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CURVATURE HOMOGENEOUS HYPERSURFACES IMMERSED IN A REAL SPACE FORM

Dedicated to Professor Morio Obata on his sixtieth birthday

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1. Introduction. Let M be an n-dimensional Riemannian manifold with the Riemannian connection ∇ and the curvature tensor R. We denote by $\nabla^k R$ the k-th covariant differential of the curvature tensor field. A linear isomorphism Φ of the tangent space T_pM onto the tangent space T_qM is naturally extended to a linear isomorphism of the tensor algebra $T(T_pM)$ onto $T(T_qM)$.

If M is locally homogeneous, i.e., for each $p, q \in M$ there exists a local isometry ϕ of a neighborhood of p onto a neighborhood of q which maps p to q, then for any integer $k \geqq 0$, the following condition $R(k)$ is satisfied:

$R(k)$: For each $p, q \in M$, there exists a linear isometry Φ of T_pM onto T_qM such that $\Phi(\nabla^i R)_p = (\nabla^i R)_q$ $i = 0, 1, \cdots, k$.

In fact Φ is given by $\Phi = d\phi_p$, where ϕ is a local isometry with maps p to q. Singer [11] dealt with the converse problem and he proved that if a complete and simply connected Riemannian manifold M satisfied the condition $R(k)$ for a certain k, then M is homogeneous. Following his proof, we see that if a connected Riemannian manifold M satisfies the condition $R(k)$ for a certain k, then M is locally homogeneous. In his theorem, the minimum of such integers k depends on M, but it is not greater than $n(n-1)/2+1$. Among others, he also posed the following question: Do there exist curvature homogeneous spaces which are not homogeneous? Here a *curvature homogeneous space* is, by definition, a Riemannian manifold satisfying the condition $R(0)$.

Many such manifolds seem to exist. The following are explicit known examples:

EXAMPLE 1 (cf. Sekigawa [10] and Takagi [13]). Sekigawa constructed 3-dimensional complete and simply connected curvature homogeneous spaces which are not homogeneous.

EXAMPLE 2. Isoparametric hypersurfaces in a unit sphere. If an immersed hypersurface in a real space form has constant principal curvatures, then it is curvature homogeneous. It is known that there exist hypersurfaces with constant principal curvatures in a unit sphere which are not homogeneous (cf. Ozeki and Takeuchi [8], Ferus, Karcher and Münzner [3]).

On the other hand, homogeneous hypersurfaces immersed in a real space form were studied by several authors and are completely classified (cf. Nagano and Takahashi [6], Ryan [9] and Takahashi [14], [15]). Here in connection with Singer's previous question, the following questions are naturally posed. In their proof, what level of homogeneity is essentially used? Is it possible to relax the condition of homogeneity to the condition of curvature homogeneity? So we consider the following problem in this paper:

Determine curvature homogeneous hypersurfaces immersed in a real space form.

Complete and simply connected Riemannian manifolds of constant curvature $\tilde{c}$ are called *real space forms*. They are as follows:

(i) $\tilde{c} = 0$: The Euclidean space E^n ($\boldsymbol{R}^n$ with the usual inner product).

(ii) $\tilde{c} > 0$: The sphere $S^n(\tilde{c})$ of radius $\tilde{c}^{-1/2}$ in the Euclidean space E^{n+1} with the metric induced from E^{n+1}.

(iii) $\tilde{c} < 0$: The hyperbolic space $H^n(\tilde{c})$. Let $\boldsymbol{L}^{n+1}$ be an $(n+1)$-dimensional Minkowsky space with the inner product $\langle x, y\rangle = \sum_{i=1}^n x^i y^i - x^{n+1}y^{n+1}$. The hyperbolic space is defined by $H^n(\tilde{c}) = \{x \in \boldsymbol{L}^{n+1}; \langle x, x\rangle = 1/\tilde{c},\ x^{n+1} > 0\}$ with the metric induced from $\boldsymbol{L}^{n+1}$.

For the Euclidean space, putting known results together, we have:

THEOREM A. *Let M^n be an $n(\geqq 3)$-dimensional connected curvature homogeneous space and let f be an isometric immersion of M^n into E^{n+1}. Then one of the following may occur:*

(1) *M^n is a flat manifold.*

(2) *M^n is locally isometric to $S^r(c)\times E^{n-r}$, $3\leqq r\leqq n$, for some $c>0$ and f is locally congruent to the isometric imbedding $\tilde{f}$ of $S^r(c)\times E^{n-r}$ into E^{n+1}.*

(3) *M^n is locally isometric to $M^2(\kappa)\times E^{n-2}$, $\kappa \neq 0$, and f is locally congruent to the product immersion $f_1\times f_2$, where $M^2(\kappa)$ denotes a surface of constant curvature κ $(\neq 0)$ and f_1 is an isometric immersion of $M^2(\kappa)$ into E^3 while f_2 is the identity map of E^{n-2} onto E^{n-2}.*

For a sphere and a hyperbolic space, we obtain the following:

THEOREM B. *Let M^n be an $n(\geqq 4)$-dimensional connected curvature homogeneous space and let f be an isometric immersion of M^n into $S^{n+1}(1)$. Then one of the following may occur:*

(1) *M^n is a Riemannian manifold of constant curvature* 1.
(2) *The immersion f has constant principal curvatures.*

THEOREM C. *Let M^n be an $n(\geqq 4)$-dimensional connected curvature homogeneous space and let f be an isometric immersion of M^n into $H^{n+1}(-1)$. Then one of the following may occur*:

(1) *M^n is a Riemannian manifold of constant curvature* -1.
(2) *M^n is a Riemannian manifold of constant curvature $c > -1$ and f is totally umbilical.*
(3) *M^n is locally isometric to $S^r(c_1)\times H^{n-r}(c_2)$, $1 \leqq r \leqq n-1$, $1/c_1 + 1/c_2 = -1$, $c_1 > 0$, $c_2 < 0$ and f is locally congruent to the isometric imbedding $\tilde{f}$ of $S^r(c_1)\times H^{n-r}(c_2)$ into $H^{n+1}(-1)$.*
(4) *$n = 4$ and M^4 is locally isometric to the example constructed in Section 4 and f is locally congruent to the isometric imbedding given in Section 4.*

The imbedding $\tilde{f}: S^r(c)\times E^{n-r} \to E^{n+1}$ in Theorem A (2) is given by $\tilde{f}((x^1,\cdots,x^{r+1})\times(y^1,\cdots,y^{n-r})) = (x^1,\cdots,x^{r+1},y^1,\cdots,y^{n-r})$, where $\sum_{i=1}^{r+1}(x^i)^2 = 1/c$. The imbedding $\tilde{f}: S^r(c_1)\times H^{n-r}(c_2) \to H^{n+1}(-1)$ in Theorem C (3) is given by $\tilde{f}((x^1, \cdots, x^{r+1})\times(y^1, \cdots, y^{n-r+1})) = (x^1, \cdots, x^{r+1}, y^1, \cdots, y^{n-r+1})$, where $\sum_{i=1}^{r+1}(x^i)^2 = 1/c_1$ and $\sum_{j=1}^{n-r}(y^j)^2 - (y^{n-r+1})^2 = 1/c_2$.

In Section 2, we review basic facts about type numbers for hypersurfaces in a real space form and show that it is essential to study the case of type number 2. In Section 3, we consider the case of type number 2 and introduce a useful operator—the conullity operator. We prove that in $S^n(1)$ ($n \geqq 5$) and in $H^n(-1)$ ($n \geqq 6$) there exists no curvature homogeneous hypersurface whose type number is equal to 2 (Corollary 3.4).

In Section 4, we construct a 4-dimensional complete curvature homogeneous space which is not homogeneous and construct an isometric imbedding of the manifold into $H^5(-1)$ whose type number is equal to 2. In Section 5, we determine curvature homogeneous hypersurfaces in $H^5(-1)$ whose type number are equal to 2 (Theorem 5.1).

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2. **Preliminaries.** Let $\tilde{M}^{n+1}(\tilde{c})$ be an $(n+1)$-dimensional real space form of constant curvature $\tilde{c}$. An n-dimensional connected Riemannian manifold M^n together with an isometric immersion f of M into $\tilde{M}^{n+1}(\tilde{c})$ is called a *hypersurface* of $\tilde{M}^{n+1}(\tilde{c})$. We assume that M is orientable. We denote by ξ a field of unit normal vectors. Let h and A be the second fundamental form and the shape operator of f corresponding to ξ, respectively. At each point p of M, the *type number* of f at p, denoted

by $t(p)$, is defined to be the rank of the linear endomorphism A of T_pM.

We summarize basic facts about type numbers. For details, see Kobayashi and Nomizu [4].

PROPOSITION 2.1 (cf. [4, Theorem 6.1, p. 42]). *For a hypersurface* (M^n, f) *immersed in* $\tilde{M}^{n+1}(\tilde{c})$,

(1) $t(p)$ *is* 0 *or* 1 *if and only if*

$$R(x, y)z = \tilde{c}\{\langle y, z\rangle x - \langle x, z\rangle y\} = \tilde{c}R_o(x, y)z\ , \quad x, y, z \in T_pM\ .$$

(2) *If* $t(p) \geqq 2$, *then* ker A *coincides with* T_p^0 *and* $t(p) = n - \dim T_p^0$, *where*

$$T_p^0 = \{x \in T_pM; (R - \tilde{c}R_o)(x, y) = 0 \quad \textit{for any}\ y \in T_pM\}\ .$$

PROPOSITION 2.2 (cf. [4, the proof of Theorem 6.2, p. 43]). *For a hypersurface* (M^n, f) *immersed in* $\tilde{M}^{n+1}(\tilde{c})$, *suppose that* $t(p) \geqq 3$ *for a* $p \in M$. *Let* $\bar{A}$ *be a symmetric linear endomorphism of* T_pM *which satisfies* $R(x, y)z - \tilde{c}R_o(x, y)z = \langle \bar{A}y, z\rangle \bar{A}x - \langle \bar{A}x, z\rangle \bar{A}y$. *Then we have* $\bar{A} = \pm A$, *where* A *denotes the shape operator of* f *at* p.

We will apply the above results to curvature homogeneous hypersurfaces in a real space form. Let M^n be an n-dimensional connected curvature homogeneous space and let f be an isometric immersion of M^n into $\tilde{M}^{n+1}(\tilde{c})$. We call (M, f) a *curvature homogeneous hypersurface* of $\tilde{M}^{n+1}(\tilde{c})$. If $t(p) \geqq 2$ at some point p of M, then by Proposition 2.1 (2), the type number is constant on M. Therefore the following three cases may occur:

(1) The type number is equal to 0 or 1 on M.

(2) The type number is equal to 2 at each point of M.

(3) The type number is constant on M and is not less than 3.

In the first case, by Proposition 2.1 (1), M is a Riemannian manifold of constant curvature $\tilde{c}$. In the third case, by Proposition 2.2 and curvature homogeneity, the principal curvatures of f are constant on M. Hence we have:

THEOREM 2.3. *For a curvature homogeneous hypersurface* (M^n, f) *immersed in* $\tilde{M}^{n+1}(\tilde{c})$, *one of the following three cases may occur*:

(1) *M is a Riemannian manifold of constant curvature* $\tilde{c}$.

(2) *The immersion f has constant principal curvatures.*

(3) *The type number is equal to 2 at each point of M.*

We note that hypersurfaces with constant principal curvatures in $\tilde{M}^{n+1}(\tilde{c})$, $\tilde{c} \leqq 0$, were completely classified by E. Cartan. Either they are totally geodesic or they are the ones in Theorem A (2) if $\tilde{c} = 0$ while they are the ones in Theorem C (2), (3) if $\tilde{c} < 0$.

In order to show Theorems A, B and C, we are left to studying curvature homogeneous hypersurfaces of the type number 2. We will discuss them in the rest of this paper.

3. The case of the type number 2—Part 1. In this section, we denote by M^n an n-dimensional connected curvature homogeneous space and by f an isometric immersion of M^n into $\tilde{M}^{n+1}(\tilde{c})$ of the type number 2. By Proposition 2.1 (2), we have the orthogonal decomposition of the tangent bundle:

$$TM = T^0 + T^1 ,$$

where $T_p^0 = \{x \in T_pM; (R - \tilde{c}R_o)(x, y) = 0 \text{ for any } y \in T_pM\}$ and $\dim T_p^1 = 2$. The shape operator A is reduced to the symmetric linear isomorphism of T_p^1. By the assumption of curvature homogeneity, for each $p, q \in M$, there exists a linear isometry Φ of T_pM onto T_qM such that $\Phi R_p = R_q$. The above decomposition of T_pM is preserved under Φ, that is, $\Phi(T_p^0) = T_q^0$ and $\Phi(T_p^1) = T_q^1$. In particular, the sectional curvature of the plane T_p^1 coincides with that of T_q^1, and is denoted by κ. We remark that $\kappa \neq \tilde{c}$. It is known that the subbundle T^0 is completely integrable and that their integral submanifolds are totally geodesic in M, that is, the subbundle T^0 is the so-called *totally geodesic foliation.*

We will define the *conullity operator* C as a smooth section of $\mathrm{Hom}(T^0, \mathrm{End}(T^1))$ (cf. Ferus [2]). We denote by ∇ the Riemannian connection of M and by $P: TM \to T^1$ the orthogonal projection. Define a linear operator C of T_p^0 into $\mathrm{End}(T_p^1)$ by

$$C_\xi x = -P(\nabla_x \Xi) \quad \text{for } x \in T_p^1,\ \xi \in T_p^0 ,$$

where Ξ is a local vector field of T^0 on M around p with $\Xi_p = \xi$. Let ∇' denote the connection of the subbundle T^1 induced from ∇.

We review basic formulas about the conullity operator. For details, see Ferus [2] and Szabó [12].

PROPOSITION 3.1. *Under the assumption of this section, the conullity operator C satisfies the following formulas*:

(1) *Let $\{\xi_3, \cdots, \xi_n\}$ be a local orthonormal frame field of the bundle T^0 around $p \in M$ and we denote by Λ_α^β, $\alpha, \beta = 3, \cdots, n$, the connection forms of Riemannian connection ∇ with respect to* $\{\xi_3, \cdots, \xi_n\}$, i.e., $\Lambda_\alpha^\beta(x) = \langle \nabla_x \xi_\alpha, \xi_\beta \rangle$ for $x \in T_pM$. *Then we have*

$$(\nabla'_x C_\alpha)(y) - (\nabla'_y C_\alpha)(x) + \sum_{\beta=3}^n \{\Lambda_\alpha^\beta(y) C_\beta(x) - \Lambda_\alpha^\beta(x) C_\beta(y)\} = 0$$

for $x, y \in T^1_p$, *where we simply write* $C_\alpha = C_{\xi_\alpha}$.

(2) *In the same notation as in* (1), *we have*

$$(\nabla_{\xi_\alpha} C_\beta)(x) = \sum_{\gamma=3}^{n} \Lambda^\gamma_\beta(\xi_\alpha) C_\gamma(x) + C_\beta C_\alpha(x) + \tilde{c}\langle \xi_\alpha, \xi_\beta \rangle x \quad \text{for } x \in T^1_p .$$

(3) *Let* γ *be a unit speed geodesic in one of the integral submanifolds of* T^0 *and let* η *be a parallel vector field along* γ *which is tangent to* T^0. *Then we have*

$$\nabla_{\dot\gamma} C_\eta = C_\eta C_{\dot\gamma} + \tilde{c}\langle \eta, \dot\gamma \rangle \mathrm{id} ,$$

where id *denotes the identity map of* T^1.

PROOF. (1) We extend $x, y \in T^1_p$ to local vector fields X, Y of T^1 around p. By assumption, $R(X, Y)\xi_\alpha = 0$. Calculating the T^1_p-component of the identity, we obtain (1).

(2) We extend $x \in T^1_p$ to a local vector field X of T^1 around p. By assumption, $R(X, \xi_\alpha)\xi_\beta = \tilde{c}\langle \xi_\alpha, \xi_\beta \rangle X$ and hence $P(R(X, \xi_\alpha)\xi_\beta) = \tilde{c}\langle \xi_\alpha, \xi_\beta \rangle X$. Calculating the left hand side of the identity, we obtain (2).

(3) follows directly from (2).

PROPOSITION 3.2. *Under the assumption of this section, the second fundamental form* h *of* f *satisfies*

$$h(C_\xi x, y) = h(x, C_\xi y) \quad \text{for } \xi \in T^0_p,\ x, y \in T^1_p .$$

PROOF. We define the covariant differential of the second fundamental form h by $\bar h(x, y, z) = zh(X, Y) - h(\nabla_z X, y) - h(x, \nabla_z Y)$, where X and Y are local vector fields with $X_p = x$ and $Y_p = y$. The equation of Codazzi implies that $\bar h(x, y, z) = \bar h(y, z, x) = \bar h(z, x, y)$. For a vector field ξ of T^0 and vector fields X, Y of T^1 we have $\bar h(\xi, Y, X) = Xh(\xi, Y) - h(\nabla_X \xi, Y) - h(\xi, \nabla_X Y) = h(C_\xi X, Y)$. Similarly we get $\bar h(\xi, X, Y) = h(X, C_\xi Y)$ and hence $h(C_\xi X, Y) = h(X, C_\xi Y)$.

PROPOSITION 3.3. *For a unit vector* ξ *of* T^0_p, *we have* $\operatorname{tr} C_\xi = 0$ *and* $\det C_\xi = \tilde{c}$.

PROOF. Let $\{e_1, e_2\}$ be a local orthonormal frame field of T^1 around p and let e_3 be a unit vector field of T^0 around p such that $(e_3)_p = \xi$. We define a tensor field S of type (1, 3) by $S(x, y)z = R(x, y)z - \tilde{c}R_o(x, y)z$. The second Bianchi identity implies that $(\nabla_{e_3} S)(e_1, e_2)e_1 + (\nabla_{e_1} S)(e_2, e_3)e_1 + (\nabla_{e_2} S)(e_3, e_1)e_1 = 0$. Calculating the left hand side, we have $-(\kappa - \tilde{c})(\Lambda^1_{13} + \Lambda^2_{23})e_2 = 0$, where $\Lambda^k_{ij} = \langle \nabla_{e_i} e_j, e_k \rangle$. Since $\kappa \neq \tilde{c}$, we have $\Lambda^1_{13} + \Lambda^2_{23} = 0$, which means that $\operatorname{tr} C_\xi = 0$.

Let $\gamma: (-\varepsilon, \varepsilon) \to M$ be a unit speed geodesic such that $\gamma(0) = p$ and

$\dot{\gamma}(0) = \xi$ and let $\{e_1, e_2\}$ be a parallel orthonormal frame field of T^1 along γ. We denote by $\alpha = (\alpha_{ij}(t))$ the 2×2-matrix which represents the conullity operator $C_{\dot{\gamma}}$ with respect to $\{e_1, e_2\}$. Then by Proposition 3.1 (3), we have

$$\frac{d\alpha}{dt} = \alpha^2 + \tilde{c}E \quad \text{on} \quad (-\varepsilon, \varepsilon)\ ,$$

where E denotes the identity matrix. Since $\operatorname{tr}\alpha = 0$ on $(-\varepsilon, \varepsilon)$, we have $\operatorname{tr}(d\alpha/dt) = (d/dt)\operatorname{tr}\alpha = 0$ and $\operatorname{tr}(\alpha^2) = -2\det\alpha$. By the above equation, we obtain $\det\alpha = \tilde{c}$ and, consequently, $\det C_\xi = \tilde{c}$.

COROLLARY 3.4. *Let M^n be an n $(\geqq 3)$-dimensional connected curvature homogeneous space. Suppose that M^n admits an isometric immersion into $\tilde{M}^{n+1}(\tilde{c})$ of the type uumber 2. If $\tilde{c} > 0$, then the dimension of M is equal to 3. If $\tilde{c} < 0$, the dimension of M is equal to 3 or 4.*

PROOF. We assume that $\tilde{c} \neq 0$. By Proposition 3.3, the conullity operator $C\colon T_p^0 \to \operatorname{End}(T_p^1)$ is injective. Therefore we have $\dim T_p^0 = \dim \operatorname{Im} C$. We define the subspaces $\mathfrak{sl}(T_p^1)$, $\operatorname{Sym}^0(T_p^1)$ and $\operatorname{Alt}(T_p^1)$ of $\operatorname{End}(T_p^1)$ by

$$\begin{aligned}
&\mathfrak{sl}(T_p^1) = \{L \in \operatorname{End}(T_p^1);\ \operatorname{tr} L = 0\}\\
&\operatorname{Sym}^0(T_p^1) = \{L \in \mathfrak{sl}(T_p^1);\ \langle Lx, y\rangle = \langle x, Ly\rangle \quad \text{for } x, y \in T_p^1\}\\
&\operatorname{Alt}(T_p^1) = \{L \in \mathfrak{sl}(T_p^1);\ \langle Lx, y\rangle + \langle x, Ly\rangle = 0 \quad \text{for } x, y \in T_p^1\}\ .
\end{aligned}$$

Then we have $\dim \mathfrak{sl}(T_p^1) = 3$, $\dim \operatorname{Sym}^0(T_p^1) = 2$, and $\dim \operatorname{Alt}(T_p^1) = 1$. The image $\operatorname{Im} C$ is contained in $\mathfrak{sl}(T_p^1)$ by Proposition 3.3. We consider the case $\tilde{c} > 0$. Suppose that $\dim \operatorname{Im} C \geqq 2$. Then since $\operatorname{Im} C \cap \operatorname{Sym}^0(T_p^1) \neq \{0\}$, there exists a unit vector $\xi \in T_p^0$ such that $C_\xi \in \operatorname{Sym}^0(T_p^1)$. On the other hand by Proposition 3.3, C_ξ has no real eigenvalue, a contradiction. Therefore we have $\dim T_p^0 = \dim \operatorname{Im} C \leqq 1$. Next consider the case $\tilde{c} < 0$. Suppose that $\dim \operatorname{Im} C = 3$. Then we have $\operatorname{Im} C \cap \operatorname{Alt}(T_p^1) \neq \{0\}$, which similarly gives rise to a contradiction. Therefore we obtain $\dim T_p^0 = \dim \operatorname{Im} C \leqq 2$.

By Theorem 2.3 and Corollary 3.4, Theorem B is proved. To prove Theorem C, we are left to studying 4-dimensional curvature homogeneous hypersurfaces of the type number 2 immersed in $H^5(-1)$. We will discuss them in Sections 4 and 5.

We devote the rest of this section to determining the curvature homogeneous hypersurfaces of the type number 2 immersed in the Euclidean space. It is known that hypersurfaces of the type number 2 in the Euclidean space are semi-symmetric, i.e., Riemannian manifolds satisfying $R(x, y)\cdot R = 0$. The local structure of such manifolds was classified by Szabó [12]. We now prove the following theorem essentially due to Szabó.

We remark that Dajczer and Gromoll [1, Theorem 3.4] prove the same result by different method.

THEOREM 3.5. *Let M^n be an n ($\geqq 3$)-dimensional connected curvature homogeneous space and let f be an isometric immersion of M^n into E^{n+1} of the type number 2. Then for each point p of M there exists an open neighborhood V of p which is isometric to the Riemannian product manifold $V_0 \times V_1$, where V_0 is an open submanifold of E^{n-2} and V_1 is a 2-dimensional Riemannian manifold of constant curvature $\kappa \neq 0$. Moreover, the immersion f restricted to V is congruent to the product immersion $f_0 \times f_1$, where f_0 is an inclusion map of V_0 into E^{n-2} while f_1 is an isometric immersion of V_1 into E^3.*

PROOF. Our aim is to show that the conullity operator vanishes. Suppose that there exists a point of M at which the conullity operator does not vanish. Evidently the conullity operator C is not zero on some neighborhood U of such point.

We first see that $\dim \operatorname{Im} C = 1$ at each point p of U. Indeed, if $\dim \operatorname{Im} C \geqq 2$, then we have $\operatorname{Im} C \cap \operatorname{Sym}^0(T_p^1) \neq \{0\}$. Hence there exists a unit vector ξ of T_p^0 such that C_ξ is contained in $\operatorname{Sym}^0(T_p^1)$ and C_ξ is not zero, a contradiction by Proposition 3.3. We denote by $\mathfrak{M}$ the one dimensional subbundle of T^0 defined over U which is an orthogonal complement of $\operatorname{Ker} C$ in T^0. Let e_3 be a local unit vector field of $\mathfrak{M}$ and let $\{e_1, e_2\}$ be a local orthonormal frame field of T^1 such that $C_{e_3}e_2 = 0$ and $C_{e_3}e_1 = be_2$, where b is a non-zero local smooth function. Then we easily see that $C_\xi e_2 = 0$ for any $\xi \in T^0$ and that $C_\xi C_\eta = 0$ for any $\xi, \eta \in T^0$. Moreover, we see that the orthonormal frame field $\{e_1, e_2\}$ is parallel along an integral submanifold of T^0. Indeed, putting $\xi_3 = e_3$ and $x = e_2$, we apply Proposition 3.1 (2). Then we have $0 = (\nabla_{\xi_\alpha} C_{e_3})(e_2) = -C_{e_3}(\nabla_{\xi_\alpha} e_2) = -\langle \nabla_{\xi_\alpha} e_2, e_1 \rangle be_2$ for $\alpha = 3, \cdots, n$ and hence $\langle \nabla_{\xi_\alpha} e_2, e_1 \rangle = 0$.

Next we investigate the form of the connection ∇' of the subbundle T^1. We denote by Λ_{ij}^k, $i, j, k = 1, 2$, the components of ∇' with respect to the orthonormal frame field $\{e_1, e_2\}$, i.e., $\Lambda_{ij}^k = \langle \nabla'_{e_i} e_j, e_k \rangle = \langle \nabla_{e_i} e_j, e_k \rangle$. Then they satisfy $\Lambda_{22}^1 = 0$, $e_2 b = b\Lambda_{11}^2$ and $e_2 \Lambda_{11}^2 = (\Lambda_{11}^2)^2 + \kappa$, where κ denotes the sectional curvature of the plane T^1. The first two identities are obtained by Proposition 3.1 (1) while the last identity follows from $\langle R(e_1, e_2)e_2, e_1 \rangle = \kappa$.

Finally, we consider the condition that M^n is isometrically immersed in E^{n+1}. We use local vector fields $\{e_1, e_2, e_3\}$ defined as above. By the equation of Gauss, we have $h(e_1, e_1)h(e_2, e_2) - h(e_1, e_2)^2 = \kappa$. From Proposition 3.2, it follows that $h(C_{e_3}e_1, e_2) = h(e_1, C_{e_3}e_2)$ and hence $bh(e_2, e_2) = 0$. Consequently, we have $h(e_2, e_2) = 0$. Therefore κ is negative and $h(e_1, e_2) =$

$\pm(-\kappa)^{1/2}$. Calculating $h(e_2, e_2, e_1) = h(e_2, e_1, e_2)$, we have $2\Lambda_{11}^2 h(e_1, e_2) = 0$ and hence $\Lambda_{11}^2 = 0$. κ vanishes by the identity $e_2\Lambda_{11}^2 = (\Lambda_{11}^2)^2 + \kappa$, a contradiction.

By the above arguments, we see that the conullity operator vanishes on M. Therefore T^0 and T^1 are both parallel distributions and by de Rham's decomposition theorem, we obtain the former part of Theorem 3.5. The reduction of the immersion f is due to Moore [5, Theorem 1].

By Theorems 2.3 and 3.5, Theorem A has been proved.

4. **Construction of an example.** In this section we will construct a 4-dimensional complete curvature homogeneous space and its isometric imbedding into $H^5(-1)$ of the type number 2. It is a hypersurface of cohomogeneity 1.

We first recall the action of $SO(3)$ on the 4-dimensional sphere S^4 in $\boldsymbol{R}^5$ and describe its orbit space. Let $G = SO(3)$ and $\mathfrak{p} = \{A \in M_3(\boldsymbol{R});\, {}^tA = A,\ \operatorname{tr} A = 0\}$, where $M_3(\boldsymbol{R})$ denotes the space of 3×3-real matrices. Then $\mathfrak{p}$ is a 5-dimensional vector space. We define the action of G on $\mathfrak{p}$ by $a(A) = aAa^{-1}$ for $a \in SO(3)$, $A \in \mathfrak{p}$. Let $(\ ,\ )$ be the G-invariant inner product on $\mathfrak{p}$ given by $(A, B) = (1/2)\operatorname{tr} AB$. Put $S^4 = \{A \in \mathfrak{p};\, (A, A) = 1/3\}$. Then S^4 has constant sectional curvature 3 with respect to the metric induced from $\mathfrak{p}$. The group G naturally acts on S^4 as a group of isometries. Define a 2-dimensional subspace $\mathfrak{a}$ of $\mathfrak{p}$ by

$$\mathfrak{a} = \left\{\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3);\, \sum_{i=1}^{3} \lambda_i = 0\right\},$$

where $\operatorname{diag}(\lambda_1, \lambda_2, \lambda_3)$ denotes the diagonal 3×3-matrix whose entries are λ_1, λ_2, and λ_3. Put $H_0 = \operatorname{diag}(-2/3, 1/3, 1/3)$ and $H_1 = \operatorname{diag}(0, 3^{-1/2}, -3^{-1/2})$, which are elements of $\mathfrak{a} \cap S^4$. Let $H(s) = \cos(3^{1/2}s)H_0 + \sin(3^{1/2}s)H_1 = \operatorname{diag}(-2(\cos(3^{1/2}s))/3, (\cos(3^{1/2}s) + 3^{1/2}\sin(3^{1/2}s))/3, (\cos(3^{1/2}s) - 3^{1/2}\sin(3^{1/2}s))/3)$. Then $H(s)$ is a unit speed geodesic of S^4 and is perpendicular to the G-orbit at each point. Moreover, $H(s)$ restricted to a closed interval $I = [0, 3^{-3/2}\pi]$ represents all G-orbits. Therefore the space S^4/G of orbits is given by the closed interval $I = [0, 3^{-3/2}\pi]$. The isotropy subgroup of G at $H(s)$ for $s \in \mathring{I} = (0, 3^{-3/2}\pi)$ is a finite subgroup of $SO(3)$ consisting of $\operatorname{diag}(1, 1, 1)$, $\operatorname{diag}(1, -1, -1)$, $\operatorname{diag}(-1, 1, -1)$, $\operatorname{diag}(-1, -1, 1)$, and the orbit of $H(s)$ under G is just $SO(3)/(\boldsymbol{Z}_2\times\boldsymbol{Z}_2)$. They are codimension one principal orbits. The isotropy subgroup of G at $H(0)$ is

$$\left\{\begin{pmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{pmatrix},\ \begin{pmatrix}-1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & \sin\theta & -\cos\theta\end{pmatrix}\right\}$$

and the orbit of $H(0)$ is the projective plane $P_2(\boldsymbol{R}) = SO(3)/O(2)$ with constant sectional curvature 1, the so-called Veronese surface. The orbit of $H(3^{-3/2}\pi)$ is also the Veronese surface. These two orbits are singular orbits.

Let M^* be the union of orbits of $H(s)$ with s running through $\mathring{I} = (0, 3^{-3/2}\pi)$ i.e., the union of all principal orbits. Then M^* is a connected, open and dense subset of S^4. We will describe the Riemannian metric on M^*. We define a smooth map $F: (G/K) \times \mathring{I} \to M^*$ by $F(aK, s) = a(H(s))$, where $G = SO(3)$, $K = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$, and $\mathring{I} = (0, 3^{-3/2}\pi)$. Then F is a G-equivariant diffeomorphism of $(G/K) \times \mathring{I}$ onto M^*. The Riemannian metric on $(G/K) \times \mathring{I}$ induced by F has the form $g_s + ds^2$, where g_s denote the G-invariant Riemannian metrics on G/K parametrized by s in $\mathring{I}$. We will write the metrics g_s explicitly. Let $\mathfrak{g} = \mathfrak{so}(3)$ be the Lie algebra of $G = SO(3)$. The tangent space $T_{eK}(G/K)$ of the homogeneous space $G/K = SO(3)/(\boldsymbol{Z}_2 \times \boldsymbol{Z}_2)$ at eK is naturally identified with $\mathfrak{g}$. We denote by $\langle\ ,\ \rangle_s$ the inner product on $\mathfrak{g}$ induced by this identification. Then we have $\langle X, Y \rangle_s = ([X, H(s)], [Y, H(s)])$ for $X, Y \in \mathfrak{g}$. Put

$$X_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad X_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad X_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then $\{X_1, X_2, X_3\}$ is a basis for $\mathfrak{g}$ and satisfies $[X_1, X_2] = X_3$, $[X_2, X_3] = X_1$, and $[X_3, X_1] = X_2$. We see that X_1, X_2, and X_3 are mutually orthogonal with respect to $\langle\ ,\ \rangle_s$ and that $\langle X_1, X_1 \rangle_s = (4/3)\sin^2(\pi/3 + 3^{1/2}s)$, $\langle X_2, X_2 \rangle_s = (4/3)\sin^2(\pi/3 - 3^{1/2}s)$, $\langle X_3, X_3 \rangle_s = (4/3)\sin^2(3^{1/2}s)$. Consequently, the inner product $\langle\ ,\ \rangle_s$ on $\mathfrak{g}$ is given by

$$\begin{aligned}\langle\ ,\ \rangle_s = {} & (4/3)\sin^2(\pi/3 + 3^{1/2}s)\omega_1 \otimes \omega_1 + (4/3)\sin^2(\pi/3 - 3^{1/2}s)\omega_2 \otimes \omega_2 \\ & + (4/3)\sin^2(3^{1/2}s)\omega_3 \otimes \omega_3 ,\end{aligned}$$

where $\{\omega_1, \omega_2, \omega_3\}$ is the basis dual to $\{X_1, X_2, X_3\}$.

We are now ready to construct a curvature homogeneous space. Removing the orbit of $H(3^{-3/2}\pi)$ from S^4, we obtain a connected open submanifold of S^4, which is denoted by M. Then M is G-invariant submanifold of S^4 and its orbit space is a semi-open interval $[0, 3^{-3/2}\pi)$. We define a new Riemannian metric g on M as follows:

$$g = a(s)g_s + a(s)^2 ds^2 \quad \text{on} \quad M^* ,$$

where $a(s) = 3(1 + 2\cos(3^{1/2}2s))^{-1}$, while at each point on the orbit of $H(0)$, g coincides with the former metric. Here when we define a metric on M^*, we use the parametrization $F: (G/K) \times \mathring{I} \to M^*$.

The new Riemannian metric g is a smooth one on M and is G-invariant. Indeed it is clear that the metric g is smooth and G-invariant on M^*. We will show that the metric g is smooth in a neighborhood of the orbit of $H(0)$. We denote by N and $T^{\perp}N$ the orbit of $H(0)$ and the normal bundle of N in S^4, respectively. Let $\exp_N$ be the exponential mapping of $T^{\perp}N$ onto S^4. Then the action of G on $T^{\perp}N$ is naturally defined and $\exp_N$ is a G-equivariant mapping. The function r on $T^{\perp}N$ is defined by $r(\xi) = \langle \xi, \xi \rangle^{1/2}$ for $\xi \in T^{\perp}N$, where $\langle\ ,\ \rangle$ denotes the metric on $T^{\perp}N$. Define the subsets U and U_0 of $T^{\perp}N$ by

$$U = \{\xi \in T^{\perp}N;\ r(\xi) < 3^{-3/2}\pi\}$$
$$U_0 = U - (\text{the zero section})\ .$$

We recall that $H(s)$ is a unit speed geodesic of S^4 and $H'(0)$ is a normal vector of N at $H(0)$. Therefore we have $\exp_N(sH'(0)) = H(s)$. We define a smooth mapping $\Phi\colon (G/K)\times \mathring{I} \to U_0$ by $\Phi(aK, s) = da(sH'(0))$, where da denotes the differential of the diffeomorphism a of S^4 for $a \in G$. Then Φ is a G-equivariant diffeomorphism and the following diagram commutes:

$$\begin{array}{ccc} (G/K)\times\mathring{I} & \xrightarrow{\ \Phi\ } & U_0 \\ & {\scriptstyle F}\searrow \quad \swarrow{\scriptstyle \exp_N} & \\ & M^* & \end{array}$$

Moreover, the function $r \circ \Phi$ coincides with the projection of $(G/K)\times\mathring{I}$ onto $\mathring{I}$. We denote by g_0 the Riemannian metric on U induced from the Riemannian metric on S^4 by $\exp_N$. Clearly, we have $\Phi^* g_0 = g_s + ds^2$. It follows that $\exp_N^* g = a(r)g_0 + (a(r)^2 - a(r))dr^2$ on U_0. Since $a(s)$ is an even function of s, $a(r)$ is a smooth function on U and since $a(s)^2 - a(s)$ is an even function of s and $a(0)^2 - a(0) = 0$, $(a(r)^2 - a(r))dr^2$ is a smooth tensor field on U. Consequently, we see that $\exp_N^* g$ is a smooth Riemannian metric on U and hence g is smooth in a neighborhood of the orbit of $H(0)$.

PROPOSITION 4.1. *The Riemannian manifold (M, g) is complete and curvature homogeneous but is not locally homogeneous.*

PROOF. To show the completeness and curvature homogeneity, we change the parametrization of M^*. Define a diffeomorphism $\phi\colon (G/K)\times\mathring{I} \to (G/K)\times \boldsymbol{R}^+$ by $\phi(x, s) = (x, t(s))$, where

$$t(s) = [\log \sin(\pi/3 + 3^{1/2}s) - \log \sin(\pi/3 - 3^{1/2}s)]/2\ .$$

Then we have $(\phi^{-1})^* F^* g = a(s(t))g_{s(t)} + dt^2$. Moreover, on the Lie algebra $\mathfrak{g}$,

$$a(s(t))\langle\ ,\ \rangle_{s(t)} = e^{2t}\omega_1 \otimes \omega_1 + e^{-2t}\omega_2 \otimes \omega_2 + (e^t - e^{-t})^2 \omega_3 \otimes \omega_3\ .$$

LEMMA 4.2. *The Riemannian manifold* $((G/K)\times \boldsymbol{R}^+, g)$ *is curvature homogeneous, where* g *is defined by*

$$e^{2t}\omega_1 \otimes \omega_1 + e^{-2t}\omega_2 \otimes \omega_2 + (e^t - e^{-t})^2\omega_3 \otimes \omega_3 + dt^2 .$$

Moreover, we have $\|\nabla\rho\|^2 = 32(e^t + e^{-t})^2$ *and hence* $((G/K)\times \boldsymbol{R}^+, g)$ *is not locally homogeneous. Here* $\nabla\rho$ *denotes the covariant differential of the Ricci tensor field* ρ.

PROOF OF LEMMA 4.2. We calculate curvature properties on the Riemannian covering space $SO(3)\times \boldsymbol{R}^+$ of $(G/K)\times \boldsymbol{R}^+$. Let $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}_3$ be left invariant vector fields on $SO(3)$ which correspond to X_1, X_2 and X_3 on $\mathfrak{g}$, respectively. We naturally extend them to vector fields on $SO(3)\times \boldsymbol{R}^+$ and use the same notation. We define vector fields e_1, e_2, e_3 and e_4 on $SO(3)\times \boldsymbol{R}^+$ by $e_1 = e^{-t}\tilde{X}_1$, $e_2 = e^t\tilde{X}_2$, $e_3 = (e^t - e^{-t})^{-1}\tilde{X}_3$, $e_4 = d/dt$. Then $\{e_1, e_2, e_3, e_4\}$ is an orthonormal frame field on $SO(3)\times \boldsymbol{R}^+$ and satisfies

$$\begin{aligned}
&[e_1, e_2] = (e^t - e^{-t})e_3 && [e_4, e_1] = -e_1\\
(4.1)\qquad &[e_2, e_3] = e^{2t}(e^t - e^{-t})^{-1}e_1 && [e_4, e_2] = e_2\\
&[e_3, e_1] = e^{-2t}(e^t - e^{-t})^{-1}e_2 && [e_4, e_3] = -(e^t + e^{-t})(e^t - e^{-t})^{-1}e_3 .
\end{aligned}$$

By direct calculation, we have

$$(4.2)\qquad \begin{aligned}
&\nabla_{e_1}e_1 = -e_4 && \nabla_{e_2}e_1 = -e^te_3\\
&\nabla_{e_1}e_2 = -e^{-t}e_3 && \nabla_{e_2}e_2 = e_4\\
&\nabla_{e_1}e_3 = e^{-t}e_2 && \nabla_{e_2}e_3 = e^te_1\\
&\nabla_{e_1}e_4 = e_1 && \nabla_{e_2}e_4 = -e_2\\
&\nabla_{e_3}e_1 = (e^t - e^{-t})^{-1}e_2 && \nabla_{e_4}e_1 = 0\\
&\nabla_{e_3}e_2 = -(e^t - e^{-t})^{-1}e_1 && \nabla_{e_4}e_2 = 0\\
&\nabla_{e_3}e_3 = -(e^t + e^{-t})(e^t - e^{-t})^{-1}e_4 && \nabla_{e_4}e_3 = 0\\
&\nabla_{e_3}e_4 = (e^t + e^{-t})(e^t - e^{-t})^{-1}e_3 && \nabla_{e_4}e_4 = 0
\end{aligned}$$

and

$$(4.3)\qquad \begin{aligned}
&R(e_1, e_2)e_i = 3\{\delta_{i2}e_1 - \delta_{i1}e_2\}\\
&R(e_i, e_3)e_j = -\{\delta_{j3}e_i - \delta_{ij}e_3\}\\
&R(e_i, e_4)e_j = -\{\delta_{j4}e_i - \delta_{ij}e_4\} , \quad i, j = 1, \cdots, 4 ,
\end{aligned}$$

where ∇ and R denote the Riemannian connection and the curvature tensor field of $(SO(3)\times \boldsymbol{R}^+, g)$. Similarly we obtain $\|\nabla\rho\|^2 = 32(e^t + e^{-t})^2$. By (4.3), the Riemannian manifold $(SO(3)\times \boldsymbol{R}^+, g)$ is curvature homogeneous.

Now we prove Proposition 4.1. Since (M, g) is curvature homogeneous on its dense subset M^* by Lemma 4.2, (M, g) is curvature homogeneous on the whole M. The projection of $((G/K)\times \boldsymbol{R}^+, g)$ onto $\boldsymbol{R}^+$ is a Riemannian submersion onto $\boldsymbol{R}^+$ with the standard metric. Thus (M, g) is complete.

Now we construct an isometric imbedding of the above Riemannian manifold (M, g) into $H^5(-1)$. We first define a G-equivariant injective map $j\colon M \to \mathfrak{p}$ by $j(a(H(s))) = a(3^{1/2}r(s)H(s))$ for $a \in G = SO(3)$, $s \in [0, 3^{-3/2}\pi)$, where

$$r(s) = \sinh^{-1}((1 + 2\cos(3^{1/2}2s))^{-1/2}) .$$

We denote by $\sinh^{-1} x$ the inverse function of $\sinh t = (e^t - e^{-t})/2$. Since $r(s)$ is an even function of s, we see that j is a smooth map. Fix a point o of $H^5(-1)$. The exponential map $\exp_o$ at o is a diffeomorphism of T_oH^5 onto H^5. We identify $\mathfrak{p}$ with T_oH^5 by a linear isometry. With this identification, $G = SO(3)$ acts on T_oH^5 as a group of linear isometries. Since any linear isometry of T_oH^5 is extended to an isometry of $H^5(-1)$, G naturally acts on $H^5(-1)$ as a group of isometries and the exponential map $\exp_o$ at o is a G-equivariant map. We define a map $f\colon M \to H^5(-1)$ by $f = \exp_o \circ j$. Then f is a G-equivariant injective smooth map. Moreover, we have:

PROPOSITION 4.3. *The map f is an isometric imbedding of (M, g) into $H^5(-1)$ and its type number is equal to* 2.

PROOF. We denote by $\tilde{g}$ the Riemannian metric on $H^5(-1)$. We will show that $f^*\tilde{g} = g$. It suffices to prove this on the dense subset M^* of M. We define a diffeomorphism $\Phi\colon S^4 \times \boldsymbol{R}^+ \to T_oH^5\setminus\{0\}$ by $\Phi((x, r)) = 3^{1/2}rx$, where S^4 denotes the sphere with radius $3^{-1/2}$ in T_oH^5. Then we have $g' = \Phi^* \exp_o^* \tilde{g} = 3(\sinh r)^2\langle\ ,\ \rangle + dr^2$, where $\langle\ ,\ \rangle$ denotes the Riemannian metric on S^4. Since $(\Phi^{-1}\circ j)(a(H(s))) = (a(H(s)), r(s))$ for $s \in \mathring{I} = (0, 3^{-3/2}\pi)$, we have

$$\begin{aligned} g'((\Phi^{-1}\circ j)_*X, (\Phi^{-1}\circ j)_*Y) &= 3(\sinh r(s))^2\langle X, Y\rangle = 3(1 + 2\cos(3^{1/2}2s))^{-1}\langle X, Y\rangle \\ &= a(s)\langle X, Y\rangle = g(X, Y) \end{aligned}$$

for vectors X and Y tangent to the orbit of $H(s)$. Since $(\Phi^{-1}\circ j)_*(H'(s)) = (H'(s), (dr/ds)(d/dr))$, we have

$$g'((\Phi^{-1}\circ j)_*X, (\Phi^{-1}\circ j)_*H'(s)) = 0$$

for a vector X tangent to the orbit of $H(s)$ and

$$\begin{aligned} g'((\Phi^{-1}\circ j)_*H'(s), (\Phi^{-1}\circ j)_*H'(s)) &= 3(\sinh r(s))^2 + (dr/ds)^2 \\ &= 9(1 + 2\cos(3^{1/2}2s))^{-2} = a(s)^2 = g(H'(s), H'(s)) . \end{aligned}$$

Consequently, j is an isometric imbedding of (M^*, g) into $(\mathfrak{p}, \exp_o{}^*\tilde{g})$ and

f is an isometric imbedding of (M, g) into $H^5(-1)$.

By Proposition 2.1 and (4.3), the type number of f is equal to 2.

5. The case of the type number 2—Part 2. In this section we will prove the following theorem and complete the proof of Theorem C.

THEOREM 5.1. *Let (M, g) be a 4-dimensional connected curvature homogeneous space and let f be an isometric immersion of (M, g) into $H^5(-1)$ of the type number 2. Then (M, g) is locally isometric to the example constructed in Section 4 and f is locally congruent to the isometric imbedding given in Section 4.*

In this section we assume that the assumption of Theorem 5.1 is satisfied and use the same notation as in Section 3.

LEMMA 5.2. *Let $\|C\|$ be the norm of the conullity operator C at $p \in M$. Then we have $\|C\|^2 \geqq 4$ and the equality holds if and only if $\operatorname{Im} C = \operatorname{Sym}^0(T_p^1)$.*

PROOF. By Proposition 3.3, we see that $\dim \operatorname{Im} C = 2$ and hence $\dim \operatorname{Im} C \cap \operatorname{Sym}^0(T_p^1) \geqq 1$. Therefore there exists a unit vector ξ of T_p^0 such that $C_\xi \in \operatorname{Sym}^0(T_p^1)$. Let $\{e_1, e_2\}$ be an orthonormal basis of T_p^1 such that $C_\xi e_1 = -e_1$ and $C_\xi e_2 = e_2$. Let η be the unit vector of T_p^0 orthogonal to ξ. Then C_η is represented as

$$C_\eta = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad a^2 + bc = 1$$

with respect to $\{e_1, e_2\}$. By Proposition 3.3, $\det(C_{(\cos\theta)\xi + (\sin\theta)\eta}) = -1$ for any θ. This implies that $a = 0$. Thus $\|C\|^2 = 2 + b^2 + c^2 \geqq 2 + 2(b^2c^2)^{1/2} = 4$. The equality holds if and only if $b = c = \pm 1$.

By Lemma 5.2, at a point of M, $\|C\|^2$ is either greater than 4 or equal to 4. Let us consider the first case. We fix a point $p \in M$ at which $\|C\|^2 > 4$. By Lemma 5.2, we have $\dim \operatorname{Im} C \cap \operatorname{Sym}^0(T^1) = 1$ on some neighborhood of p. Therefore there exists an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ defined on a neighborhood U of p which satisfies the following: $\{e_3, e_4\}$ is an orthonormal frame field of T^0 on U such that $C_{e_4} \in \operatorname{Sym}^0(T^1)$ and $\{e_1, e_2\}$ is an orthonormal frame field of T^1 such that $C_{e_4} e_1 = -e_1$ and $C_{e_4} e_2 = e_2$. By the argument in the proof of Lemma 5.2, C_{e_3} is represented with respect to $\{e_1, e_2\}$ as

$$C_{e_3} = \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \quad bc = 1,\, b \neq c\,.$$

We will determine the form of the Riemannian connection ∇ with respect to this orthonormal frame field $\{e_1, e_2, e_3, e_4\}$. We denote by Λ^k_{ij} for $i, j, k = 1, \cdots, 4$ the components of the connection, i.e., $\nabla_{e_i} e_j = \sum_{k=1}^4 \Lambda^k_{ij} e_k$.

Applying Proposition 3.1 (2), we calculate $(\nabla_{e_4} C_{e_4})(e_1)$ and $(\nabla_{e_4} C_{e_4})(e_2)$. We obtain $2\Lambda^2_{41} = -c\Lambda^3_{44}$ and $2\Lambda^1_{42} = b\Lambda^3_{44}$ and hence $\Lambda^4_{43} = \Lambda^2_{41} = 0$. This means that $\nabla_{e_4} e_1 = \nabla_{e_4} e_2 = \nabla_{e_4} e_3 = \nabla_{e_4} e_4 = 0$. Similarly calculating $(\nabla_{e_4} C_{e_3})(e_1)$ and $(\nabla_{e_4} C_{e_3})(e_2)$, we obtain $e_4 b = b$ and $e_4 c = -c$. Thus

$$e_4\|C\|^2 = e_4\{2 + b^2 + c^2\} = 2(b^2 - c^2) \neq 0 .$$

Changing the signs of e_3, e_4 and the indices of e_1, e_2, if necessary, we may assume that

$$\begin{aligned} &C_{e_4} e_1 = -e_1 , && C_{e_4} e_2 = e_2 \\ &C_{e_3} e_1 = c e_2 , && C_{e_3} e_2 = b e_1 \quad \text{with} \quad b, c < 0, bc = 1 \\ &e_4\|C\|^2 > 0 , \quad \text{i.e.,} && b^2 > c^2 \end{aligned}$$

for the orthonormal frame field $\{e_1, e_2, e_3, e_4\}$.

Using Proposition 3.1 (2), we calculate $(\nabla_{e_3} C_{e_4})(e_1)$ and $(\nabla_{e_3} C_{e_4})(e_2)$. Then we obtain $-2\Lambda^2_{31} = c(1 + \Lambda^3_{34})$ and $2\Lambda^1_{32} = b(\Lambda^3_{34} - 1)$. Hence we have $\Lambda^2_{31} = -1/(b - c)$ and $\Lambda^3_{34} = (b + c)/(b - c)$. Similarly, calculating $(\nabla_{e_3} C_{e_3})(e_1)$ and $(\nabla_{e_3} C_{e_3})(e_2)$, we obtain $e_3 b = e_3 c = 0$. By Proposition 3.1 (1), we have $\Lambda^1_{12} = (b/2)\Lambda^3_{14}$, $\Lambda^2_{21} = -(c/2)\Lambda^3_{24}$, $e_1 b = -(c(b - c)/2)\Lambda^3_{24}$, and $e_2 c = -(b(b - c)/2)\Lambda^3_{14}$. The plane spanned by $\{e_1, e_2\}$ has constant sectional curvature κ. Therefore we have

$$e_1\Lambda^1_{22} - e_2\Lambda^1_{12} - (\Lambda^1_{12})^2 - (\Lambda^1_{22})^2 = \kappa - 3 .$$

Consequently, we have

$$\begin{aligned} &\nabla_{e_1} e_1 = \Lambda^2_{11} e_2 - e_4 , && \nabla_{e_2} e_1 = \Lambda^2_{21} e_2 + b e_3 \\ &\nabla_{e_1} e_2 = \Lambda^1_{12} e_1 + c e_3 , && \nabla_{e_2} e_2 = \Lambda^1_{22} e_1 + e_4 \\ &\nabla_{e_1} e_3 = -c e_2 + \Lambda^4_{13} e_4 , && \nabla_{e_2} e_3 = -b e_1 + \Lambda^4_{23} e_4 \\ &\nabla_{e_1} e_4 = e_1 + \Lambda^3_{14} e_3 , && \nabla_{e_2} e_4 = -e_2 + \Lambda^3_{24} e_3 \\ &\nabla_{e_3} e_1 = \Lambda^2_{31} e_2 , && \nabla_{e_4} e_1 = 0 \\ &\nabla_{e_3} e_2 = \Lambda^1_{32} e_1 , && \nabla_{e_4} e_2 = 0 \\ &\nabla_{e_3} e_3 = \Lambda^4_{33} e_4 , && \nabla_{e_4} e_3 = 0 \\ &\nabla_{e_3} e_4 = \Lambda^3_{34} e_3 , && \nabla_{e_4} e_4 = 0 \\ &e_4 b = b , \quad e_3 b = 0 , && e_1 b = -(c(b - c)/2)\Lambda^3_{24} \\ &e_4 c = -c , \quad e_3 c = 0 , && e_2 c = -(b(b - c)/2)\Lambda^3_{14} \\ &\Lambda^2_{31} = -1/(b - c) , && \Lambda^1_{12} = (b/2)\Lambda^3_{14} \\ &\Lambda^3_{34} = (b + c)/(b - c) , && \Lambda^2_{21} = -(c/2)\Lambda^3_{24} \\ &e_1\Lambda^1_{22} - e_2\Lambda^1_{12} - (\Lambda^1_{12})^2 - (\Lambda^1_{22})^2 = \kappa - 3 . \end{aligned} \tag{5.1}$$

Now we consider the condition for M to be isometrically immersed in $H^5(-1)$. Since $\ker A = T^0$, we have $h(e_3, e_j) = h(e_4, e_j) = 0$ for $1 \leqq j \leqq 4$. By the equation of Gauss, we obtain $h(e_1, e_1)h(e_2, e_2) - h(e_1, e_2)^2 = \kappa + 1$. By Proposition 3.2, we have $h(C_{e_4}e_1, e_2) = h(e_1, C_{e_4}e_2)$ and hence $h(e_1, e_2) = 0$. Similarly, we have $h(C_{e_3}e_1, e_2) = h(e_1, C_{e_3}e_2)$ and hence $bh(e_1, e_1) = ch(e_2, e_2)$. Hence $\kappa + 1 = b^2h(e_1, e_1)^2$ is positive, $h(e_1, e_1) = \pm c(\kappa + 1)^{1/2}$ and $h(e_2, e_2) = \pm b(\kappa + 1)^{1/2}$. Changing the sign of the normal vector field, if necessary, we may assume that $h(e_1, e_1) = c(\kappa + 1)^{1/2}$ and $h(e_2, e_2) = b(\kappa + 1)^{1/2}$. Calculating $h(e_2, e_2, e_1) = h(e_2, e_1, e_2)$, we have $e_1b = (c - b)\Lambda^2_{21}$. In view of $e_1b = -(c(b - c)/2)\Lambda^3_{24}$ and $\Lambda^2_{21} = -(c/2)\Lambda^3_{24}$ in (5.1), we have $\Lambda^2_{21} = \Lambda^3_{24} = 0$. Similarly, by the equation $h(e_1, e_1, e_2) = h(e_1, e_2, e_1)$ and (5.1), we obtain $\Lambda^1_{12} = \Lambda^3_{14} = 0$. From these results and the last equation of (5.1) follows $\kappa = 3$. Put $t = (1/2)\cosh^{-1}((\|C\|^2 - 2)/2)$, where $\cosh^{-1} x$ denotes the inverse function of $\cosh\theta = (e^\theta + e^{-\theta})/2$. Then t is a positive smooth function on a neighborhood of p. Solving equations $b^2 + c^2 = e^{2t} + e^{-2t}$ and $bc = 1$ under the condition $b < 0$, $c < 0$ and $b^2 > c^2$, we obtain $b = -e^t$ and $c = -e^{-t}$. Since $e_1b = e_2b = e_3b = 0$ and $e_4b = b$, we see that $e_1t = e_2t = e_3t = 0$ and $e_4t = 1$. Namely, e_4 is the gradient vector field of the function t.

Consequently, we have:

LEMMA 5.3. (1) *Putting* $t = (1/2)\cosh^{-1}((\|C\|^2 - 2)/2)$, *we have* $e_1t = e_2t = e_3t = 0$ *and* $e_4t = 1$ *so that the vector field* e_4 *is the gradient vector field of the function* t.

(2) *The Riemannian connection* ∇ *is given with respect to the orthonormal frame field* $\{e_1, e_2, e_3, e_4\}$ *as follows*:

$$
\begin{aligned}
&\nabla_{e_1}e_1 = -e_4\,, && \nabla_{e_2}e_1 = -e^te_3\\
&\nabla_{e_1}e_2 = -e^{-t}e_3\,, && \nabla_{e_2}e_2 = e_4\\
&\nabla_{e_1}e_3 = e^{-t}e_2\,, && \nabla_{e_2}e_3 = e^te_1\\
&\nabla_{e_1}e_4 = e_1\,, && \nabla_{e_2}e_4 = -e_2\\
&\nabla_{e_3}e_1 = (e^t - e^{-t})^{-1}e_2\,, && \nabla_{e_4}e_1 = 0\\
&\nabla_{e_3}e_2 = -(e^t - e^{-t})^{-1}e_1\,, && \nabla_{e_4}e_2 = 0\\
&\nabla_{e_3}e_3 = -(e^t + e^{-t})(e^t - e^{-t})^{-1}e_4\,, && \nabla_{e_4}e_3 = 0\\
&\nabla_{e_3}e_4 = (e^t + e^{-t})(e^t - e^{-t})^{-1}e_3\,, && \nabla_{e_4}e_4 = 0\\
&[e_1, e_2] = (e^t - e^{-t})e_3\,, && [e_4, e_1] = -e_1\\
&[e_2, e_3] = e^{2t}(e^t - e^{-t})^{-1}e_1\,, && [e_4, e_2] = e_2\\
&[e_3, e_1] = e^{-2t}(e^t - e^{-t})^{-1}e_2\,, && [e_4, e_3] = -(e^t + e^{-t})(e^t - e^{-t})^{-1}e_3
\end{aligned}
$$

(3) *The second fundamental form* h *is given by*

$$h(e_1, e_1) = \pm 2e^{-t}\,, \quad h(e_2, e_2) = \pm 2e^t\,,$$

$$h(e_1, e_2) = h(e_3, e_j) = h(e_4, e_j) = 0$$

for the orthonormal frame field $\{e_1, e_2, e_3, e_4\}$.

Now we prove Theorem 5.1 on a neighborhood of a point $p \in M$ at which $\|C\|^2$ is greater than 4. By Lemma 5.3 (1), the level sets of the function t are hypersurfaces of M and the vector fields e_1, e_2, and e_3 are tangent to these hypersurfaces. Put $t(p) = t_0$. We take a sufficiently small connected neighborhood V of p in the hypersurface given by $t = t_0$ and take a sufficiently small $\varepsilon > 0$. Then there exists a diffeomorphism F of $V \times \mathring{I}$ into U defined by

$$F(x, t) = \phi_{t-t_0}(x) \quad \text{for} \quad x \in V \quad \text{and} \quad t \in \mathring{I} = (t_0 - \varepsilon, t_0 + \varepsilon)\ ,$$

where $\{\phi_t\}$ denotes the local one-parameter group of local transformations generated by the vector field e_4. By the definition of F, we have $t(F(x, s)) = s$. Put $\tilde{X}_1 = e^t e_1$, $\tilde{X}_2 = e^{-t} e_2$ and $\tilde{X}_3 = (e^t - e^{-t}) e_3$. Then by Lemma 5.3 (2), we have

$$\begin{aligned} [\tilde{X}_1, \tilde{X}_2] &= \tilde{X}_3\ , \quad [e_4, \tilde{X}_1] = 0 \\ [\tilde{X}_2, \tilde{X}_3] &= \tilde{X}_1\ , \quad [e_4, \tilde{X}_2] = 0 \\ [\tilde{X}_3, \tilde{X}_1] &= \tilde{X}_2\ , \quad [e_4, \tilde{X}_3] = 0\ . \end{aligned}$$

The restrictions of $\tilde{X}_1$, $\tilde{X}_2$ and $\tilde{X}_3$ to V are denoted by X_1, X_2 and X_3, respectively. Then there exists a diffeomorphism of V into $SO(3)$ which sends X_1, X_2 and X_3 to left invariant vector fields on $SO(3)$. Let ω_1, ω_2 and ω_3 be the 1-forms defined on V and dual to X_1, X_2 and X_3. By using the diffeomorphism F, the Riemannian metric g on $F(V \times \mathring{I})$ is represented as

$$g = e^{2t}\omega_1 \otimes \omega_1 + e^{-2t}\omega_2 \otimes \omega_2 + (e^t - e^{-t})^2 \omega_3 \otimes \omega_3 + dt^2\ .$$

Thus $F(V \times \mathring{I})$ in U is locally isometric to the example in Section 4. By Lemma 5.3 (3), the immersion f is rigid. Therefore Theorem 5.1 has been proved in our case.

Next we consider Theorem 5.1 on a neighborhood of a point of M at which $\|C\|^2$ is equal to 4. We note that $\|C\|^2$ is equal to 4 at points on the orbit of $H(0)$ for the example in Section 4. We fix a point $p \in M$ at which $\|C\|^2$ is equal to 4. Let $\gamma\colon [0, \delta] \to M$ be a unit speed geodesic such that $\gamma(0) = p$ and $\dot{\gamma}(0) \in T_p^0$. By Lemma 5.2, we have $C_{\dot{\gamma}(0)} \in \mathrm{Sym}^0(T_p^1)$. Let $\{f_1, f_2\}$ be an orthonormal basis of T_p^1 such that $C_{\dot{\gamma}(0)} f_1 = -f_1$ and $C_{\dot{\gamma}(0)} f_2 = f_2$, and let $\{f_1(t), f_2(t)\}$ be a parallel orthonormal frame field of T^1 along γ such that $f_i(0) = f_i$, $i = 1, 2$. We represent $C_{\dot{\gamma}(t)}$ as a 2×2-matrix with respect to $\{f_1(t), f_2(t)\}$. Then by Proposition 3.1 (3), we obtain the following ordinary differential equation:

$$\frac{d}{dt}C_{\dot{\gamma}} = 0$$

$$C_{\dot{\gamma}(0)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We immediately have

$$C_{\dot{\gamma}(t)} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $\eta(t)$ be a parallel vector field along γ such that $\eta(0)$ is a unit vector of T_p^0 and orthogonal to $\dot{\gamma}(0)$. By the argument in the proof of Lemma 5.2, we have

$$C_{\eta(0)} = \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}$$

with respect to $\{f_1, f_2\}$. We may assume that

$$C_{\eta(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Solving the equation

$$\frac{d}{dt}C_{\eta} = C_{\eta}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$C_{\eta(0)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

given by Proposition 3.1 (3), we have

$$C_{\eta(t)} = \begin{pmatrix} 0 & e^{t} \\ e^{-t} & 0 \end{pmatrix}.$$

Thus $\|C\|^2 = 2 + e^{2t} + e^{-2t}$ at $\gamma(t)$. In particular, $\|C\|^2$ is greater than 4 at $\gamma(t)$, $t > 0$.

We fix t_0, $0 < t_0 < \delta$. Put $\gamma(t_0) = p_0$. Since $\|C\|^2 > 4$ at p_0, a neighborhood of p_0 is locally isometric to the example in Section 4. In particular, the sectional curvature of the T^1-plane is equal to 3 and there exists an orthonormal frame field $\{e_1, e_2, e_3, e_4\}$ on some neighborhood U of p_0 such that Lemma 5.3 holds. We note that $\dot{\gamma}(t) = e_4$ on $\gamma(t) \cap U$. The subset K in U consisting of points at which $\|C\|^2 = 2 + e^{2t_0} + e^{-2t_0}$ is a hypersurface of U and the vector fields e_1, e_2, and e_3 are tangent to K. Evidently K contains p_0. We define a smooth map $\phi\colon K \to M$ by $\phi(q) = \exp_q(-t_0 e_4|_q)$ for $q \in K$, which is well-defined on a suitable neighborhood of p_0 in K. Put $\phi(K) = N$. Then we have

LEMMA 5.4. *N is a 2-dimensional submanifold of M containing p. Moreover, N is an integral submanifold of the distribution T^1.*

PROOF OF LEMMA 5.4. p is contained in N, since $\phi(p_0) = p$. We compute the differential $d\phi$ of ϕ at $q \in K$. Let σ: $[0, t_0] \to M$ be a geodesic such that $\sigma(0) = q$ and $\dot{\sigma}(0) = -e_4|_q$. Let X be a Jacobi field along σ such that $X(0) = x \in T_qK$ and $X'(0) = A_{e_4}x$, where A denotes the shape operator of the submanifold K in M. Then we have $d\phi(x) = X(t_0)$. Let $f_i(t)$ be a parallel vector field along σ such that $f_i(0) = e_i|_q$, $i = 1, 2, 3$. We note that $f_1(t)$ and $f_2(t)$ are tangent to T^1 while $f_3(t)$ is tangent to T^0, and that $R(f_i, \dot{\sigma})\dot{\sigma} = -f_i$, $i = 1, 2, 3$. By Lemma 5.3 (2), we have $A_{e_4}e_1 = -e_1$, $A_{e_4}e_2 = e_2$ and $A_{e_4}e_3 = -(e^{t_0} + e^{-t_0})(e^{t_0} - e^{-t_0})^{-1}e_3$ at $q \in K$. Let E_1, E_2 and E_3 be Jacobi fields along σ whose initial conditions are given by $E_1(0) = e_1|_q$, $E_1'(0) = -e_1|_q$ and $E_2(0) = e_2|_q$, $E_2'(0) = e_2|_q$ and $E_3(0) = e_3|_q$, $E_3'(0) = -(e^{t_0} + e^{-t_0})(e^{t_0} - e^{-t_0})^{-1}e_3|_q$, respectively. Solving the equations of Jacobi fields, we obtain $E_1(t) = e^{-t}f_1(t)$, $E_2(t) = e^{t}f_2(t)$ and $E_3(t) = \{(e^t + e^{-t})/2 - (e^{t_0} + e^{-t_0})(e^t - e^{-t})/2(e^{t_0} - e^{-t_0})\}f_3(t)$. Thus $d\phi(e_1|_q) = e^{-t_0}f_1(t_0)$, $d\phi(e_2|_q) = e^{t_0}f_2(t_0)$ and $d\phi(e_3|_q) = 0$. Hence Lemma 5.4 has been proved.

We will investigate the properties of the submanifold N in more detail. The subspace T_q^0 at $q \in N$ is just the normal space $T_q^{\perp}N$ of the submanifold N in M and the conullity operator C is just the shape operator of N in M. Therefore C_ξ for $\xi \in T_q^0$, $q \in N$, is a symmetric linear endomorphism of $T_q^1 = T_qN$. By Lemma 5.2, we have $\|C\|^2 = 4$ at $q \in N$. For an arbitrary orthonormal basis $\{e_3, e_4\}$ of $T_q^0 = T_q^{\perp}N$, there exists an orthonormal basis $\{e_1, e_2\}$ of T_qN with respect to which C_{e_3} and C_{e_4} are represented as

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

respectively (see the argument in the proof of Lemma 5.2). By the equation of Gauss, we see that N is a surface with constant curvature 1 with respect to the induced Riemannian metric.

Now we fix some notion on the symmetric tensor product. Let V be a 2-dimensional Euclidean space with the inner product $\langle\ ,\ \rangle$. We denote by $S^2(V)$ the *symmetric tensor product* of V. The space $S^2(V)$ is linearly spanned by $x \cdot y = (x \otimes y + y \otimes x)/2$ with x, y running through V. We introduce an inner product $(\ ,\ )$ on $S^2(V)$ by $(x \cdot y, u \cdot v) = \{\langle x, u\rangle\langle y, v\rangle + \langle x, v\rangle\langle y, u\rangle\}/2$. We identify $S^2(V)$ with the space of all symmetric linear endomorphisms of V, defining the linear endomorphism $x \cdot y$ by $(x \cdot y)(u) = \{\langle x, u\rangle y + \langle y, u\rangle x\}/2$ for x, y, $u \in V$. Then we note that $\langle \xi(x), y\rangle = (x \cdot y, \xi)$

for $\xi \in S^2(V)$, $x, y \in V$. Put $e = e_1 \cdot e_1 + e_2 \cdot e_2$ for some orthonormal basis $\{e_1, e_2\}$ of V. This definition is independent of the choice of an orthonormal basis. Then we have $e(x) = x$ and $(x \cdot y, e) = \langle x, y \rangle$. With $H^2(V) = \{\xi \in S^2(V); (\xi, e) = 0\}$, we have an orthogonal decomposition:

$$S^2(V) = H^2(V) + \boldsymbol{R}e\ .$$

We may take $\{2^{-1/2}(e_1 \cdot e_1 - e_2 \cdot e_2), 2^{1/2} e_1 \cdot e_2\}$ as an orthonormal basis of $H^2(V)$ for some orthonormal basis $\{e_1, e_2\}$ of V. We apply the above arguments to the tangent space T_qN at $q \in N$. We denote by $S^2(TN)$ the tensor bundle on N consisting of the symmetric tensor products $S^2(T_qN)$ and by $H^2(TN)$ the subbundle of $S^2(TN)$ consisting of $H^2(T_qN)$. The Riemannian connection of N with respect to the induced Riemannian metric is denoted by the same notation ∇ as that of M. The connection ∇ on TN is naturally extended to $S^2(TN)$ and the subbundle $H^2(TN)$ is parallel with respect to this connection.

The following holds:

LEMMA 5.5. *Let α be the second fundamental form of the submanifold N in M.*

(1) *α is a linear isomorphism of $H^2(T_qN)$ onto $T_q^{\perp}N = T_q^0$. Moreover, we have $\langle \alpha(\xi), \alpha(\eta) \rangle = 2(\xi, \eta)$ for $\xi, \eta \in H^2(T_qN)$ and $C_{\alpha(\xi)}x = 2\xi(x)$, where $\xi(x)$ means the action of ξ on x as a symmetric linear endomorphism.*

(2) *The second fundamental form α is parallel and hence is a bundle isomorphism of $H^2(TN)$ onto $T^{\perp}N$ which preserves the connections, where $T^{\perp}N$ is equipped with the normal connection $\nabla^{\perp}$.*

PROOF OF LEMMA 5.5. (1) Let $\{e_1, e_2\}$ and $\{e_3, e_4\}$ be the orthonormal bases of T_qN and $T_q^{\perp}N$, respectively such that

$$C_{e_3} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad C_{e_4} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to $\{e_1, e_2\}$. Then we have $\alpha(e_1, e_1) = -e_3$, $\alpha(e_2, e_2) = e_3$ and $\alpha(e_1, e_2) = e_4$. Put $\xi = 2^{-1/2}\{e_1 \cdot e_1 - e_2 \cdot e_2\}$ and $\eta = 2^{1/2} e_1 \cdot e_2$. Then $\{\xi, \eta\}$ is an orthonormal basis of $H^2(T_qN)$. We see that $\alpha(e) = \alpha(e_1 \cdot e_1 + e_2 \cdot e_2) = 0$, $\alpha(\xi) = -2^{1/2} e_3$ and $\alpha(\eta) = 2^{1/2} e_4$.

(2) By (1), we have $\langle \alpha(y, z), \alpha(u, v) \rangle = \langle y, u \rangle \langle z, v \rangle + \langle y, v \rangle \langle z, u \rangle - \langle y, z \rangle \langle u, v \rangle$. Thus $\langle (\bar{\nabla}_x \alpha)(y, z), \alpha(u, v) \rangle + \langle \alpha(y, z), (\bar{\nabla}_x \alpha)(u, v) \rangle = 0$. By Proposition 3.1 (1), we have $(\bar{\nabla}_x \alpha)(y, z) = (\bar{\nabla}_y \alpha)(x, z)$. Using the above equations, we have $\langle (\bar{\nabla}_x \alpha)(y, z), \alpha(u, v) \rangle = -\langle \alpha(y, z), (\bar{\nabla}_x \alpha)(u, v) \rangle = \langle (\bar{\nabla}_u \alpha)(y, z), \alpha(x, v) \rangle = -\langle \alpha(u, z), (\bar{\nabla}_y \alpha)(x, v) \rangle = \langle (\bar{\nabla}_v \alpha)(u, z), \alpha(x, y) \rangle = -\langle \alpha(u, v), (\bar{\nabla}_z \alpha)(x, y) \rangle = -\langle (\bar{\nabla}_x \alpha)(y, z), \alpha(u, v) \rangle$ and hence $\langle (\bar{\nabla}_x \alpha)(y, z), \alpha(u, v) \rangle =$

0. Consequently, we have $(\bar{\nabla}_x\alpha)(y, z) = 0$, that is, $\nabla_x^{\perp}(\alpha(y, z)) = \alpha(\nabla_x y, z) + \alpha(y, \nabla_x z) = \alpha(\nabla_x(y\cdot z))$ holds. This equation means that α preserves the connections.

We will show that there exists a local isometry of a tubular neighborhood of N into the example of Section 4. For this we need a general lemma on equivalence problem. Let M be an m-dimensional Riemannian manifold and N an n-dimensional submanifold imbedded in M $(0 \leqq n \leqq m-1)$. We denote by $T^{\perp}N$ the normal bundle of N. The exponential map $\exp_N$ of the normal bundle maps a neighborhood U_N of the zero section of $T^{\perp}N$ into M. We denote by U_N^r the neighborhood of the zero section of $T^{\perp}N$ with radius r and put $U_r(N) = \exp_N(U_N^r)$. For a unit vector ξ of T_pM, let γ be a geodesic of M such that $\gamma(0) = p$ and $\dot{\gamma}(0) = \xi$. We denote by $\tau_0^t\colon T_pM \to T_{\gamma(t)}M$ the parallel translation along γ and denote by $\hat{R}_{(t,\xi)}$ the tensor of type (1, 3) on T_pM defined by

$$\hat{R}_{(t,\xi)}(x, y)z = (\tau_0^t)^{-1}(R(\tau_0^t x, \tau_0^t y)\tau_0^t z) \quad \text{for} \quad x, y, z \in T_pM\,.$$

Let $\bar{M}$ be another m-dimensional Riemannian manifold and let $\bar{N}$ be an n-dimensional submanifold imbedded in $\bar{M}$.

We prove a lemma on equivalence problem under the following assumption: For some $r > 0$, the exponential maps $\exp_N\colon U_N^r \to U_r(N)$ and $\exp_{\bar{N}}\colon U_{\bar{N}}^r \to U_r(\bar{N})$ are both diffeomorphisms. There exists an isometry ϕ of N onto $\bar{N}$ and a bundle isomorphism f of $T^{\perp}N$ onto $T^{\perp}\bar{N}$ which satisfy the following conditions:

(1) The following diagram is commutative;

$$\begin{array}{ccc} T^{\perp}N & \xrightarrow{f} & T^{\perp}\bar{N} \\ \pi\downarrow & & \downarrow\bar{\pi} \\ N & \xrightarrow{\phi} & \bar{N} \end{array}$$

(2) The isomorphism f preserves the metrics and the normal connections of the normal bundles $T^{\perp}N$ and $T^{\perp}\bar{N}$.

(3) The differential $d\phi$ of ϕ and f preserve the shape operators of the submanifolds N and $\bar{N}$ in M and $\bar{M}$, respectively, i.e.,

$$d\phi A_{\xi} = A_{f(\xi)}d\phi \quad \text{for} \quad \xi \in T_p^{\perp}N\,.$$

(4) We denote by $F = d\phi + f$ the bundle isomorphism of $TM|_N$ onto $T\bar{M}|_{\bar{N}}$ defined by $d\phi$ and f. Then F satisfies

$$F\hat{R}_{(t,\xi)} = \hat{R}_{(t,f(\xi))}\,, \quad |t| < r$$

for any point $p \in N$ and any unit vector $\xi \in T_p^{\perp}N$.

LEMMA. *Under the above assumption, the diffeomorphism Φ of the tubular neighborhood $U_r(N)$ onto $U_r(\bar{N})$ defined by $\Phi = \exp_{\bar{N}} \circ f \circ \exp_N^{-1}$ is an isometry.*

This lemma is seen by using the arguments of Jacobi fields. We sketch the proof (for reference [7, Chapter 2]).

We first calculate the differential $d(\exp_N)$ of $\exp_N$ at a point (p, ξ) of $T^\perp N$, where (p, ξ) means a normal vector ξ at $p \in N$. We denote by $\mathscr{H}_{(p,\xi)}$ the horizontal subspace of the tangent space $T_{(p,\xi)}T^\perp N$ with respect to the normal connection $\nabla^\perp$ and by $\mathscr{V}_{(p,\xi)}$ the vertical subspace, i.e., the kernel of the differential $d\pi$ of the natural projection $\pi: T^\perp N \to N$. Then we have the direct sum decomposition:

$$T_{(p,\xi)}T^\perp N = \mathscr{H}_{(p,\xi)} + \mathscr{V}_{(p,\xi)} .$$

We naturally identify $\mathscr{V}_{(p,\xi)}$ with $T_p^\perp N$. For $x \in \mathscr{H}_{(p,\xi)}$, put a curve $c(s)$ of N such that $c(0) = p$, $\dot{c}(0) = d\pi(x)$ and let $\xi(s)$ denote the parallel normal vector field through ξ along $c(s)$ with respect to the normal connection $\nabla^\perp$. Then we have $\dot{\xi}(0) = x$. Put $\alpha(t, s) = \exp_{c(s)} t\xi(s)$. Then $\alpha(t, s)$ is a variation of the geodesic $\gamma(t) = \exp_p t\xi$ and its variational vector field $X(t)$ is a Jacobi field along $\gamma(t)$ and satisfies $X(0) = d\pi(x)$ and $X'(0) = -A_{\dot{\gamma}(0)}X(0) = -A_\xi d\pi(x)$. In particular, $d(\exp_N)_{(p,\xi)}(x) = X(1)$. On the other hand, for $y \in \mathscr{V}_{(p,\xi)}$, put $\alpha(t, s) = \exp_p t(\xi + sy)$. Then $\alpha(t, s)$ is a variation of γ and its variational vector field $Y(t)$ is a Jacobi field along $\gamma(t)$ and satisfies $Y(0) = 0$ and $Y'(0) = y$. Moreover, we have $d(\exp_N)_{(p,\xi)}(y) = Y(1)$. For an arbitrary vector $v \in T_{(p,\xi)}T^\perp N$, let y be the vertical component of v and let $V(t)$ be the Jacobi field along the geodesic $\gamma(t) = \exp_p t\xi$ whose initial conditions are $V(0) = d\pi(v)$ and $V'(0) = y - A_\xi d\pi(v)$. Then the above argument implies $d(\exp_N)_{(p,\xi)}(v) = V(1)$.

Given $v \in T_{(p,\xi)}T^\perp N$, $(p, \xi) \in U_N^r$, let $V(t)$ be the Jacobi field along $\gamma(t) = \exp_p t\xi$ such that $V(0) = d\pi(v)$ and $V'(0) = y - A_\xi d\pi(v)$, where y denotes the vertical component of v under the identification of $\mathscr{V}_{(p,\xi)}$ with $T_p^\perp N$. Define a vector field $\bar{V}(t)$ along $\bar{\gamma}(t) = \exp_{\phi(p)} tf(\xi)$ by $\bar{V}(t) = \bar{\tau}_0^t F(\tau_0^t)^{-1} V(t)$, where τ_0^t and $\bar{\tau}_0^t$ denote the parallel translations along γ and $\bar{\gamma}$, respectively. Then by the assumption of Lemma, $\bar{V}(t)$ is a Jacobi field along $\bar{\gamma}$. Moreover, we have $\bar{V}(0) = FV(0) = d\phi d\pi(v) = d\bar{\pi} df(v)$ and $\bar{V}'(0) = FV'(0) = f(y) - d\phi A_\xi d\pi(v) = f(y) - A_{f(\xi)} d\phi d\pi(v) = f(y) - A_{f(\xi)} d\bar{\pi} df(v)$. Since f preserves the normal connections, $f(y)$ coincides with the vertical component of $df(v)$ under the identification of $\mathscr{V}_{(\phi(p),f(\xi))}$ with $T_{\phi(p)}^\perp \bar{N}$. Therefore we have $d(\exp_{\bar{N}})_{(\phi(p),f(\xi))}(df(v)) = \bar{V}(1)$. Since $\|V(1)\| = \|\bar{V}(1)\|$, we see that $\|d(\exp_N)_{(p,\xi)}(v)\| = \|d(\exp_{\bar{N}})_{(\phi(p),f(\xi))}(df(v))\|$. Consequently, Φ is an isometry.

Now let us return to the proof of Theorem 5.1 on a neighborhood of a point at which $\|C\|^2$ is equal to 4. We fix a point $\bar{p}$ of the example in section 4 at which $\|C\|^2$ is equal to 4. Then there exists an integral submanifold $\bar{N}$ of the distribution T^1 containing $\bar{p}$. Since both N and $\bar{N}$ are surfaces of constant curvature 1, there exists an isometry ϕ of N onto $\bar{N}$. Here we take N and $\bar{N}$ sufficiently small. Let α and $\bar{\alpha}$ be the second fundamental forms of N and $\bar{N}$, respectively. Then by Lemma 5.5, α: $H^2(TN) \to T^{\perp}N$ and $\bar{\alpha}$: $H^2(T\bar{N}) \to T^{\perp}\bar{N}$ are both bundle isomorphisms. We extend the differential $d\phi$ of the isometry ϕ to a bundle isomorphism of $H^2(TN)$ onto $H^2(T\bar{N})$ and define a bundle isomorphism f of $T^{\perp}N$ onto $T^{\perp}\bar{N}$ by $f = \bar{\alpha} \circ d\phi \circ \alpha^{-1}$. We note that the curvature tensor field R is parallel along the geodesic tangent to T^0. Then the isometry ϕ and the bundle isomorphism f satisfy the assumption of the above lemma. Hence there exists an isometry of a tubular neighborhood of N onto a tubular neighborhood of $\bar{N}$.

By Proposition 3.2, we see that the shape operator A of the immersion of M into $H^5(-1)$ has the form $A = \pm 2$ id on T^1_p at a point $p \in M$ at which $\|C\|^2$ is equal to 4, where id denotes the identity map of T^1_p. On the other hand, Lemma 5.3 (3) holds at a point of M at which $\|C\|^2$ is greater than 4. Thus the immersion of M into $H^5(-1)$ is rigid. Hence Theorem 5.1 has been proved.

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