# Curvature-bounded traversals of narrow corridors

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## ABSTRACT

We consider the existence and efficient construction of bounded curvature paths traversing constant-width regions of the plane, called *corridors*. We make explicit a width threshold $\tau$ with the property that (a) all corridors of width at least $\tau$ admit a unit-curvature traversal and (b) for any width $w < \tau$ there exist corridors of width $w$ with no such traversal. Applications to the design of short, but not necessarily shortest, and high clearance, but not necessarily maximum clearance, curvature- bounded paths in general polygonal domains, are also discussed.

## 1. INTRODUCTION

We are interested in studying the existence and construction of paths of bounded curvature (e.g. the trace of the midpoint of the rear axle of a car), in the presence of obstacles in the plane. The curvature of any smooth path is defined as the inverse of its minimum curvature radius over all points on the path. Thus, a $\kappa$-curvature path has curvature radius at least $1/\kappa$ at every point. Both 0-curvature (straight) and $\infty$-curvature (polygonal) paths are (relatively) simple special cases. Furthermore, a $\kappa$-curvature path becomes a unit-curvature path under scaling by $\kappa$. Thus, it is natural to restrict attention to unit-curvature paths.

The focus of our present study is the problem of constructing "good" (which could be interpreted as "short" or "high clearance", or some combination of these), but not necessarily optimal, curvature-bounded paths in connected, but not necessarily simply connected, polygonal regions. Such a region $\mathcal{P}$, called a *passageway*, is described by (i) a simple polgonal curve $\mathcal{B}$ (the *boundary* of $\mathcal{P}$), (ii) two distinguished edges of $\mathcal{B}$ (the initial and final *gates* of $\mathcal{P}$), and (iii) zero or more disjoint polygonal obstacles in the interior of $\mathcal{P}$. We seek unit-curvature *traversals* of $\mathcal{P}$: unit-curvature paths in the interior of $\mathcal{P}$ that connect the initial and final gates and avoid all obstacles (cf. Fig. 1 (a)).

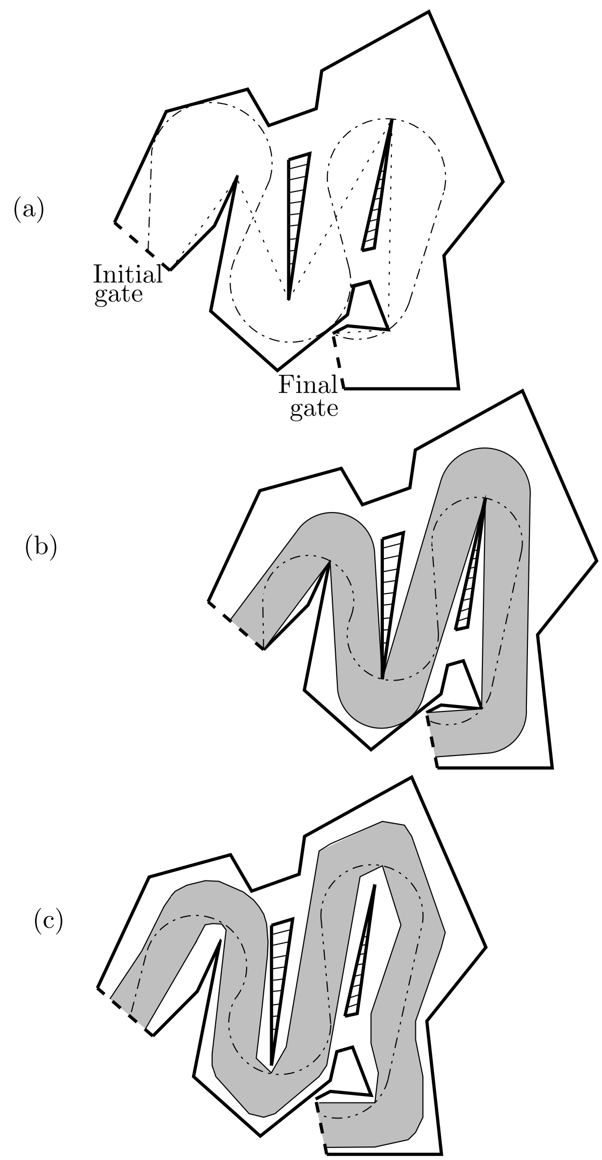


**Figure 1: (a) passageway $\mathcal{P}$ with bounded-curvature (dot-dashed) and $\infty$-curvature (dotted) traversals; (b) a minimum length corridor (shaded), and associated bounded-curvature traversal, in $\mathcal{P}$; and (c) maximum clearance corridor (shaded) and associated bounded-curvature traversal, in $\mathcal{P}$.**

In particular, we want to determine, for a given passageway $\mathcal{P}$, whether there exists a unit-curvature traversal of $\mathcal{P}$. Furthermore, if such a traversal exists, we would like to construct one whose length (or other criterion of quality) is close to optimal among all such traversals and, if such a traversal does not exist, we would like to determine the smallest curvature bound greater than 1 for which such a traversal does exist.

## 1.1 Related work

Curvature-bounded motion-planning problems have received considerable attention in the robotics literature [3, 6, 7, 11, 18, 25]. For surveys the reader may consult [17, 19, 24]. In general, path planning problems (given a collection of obstacles and specified initial and final configurations) in dimensions higher than 2 are hard, even if the curvature bound $\kappa$ is set to $\infty$. Specifically if the dimension is not fixed the path existence problem is PSPACE-complete [8, 23]. In three dimensions, Canny and Reif [9] proved that the shortest (polygonal) path problem is NP-hard. In two dimensions, Reif and Wang [22] showed that the problem of finding a shortest unit-curvature path in a general polygonal domain (a polygonal region with polygonal obstacles or holes) is NP-hard.

In contrast, efficient approximation algorithms are known [16, 16, 26, 2]. Boissonnat and Lazard [6] give an $O(n^2 \log n)$ algorithm for finding shortest unit-curvature paths amid disjoint moderate obstacles (convex obstacles bounded by segments of zero or unit curvature). Agarwal et al. [1] obtain an $O(n^2 \log n)$ algorithm for finding shortest unit-curvature paths in a convex polygonal region. Boissonnat et al. [5] present a linear time algorithm for finding a convex unit-curvature path (not necessarily shortest), if one exists, in a simple polygon. The construction of smooth paths, using spline functions, in simple polygons (channels) defined by two polgonal chains, is studied in [4, 20].

A shortest polygonal (i.e. $\infty$-curvature) path $\pi_\infty$, with specified endpoints, can be determined efficiently in a simple polygon [12, 13], in the presence of polygonal obstacles [15], and with a specified homotopy [14] or minimum clearance (using Minkowski expansion of all barriers). Furthermore, using the idea of retraction motion planning [21] a path $\pi_{max}$ of maximum clearance (within a specified homotopy class) can be determined efficiently from the generalized Voronoi diagram of the domain.

Taken together these allow us to construct "good" paths of specified minimum clearance within a fixed homotopy class in a specified polygonal domain (cf. Fig. 1 (b),(c)). This motivates the study of curvature bounded paths in constant-width passageways.

## 1.2 Overview and summary of results

One of the things that distinguishes our approach is its departure from the conventional focus on minimum length curvature-bounded paths (or close approximations thereof), which appears to be motivated at least as much by the desire to restrict the search space as the need to optimize the path length. Instead, as we have illustrated, we seek a unit-curvature path in the neighbourhood of some specified feasible path $\pi$ that has some specified clearance. In so doing we trade a modest (potential) increase in path length for a guarantee (based on global, but efficiently computable, properties of the domain) of the existence of a unit-curvature path that approximates (and, hence, inherits some of the properties of) $\pi$, together with an efficient algorithm for its construction.

The existence of a unit-curvature approximation of a path $\pi$ is thus reduced to the existence of a unit-curvature traversal of a fixed width *corridor*, the subset of a passageway's freespace formed by sweeping a disk of some fixed width $w$ along $\pi$. Note that corridors formed in this way are connected but not necessarily simply connected (cf. Fig 2). (Corridors and their properties are described more precisely in the next section.)

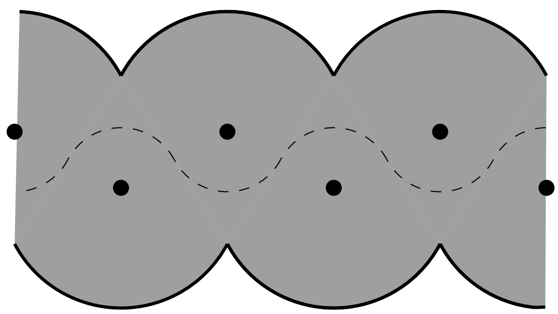

**Figure 2: A non-simply-connected corridor and its centerline**

Obviously there is a tradeoff between the width of corridors and the maximum curvature necessary for their traversal since, in general, wider corridors admit gentler turns. It is interesting to ask what one can say about the (unit-curvature) traversability of a corridor knowing only its width. For example, it is easy to see that every corridor of width at least 2 admits a unit-curvature traversal (in fact the path traced by the center of a disk of diameter 2 as it rolls along the boundary of such a corridor has curvature at most 1). Furthermore, corridors of width 1 are not guaranteed to admit such a traversal. (The reader is invited to confirm that the corridor illustrated in Fig. 3 has this property.)

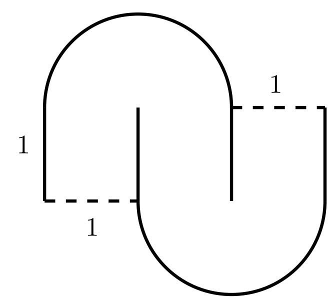


**Figure 3: A corridor with no unit-curvature traversal.**

One of the main contributions of this paper is to make this tradeoff explicit. Specifically, let $\tau$ be the unique root of the cubic equation $w^3 - 5w^2 - 16w + 32 = 0$ in the interval $[1, 2]$ ($\tau \approx 1.50515$). We prove the following:

Theorem 1. *(a) For any $w < \tau$ there exist corridors of width $w$ that do not admit a unit-curvature traversal;*
*(b) Every corridor $\mathcal{C}$ of width $w \geq \tau$ admits a unit-curvature traversal. Furthermore, such a traversal can be constructed in time linear in the complexity of $\mathcal{C}$.*

In Section 2 we provide a more precise definition of corridors and describe some of their properties. We also introduce a special class of corridors, called *switchbacks* that play a fundamental role in the proof of Theorem 1(a). Section 3 describes a normal form for corridor traversals. For the special case of switchbacks this allows us to describe unit-curvature traversals as annotations of the shortest (unrestricted curvature) traversal. This leads naturally to the specification of a critical switchback that embodies the definition of (and intuition behind) the threshold constant $\tau$. The proof of Theorem 1(a) itself is outlined in Section 5. The proof of Theorem 1(b), for the special case of switchback corridors, is presented in Section 6; the proof for general corridors is sketched in Section 7. Section 8 mentions some related results and open questions.

## 2. CORRIDORS, REDUCED CORRIDORS AND SWITCHBACKS

As we have defined them, corridors of width $w$ can be expressed as the Minkowski sum $\pi \oplus D$ of a path $\pi$ with a disk $D$ of diameter $w$. Such a corridor is said to be *reduced* if it does not properly contain another corridor $\pi' \oplus D$, where $\pi'$ has the same endpoints as $\pi$ (cf. Fig. 4).

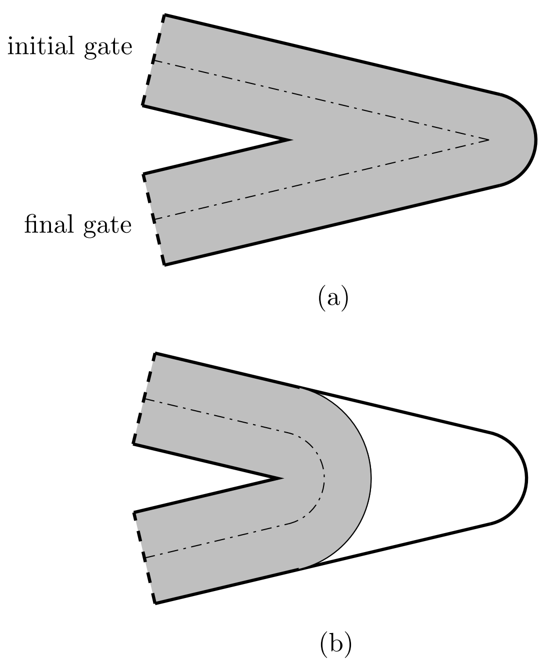


**Figure 4: (a) a corridor and (b) a corresponding reduced corridor.**

It is straightforward to confirm that the centerline $\pi$ of a reduced corridor of width $w$ forms a simple smooth curve of curvature at least $w/2$. In fact if $\pi$ is the centerline of a minimum length corridor, not only is the associated corridor reduced but $\pi$ is composed of straight segments and arcs of circles of radius exactly $w/2$. Since an arbitrary smooth curve of curvature at least $w/2$ can be approximated arbitrarily closely by a curve of this form, we will assume hereafter that any given reduced corridor is expressed in this way.

Any point $p$ on the centerline $\pi$ defines two points $p_L$ and $p_R$ at distance $w/2$ from $p$ in directions normal to $\pi$ at $p$. The locus of all such points describes two (not necessarily simple) curves $\pi_L$ and $\pi_R$ that serve to restrict the corridor locally. The locus of the segment $p_L p_R$ traces a continuously changing diameter that does not locally self-intersect. For this reason, we can view reduced corridors as a fixed sequence of straight sections (*pipes*) and wedges (*elbows*) (cf. Fig 5).

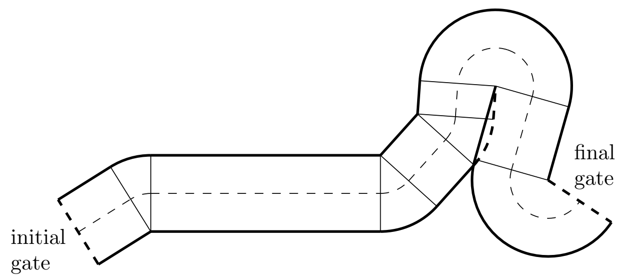


**Figure 5: A corridor formed from pipes and elbows.**

Viewed in this way, reduced corridors express the constraints associated with real-world roadways that can cross over themselves and spiral while remaining locally planar.

Since corridors of width at least 2 admit straightforward unit-curvature traversals, it suffices to restrict our attention to corridors of width $w < 2$, what we call *narrow* corridors. Narrow corridors are a particularly interesting special case of simply connected passageways in that they admit only *monotone* traversals (u-turns are impossible). It follows from this that any curvature-bounded traversal is at most some constant factor longer than the corridor centerline.

A *long* elbow is a corridor segment whose associated arc spans more than $180°$. A corridor that is composed of a (necessarily alternating) sequence of left and right long elbows (i.e. all of its straight sections have length zero) is called a *switchback*. Switchbacks are interesting because they constitute what are intuitively (and, as we shall see, provably) the most difficult corridors to traverse. It is not hard to see that switchbacks are completely characterized by their shortest ($\infty$-curvature) traversals (which describes a zig-zag path, each link of length $w$, between successive corners). Seriously mixing metaphors, we refer to this path as the *spine* of the switchback. Fig. 6 illustrates a switchback and its associated spine, providing a different view of the same corridor illustrated in Fig. 2.

## 3. NORMAL PATHS IN CORRIDORS AND SWITCHBACKS

If there is a curvature-bounded path joining two configurations $I$ and $F$ then there is a corresponding Dubins path [10], a smooth curve consisting of line segments and arcs of ra-

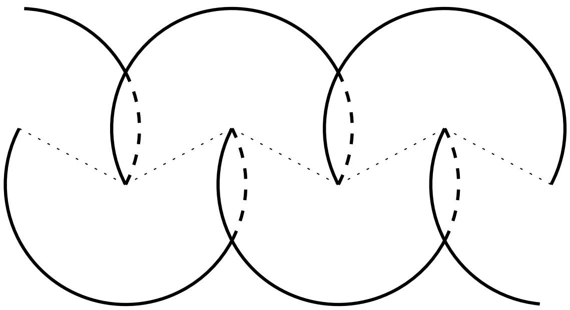

**Figure 6: A switchback and its spine (dotted)**

dius 1. Furthermore, Fortune and Wilfong [11] show that, in presence of obstacles, a path can be chosen to have an even more restrictive normal form. An arc of a path is *short* (resp. *long*) if its angle is less than (resp. at least) 180°. A short arc of a path is *supported* if it touches an obstacle inside of the arc. A long arc is supported if it touches an obstacle outside of the arc. A path is *fully supported* if all its arcs are supported.

LEMMA 2 (FORTUNE AND WILFONG [11]). *If there is a curvature-bounded path from $I$ to $F$ in a bounded region with obstacles, then there is a fully supported path from $I$ to $F$.*

Among other things this allows us to assume that every *long elbow* in a reduced corridor is traversed using either a *wide turn* (touching the outer boundary of the elbow) or a *shallow turn* (touching the elbow centre). It follows that if a switchback is traversable at all it is traversable by a path consisting of a sequence of wide and shallow turns, one for each elbow. The specific wide (resp. shallow) turn used within a given elbow with center point $O$ is neatly specified by a line segment of length $w-1$ (respectively 1) joining $O$ to the center $A$ (resp. $B$) of the turning circle (cf. Fig. 7).

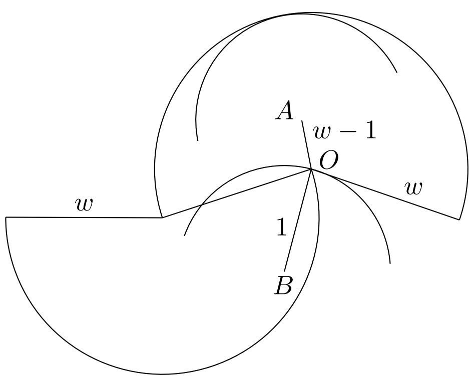


**Figure 7: A switchback elbow with wide and shallow turns.**

In this way, a full traversal of a switchback is specified by annotating its spine with appropriate wide or shallow turn segments at each of its corners. Similarly, the feasibility of a traversal can be determined by determining feasible wide and shallow turn segments for each corner in its spine. Of course the feasibility of a transition from a turn in one elbow to a turn in its successor requires that the centers of the associated turning circles be at least distance 2 apart.

Within a single wide elbow with center $O$, a wide (resp. shallow) turn arc with center $A$ is *earlier* than a wide (resp. shallow) turn with center $A'$ if the angle formed by line segment $OA$ and the spine edge entering $O$ is smaller than that formed by segment $OA'$. Since earlier turns provide more flexibility for turns in subsequent elbows, switchback traversals admit an even more restrictive normal form: each turn in the traversal is the earliest feasible turn of its type within its elbow. We refer to such traversals as *greedy* traversals.

## 4. CRITICAL WIDTH SWITCHBACKS

A typical switchback will admit many distinct traversals even in our most restrictive (greedy) normal form. In this section we show how to construct a switchback whose structure is critical in the sense that, while it admits exponentially many distinct greedy traversals, any one of its greedy turn transitions can be made infeasible by a small perturbation. The width of this critical corridor defines our width threshold $\tau$.

Consider the configuration of turning circle centers illustrated in Figure 8. The points $A$ and $B$ represent the centers

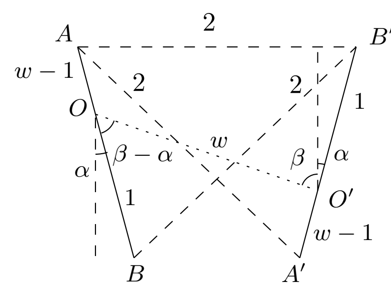


**Figure 8: Configuration specifying $\tau$.**

of aligned wide and shallow turning circles around $O$. Similarly, the points $A'$ and $B'$ represent the centers of aligned wide and shallow turning circles around $O'$. If $|AA'|$, $|BB'|$ and $|AB'|$ are all constrained to be equal to 2, the quadralateral $AB'A'B$, drawn with $AB'$ horizontal, has vertical symmetry. Thus $BA'$ is also horizontal and the segments $AB$ and $A'B'$ have complementary slopes. Let $\alpha$ denote the angle of deviation of segment $AB$ from the vertical, and let $\beta$ denote the angle of deviation of the segment $OO'$ from the vertical. We denote by $\tau$ the unique value of the width variable $w$ implied by all of the separation constraints.

LEMMA 3. *The width $\tau$ is the only root of the cubic equation $w^3 - 5w^2 - 16w + 32 = 0$ in the interval $[1, 2]$ ($\tau \approx 1.50515$).*

PROOF. The condition $|AA'| = 2$ can be written as

$$(w \sin\beta)^2 + (w\cos\beta + 2(w-1)\cos\alpha)^2 = 4. \qquad (1)$$

Since the segment $AB'$ is horizontal, the condition $|AB'| = 2$

can be written as

$$w \sin\alpha + w \sin\beta = 2 \quad (2)$$

$$(w-1)\cos\alpha + w\cos\beta = \cos\alpha \quad (3)$$

The second equation simplifies to

$$w\cos\beta = (2-w)\cos\alpha. \quad (4)$$

Substituting $w\cos\beta$ in Equation (1) we get $(w\sin\beta)^2 + (w\cos\alpha)^2 = 4$ or

$$(w\sin\beta)^2 + w^2(1-\sin^2\alpha) = 4.$$

Combining this with Equation (2) we obtain $(2-w\sin\alpha)^2 + w^2(1-\sin^2\alpha) = 4$ or

$$w = 4\sin\alpha. \quad (5)$$

Thus

$$\begin{aligned} w^2/4 &= (w\sin\beta)^2/4 + (w\cos\beta)^2/4 \\ &= (4 - w^2\sin^2\alpha)/4 + ((2-w)\cos\alpha)^2/4 \\ &= 1-(w-1)\cos^2\alpha = 1-(w-1)(1-\sin^2\alpha) \\ &= 1-(w-1)(1-w^2/16). \end{aligned}$$

This simplifies to the cubic equation $p(w) = 0$ where $p(w) = w^3 - 5w^2 - 16w + 32$. Since $p(1) = 12$ and $p(2) = -16$ there is a unique root $\tau$ of the polynomial $p(w)$ in the interval $[1,2]$. The approximate value of $\tau$ is 1.50515. $\square$

The angles $\alpha$ and $\beta$, computed from Equation (5) and Equation (4), satisfy $\alpha \approx 22.103°$ and $\beta \approx 72.265°$.

If we take the configuration illustrated in Figure **??** and compose it with copies of itself and its horizontal reflection in an alternating sequence we form the annotated spine of a switchback (cf. Fig 9).

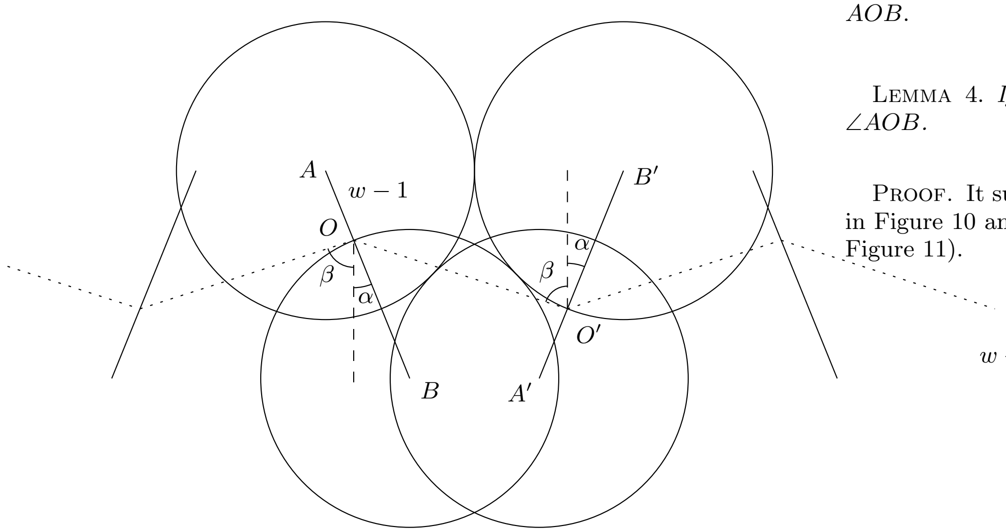


**Figure 9: Critical switchback.**

It is easy to confirm that for each successive pair of elbows in the associated switchback there are *kissing* transitions (direct transitions between successive turning circles) of three different forms: *wide-wide* (from the circle at $A$ to the circle at $A'$), *shallow-shallow* (from the circle at $B$ to the circle at $B'$), and *wide-shallow* (from the circle at $A$ to the circle at $B'$). If the angles $\alpha$ or $\beta$ are changed, or the width $w$ is reduced below $\tau$, then at least one of these is impossible.

## 5. PROOF OF THEOREM 1(A)

Suppose $w < \tau$. We outline the construction of a switchback of width $w$ that does not admit a unit-curvature traversal, thereby establishing Theorem 1(a). As was the case for the critical width switchback, it is easiest to describe the construction in terms of the switchback spine.

Consider two successive spine corners and the earliest feasible wide and shallow turns associated with each corner (cf. Fig. 10).

*******remark on initial state?******

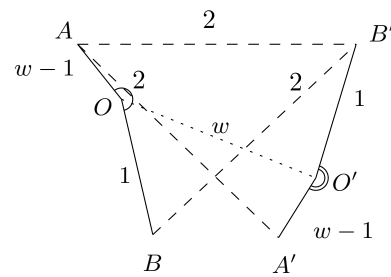


**Figure 10: Tight transition configuration.**

Since $A'$ and $B'$ denote the positions of earliest wide and shallow turns around corner $O'$, it follows that $\max\{|AB'|, |BB'|\} = 2$ and $\max\{|AA'|, |BA'|\} = 2$. If we choose $|AB'| = |BB'| = |AA'| = 2$, then the positions of $A'$,$B'$ and $O'$ are completely determined relative to $A$,$B$ and $O$. Furthermore, the (clockwise) angle $B'O'A'$ must be less than the (clockwise) angle $AOB$.

Lemma 4. *If $w < \tau$ and $\angle AOB \geq 180°$ then $\angle B'O'A' < \angle AOB$.*

Proof. It suffices to relax the constraint that $|OO'| = w$ in Figure 10 and insist instead that $\angle B'O'A' = \angle AOB$ (cf. Figure 11).

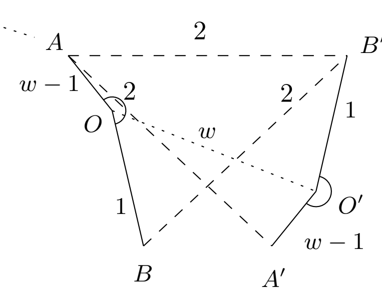


**Figure 11: Proof of Lemma 4.**

Once again the positions of $A'$,$B'$ and $O'$ are completely determined relative to $A$,$B$ and $O$. By exploiting the symmetry of the configuration, it is easy to show that $|OO'| > w$

(in fact, this holds when $\angle AOB = 180°$ and $|OO'|$ increases monotonically with $\angle AOB$, for $\angle AOB > 180°$. It follows that if we reduce the length of segment $OO'$ to $w$, the angle $\angle B'O'A'$ must decrease (since point $B'$ remains fixed and point $A'$ must remain on the circle of radius 2 centered at $A$).

□

A completely symmetric argument (essentially viewing Figure 10 upside down, and reversing the roles of $A$ and $A'$ etc.) shows that, assuming $w < \tau$, $\angle B'O'A' < \angle AOB$ in the case where $\angle AOB < 180°$ as well.

Thus, if we start with any initial configuration of the turn segments (describing the earliest feasible turns in an initial elbow) and we chain together, in an alternating sequence, many copies of the construction of Figure 10 and its vertical reflection, adjusting the turn segments to reflect the deviations in successive elbows, we produce an annotated spine (and, implicitly, a switchback) with the property that the forward angle at the final corner can be made arbitrarily small.

It is easy to confirm that as long as the forward angle $\angle AOB$ is no smaller than some critical value $\Theta$ the point $B'$ is well defined (i.e. $|AB'| = |BB'| = 2$ is satisfiable). However, when $\angle AOB = \Theta$ it is easy to check that the point $A'$ does not exist. Thus, at some point in our construction a wide turn is no longer feasible. If we break the sequence at this point and add one last very wide elbow (with the spine edge passing through the center of the last shallow turning circle, for example) then a continuation from the last shallow turn becomes impossible. Since neither a wide or shallow turn is possible at this point, the resulting switchback is not traversable.

## 6. PROOF OF THEOREM ??(B) FOR SWITCHBACKS

We first outline a proof of Theorem 1(b) for the special case of switchbacks. (Of course, the bound is tight even in this case by the result of the preceeding section.)

Let $S$ be an arbitrary switchback of width $w = \tau$ and let $p_1, p_2, \ldots, p_n$ be its spine. The proof of the lower bound depended critically on the property of the configuration defining $\tau$ that the angles $\alpha$ and $\gamma$, describing sustainable wide and shallow turns, are equal. In effect the wide and shallow turn circles are *aligned* with their associated corner. The key idea of the proof is that for each corner in succession it is always possible to find such an aligned pair of wide and shallow turns such that each turn associated with corner $p_i$ is reachable from at least one of the turns associated with corner $p_{i-1}$.

LEMMA 5. *Let $A_{i-1}B_{i-1}$ be an aligned turn configuration at corner $p_{i-1}$. Then there exists an aligned turn configuration $A_iB_i$ at corner $p_i$.*

PROOF. By the switchback property $|p_{i-1}p_i| = \tau$. There are two configurations $A_i'B_i'$ and $A_i''B_i''$, at points $p_i'$ and

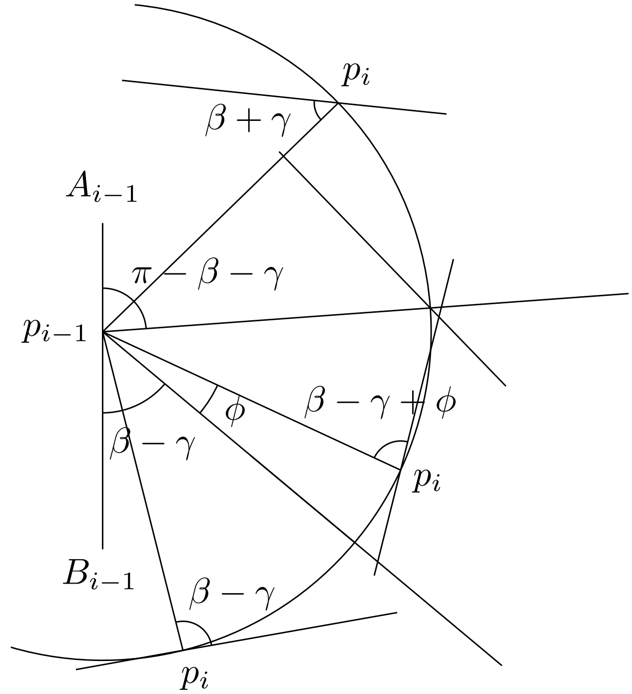


**Figure 12: Strategies for next feasible turns.**

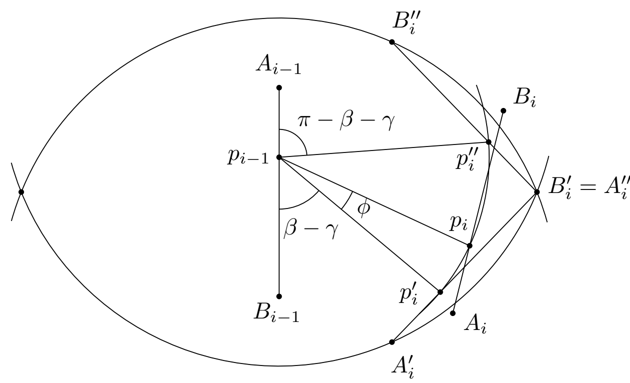


**Figure 13: Proof of the range property.**

$p_i''$ respectively, in Figure 12 that correspond to the definition of $\tau$, see Fig. 7. They share the same point $B_i' = A_i''$ at the corner of the lune formed by two circles of radius 2 with centers $A_{i-1}$ and $B_{i-1}$. The angle $\angle p_i'p_{i-1}p_i''$ is equal to $2\gamma$ and $\angle p_{i-1}p_i'B_i' = \beta + \gamma$ and $\angle p_{i-1}p_i''B_i'' = \beta - \gamma$. If the point $p_i$ moves upward from $p_i''$ and the points $A_i''$ and $B_i''$ remain on the lune keeping their distances from $p_i$ then the angle $B_i''p_iA_i''$ increases and becomes larger than $\pi$. Similarly, if the point $p_i$ moves downward from $p_i'$ then the angle $B_i'p_iA_i'$ increases and becomes larger than $\pi$. It remains to consider the locations of the point $p_i$ in between $p_i'$ and $p_i''$.

Note that the segment $p_{i-1}p_i$ rotates by the angle $2\gamma$ when $p_i$ moves from $p_i'$ to $p_i''$. The segment $A_i'B_i'$ rotates to the segment $A_i''B_i''$ by the angle $2\gamma$ with respect to the rotating segment $p_{i-1}p_i$. So, in general, we define two points $A_i$ and $B_i$ as endpoints of the continuously moving segment, see

Fig. 12. Let $\phi$ be the angle $p_i p_{i-1} p_i'$. Then $\angle p_i p_{i+1} B_i = \beta + \gamma - \phi$.

Let $x = \angle B_{i-1} p_{i-1} p_i = \beta - \gamma + \phi$ and $y = \angle p_{i-1} p_i B_i - \angle B_{i-1} p_{i-1} p_i = 2\gamma - 2\phi$. The squared distance $|A_{i-1}A_i|$ can be written as

$$\begin{aligned} &|A_{i-1}A_i|^2 = \\ &(w\sin x - (w-1)\sin y)^2 + ((w-1) + w\cos x + (w-1)\cos y)^2 \\ &= w^2 + 2(w-1)^2 + 2w(w-1)(-\sin x \sin y + \cos x \cos y + \cos x) \\ &\qquad + 2(w-1)^2 \cos y \\ &= w^2 + 2(w-1)^2 + 2w(w-1)(\cos(x+y) + \cos x) + 2(w-1)^2 \cos y. \end{aligned}$$

Note that

$$\cos(x+y) + \cos x = 2\cos(x + y/2)\cos(y/2) = 2\cos\beta\cos(\gamma - \phi).$$

Hence

$$\begin{aligned} &|A_{i-1}A_i|^2 = w^2 + 2(w-1)^2 \\ &+ 4w(w-1)\cos\beta\cos(\gamma-\phi) + 2(w-1)^2\cos(2(\gamma-\phi)). \end{aligned}$$

The minimum value of $|A_{i-1}A_i|$ is achieved when $\phi = 0$ and is equal to 2.

Let $x$ and $y$ be as above and let $z = \cos(\gamma - \phi)$. The squared distance $|B_{i-1}B_i|$ can be written as a quadratic equation in terms of $z$

$$\begin{aligned} |B_{i-1}B_i|^2 &= (w\sin x + \sin y)^2 + (1 - w\cos x + \cos y)^2 \\ &= w^2 + 2 + 2w(\sin x \sin y - \cos x \cos y - \cos x) + 2\cos y \\ &= w^2 + 2 - 2w(\cos(x+y) + \cos x) + 2\cos y \\ &= w^2 + 2 - 2w\cos\beta\cos(\gamma-\phi) + 4\cos^2(\gamma-\phi) - 2 \\ &= 4z^2 - 2(w\cos\beta)z + w^2. \end{aligned}$$

Since $\phi \in [0, 2\gamma]$, the angle $\gamma - \phi$ varies from $\gamma$ to $-\gamma$. Then $z \in [\cos\gamma, 1]$. Let $z_0 = w\cos\beta/4$ be the value of $z$ that minimizes $|B_{i-1}B_i|^2$. We show that $z > z_0$ for any $\phi[0, 2\gamma]$. It means $\cos\gamma > (w\cos\beta)/4$. By Equation (4) it reduces to $4\cos\gamma > (2-w)\cos\gamma$ or $4 > 2 - w$ or $w < 2$.

The minimum of $|B_{i-1}B_i|^2$ is achieved when $\phi \in \{0, 2\gamma\}$. Then $|B_{i-1}B_i| = 2$. Therefore $|B_{i-1}B_i| \geq 2$ for any $\phi \in [0, 2\gamma]$. $\square$

## 7. UPPER BOUND FOR GENERAL CORRIDORS

In this section we outline the proof of Theorem 1(b) in its full generality exploiting the similarities with the proof of the special case (switchbacks) from the preceding section.

As in the case of switchbacks, we construct a unit-curvature traversal of an arbitrary corridor $\mathcal{C}$ of width $w \geq \tau$ incrementally, using the shortest path $p_1, p_2, \ldots, p_n$ (of unbounded curvature) through $\mathcal{C}$ (the analogue of the switchback spine) as a template. We show that at each of a subsequence of points (called *critical points*) along this path we can define aligned turning configurations with the property (as before) that both turning circles associated with the $i$th critical point can be reached from at least one of the turning circles associated with the $i-1$st critical point.

Two considerations make the general proof considerably more involved:
(i) successive points on the shortest path may belong to the same or opposite side of the corridor;
(ii) the length of individual segments on the shortest path may be less than $\tau$ or greater than $\tau$.

A point $p_i$ on the shortest path through $\mathcal{C}$ is *critical* if either (i) $i = 1$, or (ii) $p_{i-1}$ and $p_i$ lie on the opposite sides of the corridor, or (iii) $p_i$ is the first vertex after a critical point $p_j, j < i$ which lies outside at least one of the disks, shallow turn or wide turn at $p_j$, see Fig. 14.

The intuition is that non-critical vertices may not be suitable as "checkpoints" at which we can establish a new configuration with aligned turning circles. (Note that if $p_i$ lies in the intersection of the shallow and wide turn disks of its critical predecessor $p_j$ then a shallow turn at $p_i$, which passes through $p_i$, may not be possible.)

A *transition* is a path between turn circles of two consecutive critical points. A $s-s$ *transition* corresponds to the path between shallow turn circles of two consecutive critical points. There are also $s-w$, $w-w$, and $w-w$ transitions. In order to represent two paths (reaching shallow and wide turns) we use notation: $s-s, w-w$ transition, $s-sw$ transition ($s-s$ and $s-w$ paths), and $w-sw$ transition ($w-s$ and $w-w$ paths). A *same side transition* is defined by critical points on the same side, otherwise the transition is called an *opposite side* transition.

We show that a transition always exists. The transition depends on the location of next critical point (and its side). Our argument involves a large number of cases.

For same-side transitions we distinguish short (distance less than $d_{max}$, defined to be the distance $|OP|$ in Figure 15) and long transitions.

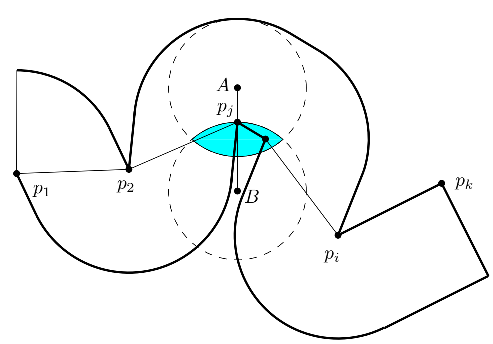


**Figure 14: Critical points. $p_i$ is the first point of $\Pi$ outside the shaded lune.**

Figure 17 (respectively, Figures 18 and 19) illustrates the coverage of short $s-s, w-w$ transitions (respectively, $s-sw$ transitions and $w-sw$ transitions). (As the Figures suggest, the details are somewhat involved.)

Sufficiently long transitions of either type (same or opposite side) are achieved by essentially the same mechanism. A transition is made from the initial configuration to the next

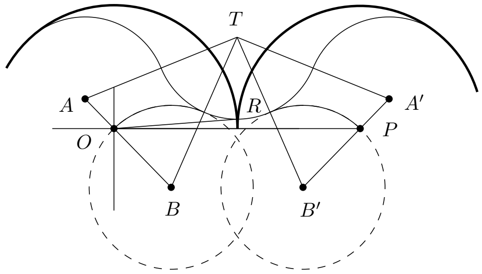


**Figure 15: Radius of large circle $R_{\max}$.**

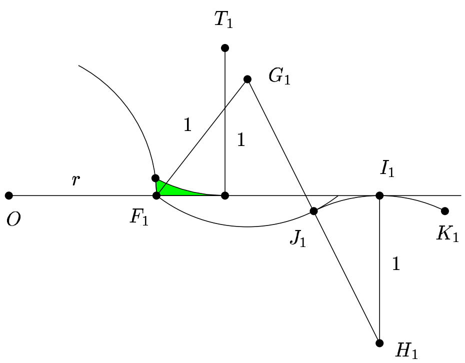


**Figure 16: Case with a barrier in shaded region.**

edge on the shortest path. This edge is then followed until the path reaches the neighbourhood of the target configuration at which point a transition is made to each of the two turn disks defining that configuration. Figure 16 illustrates some of the subtlety needed for long transitions (especially at the boundary between short and long transitions).

# 8. CONCLUSION

We have shown how to construct unit-curvature traversals of corridors of width $\tau$, the minimum width for which such traversals are guaranteed to exist. (An efficient implementation is discussed in the full paper.) It remains to demonstrate an efficient algorithm to determine if a unit-curvature traversal exists for specific corridors of width less than $\tau$.

Similar results (with a slightly smaller width threshold $\tau_0 = 2\sqrt{3} - 2 \approx 1.46410$ can be shown for the restricted case of simple (non-self-intersecting) corridors. It also turns out that our algorithms and analysis of the curvature-bounded traversal problem for corridors have application to other natural problems that do not relate directly to motion planning. One of these, the *curvature-bounded separation problem*, takes as input two collections of polygonal objects (say red objects and blue objects) together with a curve $C$ that divides the plane into two unbounded regions containing red or blue objects exclusively, and asks for a curvature-bounded homotope of $C$. As before the existence of such a separator depends on the placement (and in particular the proximity) of the objects. We are able to provide an exact specification of the minimum object separation that guarantees the existence of a curvature-bounded separator.

An additional application concerns the *curvature-bounded polyline approximation* problem: given a polygonal chain $A$ and a bound $\delta > 0$, find a curvature-bounded curve $\tilde{A}$ such that (i) there exists a continuous bijection $\lambda$ from $A$ to $\tilde{A}$ satisfying $|\lambda(p) - p| \leq \delta$, and (ii) $\tilde{A}$ respects the corners of $A$ in the sense that if $A$ turns right (respectively, left) at corner $p$ then $p$ lies to the right (respectively, left) of $\tilde{A}$. Once again the existence of such an approximation depends on the shape of $A$ and the parameter $\delta$, but we are able to specify a minimum value $\Delta$ such that $\delta \geq \Delta$ guarantees an approximation regardless of the shape of $A$.

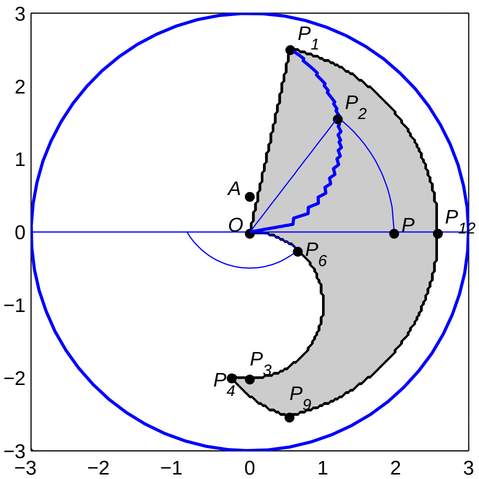


**Figure 17: Region $R_{ss-ww}$.**

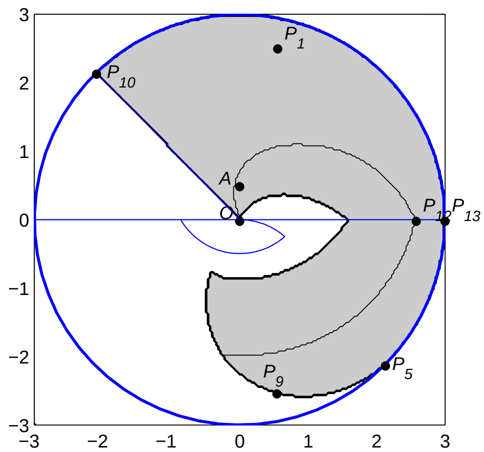


**Figure 18: Region $R_{s-sw}$.**

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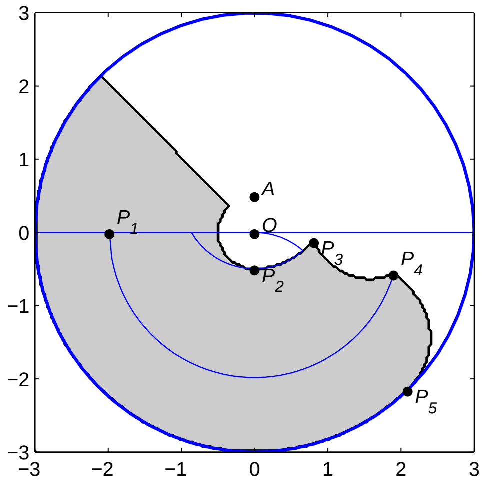


**Figure 19: Region $R_{w-sw}$.**

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