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Curvature Invariants of Random Interface Generated by Gaussian Fields

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We consider random interfaces generated by level crossing of a Gaussian random field. Transition from an isolated pattern to a percolating one is discussed by calculating the Euler characteristics of the interface and the number of local minimum of the generating Gaussian field. These topological invariants are shown to be universal.

The picture of random interfaces is very useful in investigating the phenomena of pattern formations, e.g., the ordering process in a quenched binary system. In the ordering process the order parameter field obeys a nonlinear dynamics. But, if one incorporates the nonlinearity by a nonlinear transformation of the order parameter, e.g.,

$$s = \text{sign } u\,, \tag{1}$$

the fictitious field u may be treated as a linear field, i.e., a Gaussian field, and the original field s consists of binary phases separated by random interfaces defined by contour surfaces of u. This scheme has been applied successfully to the order-disorder transition problem.[1)~3)]

The purpose of this paper is to study geometrical statistics of random interfaces. A statistically tractable model is given by the theory of excursion set of random fields.[4)] Restricting the problem within Gaussian fields, some expectations of significant geometrical quantities of interfaces are calculable as is shown in the following.

Let $\{u(\boldsymbol{r})\}$ be a homogeneous, isotropic and non-erratic Gaussian random field in d-dimensional Euclidean space with expectations

$$\langle u(\boldsymbol{r})\rangle = 0 \quad \text{and} \quad \langle u(0)\,u(\boldsymbol{r})\rangle = \sigma(r)\,, \tag{2}$$

where the covariance, or the correlation function $\sigma(r)$ is assumed to be regular, i.e.,

$$\sigma(r) = \sigma(0) - |\sigma''(0)|r^2/2! + \sigma^{(4)}(0)\,r^4/4! + \cdots . \tag{3}$$

The characteristic parameters of the interface are only $\sigma(0)$, $\sigma''(0)$ and $\sigma^{(4)}(0)$ for our purpose in the following. Let us use the following parameters instead of them for simplicity,

$$\lambda = [2\pi\sigma(0)/|\sigma''(0)|]^{1/2}\,,$$

$$\gamma^2 = 3\sigma''(0)^2/\sigma(0)\sigma^{(4)}(0)\,, \tag{4}$$

and let

$$\sigma(0) = 1 \quad \text{and} \quad \lambda = 1\,,$$

which define new scales of the unit of length and the field variable u. The parameter γ characterizes the nature of the correlation of the random field and then the pattern of the interface, and is shown to satisfy the following inequality in the isotropic system:

$$\gamma^2 < (d+2)/d\,. \tag{5}$$

The quantities required to calculate curvatures of the level crossing set, i.e., the contour surface defined by

$$\partial U = \{\boldsymbol{r}\,|\,u(\boldsymbol{r}) = U\} \tag{6}$$

are the derivative fields $\boldsymbol{v} = \{\partial_i u\}$ and $\{u_{ij}\} = \{\partial_i\partial_j u\}$ at the same point on the surface, where $\partial_i = \partial/\partial x_i$. These field variables obey a multi-dimensional Gaussian distribution with the following expectations:

$$\begin{aligned}
&\langle u\rangle = \langle v_i\rangle = \langle u_{ij}\rangle = 0, \quad \langle u^2\rangle = 1\,,\\
&\langle uv_i\rangle = 0\,, \quad \langle v_i v_j\rangle = |\sigma''(0)|\delta_{ij}\,,\\
&\langle uu_{ij}\rangle = -|\sigma''(0)|\delta_{ij}\,,\\
&\langle u_{ij}u_{kl}\rangle = (\sigma^{(4)}(0)/3)\,[\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}]\,.
\end{aligned} \tag{7}$$

Note that the gradient field $\boldsymbol{v} = \nabla u$ is independent of u and $\{u_{ij}\}$.

A simple definition of the curvatures is given by using a new coordinate set at each point on the contour surface; one component, say, the d-th component, is to be in the normal direction $\boldsymbol{n} = \nabla u/|\nabla u|$ and others in the tangential space. Let $\{T_{ij}\}$ be the coefficient matrix of the orthogonal transformation into the new coordinate set.

Then the derivative field variables are transformed as

$$\tilde{v}_i = T_{ii'} v_{i'} = |\nabla u| \delta_{id}$$

and

$$\tilde{u}_{ij} = T_{ii'} T_{jj'} u_{i'j'} . \tag{8}$$

In this frame the principal curvatures $\{R_i^{-1}\}(i=1, 2, \cdots, d-1)$ are the eigenvalues of the matrix

$$\Lambda = \{\tilde{u}_{ij}/|\nabla u|\}, \ (i, j = 1, 2, \cdots, d-1) \tag{9}$$

e.g., the mean curvature is given by

$$H = \mathrm{Tr}\ \Lambda/(d-1) , \tag{10}$$

and the Gauss' total curvature by

$$K = \det \Lambda . \tag{11}$$

Define other invariants related to the principal minors of Λ by

$$K(n) = \frac{1}{{}_{d-1}C_n} \Sigma_S \frac{1}{R_1 R_2 \cdots R_n} , \tag{12}$$

where Σ_S denotes the summation over ${}_{d-1}C_n$ symmetric combinations of n different components of $\{R_i^{-1}\}$.

Note that it is always possible to select the matrix $\{T_{ij}\}$ so that it does not depend on u and $\{u_{ij}\}$ and then the covariance relations (7) are all conserved, e.g.,

$$\begin{aligned}\langle u\tilde{u}_{ij}\rangle &= \langle T_{ii'} T_{jj'}\rangle\langle u u_{i'j'}\rangle \\ &= -|\sigma''(0)|\langle T_{ii'} T_{jj'}\rangle \delta_{i'j'} \\ &= -|\sigma''(0)|\delta_{ij} . \end{aligned} \tag{13}$$

Now it is straightforward to calculate the Gaussian expectations of the invariants (10) ~ (12) with the condition $u(r) = U$.

Physically interesting quantities are the spatial averages over the whole system of volume V, which can be replaced by the conditional ensemble averages in the homogeneous system, i.e.,

$$\bar{Q} = \frac{1}{V}\int_{\partial U} Q\ da = \langle Q|\nabla u|\delta(U-u)\rangle , \tag{14}$$

where Q is one of the invariants of curvatures and $\langle\cdots\rangle$ denotes the ensemble average by multidimensional Gaussian distribution. The simplest example is the case $Q=1$, which defines the interface density[1),5)]

$$\bar{A}(U) = \frac{1}{V}\int_{\partial U} da = \langle|\nabla u|\delta(U-u)\rangle . \tag{15}$$

The average defined by Eq. (14) should be distinguished from the average over the interface defined by

$$\int_{\partial U} Q\ da / \int_{\partial U} da .$$

The latter is not tractable in general because the denominator is a fluctuating quantity, though one may approximate it by $V\bar{A}(U)$.

By using a recurrence formula for the determinants, one obtains

$$\overline{K(n)} = \left[2\pi^{(n+1)/2} \times \Gamma\left(\frac{d-n+1}{2}\right)/\Gamma\left(\frac{d}{2}\right)\right]\phi^{(n)}(-U) , \tag{16}$$

where $\Gamma(\nu)$ is the Gamma function and $\phi^{(n)}(x)$ is related to the Hermite polynomials $H_n(x)$ as

$$\begin{aligned}\phi^{(n)}(x) &= (-d/dx)^n e^{-x^2/2}/\sqrt{2\pi} \\ &= H_n(x) e^{-x^2/2}/\sqrt{2\pi} . \end{aligned} \tag{17}$$

Examples of Eq. (16) are as follows:

$$\begin{aligned}\bar{A} &= \overline{K(0)} \\ &= \left[2\pi^{1/2}\Gamma\left(\frac{d+1}{2}\right)/\Gamma\left(\frac{d}{2}\right)\right] e^{-U^2/2}/\sqrt{2\pi} , \\ \bar{H} &= \overline{K(1)} = -2\pi\ U e^{-U^2/2}/\sqrt{2\pi} \end{aligned} \tag{18}$$

and

$$\bar{K} = \overline{K(d-1)} = S_d \phi^{(d-1)}(-U) , \tag{19}$$

where

$$S_d = 2\pi^{d/2}/\Gamma(d/2)$$

is the surface area of d-dimensional unit sphere. The expression (18) for $\bar{A}$ has been obtained by several authors.[1)~3),6)] Note that these expectations are universal quantities, i.e., do not include the pattern parameter γ.

Expectations of another series of invariants are obtained by a kind of partition function method,

$$\overline{H^n} = \left[2\pi^{(n+1)/2}\Gamma\left(\frac{d-n+1}{2}\right)/\Gamma\left(\frac{d}{2}\right)\right] \times (i/\alpha)^n H_n(i\alpha U) e^{-U^2/2}/\sqrt{2\pi} , \tag{20}$$

where

$$\alpha^{-2} = (d+1)/(d-1)\gamma^2 - 1 . \tag{21}$$

An example is the averaged square of mean curvature[5)]

$$\overline{H^2} = \left[2\pi^{3/2}\Gamma\left(\frac{d-1}{2}\right)/\Gamma\left(\frac{d}{2}\right)\right] \times (U^2 + \alpha^{-2}) e^{-U^2/2}/\sqrt{2\pi} , \tag{22}$$

which is non-universal because of α^{-2} term. An application of it is the averaged square of Euler's normal curvature which appears in the expansion of the correlation function of the nonlinear field s defined by Eq. (1).[7] With use of the principal curvatures it is expressed as

$$R^{-2}=[2\sum R_i^{-2}+(\sum R_i^{-1})^2]/(d^2-1)\,,$$

then its expectation is given by

$$\overline{R^{-2}}=\left[2\pi^{3/2}\Gamma\left(\frac{d-1}{2}\right)/\Gamma\left(\frac{d}{2}\right)\right]\times(U^2-1+3/\gamma^2)\,e^{-U^2/2}/\sqrt{2\pi}$$

which is in agreement with that obtained by a correlation function method,[6] if it is divided by $\bar{A}$.

Expectations of mixed type invariants $\overline{H^nK(m)}$ are also calculable, but expressions of them are complicated and are neglected here.

Now let us discuss the geometry of the random interface introduced above: Define the expectation of a topological characteristic by

$$\bar{\chi}_d=\frac{1}{V}\int_{\partial U}K(d-1)\,da/S_d\,, \tag{23}$$

which is related to Euler's index by the Gauss-Bonnet theorem.[8] For example, in a droplet system where the interface consists of simple closed surfaces only, $\bar{\chi}_d$ has a simple meaning of the number density of droplets. In fact, it has been shown that $\bar{\chi}_d$ coincides with the density of local minimum under the level U when $U\to-\infty$.[4] This problem is discussed later.

The expectation of $\bar{\chi}_d$ is simply given by

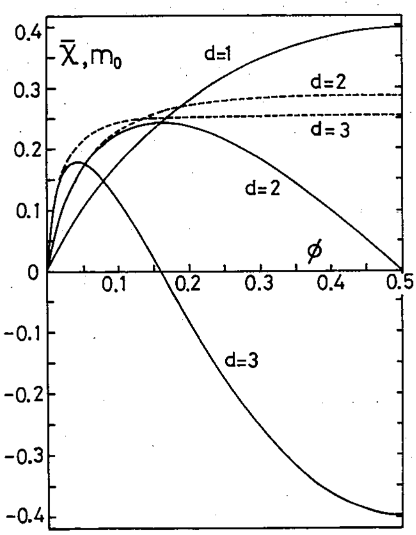

Fig. 1. The topological characteristic $\bar{\chi}_d(U)$ (solid lines) and the lower bound $m_0(U)$ (broken lines) of the density of local minimum in the unit λ^{-d}. For $d=1$, both curves coincide with each other in the range $\phi<1/2$.

$$\bar{\chi}_d=\phi^{(d-1)}(-U)\,, \tag{24}$$

which is in agreement with the expansion form given by Adler.[4] The behaviour of $\bar{\chi}_d$ for $d=1$, 2, 3 is shown in Fig. 1, where we have used the *volume fraction* defined by

$$\phi=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{U}e^{-u^2/2}du\,, \tag{25}$$

instead of the level height U. The parts for $\phi>1/2$ are neglected because of the symmetry (or anti-symmetry for $d=2$). Note that $\bar{\chi}_1$ and $\bar{\chi}_2$ also represent $\bar{A}$ and $\bar{H}$ for higher dimensions except for numerical factors. For $d=2$ and 3 the *droplet density* $\bar{\chi}_d$ increases rapidly with ϕ, but soon turns to decreasing. The reason of this decreasing is not only the coalescence between droplets but also the development of topological complexity of droplets. At $\phi=1/2$ one finds $\bar{\chi}_2=0$ (or $\bar{H}=0$). This fact with finite $\bar{A}$ corresponds to the symmetric distribution of the positive and negative curvatures on the interface. In $d=3$, the interface becomes dominated by hyperbolic points because $\bar{\chi}_3<0$ (or $\bar{K}=\overline{(R_1R_2)^{-1}}<0$) at $\phi=1/2$. Such system has been shown to be *mutually percolating* because of its symmetry.[6] It should be noted that this transition from an isolated droplet system to a mutually percolating pattern with increasing volume fraction is a universal conclusion and does not depend on the detailed nature of the Gaussian field, though we cannot say strictly where the transition occurs.

Another interesting characteristics is the density of local minimum under a given level U, which is given by

$$m(U)=\langle\delta(\boldsymbol{v})\,\theta(U-u)\,\Theta(\{u_{ij}\})\det\{u_{ij}\}\rangle\,, \tag{26}$$

where δ and θ are the ordinary delta- and the step-function and $\Theta(\{u_{ij}\})$ denotes the concave condition, i.e.,

$$\Theta(\{u_{ij}\})=\begin{cases}1, & \text{if } \{u_{ij}\} \text{ is positive definite},\\ 0, & \text{otherwise}.\end{cases}$$

The determinant det $\{u_{ij}\}$ is introduced here as a Jacobian between the components of $\boldsymbol{v}=\nabla u$ and $\boldsymbol{r}$. This quantity $m(U)$ is not universal in general, i.e., it depends on the pattern parameter γ. For example, it can be easily shown from Eq. (26) the total number of minimum $m(+\infty)$ is written as $m(+\infty)=c/\gamma^d$, where c is a constant independent of γ, i.e., $c=\sqrt{3/2\pi}\,(d=1)$, $1/\sqrt{3}\,(d=2)$ and

$0.55\cdots(d=3)$, the result for $d=3$ being estimated by numerical computations.

The opposite limit $U=-\infty$, however, is universal, i.e., it has been shown that for $-U\gg 1$ we have[4)]

$$m(U)\sim|U|^{(d-1)}e^{-U^2/2}/\sqrt{2\pi}$$
$$\sim\bar{\chi}_d(U)\,, \tag{27}$$

by using the relation

$$dm/dU=\langle\delta(U-u)\,\Theta(\{u_{ij}\})\det\{u_{ij}\}\rangle$$

and calculus similar to that used for $\det\Lambda=\det\{\tilde{u}_{ij}/|\nabla u|\}$.

It can also be shown that the inequality (5) gives the lower bound $m_0(U)$ for $m(U)$. This is a universal quantity, too. Though it is difficult to find an analytical form of it for $d\geqq 2$, numerical estimations are possible. The results by the Monte-Carlo method are shown in Fig. 1 by broken lines, where the parts for $\phi>1/2$ are neglected because $m_0(U>0)=m_0(0)$. At small ϕ region, $\bar{\chi}_d(U)$ and $m_0(U)$ are almost indistinguishable. Thus Eq. (27) is confirmed numerically.

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