PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 72, Number 1, October 1978

CURVATURE OF PRODUCT 3-MANIFOLDS

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ABSTRACT. Let M be a compact product 3-manifold without boundary. Let g be a Riemannian metric on M. If g has everywhere nonpositive sectional curvature, then g is locally diffeomorphic to a product metric. The proof is by the method of pseudoframes.

1. Introduction. A. Preissmann [**2**] proved compact product manifolds do not admit Riemannian metrics with everywhere negative sectional curvature. Here we prove the following refinement for dimension three:

1.1 THEOREM. *Let M be a compact product 3-manifold without boundary. Let g be a Riemannian metric on M. If g has everywhere nonpositive sectional curvature, then g is locally diffeomorphic to a product metric.*

Our proof will be by the method of pseudoframes. We begin with a brief exposition of this theory.

2. Pseudoframes. Let g and $\bar{g}$ be Riemannian metrics on a smooth manifold M of dimension m. At each point x of M, we may find an automorphism F of the tangent space T_xM such that, for all $X, Y \in T_xM$, $g(X, Y) = \bar{g}(FX, FY)$. If $\omega^1, \dots, \omega^m$ are a coframe at x, then we may write $g = g_{ij}\omega^i \otimes \omega^j$, and $\bar{g} = \bar{g}_{ij}\omega^i \otimes \omega^j$, where here and always we sum over repeated indices from 1 to m. If F has matrix representation F^i_j in this frame, then $g_{ij} = F^i_s\bar{g}_{st}F^t_j$. F is determined in any case up to left translation by elements of the orthogonal group for $\bar{g}$. If we require that $F^i_j = F^j_i$ in frames orthonormal for g, and all eigenvalues of F be positive, then F is unique. The symmetric F determined in this way at each point gives rise to a global tensor field of type (1,1) which determines an automorphism of the tangent bundle. We use such an object to mimic the effect of a global change of frame. For this reason we call it a *pseudoframe*.

REMARK. The symmetry condition on F is important only to establish global existence of the tensor field. In what follows, we shall not assume F to be symmetric.

Let $\mathbf{F}(M)$ be the frame bundle of M, p: $\mathbf{F}(M) \to M$ the natural projection. Given a standard basis of $\mathbf{R}^m$, we can consider each $u \in \mathbf{F}(M)$ as a linear isomorphism u: $\mathbf{R}^m \to T_{p(u)}M$. Then the natural right action of $GL(m)$ on

Presented to the Society, October 22, 1977; received by the editors September 9, 1977 and, in revised form, February 2, 1978.

AMS (MOS) subject classifications (1970). Primary 53C20.

Key words and phrases. Manifold, Riemannian metric, sectional curvature, pseudoframe.

$\mathbf{F}(M)$ is given by $R_a u = ua$ where ua: $\mathbf{R}^m \to T_{p(u)}$ is the composition $\mathbf{R}^m \xrightarrow{a} \mathbf{R}^m \xrightarrow{u} T_{p(u)}M$ for $a \in GL(m)$.

DEFINITION. A diffeomorphism f: $\mathbf{F}(M) \to \mathbf{F}(M)$ is a *bundle automorphism* if for all $u \in \mathbf{F}(M)$, $p(f(u)) = p(u)$, and, for all $a \in GL(m)$, $f(ua) = (f(u))a$.

If g is a Riemannian metric on M, let $O(g)$ denote the subbundle of frames orthonormal for g.

Let F be a pseudoframe on M such that $g(X, Y) = \bar{g}(FX, FY)$ for two Riemannian metrics g and $\bar{g}$. Define f: $\mathbf{F}(M) \to \mathbf{F}(M)$ by $f(u)(A) = F(u(A))$ for all $u \in \mathbf{F}(M)$ and all $A \in \mathbf{R}^m$. Then $p(f(u)) = p(u)$. For $a \in GL(m)$, $A \in \mathbf{R}^m$, $ua(A) = u(a(A))$. Then $f(ua)(A) = F(ua(A)) = f(u)a(A)$. So $f(ua) = (f(u))a$. A frame u is in $O(g)$ if and only if $(A, B) = g(uA, uB)$ for all $A, B \in \mathbf{R}^m$. But $g(uA, uB) = \bar{g}(F(uA), F(uB)) = \bar{g}(f(u)(A), f(u)(B))$. Thus f is a bundle automorphism, and $f(O(g)) = O(\bar{g})$.

Given a bundle automorphism f, we define an associated function $\bar{F}$: $\mathbf{F}(M) \to GL(m)$ by $\bar{F}(u)(A) = u^{-1}(f(u)(A))$ for all $A \in \mathbf{R}^m$. Then $f(u) = u\bar{F}(u)$ and $\bar{F}(ua) = a^{-1}\bar{F}(u)a$. In matrix coordinates we have

$$\bar{F}_j^i(ua) = a_s^{-1i}\bar{F}_t^s(u)a_j^t. \tag{2.1}$$

If ϕ is a connection on $\mathbf{F}(M)$, we define the covariant derivative $D_\phi \bar{F}_j^i$ by

$$d\bar{F}_j^i = D_\phi \bar{F}_j^i - \phi_s^i \bar{F}_t^s + \bar{F}_s^i \phi_t^s. \tag{2.2}$$

For fixed $a \in GL(m)$, define R_a: $\mathbf{F}(M) \to \mathbf{F}(M)$ by $R_a(u) = ua$, and for fixed $u \in \mathbf{F}(M)$, L_u: $GL(m) \to p^{-1}(p(u))$ by $L_u(a) = ua$. Then for $X \in T_u\mathbf{F}(M)$,

$$f_*(X) = R_{\bar{F}(U)*}(X) + L_{f(u)*}\big(\bar{F}^{-1}(u)d\bar{F}(X)\big) \tag{2.3}$$

where $\bar{F}^{-1}(u)d\bar{F}(x) \in T_{id}GL(m)$.

Now suppose that $f(O(g)) = O(\bar{g})$ and let ϕ be a connection on $O(\bar{g})$. Then $f^*\phi_j^i(X) = \phi_j^i(f_*X) = \phi_j^i(R_{\bar{F}*}X) + \phi_j^i(L_{f*}\bar{F}^{-1}d\bar{F}(X))$. But $R_{a*}\phi_j^i = a_s^{-1i}\phi_t^s a_j^t$. Thus

$$f^*\phi_j^i = \bar{F}_s^{-1i}d\bar{F}_j^s + \bar{F}_s^{-1i}\phi_t^s\bar{F}_j^t. \tag{2.4}$$

Let θ be the canonical horizontal R^m-valued one-form on $\mathbf{F}(M)$, $\theta(X) = u^{-1}(p_*(X))$ for $X \in T_u\mathbf{F}(M)$. Since θ vanishes on vectors tangent to the fiber $p^{-1}(x)$, $f^*\theta = R_{\bar{F}}^*\theta$. If θ^i is the ith component of θ with respect to the standard basis of R^m, then $R_a^*\theta^i + a_s^{-1i}\theta^s$. Thus

$$f^*\theta^i = \bar{F}_s^{-1i}\theta^s. \tag{2.5}$$

Note that these formulae are similar to those induced by a change of frame.

We may use f to pull back a connection on $O(\bar{g})$ to a connection on $O(g)$. For geometric purposes, we are most interested in what happens to the Levi-Civita connection under such an operation. Let ω and ϕ be the Levi-Civita connections on $O(g)$ and $O(\bar{g})$ respectively. We define the *transition forms* (f^g) by

$$(f^g)^i_j = \omega^i_j - f^*\phi^i_j. \tag{2.6}$$

Note that each $(f^g)^i_j$ is a horizontal one-form. If $(f^g)^i_j = (f^g)^i_{jk} f^*\theta^k$, then $(f^g)^i_{jk}$ is given explicitly by the formula

$$\begin{aligned}2(f^g)^i_{jk} = &\left(D_\omega \bar{F}_s^{-1i}\left(\bar{E}_t\right)\right)\left(\bar{F}^s_j\bar{F}^t_k - \bar{F}^s_k\bar{F}^t_j\right) - \left(D_\omega \bar{F}_s^{-1j}\left(\bar{E}_t\right)\right)\left(\bar{F}^s_i\bar{F}^t_k - \bar{F}^s_k\bar{F}^t_i\right)\\ &- \left(D_\omega \bar{F}_s^{-1k}\left(\bar{E}_t\right)\right)\left(\bar{F}^s_i\bar{F}^t_j - \bar{F}^s_j\bar{F}^t_i\right)\end{aligned} \tag{2.7}$$

where $\bar{E}_t \in T_u\mathbf{F}(M)$ such that $p_*\bar{E}_t = u(E_t)$, $E_t \in \mathbf{R}^m$ the tth leg of the standard basis.

3. Proof of Theorem 1.1. Let N be a compact, oriented 2-manifold without boundary, and S^1 the unit circle; g a product metric on $N \times S^1$. Let $R(g)$ be the subbundle of $O(g)$ consisting of frames such that $u(E_1)$, $u(E_2)$ are an oriented basis of $T_{p(u)}N$. Let ω be the Levi-Civita connection on $O(g)$.

If $\pi: M \to N$ is the natural projection, let $\bar{g}$ be a Riemannian metric on M such that, for $u \in R(g)$, $\bar{g}_{33} = \bar{g}(u(E_3), u(E_3)) = \pi^*\lambda$, for some positive function λ on N.

3.1 LEMMA. *There exists a sequence of bundle automorphisms* $O(g) \xrightarrow{f} O(\hat{g}) \xrightarrow{h} O(\bar{g})$ *such that for* $u \in R(g)$,

(A) $\bar{F}_3^{-11} = \bar{F}_3^{-12} = \bar{F}_1^{-13} = \bar{F}_2^{-13} = 0$,

(B) $\bar{F}_3^{-13} = 1$,

and for $v \in R(\hat{g}) = f(R(g))$

(C) $\bar{H}_3^{-11} = \bar{H}_3^{-12} = 0$,

(D) $\bar{H}_1^{-11} = \bar{H}_2^{-12} = \bar{H}_3^{-13}$,

(E) $\bar{H}_1^{-12} = \bar{H}_2^{-11} = 0$.

PROOF. If we require additionally that $\bar{F}_2^{-11} = \bar{F}_1^{-12}$, then f and h are determined uniquely.

Note that for $u \in R(g)$, $\hat{g}(u(E_3), u(E_3)) = 1$, $\hat{g}(u(E_3), u(E_i)) = \theta$, $i = 1, 2$. Also, $\bar{H}_3^{-13} = \mu = (\pi^*\lambda)^{-1/2}$.

Let ϕ (ψ) be the Levi-Civita connection on $O(\bar{g})$ ($O(\hat{g})$), Φ (Ψ) its curvature form. If $\Phi^i_j = \Phi^i_{jst}\theta^s \wedge \theta^t$ then the sectional curvature $\bar{\sigma}_{ij}$ of the plane spanned by $u(E_i)$ and $u(E_j)$, $u \in O(\bar{g})$ is given by Φ^i_{jij}.

Let $\psi^i_j = \psi^i_{jk}h^*\theta^k$. Then by (2.7), on $R(\hat{g})$,

$$\psi^i_{33} = 0, \quad \psi^3_{12} = \psi^3_{21}, \tag{3.1}$$

$$(h^g)^3_{11} = (h^g)^3_{22} = 0, \quad (h^g)^3_{12} = -(h^g)^3_{21}. \tag{3.2}$$

(Note that $\bar{g}_{33} = \pi^*\lambda$ is required to prove (3.2).)

On $R(\hat{g})$, let $dv_{\bar{g}} = h^*\theta^1 \wedge h^*\theta^2 \wedge h^*\theta^3 \wedge \psi^1_2$, $dv_{\hat{g}} = \theta^1 \wedge \theta^2 \wedge \theta^3 \wedge \psi^1_2$.

3.2 LEMMA.

$$\int_{R(\hat{g})} h^*\Phi^1_{313} dv_{\bar{g}} = \int_{R(\hat{g})} \mu\Psi^1_{313} dv_{\hat{g}} - \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^3_{12}\left(h^{\hat{g}}\right)^3_{21} dv_{\bar{g}}. \tag{3.3}$$

PROOF.

$$\begin{aligned}\int_{R(\hat{g})} h^*\Phi^1_{313} dv_{\bar{g}} &= -\int_{R(\hat{g})} h^*\Phi^1_3 \wedge h^*\theta^2 \wedge \psi^1_2 \\ &= -\int_{R(\hat{g})} d\psi^1_3 \wedge h^*\theta^2 \wedge \psi^1_2 + \int_{R(\hat{g})} d\left(h^{\hat{g}}\right)^1_3 \wedge h^*\theta^2 \wedge \psi^1_2 \\ &\quad -\int_{R(\hat{g})} h^*\phi^1_2 \wedge h^*\phi^2_3 \wedge h^*\theta^2 \wedge \psi^1_2 \\ &= \int_{R(\hat{g})} \mu\Psi^1_{313} dv_{\hat{g}} + \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^1_3 \wedge h^*\theta^1 \wedge h^*\phi^2_1 \wedge \psi^1_2 \\ &\quad + \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^1_3 \wedge h^*\theta^3 \wedge h^*\phi^2_3 \wedge \psi^1_2 \\ &\quad + \int_{R(\hat{g})} \left(h^{\hat{g}}\right)^1_3 \wedge h^*\phi^2_3 \wedge h^*\theta^2 \wedge \psi^1_2.\end{aligned}$$

Now, $\int_{R(\hat{g})} h^*\Phi^1_{313} dv_{\bar{g}} = \int_{R(\hat{g})} h^*\Phi^2_{323} dv_{\bar{g}}$, employing the same derivation for Φ^2_{323} we obtain the desired result by cancelling cross-terms using (3.1) and (3.2).

Note that by (3.2), the second term of (3.3) is nonnegative.

3.3 LEMMA. $0 \leqslant \int_{R(\hat{g})} \mu\Psi^1_{313} dv_{\hat{g}}$. *The equality holds only if $\hat{g}$ is a product metric.*

PROOF. Let C be a simple closed curve on N. Let $\sigma: C \times S^1 \to R(g)$ be a section such that, at each $x \in C$, $\sigma(x)(E_1)$ is the oriented tangent vector to C, and $\sigma(x)(E_2)$ the outward normal. Now, near C, we can choose f so, in addition to conditions (A) and (B), we have $\bar{F}_1^{-12} = \bar{F}_3^{-12} = 0$ on the image of σ. Since $\sigma^*\theta^2 = 0$, and $f^*\theta^2 = \bar{F}_j^{-12}\theta^j$, we have $\sigma^* f^*\theta^2 = 0$. Thus the $\hat{g}$ volume element on $C \times S^1$ is $dv = \sigma^* f^*(\theta^1 \wedge \theta^3)$. Then

$$\int_{C\times S^1} \sigma^* f^* \Psi^1_{313} dv = \int_{C\times S^1} \sigma^* f^* \Psi^1_3 = \int_{C\times S^1} (f^g)^2_{13}(f^g)^2_{31} - (f^g)^2_{33}(f^g)^2_{11}\, dv. \tag{3.5}$$

Since $d\sigma^* f^{*2} = 0$, $(f^g)^2_{31} = (f^g)^2_{13}$, and by (2.7), $(f^g)^2_{33} = 0$. This must be true for every curve C on N, and μ is a function on N only. Thus we have the desired inequality. The equality holds only if $(f^g)^2_{13}$ vanishes pointwise on $R(g)$. By (2.7), this implies that $\bar{F}_2^{-11} D_\omega \bar{F}_2^{-12}(\bar{E}_3) = 0$. Since we may replace g with any other product metric, we conclude that the $\bar{F}^i_j$, $i, j = 1, 2$, are functions on N only, so that $\hat{g}$ is itself a product metric.

3.4 PROPOSITION. *If the sectional curvature of $\bar{g}$ is nonpositive, $\bar{g}$ is locally diffeomorphic to a product metric.*

PROOF. By Lemmas 3.2 and 3.3, we may assume that $\hat{g} = g$. Then, by

Lemma 3.1, $(h^g)^3_{11} = (h^g)^3_{22} = (h^g)^3_{12} = (h^g)^3_{21} = 0$ on $R(g)$. By (2.7), this also means that $(h^g)^1_{23} = 0$. Since μ is a function on N only, we can alter the product metric g so that $\bar{F}_1^{-11} = \bar{F}_2^{-12} = 1$. (The $(h^g)^3_{ij}$ and $(h^g)^i_{j3}$, $i, j = 1, 2$, will remain equal to zero when this is done.) Then $(h^g)^1_2 = 0$. Now, by Lemma 3.2, $h^*\Phi^1_{313}$ must vanish on $R(g)$ if it is to be nonpositive. But this implies that $d(h^g)^1_{33}(E_3) - ((h^g)^1_{33})^2 = 0$. Thus $(h^g)^1_{33} = (h^g)^2_{33} = 0$, and $(h^g)^i_j = 0$ for all i, j. It now follows from the De Rham Decomposition Theorem that, on each simply-connected open set U of M, $\bar{g}$ is a product metric for some product structure on U. Note that the product structure may differ from the original one induced by the inclusion $i\colon U \to N \times S^1$ by a diffeomorphism.

We now remove the restriction that $\bar{g}_{33}$ be a function on N only.

3.5 PROPOSITION. *Let N be a surface, S^1 the unit circle, $\pi\colon N \times S^1 \to N$ the natural projection. Let t be a unit-length parameter on S^1 (i.e., $t = 0$ and $t = 1$ are identified). Then, for any Riemannian metric g on $N \times S^1$, there exists a diffeomorphism $\phi\colon N \times S^1 \to N \times S^1$ such that $\phi^*g(d/dt, d/dt) = \pi^*\lambda$, for some positive function λ on N.*

PROOF. Let (x_1, x_2, t) be a local product coordinate chart on $N \times S^1$, and let $\mu^{-2} = g(d/dt, d/dt)$. Define ϕ by

$$\phi_1(x_1, x_2, t) = x_1, \quad \phi_2(x_1, x_2, t) = x_2,$$

$$\phi_3(x_1, x_2, t) = K(x_1, x_2)\int_0^t \mu(x_1, x_2, s)ds,$$

where $K^{-1} = \int_0^1 \mu(x_1, x_2, t)\, dt$. It is clear that ϕ is a diffeomorphism, and it is easy to calculate that $\phi^*g_{33} = K^2$. But by construction, $K = \pi^*\lambda$ for some positive function λ on N. For compact, oriented 3-manifolds, Theorem 1.1 now follows from Propositions 3.4 and 3.5. If a product 3-manifold M is compact, but not oriented, we may apply our results to the orientation covering $\bar{M}$; the local diffeomorphism found there will project to a local diffeomorphism of M.

REFERENCES

1. S. Kobayashi and K. Nomizu, *Foundations of differential geometry*, Vol. I, Interscience, New York, 1963.

2. A. Preissmann, *Quelques propriétés des espaces de Riemann*, Comment. Math. Helv. **15** (1942), 175–216.

3. J. R. Wason, *Comparison of Riemannian structures by the method of pseudoframes*, Thesis, Columbia Univ., 1973.

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