frac{1031}{21}\right)c|R|^2+\left(-\frac{8}{7}n-\frac{410}{7}\right)R_{abcd}R_{abuv}R_{cduv}$$

$$+\left(-\frac{65}{28}n-\frac{341}{84}\right)|\nabla R|^2+\left(-\frac{5}{7}n+\frac{5801}{168}\right)\Delta|R|^2\Big\}\,.$$

From (22) through (27) we have,

$$u_{1,3}(m)=\frac{1}{6!}\Big\{\left(\frac{5}{9}n^4-\frac{32}{3}n^3+\frac{4016}{63}n^2-120n\right)c^3$$

$$(28)\qquad +\left(\frac{2}{3}n^2-\frac{890}{63}n+56\right)c|R|^2+\left(\frac{8}{21}n-6\right)R_{abcd}R_{abuv}R_{cduv}$$

$$+\left(-\frac{1}{9}n+2\right)|\nabla R|^2+\left(\frac{44}{63}n-9\right)\Delta|R|^2\Big\}\,.$$

Integrating $u_{1,3}$ over M, we have our main lemma.

LEMMA 2. *Let M be an n-dimensional compact Einstein space of constant scalar curvature nc. Then the coefficient $a_{1,3}$ in* (3) *for $p=1$ is given as follows*:

$$a_{1,3}=\frac{1}{6!}\Big\{\left(\frac{5}{9}n^4-\frac{32}{3}n^3+\frac{4016}{63}n^2-120n\right)c^3\operatorname{vol}(M)$$

$$(29)\qquad +\left(\frac{2}{3}n^2-\frac{890}{63}n+56\right)c\int_M|R|^2$$

$$+\left(\frac{8}{21}n-6\right)\int_M R_{abcd}R_{abuv}R_{cduv}+\left(-\frac{1}{9}n+2\right)\int_M|\nabla R|^2\Big\}\,.$$

For the proof of our theorem, we employ the following two lemmas.

LEMMA 3. (V. K. Patodi) *Let M be an n-dimensional compact Einstein space of constant scalar curvature nc, and M' a compact Riemannian manifold. Suppose that*

$$\mathrm{Spec}^0(M) = \mathrm{Spec}^0(M') \quad \textit{and} \quad \mathrm{Spec}^1(M) = \mathrm{Spec}^1(M'),$$

then M' is also an n-dimensional Einstein space of constant scalar curvature nc.

For the proof, see Patodi [3]. Under the same condition as Lemma 3, the following are well known:

$$\mathrm{vol}(M) = \mathrm{vol}(M'), \tag{30}$$

$$\int_M |R|^2 = \int_{M'} |R'|^2. \tag{31}$$

LEMMA 4. (T. Sakai) *For an n-dimensional compact Einstein space M of constant scalar curvature nc, the coefficient $a_{0,3}$ in* (3) *for $p = 0$ is given as follows*:

$$\begin{aligned} a_{0,3} = \frac{1}{6!}\Big\{\Big(\frac{5}{9}n^3 - \frac{2}{3}n^2 - \frac{16}{63}n\Big)c^3 \mathrm{vol}(M) + \Big(\frac{2}{3}n - \frac{8}{63}\Big)c\int_M |R|^2 \\ + \frac{8}{21}\int_M R_{abcd}R_{abuv}R_{cduv} - \frac{1}{9}\int_M |\nabla R|^2\Big\}. \end{aligned} \tag{32}$$

For the proof, see Sakai [5]. From Lemma 3, (30), (31), and Lemma 4, we have,

$$\int_M |\nabla R|^2 = \int_{M'} |\nabla R'|^2,$$

which completes the proof of our theorem.

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