

CURVATURE ESTIMATES FOR MINIMAL HYPERSURFACES

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In [12] J. Simons initiated a study of minimal cones from a more differential geometric point of view than had previously been attempted. One of Simons' main results was an identity for the Laplacian of the second fundamental form of minimal hyper-surfaces. Coupling this identity with an analysis of the first eigenvalue of a certain differential operator, he was able to prove that no non-trivial n -dimensional stable minimal cones exist in \mathbf{R}^{n+1} for $n \leq 6$. He was thus able to demonstrate that any boundary of least area in \mathbf{R}^{n+1} , $n \leq 6$, must in fact be a hyperplane, because Fleming [7] had demonstrated that the non-existence of non-trivial stable minimal cones in \mathbf{R}^n implies the result that the only boundaries of least area in \mathbf{R}^n are the hyperplanes.

Simons was in fact able to deduce that, for $n \leq 7$, the only entire solutions of the minimal surface equation

$$(1 + |\nabla u|^2) \sum_{j=1}^n \frac{\partial^2 u}{\partial x_j^2} - \sum_{i,j=1}^n \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0 \quad (*)$$

are linear functions, because De Giorgi [6] had improved Fleming's result in the non-parametric case, by showing that the non-existence of non-trivial stable minimal cones in \mathbf{R}^n implies that the only *non-parametric* boundaries of least area in \mathbf{R}^{n+1} are the hyperplanes. The conjecture that the only entire solution of (*) are linear functions was known as the Bernstein conjecture, after Bernstein [2]. Prior to Simons' paper, it had been settled in the case $n=2$ by Bernstein [2], $n=3$ by De Giorgi [6] and $n=4$ by Almgren [1]. Subsequent to Simons' paper the conjecture was finally completely settled; it was shown to be false for $n > 7$ by Bombieri, De Giorgi and Giusti [3].

In the case $n=2$ Heinz [8] considered solutions of (*) which were defined over a disc $\{x \in \mathbf{R}^2: |x - x_0| < R\}$. He proved there is an absolute constant β such that

$$(\kappa_1^2 + \kappa_2^2)(x_0) \leq \beta/R^2, \quad (**)$$

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where κ_1, κ_2 are principal curvatures of the graph of the solution u of (*). In the case when u was an entire solution of (*) Heinz let $R \rightarrow \infty$ in (**) and hence proved $\kappa_1 = \kappa_2 \equiv 0$; i.e., (**) implies Bernstein's theorem in the case $n=2$. The result (**) and its proof have been refined by various authors, and a parametric version was obtained by Osserman (See [11]). However, the methods used were all strictly 2-dimensional.

In this paper we will use Simons' identity for the Laplacian of the second fundamental form for minimal hypersurfaces to obtain a number of new estimates for the curvatures of stable minimal hypersurfaces M which are immersed in a Riemannian manifold N . Under suitable restrictions on N , we will in fact obtain (see Theorem 3) a pointwise bound for the principal curvatures of M , provided $\dim(M) \leq 5$. In the special case when $N = \mathbf{R}^{n+1}$, when M is an area minimizing hypersurface with boundary outside the ball $\{x \in \mathbf{R}^{n+1}: |x - x_0| < R\}$ and when $n \leq 5$, Theorem 3 gives the inequality (cf. (**) above)

$$\sum_{i=1}^n \kappa_i^2(x_0) \leq \beta/R^2,$$

where $\kappa_1, \dots, \kappa_n$ are the principal curvatures of M and β is an absolute constant. When $\partial M = \phi$, we can let $R \rightarrow \infty$ and deduce $\kappa_i \equiv 0$, $i=1, \dots, n$; i.e., we obtain a proof of Bernstein's Theorem for $n \leq 5$. A Bernstein-type result which is valid in a more general setting is given in Theorem 2.

In the final section of the present paper we give a simplified proof of Simons' result that there are no non-trivial 6-dimensional stable minimal cones in \mathbf{R}^7 .

§ 1. Notation and preliminary results

In this section, we set up our terminology and record Chern's [4] computation of Simons' inequality for minimal hypersurfaces. (See inequality (1.20) below.) We then demonstrate that this inequality gives a better lower bound (inequality (1.34)) for the Laplacian of $|A|$ (A = second fundamental form of M) than had previously been realized.

Let M be an oriented n -dimensional manifold immersed in an oriented $(n+1)$ -dimensional Riemannian manifold N . We choose a local field of orthonormal frames e_1, \dots, e_{n+1} in N such that, restricted to M , the vectors e_1, \dots, e_n are tangent to M . With respect to this frame field of N , let $\omega_1, \dots, \omega_{n+1}$ be the field of dual frames. Then the structure equations of N are given by

$$d\omega_i = - \sum_{j=1}^{n+1} \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0 \quad (1.1)$$

$$d\omega_{ij} = - \sum_{k=1}^{n+1} \omega_{ik} \wedge \omega_{kj} + \Omega_{ij}, \quad (1.2)$$

where

$$\Omega_{ij} = \frac{1}{2} \sum_{k,l=1}^{n+1} K_{ijkl} \omega_k \wedge \omega_l$$

and

$$K_{ijkl} + K_{ijlk} = 0.$$

We restrict these forms to M . Then

$$\omega_{n+1} = 0. \tag{1.3}$$

Since $0 = d\omega_{n+1} = -\sum_{i=1}^n \omega_{n+1,i} \wedge \omega_i$, by Cartan's lemma we can write

$$\omega_{n+1,i} = \sum_j h_{ij} \omega_j, \quad h_{ij} = h_{ji}. \tag{1.4}$$

Here and in what follows, the range of summation is from 1 to n .

By using (1.1)-(1.4), we obtain

$$d\omega_i = -\sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{1.5}$$

$$d\omega_{ij} = -\sum_k \omega_{ik} \wedge \omega_{kj} + \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{1.6}$$

where

$$R_{ijkl} = K_{ijkl} + h_{ik}h_{jl} - h_{il}h_{jk}. \tag{1.7}$$

The form $\sum_{i,j} h_{ij} \omega_i \omega_j$ and the scalar $(1/n) \sum_i h_{ii} = H$ are called respectively the second fundamental form and the mean curvature of the immersed manifold M . If H is identically zero, M is said to be minimal.

Now exterior differentiate (1.4) and define h_{ijk} by

$$\sum_k h_{ijk} \omega_k = dh_{ij} - \sum_k h_{ik} \omega_{kj} - \sum_k h_{kj} \omega_{ki}. \tag{1.8}$$

Then

$$\sum_j (h_{ijk} + \frac{1}{2} K_{n+1,ijk}) \omega_j \wedge \omega_k = 0, \tag{1.9}$$

$$h_{ijk} - h_{ikj} = K_{n+1,ikj} = -K_{n+1,ijk}. \tag{1.10}$$

Next, we exterior differentiate (1.8) and define h_{ijkl} by

$$\sum_l h_{ijkl} \omega_l = dh_{ijk} - \sum_l h_{iljk} \omega_{lj} - \sum_l h_{ijlk} \omega_{li} - \sum_l h_{ijkl} \omega_{lk} - \sum_l h_{iljk} \omega_{lj}. \tag{1.11}$$

Then

$$\sum_{k,l} (h_{ijkl} - \frac{1}{2} \sum_m h_{im} R_{m,kl} - \frac{1}{2} \sum_m h_{mj} R_{mikl}) \omega_k \wedge \omega_l = 0, \tag{1.12}$$

and

$$h_{,jkl} - h_{ijlk} = \sum_m h_{im} R_{mjkl} + \sum_m h_{mj} R_{mikl}. \tag{1.13}$$

Let us now denote the covariant derivative of K_{ijk} , as a curvature tensor of N , by $K_{ijk;l:m}$. Then restricting to M , we obtain

$$K_{n+1,ijk;l} = K_{n+1,ijkl} - K_{n+1,i,n+1,k}h_{jl} - K_{n+1,ij,n+1}h_{kl} + \sum_m h_{mi}K_{mjlk}, \tag{1.14}$$

where

$$\sum_i K_{n+1,ijk} \omega_i = dK_{n+1,ijk} - \sum_m K_{n+1,mjk} \omega_{mi} - \sum_m K_{n+1,imk} \omega_{mj} - \sum_m K_{n+1,ijm} \omega_{mk}. \tag{1.15}$$

The Laplacian Δh_{ij} of the second fundamental form h_{ij} is defined by

$$\Delta h_{ij} = \sum_k h_{ijkk}. \tag{1.16}$$

From (1.10), we obtain

$$\Delta h_{ij} = \sum_k h_{ikjk} - \sum_k K_{n+1,ijk} = \sum_k h_{kij} - \sum_k K_{n+1,ijk}. \tag{1.17}$$

Also, from (1.13) we obtain

$$h_{kij} = h_{kikj} + \sum_m h_{km}R_{mijk} + \sum_m h_{mi}R_{mkjk}. \tag{1.18}$$

Then if we replace h_{kikj} in (1.18) by $h_{kij} - K_{n+1,kikj}$ (by (1.10)) and if we substitute the right hand side of (1.18) into h_{kij} of (1.17), we obtain

$$\Delta h_{ij} = \sum_k (h_{kij} - K_{n+1,kikj} - K_{n+1,ijk}) + \sum_k (\sum_m h_{km}R_{mijk} + \sum_m h_{mi}R_{mkjk}). \tag{1.19}$$

From (1.7), (1.14) and (1.19) we then obtain

$$\begin{aligned} \Delta h_{ij} &= \sum_k h_{kicij} - \sum_k K_{n+1,kik;j} - \sum_k K_{n+1,ijk;k} + \sum_k (-h_{ik}K_{n+1,ij,n+1} - h_{ij}K_{n+1,k,n+1,k}) \\ &\quad + \sum_{m,k} (h_{mj}K_{mkitk} + h_{mi}K_{mkjlk} + 2h_{mk}K_{mijl}) \\ &\quad + \sum_{m,k} (h_{mi}h_{mj}h_{ik} + h_{icm}h_{ci}h_{mj} - h_{icm}h_{icm}h_{ij} - h_{mi}h_{mk}h_{kj}). \end{aligned} \tag{1.20}$$

Now assuming M is minimal in N , so that $\sum_k h_{kk} = 0$, we obtain

$$\begin{aligned} \sum_{i,j} h_{ij} \Delta h_{ij} &= - \sum_{i,j,k} h_{ij} K_{n+1,kik;j} - \sum_{i,j,k} h_{ij} K_{n+1,ijk;k} - \sum_{i,j,k} h_{ij}^2 K_{n+1,k,n+1,k} \\ &\quad + \sum_{m,i,j,k} (2h_{mj}h_{ij}K_{mkitk} + 2h_{mk}h_{ij}K_{mijl}) - (\sum_{i,j} h_{ij}^2)^2. \end{aligned} \tag{1.21}$$

Up to now, we have followed the exposition in [5]. In order to proceed, we assume that the sectional curvatures of N are bounded between K_1 and K_2 and

$$|\nabla K|^2 = \sum_{i,j,k,l,m} K_{ijkl}^2; m \leq c^2. \tag{1.22}$$

For any point $p \in M$, we can choose our frame $\{e_1, \dots, e_n\}$ at that point so that

$$h_{ij} = \lambda_i \delta_{ij}. \tag{1.23}$$

At such a point we have

$$\begin{aligned} \sum_{m,i,j,k} (2h_m h_{ij} K_{mki} + 2h_{nk} h_{ij} K_{mij}) &= \sum_{i,k} (2\lambda_i^2 K_{ikik} + 2\lambda_k \lambda_i K_{kiki}) = \sum_{i,k} (\lambda_i^2 - 2\lambda_i \lambda_k + \lambda_k^2) K_{ikik} \\ &= \sum_{i,k} (\lambda_i - \lambda_k)^2 K_{ikik} \geq K_2 \sum_{i,k} (\lambda_i - \lambda_k)^2 = 2nK_2 \sum_i \lambda_i^2 - 2K_2 (\sum_i \lambda_i)^2 = 2nK_2 \sum_i \lambda_i^2. \end{aligned} \tag{1.24}$$

It then follows from (1.21), (1.22), (1.23) and (1.24) that

$$\begin{aligned} \sum_{i,j} h_{ij} \Delta h_{ij} &\geq -2\sqrt{\sum_{i,j} h_{ij}^2} \sqrt{\sum_{i,j,k,l,m} K_{ijklm}^2} - nK_1 (\sum_{i,j} h_{ij}^2) + 2nK_2 (\sum_{i,j} h_{ij}^2) - (\sum_{i,j} h_{ij}^2)^2 \\ &\geq -2c\sqrt{\sum_{i,j} h_{ij}^2} + n(2K_2 - K_1) (\sum_{i,j} h_{ij}^2) - (\sum_{i,j} h_{ij}^2)^2 \end{aligned} \tag{1.25}$$

Now let

$$|A|^2 = \sum_{i,j} h_{ij}^2. \tag{1.26}$$

Then (1.25) shows that, at all points where $|A| \neq 0$,

$$\begin{aligned} 2|A|\Delta|A| + 2(\nabla|A|)^2 &= \Delta|A|^2 = 2 \sum_{i,j,k} h_{ij}^2 h_{jk} + 2 \sum_{i,j} h_{ij} \Delta h_{ij} \\ &\geq 2 \sum_{i,j,k} h_{ij}^2 h_{jk} - 4c|A| + 2n(2K_2 - K_1)|A|^2 - 2|A|^4. \end{aligned} \tag{1.27}$$

The crucial point now is to give a lower bound for $\sum_{i,j,k} h_{ij}^2 h_{jk}$ in terms of $|\nabla|A|^2|$. First, by using (1.10) together with the inequality

$$2|K_{n+1,ij}| \leq K_1 - K_2,$$

we obtain

$$\begin{aligned} \sqrt{\sum_{i \neq j} h_{ij}^2} &\geq \sqrt{\sum_{i \neq j} h_{ii}^2} - \sqrt{\sum_{i,j} (h_{ij} - h_{ii})^2} \geq \sqrt{\sum_{i \neq j} h_{ii}^2} - \sqrt{\sum_{i,j} K_{n+1,ij}^2} \\ &\geq \sqrt{\sum_{i \neq j} h_{ii}^2} - \frac{1}{2} \sqrt{n(n-1)} (K_1 - K_2). \end{aligned} \tag{1.28}$$

Also,

$$\begin{aligned} \sum_{i,j,k} h_{ij}^2 h_{jk} - |\nabla|A|^2| &= [(\sum_{i,j} h_{ij}^2) (\sum_{i,j,k} h_{ij}^2 h_{jk}) - \sum_k (\sum_{i,j} h_{ij} h_{ijk})^2] (\sum_{i,j} h_{ij}^2)^{-1} \\ &= \frac{1}{2} \sum_{i,j,s,t,k} (h_{ij} h_{stk} - h_{st} h_{ijk})^2 (\sum_{i,j} h_{ij}^2)^{-1}, \end{aligned} \tag{1.29}$$

and, using (1.23),

$$\begin{aligned} \sum_{i,j,s,t,k} (h_{ij} h_{stk} - h_{st} h_{ijk})^2 &= \sum_{i,s,t,k} (h_{ii} h_{stk} - h_{st} h_{iik})^2 + (\sum_{s,t} h_{st}^2) (\sum_{i \neq j} h_{ij}^2) \\ &\geq (\sum_i h_{ii}^2) \sum_{s \neq t} h_{stk}^2 + \sum_{s,t} h_{st}^2 (\sum_{i \neq j} h_{ij}^2) = 2(\sum_{s,t} h_{st}^2) (\sum_{i \neq j} h_{ij}^2). \end{aligned} \tag{1.30}$$

But by using (1.28) we obtain

$$\sum_{i \neq j} h_{ij}^2 \geq \sum_{i \neq j} h_{ii}^2 + \sum_{i \neq j} h_{ij}^2 = 2 \sum_{i \neq j} h_{ii}^2 \geq \frac{2}{1 + \varepsilon} (\sum_{i \neq j} h_{ii}^2) - \frac{n(n-1)}{2\varepsilon} (K_1 - K_2)^2 \tag{1.31}$$

for all $\varepsilon > 0$. Here we have used the fact that

$$\sqrt{A} \geq \sqrt{B} - \sqrt{C} \text{ implies } A \geq \frac{B}{1 + \varepsilon} - \frac{C}{\varepsilon}$$

for any non-negative A, B, C and any $\varepsilon > 0$.

Then since

$$\begin{aligned} |\nabla|A||^2 &= \sum_k (\sum_{i,j} h_{ij} h_{ijk})^2 (\sum_{i,j} h_{ij}^2)^{-1} = \sum_k (\sum_i h_{ii} h_{iik})^2 (\sum_i h_{ii}^2)^{-1} \leq \sum_{i,k} h_{iik}^2 = \sum_{i \neq k} h_{iik}^2 + \sum_i h_{iii}^2 \\ &= \sum_{i \neq k} h_{iik}^2 + \sum_i (\sum_{j \neq i} h_{jji})^2 \leq \sum_{i \neq k} h_{iik}^2 + (n-1) \sum_{i=j} h_{jji}^2 = n \sum_{i \neq j} h_{jji}^2, \end{aligned} \tag{1.32}$$

we can conclude from (1.29), (1.30) and (1.31) that

$$\begin{aligned} \sum_{i,j,k} h_{ijk}^2 - |\nabla|A||^2 &\geq \sum_{i \neq j} h_{ijk}^2 \geq \frac{2}{1 + \varepsilon} (\sum_{i \neq j} h_{ijj}^2) - \frac{n(n-1)}{2\varepsilon} (K_1 - K_2)^2 \\ &\geq \frac{2}{(1 + \varepsilon)n} |\nabla|A||^2 - \frac{n(n-1)}{2\varepsilon} (K_1 - K_2)^2. \end{aligned} \tag{1.33}$$

Combining (1.27) and (1.33), we then have

$$|A|\Delta|A| + |A|^4 \geq \frac{2}{(1 + \varepsilon)n} |\nabla|A||^2 - \frac{n(n-1)}{2\varepsilon} (K_1 - K_2)^2 - 2c|A| + n(2K_2 - K_1)|A|^2. \tag{1.34}$$

at all points where $|A| \neq 0$. Actually since $|A|\Delta|A| = \frac{1}{2}\Delta|A|^2 - |\nabla|A||^2$ we can in fact see that this inequality must be globally true in the distribution sense, even if $|A|$ vanishes at various points.

The next important inequality is the stability inequality. Recall that a minimal hypersurface is called stable if and only if the second variation of the area functional is non-negative for all compactly supported deformations. Of course it is true that area minimizing hypersurfaces are stable in this sense. A direct computation (cf. [4]) shows that, if M is stable, then

$$\int_M [f\Delta f + (\sum_i K_{n+1,i,n+1,i} + |A|^2)f^2] \leq 0$$

for any smooth function f with compact support in M . Therefore

$$\int_M [f\Delta f + (nK_2 + |A|^2)f^2] \leq 0;$$

that is

$$\int_M (nK_2 + |A|^2)f^2 \leq \int_M |\nabla f|^2. \tag{1.35}$$

Replacing f by $|A|^{1+q}f$ ($q \geq 0$) this gives

$$\begin{aligned} \int_M (nK_2|A|^{2+2q} + |A|^{4+2q})f^2 &\leq \int_M (1+q)^2|A|^{2q}|\nabla|A||^2f^2 + |A|^{2+2q}|\nabla f|^2 \\ &+ (2+2q)\int_M |A|^{1+2q}f(\nabla|A|) \cdot (\nabla f). \end{aligned} \tag{1.36}$$

On the other hand if we multiply the inequality (1.34) by $f^2|A|^{2q}$ and integrate over M we obtain (after integrating by parts in $\int_M |A|^{1+2q}(\Delta|A|)f^2$)

$$\begin{aligned} \frac{2}{(1+\varepsilon)n} \int_M |A|^{2q}|\nabla|A||^2f^2 &\leq -(1+2q)\int_M |A|^{2q}|\nabla|A||^2f^2 \\ &+ \int_M \left\{ n(K_1 - 2K_2)|A|^2 + |A|^4 + |\nabla|A||^2 + 2c|A| + \frac{n(n-1)}{2\varepsilon}(K_1 - K_2)^2 \right\} |A|^{2q}f^2 \\ &- 2\int_M |A|^{1+2q}f(\nabla|A|) \cdot (\nabla f). \end{aligned} \tag{1.37}$$

By adding (1.36), (1.37) and using the inequality

$$2q|A|^{1+2q}f(\nabla|A|) \cdot (\nabla f) \leq \varepsilon q^2 f^2 |A|^{2q} |\nabla|A||^2 + \varepsilon^{-1} |A|^{2+2q} |\nabla f|^2,$$

we then have

$$\begin{aligned} &\left(\frac{2}{(1+\varepsilon)n} - (1+\varepsilon)q^2 \right) \int_M |A|^{2q}|\nabla|A||^2f^2 \\ &\leq \int_M (1+\varepsilon^{-1})|A|^{2q} \left\{ |A|^2|\nabla f|^2 + n(K_1 - 3K_2)|A|^2f^2 + 2c|A|f^2 + \frac{n(n-1)}{2\varepsilon}(K_1 - K_2)^2f^2 \right\}. \end{aligned} \tag{1.38}$$

Inequality (1.38) will be of central importance in what follows.

§ 2. Main results

Throughout this section we will assume M is a stable immersion, so that all the inequalities of § 1 are valid.

First of all, we obtain an L_p estimate for A by using (1.38) together with (1.36).

THEOREM 1. *For each $p \in [4, 4 + \sqrt{8/n})$ and for each non-negative smooth function f with compact support in M , we have*

$$\int_M |A|^p f^p \leq \beta \int_M [|\nabla f|^p + (c^{2/3} + K_1 - K_2 + \max\{-K_2, 0\})^{p/2} f^p],$$

where β is a constant depending only on n and p .

Proof. Let $q = (p - 4)/2$, so that $q > 0$ and $q^2 < 2/n$. By using (1.38) with ε chosen small enough to ensure $2/[(1 + \varepsilon)n] - (1 + \varepsilon)q^2 > 0$, we have

$$\int_M |A|^q |\nabla |A||^2 f^2 \leq \beta_1 \int_M (|A|^{p-2} |\nabla f|^2 + (K_1 - 3K_2) |A|^{p-2} f^2 + c |A|^{p-3} f^2 + (K_1 - K_2)^2 |A|^{p-4} f^2), \tag{2.1}$$

where β_1 depends only on n, p .

On the other hand, (1.36) says

$$\begin{aligned} \int_M |A|^p f^2 &\leq \int_M ((1 + q)^2 |A|^{2q} |\nabla |A||^2 f^2 + 2(1 + q) (|A|^q |\nabla |A||) \cdot (|A|^{p/2-1} \nabla f)) \\ &\quad + \int_M (|A|^{p-2} |\nabla f|^2 - nK_2 |A|^{p-2} f^2). \end{aligned} \tag{2.2}$$

By the Cauchy inequality we have

$$(|A|^q |\nabla |A||) \cdot (|A|^{p/2-1} \nabla f) \leq \frac{1}{2} |A|^{2q} |\nabla |A||^2 + \frac{1}{2} |A|^{p-2} |\nabla f|^2, \tag{2.3}$$

and by using Young's inequality we have the following for each $\varepsilon > 0$:

$$c |A|^{p-3} \leq \varepsilon |A|^p + \beta_2 c^{p/3}, \tag{2.4}$$

$$|A|^{p-2} |\nabla f|^2 = f^2 (|A|^{p-2} (|\nabla f|^2/f^2)) \leq \varepsilon |A|^p f^2 + \beta_3 (|\nabla f|^p/f^{p-2}), \tag{2.5}$$

$$\max \{ (K_1 - 3K_2) |A|^{p-2}, -nK_2 |A|^{p-2} \} \leq \varepsilon |A|^p + \beta_4 (\max \{ K_1 - 3K_2, -nK_2, 0 \})^{p/2} \tag{2.6}$$

$$(K_1 - K_2)^2 |A|^{p-4} \leq \varepsilon |A|^p + \beta_5 (K_1 - K_2)^{p/2}, \tag{2.7}$$

where β_2, \dots, β_5 are determined by ε, p .

Now let M_+ be defined by

$$M_+ = \{x \in M : f \neq 0\}.$$

Then using (2.1), (2.3)–(2.7) in the inequality (2.2) we obtain

$$\begin{aligned} (1 - \beta_6 \varepsilon) \int_M |A|^p f^2 &\leq \beta_7 \int_{M_+} \{ |\nabla f|^p / f^{p-2} + (c^{p/3} + (K_1 - K_2)^{p/2}) \\ &\quad + (\max \{ K_1 - 3K_2, -nK_2, 0 \})^{p/2} f^2 \} \end{aligned} \tag{2.8}$$

where β_6 depends on n, p and β_7 depends on ε, n, p .

If we now take $\varepsilon = 1/(2\beta_6)$ and replace f by $f^{p/2}$ the required inequality easily follows.

Suppose now that we have a constant R_0 with $0 < R_0 \leq \infty$ and a family of subsets $\{B_R\}_{R \in (0, R_0)}$ defined by

$$B_R = \{x \in M : r(x) \leq R\},$$

where r is a given Lipschitz function on M with

$$|\nabla r| \leq 1 \quad \text{a.e. on } M.$$

Suppose also that each B_R is compact and

$$M = \bigcup_{R \in (0, R_0)} B_R.$$

We have in mind the particular cases where the B_R are either geodesic balls of radius R in M or the intersection with M of geodesic balls of radius R in N . (We note that in the former case, the immersion of M into N need not even be proper.)

Now let $f = \gamma \circ r$, where γ is the Lipschitz function on R with $\gamma(t) \equiv 1$ for $t \leq \theta R$, ($\theta \in (0, 1)$ a given constant) $\gamma(t) \equiv 0$ for $t > R$ and with $\gamma(t)$ decreasing linearly for $t \in (\theta R, R)$. It is then not difficult to see that Theorem 1 implies

$$\int_{B_{\theta R}} |A|^p \leq \tilde{\beta} R^{-p} |B_R|, \quad R \in (0, R_0), \quad \theta \in (0, 1), \quad p \in (0, 4 + \sqrt{8/n}), \tag{2.9}$$

where $\tilde{\beta} = \beta \{ (1 - \theta)^{-p} + [R^2(c^{2/3} + K_1 - K_2 + \max\{-K_2, 0\})]^{p/2} \}$ (β as in Lemma 1) and $|B_R|$ is the n -dimensional volume of B_R .

Note that if $c = 0$ and $K_1 = K_2 \geq 0$, then (2.9) gives

$$\int_{B_{\theta R}} |A|^p \leq \frac{\beta}{(1 - \theta)^p} R^{-p} |B_R|.$$

Thus if $\lim_{R \rightarrow \infty} R^{-p} |B_R| = 0$ for some $p \in (0, 4 + \sqrt{8/n})$, we must then have $|A| = 0$; that is, we have the Bernstein type result stated in the following theorem.

THEOREM 2. *Suppose $K_1 = K_2 \geq 0$, $c = 0$ and $\lim_{R \rightarrow \infty} R^{-p} |B_R| = 0$ for some $p \in (0, 4 + \sqrt{8/n})$. Then M is totally geodesic.*

Remarks. 1. Suppose N is complete and M is a boundary of least area in N in the sense that $M = \partial U = \partial \tilde{U}$ for some open $U \subset N$ and $\text{Vol}(\partial U \cap A) \leq \text{Vol}(\partial \tilde{U} \cap A)$ for each open $A \subset N$ with compact closure. Then we can take r to be geodesic distance in N and prove that

$$|B_R| \leq \frac{1}{2} \text{Vol}(S_R),$$

where S_R is the geodesic sphere of radius R in N .

In particular, if N is flat we deduce that $|B_R|$ has order at most R^n and hence there is a p satisfying the conditions of Theorem 2 provided $n < 4 + \sqrt{8/n}$, that is, $n \leq 5$. Thus we deduce that any boundary of least area in N is totally geodesic if $n \leq 5$. In particular, we deduce Bernstein's theorem for minimal graphs in \mathbf{R}^{n+1} when $n \leq 5$.

The dimensional restriction $n \leq 5$ can be relaxed if the volume growth of N is small. For example if N is the product of the $(n - 4)$ -dimensional torus and the 5-dimensional Euclidean space, then all boundaries of least area in N are totally geodesic.

In the case $n \leq 5$ we now show that one can actually obtain a pointwise bound for $|A|$, provided appropriate restrictions are imposed on N .

We here continue to use the family of subsets $\{B_R\}_{R \in (0, R_0)}$ introduced above.

THEOREM 3. *Suppose N is simply connected, complete and has non-positive curvature ($K_1 \leq 0$). Then if $n \leq 5$ and*

$$R^2(c^{2/3} + |K_2|) + R^{-n}|B_R| \leq \beta_0,$$

we have

$$\sup_{B_{\theta R}} |A| \leq \beta R^{-1}$$

for each $\theta \in (0, 1)$, where β is a constant depending only on θ , n and β_0 .

Remarks. We note that in the special case when $N = \mathbf{R}^{n+1}$, if M is a boundary of least area in $\{x \in \mathbf{R}^{n+1} : |x - x_0| < R_0\}$ and if B_R is the intersection with M of the ball $\{x \in \mathbf{R}^{n+1} : |x - x_0| \leq R\}$, then (because $R^{-n}|B_R| \leq (n+1)\omega_{n+1}/2$ —see Remark 1 after Theorem 2) the inequality of the theorem implies

$$|A|(x_0) \leq \beta_1/R, \quad R < R_0, \tag{2.10}$$

where β_1 is an absolute constant. (Note that β_1 can be computed explicitly.) If $R_0 = \infty$ we can let $R \rightarrow \infty$ in (2.10) and obtain another proof of Bernstein's theorem for $n \leq 5$.

It is an open question whether or not an inequality like (2.10) is true in the case $n = 6$. In the case $n = 2$ an inequality of the form (2.10) was first established for non-parametric surfaces in [8]. An analogous result, also in the case $n = 2$, was established by Osserman (see [11]). Osserman's result was proved subject to the assumption that the Gauss map omits a neighborhood of S^2 ; no stability condition was assumed. However, since it is not clear whether or not a 2-dimensional boundary of least area must have a Gauss map which omits a neighborhood of S^2 (at least when we restrict the Gauss map to B_R , $R < R_0$), our inequality seems to be of some interest even in the case $n = 2$.

Proof of Theorem 3. By (1.27) it is not difficult to check that the function $u = R^{-2}\beta_0^2 + |A|^2$ satisfies an inequality of the form

$$\Delta u + \beta_1(R^{-2} + |A|^2)u \geq 0, \tag{2.11}$$

where β_1 is a constant depending only on n .

We now need to recall a well known result from the theory of elliptic equations (see for example [10], Theorem 5.3.1): Suppose ϕ is a non-negative function satisfying

$$\Delta \phi + c\phi \geq 0$$

on some ball K_R of radius R in \mathbf{R}^n . Then for each $\varepsilon > 0$ and each $\theta \in (0,1)$

$$\sup_{K_{\theta R}} \phi \leq c_1 \left\{ R^{-n} \int_{K_R} \phi^2 dx \right\}^{1/2}, \tag{2.12}$$

where c_1 depends only on n, ε, θ and $R^\varepsilon \int_{K_R} |c|^{(n+\varepsilon)/2} dx$.

The same argument can be used to bound functions u satisfying (2.11) on M . The only difficulty in modifying the proof from \mathbf{R}^n to the present manifold setting is that one needs a suitable Sobolev inequality. Under the hypotheses stated in Theorem 3 such an inequality has been proved in [9]. In fact it is proved in [9] that if N is simply connected, complete and has non-positive curvature then

$$\left\{ \int_M f^{n/(n-1)} \right\}^{(n-1)/n} \leq c_2 \int_M |\nabla f|$$

for any smooth f with compact support in M , where c_2 depends only on n .

Thus we can copy the \mathbf{R}^n proof of (2.12) and obtain

$$\sup_{B_{\theta R}} |A|^2 \leq c_3 \left\{ R^{-n} \int_{B_R} (R^{-2} \rho_0^2 + |A|^2)^2 \right\}^{1/2},$$

where c_3 depends on $R^\varepsilon \int_{B_R} (R^{-2} + |A|^2)^{(n+\varepsilon)/2}$, n, ε and θ . Choosing $\varepsilon > 0$ such that $n + \varepsilon < 4 + \sqrt{8/n}$ (which can be done for $n \leq 5$) and using (2.9), we then have Theorem 3.

§ 3. Minimal cones in \mathbf{R}^{n+1}

We conclude this paper with a simplified proof of Simons' theorem concerning the non-existence of stable 6-dimensional minimal cones in \mathbf{R}^7 .

We let C be an n -dimensional stable minimal cone in \mathbf{R}^{n+1} with vertex 0. That is, C is a union of rays emanating from 0 such that $C - \{0\}$ is a n -dimensional C^∞ stable minimal submanifold of \mathbf{R}^{n+1} .

Using (1.27) and (1.29) together with the fact that $c = K_1 = K_2 = 0$ (since $N = \mathbf{R}^{n+1}$ in this case), we have

$$|A| \Delta |A| + |A|^4 = \frac{1}{2} |A|^{-2} \sum_{i,j,r,s,k} \sigma_{ijrsk}^2$$

at all points of $C - \{0\}$ for which $|A| \neq 0$, where

$$\sigma_{ijrsk} = h_{ij} h_{rsk} - h_{rs} h_{ijk}, \quad i, j, r, s, k = 1, \dots, n.$$

Then clearly, since $h_{ij} = h_{ji}$ and $h_{ijk} = h_{ikj}$ (by (1.10)), we then have

$$|A| \Delta |A| + |A|^4 \geq 2 |A|^{-2} \sum_{k=1}^n \sum_{j \neq n, r \neq n, s \neq n} \sigma_{n j r s k}^2. \tag{3.1}$$

If we now choose a frame e_1, \dots, e_n at a given point x in such a way that h_{ij} is diagonal and such that e_n is in the radial direction $x/|x|$, then we have

$$h_{ij} = 0, i \neq j, h_{nn} = 0 \quad \text{and} \quad h_{i,n} = -|x|^{-1}h_{i,j}, i, j = 1, \dots, n.$$

Then

$$\sigma_{nfrsk} = -h_{rs}h_{njk} = |x|^{-1}h_{rs}h_{jk}.$$

Thus, since $h_{nn} = 0$ and $h_{ij} = 0$ for $i \neq j$, (3.1) gives

$$|A|\Delta|A| + |A|^4 \geq 2|A|^{-2}|x|^{-2}|A|^4 = 2|x|^{-2}|A|^2 \tag{3.2}$$

As with inequality (1.34), this inequality holds globally in the distribution sense even if $|A|$ vanishes at various points.

We will also need the following formula (which is a special case of the co-area formula) for integration over C :

$$\int_C \phi dV_n = \int_0^\infty \int_{\partial B_R} \phi(x) dV_{n-1}(x) dR = \int_0^\infty R^{n-1} \int_{\partial B_1} \phi(R\xi) dV_{n-1}(\xi) dR. \tag{3.3}$$

Here ϕ is an arbitrary summable function on C and B_R denotes the intersection with C of the ball in \mathbb{R}^{n+1} with radius R and center 0.

We now take f to be a C^1 function on $C - \{0\}$ with compact support in $C - \{0\}$. Then, multiplying by f^2 in (3.2) and integrating by parts, we have

$$2 \int_C |A|^2 f^2 r^{-2} \leq \int_C (|A|^4 - |\nabla|A||^2) f^2 - 2 \int_C |A| f (\nabla f) \cdot (\nabla|A|), \tag{3.4}$$

where r is defined on C by

$$r(x) = |x|.$$

On the other hand if we use (1.36) with $f|A|$ in place of f (and with $K_2 = 0$), we have

$$\int_C |A|^4 f^2 \leq \int_C |\nabla(f|A|)|^2 = \int_C |\nabla|A||^2 f^2 + \int_C |A|^2 |\nabla f|^2 + 2 \int_C |A| f (\nabla f) \cdot (\nabla|A|). \tag{3.5}$$

Combining (3.4) and (3.5) we then have

$$2 \int_C |A|^2 f^2 r^{-2} \leq \int_C |A|^2 |\nabla f|^2. \tag{3.6}$$

We now assert that (3.6) is valid even if f does not have compact support in $C - \{0\}$, provided that

$$\int_C |A|^2 f^2 r^{-2} < \infty. \tag{3.7}$$

This is proved by applying (3.6) to the function $\gamma_\epsilon f$, where γ_ϵ is any smooth function on

$C - \{0\}$ with

$$\begin{aligned} \gamma_\varepsilon(x) &\equiv 1 \quad \text{for } \varepsilon \leq |x| \leq \varepsilon^{-1}, \quad |\nabla \gamma_\varepsilon(x)| \leq 2/|x| \quad \text{for all } x, \\ &\text{and } \gamma_\varepsilon(x) \equiv 0 \quad \text{for } |x| \leq \varepsilon/2 \quad \text{or } |x| \geq 2\varepsilon^{-1}, \end{aligned}$$

and then letting $\varepsilon \rightarrow 0$.

We now show that, if $n \leq 6$, (3.6) cannot possibly hold for all f satisfying (3.7) unless $|A| \equiv 0$. To prove this we take $\varepsilon \in (0, \frac{1}{2})$ and take

$$f = r^{1+\varepsilon} r_1^{1-\frac{n}{2}-2\varepsilon},$$

where r_1 is defined by

$$r_1 = \max \{1, r\}.$$

This choice of f is valid because, using (3.3) together with the fact that $|A(x)| = |x|^{-1} |A(x/|x|)|$, one can easily check that (3.7) holds. Then (3.6) gives

$$2 \int_C |A|^2 r^{2\varepsilon} r_1^{2-n-4\varepsilon} \leq \left(\frac{n}{2} - 2 + \varepsilon\right)^2 \int_{C\{|x|>1\}} |A|^2 r^{2-n-2\varepsilon} + (1+\varepsilon)^2 \int_{C\{|x|<1\}} |A|^2 r^{2\varepsilon}. \quad (3.3)$$

Now for $n \leq 6$ we can choose ε such that $(\frac{1}{2}n - 2 + \varepsilon)^2 < 2$ and $(1 + \varepsilon)^2 < 2$. (3.3) then gives

$$\int_C |A|^2 r^{2\varepsilon} r_1^{2-n-4\varepsilon} = 0;$$

that is $|A| \equiv 0$ as required.

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