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CURVATURE PROPERTIES OF RIEMANNIAN METRICS OF THE FORM ${}^S g_f + {}^H g$ ON THE TANGENT BUNDLE OVER A RIEMANNIAN MANIFOLD (M, g)

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Abstract. In this paper, we define a special new family of metrics which rescale the horizontal part by a nonzero differentiable function on the tangent bundle over a Riemannian manifold. We investigate curvature properties of the Levi-Civita connection and another metric connection of the new Riemannian metric.

1. Introduction

The research in the topic of differential geometry of tangent bundles over Riemannian manifolds has begun with S. Sasaki. In his original paper [17] of 1958, he constructed a Riemannian metric ${}^S g$ on the tangent bundle TM of a Riemannian manifold (M, g), which depends closely on the base metric g. Although the Sasaki metric is *naturally* defined, it was shown in many papers that the Sasaki metric presents a kind of rigidity. In [10], O. Kowalski proved that if the Sasaki metric ${}^S g$ is locally symmetric, then the base metric g is flat and therefore ${}^S g$ is also flat. In [12], E. Musso and F. Tricerri demonstrated an extreme rigidity of ${}^S g$ in the following sense: if $(TM, {}^S g)$ is of constant scalar curvature, then (M, g) is flat. They also defined a new Riemannian metric g_{CG} on the tangent bundle TM which they called the Cheeger Gromoll metric. Given a Riemannian metric g on a differentiable manifold M, there are well known classical examples of metrics on the tangent bundle TM which can be constructed from a Riemannian metric g, namely the Sasaki metric, the horizontal lift and the vertical lift. The three classical constructions of metrics on tangent bundles are given as follows:

(a) The Sasaki metric ${}^S g$ is a (positive definite) Riemannian metric on the tangent bundle TM which is derived from the given Riemannian metric on M as

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follows:

$$\begin{aligned} {}^{S}g\left({}^{H}X,{}^{H}Y\right) &= g\left(X,Y\right) \\ {}^{S}g\left({}^{H}X,{}^{V}Y\right) &= {}^{S}g\left({}^{V}X,{}^{H}Y\right)=0 \\ {}^{S}g\left({}^{V}X,{}^{V}Y\right) &= g\left(X,Y\right) \end{aligned}$$

for all $X,Y\in\Im_{0}^{1}(M)$.

(b) The horizontal lift ${}^{H}g$ of g is a pseudo-Riemannian metric on the tangent bundle TM with signature (n,n) which is given by

$$\begin{aligned} {}^{H}g\left({}^{H}X,{}^{H}Y\right) &= 0, \\ {}^{H}g\left({}^{H}X,{}^{V}Y\right) &= {}^{H}g\left({}^{V}X,{}^{H}Y\right)=g\left(X,Y\right), \\ {}^{H}g\left({}^{V}X,{}^{V}Y\right) &= 0 \end{aligned}$$

for all $X,Y\in\Im_{0}^{1}(M)$.

(c) The vertical lift ${}^{V}g$ of g is a degenerate metric of rank n on the tangent bundle TM which is given by

$$\begin{aligned} {}^{V}g\left({}^{H}X,{}^{H}Y\right) &= 0 \\ {}^{V}g\left({}^{H}X,{}^{V}Y\right) &= {}^{V}g\left({}^{V}X,{}^{H}Y\right)=0 \\ {}^{V}g\left({}^{V}X,{}^{V}Y\right) &= g\left(X,Y\right) \end{aligned}$$

for all $X,Y\in\Im_{0}^{1}(M)$.

Another classical construction is the complete lift of a tensor field to the tangent bundle. It is well known that the complete lift ${}^{C}g$ of a Riemannian metric g coincides with the horizontal lift ${}^{H}g$ given above. A "nonclassical" example is the Cheeger-Gromoll metric g_{CG} on the tangent bundle TM. Other metrics on the tangent bundle TM can be constructed by using the three classical lifts ${}^{S}g$, ${}^{H}g$ and ${}^{V}g$ of the metric g (for example, see [7, 19]).

V. Oproiu and his collaborators constructed natural metrics on the tangent bundles of Riemannian manifolds possessing interesting geometric properties ([13, 14, 15, 16]). All the preceding metrics belong to a wide class of the so-called *g-natural metrics* on the tangent bundle, initially classified by O. Kowalski and M. Sekizawa [11] and fully characterized by M.T.K Abbassi and M. Sarih [1, 2, 3] (see also [9] for other presentation of the basic result from [11] and for more details about the concept of naturality).

In [20](see also [21, 22], B. V. Zayatuev introduced a Riemannian metric ${}^{S}\overline{g}$ on the tangent bundle TM given by

$$\begin{aligned} {}^{S}g_{f}\left({}^{H}X,{}^{H}Y\right) &= fg\left(X,Y\right), \\ {}^{S}g_{f}\left({}^{H}X,{}^{V}Y\right) &= {}^{S}g_{f}\left({}^{V}X,{}^{H}Y\right)=0, \\ {}^{S}g_{f}\left({}^{V}X,{}^{V}Y\right) &= g\left(X,Y\right), \end{aligned}$$

where $f>0$, $f\in C^{\infty}(M)$ (see also, [5, 18]). For $f=1$, it follows that ${}^{S}g_{f}={}^{S}g$, i.e. the metric ${}^{S}g_{f}$ is a generalization of the Sasaki metric ${}^{S}g$. For the rescaled Sasaki type metric on the cotangent bundle, see [6].

Our purpose is to study some properties of a special new family of metrics on the tangent bundle constructed from the base metric, and generated by positive functions on M, which the metric is in the form ${}^{f}\widetilde{G}={}^{S}g_{f}+{}^{H}g$. The paper can be considered as a contribution in the topic, considering for study a special new family of metrics on the tangent bundle. It is worth mentioning that a metric from this new family is g-natural only if the generating function is constant. So the considered family is far from being a subfamily of the class of g-natural metrics, and its study could be of interest in some sense.

The present paper is organized as follows: In section 2, we review some introductory materials concerning with the tangent bundle TM over an n-dimensional Riemannian manifold M and also introduce the adapted frame in the tangent bundle TM. In section 3, we present a Riemannian metric of the form ${}^{f}\widetilde{G} = {}^{S}g_f + {}^{H}g$ defined by

$$\begin{aligned} {}^{f}\widetilde{G}\left({}^{H}X, {}^{H}Y\right) &= fg\left(X,Y\right) \\ {}^{f}\widetilde{G}\left({}^{H}X, {}^{V}Y\right) &= {}^{f}\widetilde{G}\left({}^{V}X, {}^{H}Y\right) = g\left(X,Y\right) \\ {}^{f}\widetilde{G}\left({}^{V}X, {}^{V}Y\right) &= g\left(X,Y\right) \end{aligned}$$

for all $X, Y \in \Im_0^1(M)$, where $f > 1$, $f \in C^\infty(M)$ and compute the Christoffel symbols of the Levi-Civita connection ${}^{f}\widetilde{\nabla}$ of ${}^{f}\widetilde{G}$ with respect to the adapted frame. In section 4 and 5, we compute all kinds of curvatures of the metric ${}^{f}\widetilde{G}$ with respect to the adapted frame and give some geometric results concerning them. In section 5, we give conditions for which the metric ${}^{f}\widetilde{G}$ is locally conformally flat. Section 6 deals with another metric connection with torsion of the metric ${}^{f}\widetilde{G}$.

Throughout this paper, all manifolds, tensor fields and connections are always assumed to be differentiable of class C^∞. Also, we denote by $\Im_q^p(M)$ the set of all tensor fields of type (p,q) on M, and by $\Im_q^p(TM)$ the corresponding set on the tangent bundle TM.

2. Preliminaries

2.1. **The tangent bundle.** Let TM be the tangent bundle over an n-dimensional Riemannian manifold (M,g), and π be the natural projection $\pi : TM \to M$. Let the manifold M be covered by a system of coordinate neighborhoods (U, x^i), where (x^i), $i = 1, ..., n$ is a local coordinate system defined in the neighborhood U. Let (y^i) be the Cartesian coordinates in each tangent space $T_P M$ at $P \in M$ with respect to the natural basis $\left\{ \frac{\partial}{\partial x^i} |_P \right\}$, where P is an arbitrary point in U with coordinates (x^i). Then we can introduce local coordinates (x^i, y^i) on the open set $\pi^{-1}(U) \subset TM$. We call such coordinates as *induced coordinates* on $\pi^{-1}(U)$ from (U, x^i). The projection π is represented by $(x^i, y^i) \to (x^i)$. The indices $I, J, ...$ run from 1 to $2n$, while $\bar{i}, \bar{j}, ...$ run from $n+1$ to $2n$. Summation over repeated indices is always implied.

Let $X = X^i \frac{\partial}{\partial x^i}$ be the local expression in U of a vector field X on M. Then the vertical lift ${}^{V}X$ and the horizontal lift ${}^{H}X$ of X are given, with respect to the induced coordinates, by

$$ {}^{V}X = X^i \partial_{\bar{i}}, \tag{2.1}$$

and

$$ {}^{H}X = X^i \partial_i - y^s \Gamma_{sk}^i X^k \partial_{\bar{i}}, \tag{2.2}$$

where $\partial_i = \frac{\partial}{\partial x^i}$, $\partial_{\bar{i}} = \frac{\partial}{\partial y^i}$ and Γ_{jk}^i are the coefficients of the Levi-Civita connection ∇ of g.

Explicit expressions for the Lie bracket $[,]$ of TM are given by Dombrowski in [4]. The bracket operation of vertical and horizontal vector fields is given by the formulas

$$\left\{ \begin{array}{l} \left[{}^{H}X, {}^{H}Y\right] = {}^{H}[X,Y] - {}^{V}(R(X,Y)u) \\ \left[{}^{H}X, {}^{V}Y\right] = {}^{V}(\nabla_X Y) \\ \left[{}^{V}X, {}^{V}Y\right] = 0 \end{array} \right. \tag{2.3}$$

for all vector fields X and Y on M, where R is the Riemannian curvature of g defined by $R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ (for details, see [19]).

2.2. **The adapted frame.** We insert the adapted frame which allows the tensor calculus to be efficiently done in TM. With the connection ∇ of g on M, we can introduce adapted frames on each induced coordinate neighborhood $\pi^{-1}(U)$ of TM. In each local chart $U \subset M$, we write $X_{(j)} = \dfrac{\partial}{\partial x^j}, j = 1, ..., n$. Then from (2.1) and (2.2), we see that these vector fields have, respectively, local expressions

$$ {}^{H}X_{(j)} = \delta_j^h \partial_h + (-y^s \Gamma_{sj}^h)\partial_{\overline{h}}$$

$$ {}^{V}X_{(j)} = \delta_j^h \partial_{\overline{h}}$$

with respect to the natural frame $\left\{\partial_h, \partial_{\overline{h}}\right\}$, where δ_j^h denotes the Kronecker delta. These $2n$ vector fields are linearly independent and they generate the horizontal distribution of ∇_g and the vertical distribution of TM, respectively. We call the set $\left\{{}^{H}X_{(j)}, {}^{V}X_{(j)}\right\}$ the frame adapted to the connection ∇ of g in $\pi^{-1}(U) \subset TM$. By denoting

$$\begin{array}{rcl} E_j & = & {}^{H}X_{(j)}, \\ E_{\overline{j}} & = & {}^{V}X_{(j)}, \end{array} \tag{2.4}$$

we can write the adapted frame as $\{E_\beta\} = \left\{E_j, E_{\overline{j}}\right\}$.

Using (2.1), (2.2) and (2.4), we have

$$ {}^{V}X = \begin{pmatrix} 0 \\ X^h \end{pmatrix} = \begin{pmatrix} 0 \\ X^j \delta_j^h \end{pmatrix} = X^j \begin{pmatrix} 0 \\ \delta_j^h \end{pmatrix} = X^j E_{\overline{j}}, \tag{2.5}$$

and

$$ {}^{H}X = \begin{pmatrix} X^j \delta_j^h \\ -X^j \Gamma_{sj}^h y^s \end{pmatrix} = X^j \begin{pmatrix} \delta_j^h \\ -\Gamma_{sj}^h y^s \end{pmatrix} = X^j E_j \tag{2.6}$$

with respect to the adapted frame $\{E_\beta\}$ (see [19]).

3. The Riemannian metric and its Levi-Civita connection

Let (M, g) be a Riemannian manifold. A Riemannian metric ${}^{f}\widetilde{G}$ is defined on TM by the following three equations

$$\begin{array}{rcl} {}^{f}\widetilde{G}({}^{H}X, {}^{H}Y) & = & fg(X,Y), \\ {}^{f}\widetilde{G}({}^{H}X, {}^{V}Y) & = & {}^{f}\widetilde{G}({}^{V}X, {}^{H}Y) = g(X,Y), \\ {}^{f}\widetilde{G}({}^{V}X, {}^{V}Y) & = & g(X,Y) \end{array} \tag{3.1}$$

for all X, $Y \in \Im_0^1(M)$, where $f > 1$ and $f \in C^\infty(M)$.

From the equations (3.1), by virtue of (2.5) and (2.6), The metric ${}^f\widetilde{G}$ and its inverse ${}^f\widetilde{G}^{-1}$ respectively have the following components with respect to the adapted frame $\{E_\beta\}$:

$$(3.2)\qquad {}^f\widetilde{G} = ({}^f\widetilde{G}_{\alpha\beta}) = \begin{pmatrix} fg_{ij} & g_{ij} \\ g_{ij} & g_{ij} \end{pmatrix}$$

and

$$(3.3)\qquad {}^f\widetilde{G}^{-1} = ({}^f\widetilde{G}^{\alpha\beta}) = \begin{pmatrix} \frac{1}{f-1}g^{ij} & -\frac{1}{f-1}g^{ij} \\ -\frac{1}{f-1}g^{ij} & \frac{f}{f-1}g^{ij} \end{pmatrix}.$$

We now consider local 1-forms ω^λ in $\pi^{-1}(U)$ defined by $\omega^\lambda = \tilde{A}^\lambda{}_B dx^B$, where

$$(3.4)\qquad A^{-1} = \tilde{A}^\lambda{}_B = \begin{pmatrix} \tilde{A}^h{}_j & \tilde{A}^h{}_{\bar{j}} \\ \tilde{A}^{\bar{h}}{}_j & \tilde{A}^{\bar{h}}{}_{\bar{j}} \end{pmatrix} = \begin{pmatrix} \delta^h_j & 0 \\ y^s\Gamma^h_{sj} & \delta^h_j \end{pmatrix}$$

is the inverse matrix of the matrix

$$(3.5)\qquad A = A_\beta{}^A = \begin{pmatrix} A_j{}^h & A_{\bar{j}}{}^h \\ A_j{}^{\bar{h}} & A_{\bar{j}}{}^{\bar{h}} \end{pmatrix} = \begin{pmatrix} \delta^h_j & 0 \\ -y^s\Gamma^h_{sj} & \delta^h_j \end{pmatrix}$$

of the transformation $E_\beta = A_\beta{}^A\partial_A$. We easily see that the set $\{\omega^\lambda\}$ is the coframe dual to the adapted frame $\{E_\beta\}$, e.i. $\omega^\lambda(E_\beta) = \tilde{A}^\lambda{}_B A_\beta{}^B = \delta^\lambda_\beta$.

Since the adapted frame field $\{E_\beta\}$ is non-holonomic, we put

$$[E_\alpha, E_\beta] = \Omega_{\alpha\beta}{}^\gamma E_\gamma$$

from which we have

$$\Omega_{\gamma\beta}{}^\alpha = (E_\gamma A_\beta{}^A - E_\beta A_\gamma{}^A)\tilde{A}^\alpha{}_A.$$

According to (2.4), (3.4) and (3.5), the components of non-holonomic object $\Omega^\alpha_{\gamma\beta}$ are given by

$$(3.6)\qquad \begin{cases} \Omega_{i\bar{j}}{}^{\bar{k}} = -\Omega_{\bar{j}i}{}^{\bar{k}} = \Gamma^k_{ji} \\ \Omega_{ij}{}^{\bar{k}} = -\Omega_{ji}{}^{\bar{k}} = -y^s R_{ijs}{}^k \end{cases}$$

all the others being zero, where $R_{ijs}{}^k$ are local components of the Riemannian curvature tensor R of the Riemannian manifold (M, g).

Let ${}^f\widetilde{\nabla}$ be the Levi-Civita connection of the Riemannian metric ${}^f\widetilde{G}$. Putting ${}^f\widetilde{\nabla}_{E_\alpha}E_\beta = {}^f\widetilde{\Gamma}^\gamma_{\alpha\beta}E_\gamma$, from the equation ${}^f\widetilde{\nabla}_{\widetilde{X}}\widetilde{Y} - {}^f\widetilde{\nabla}_{\widetilde{Y}}\widetilde{X} = \left[\widetilde{X}, \widetilde{Y}\right]$, $\forall \widetilde{X}, \widetilde{Y} \in \Im^1_0(TM)$, we have

$$(3.7)\qquad {}^f\widetilde{\Gamma}^\alpha_{\gamma\beta} - {}^f\widetilde{\Gamma}^\alpha_{\beta\gamma} = \Omega_{\gamma\beta}{}^{\overline{\alpha}}.$$

The equation $({}^f\widetilde{\nabla}_{\widetilde{X}}\ {}^f\widetilde{G})(\widetilde{Y}, \widetilde{Z}) = 0$, $\forall \widetilde{X}, \widetilde{Y}, \widetilde{Z} \in \Im^1_0(TM)$ has the form

$$(3.8)\qquad E_\alpha\ {}^f\widetilde{G}_{\gamma\beta} - {}^f\widetilde{\Gamma}^\varepsilon_{\delta\gamma}\ {}^f\widetilde{G}_{\varepsilon\beta} - {}^f\widetilde{\Gamma}^\varepsilon_{\delta\beta}\ {}^f\widetilde{G}_{\gamma\varepsilon} = 0$$

with respect to the adapted frame $\{E_\beta\}$. Thus we have from (3.7) and (3.8)

$$(3.9)\qquad {}^f\widetilde{\Gamma}^\alpha_{\beta\gamma} = \frac{1}{2}\ {}^f\widetilde{G}^{\alpha\varepsilon}(E_\beta\ {}^f\widetilde{G}_{\varepsilon\gamma} + E_\gamma\ {}^f\widetilde{G}_{\beta\varepsilon} - E_\varepsilon\ {}^f\widetilde{G}_{\beta\gamma}) + \frac{1}{2}(\Omega_{\beta\gamma}{}^\alpha + \Omega^\alpha{}_{\beta\gamma} + \Omega^\alpha_{\gamma\beta}),$$

where $\Omega^{\alpha}{}_{\gamma\beta} = {}^{f}\widetilde{G}^{\alpha\varepsilon}\, {}^{f}\widetilde{G}_{\delta\beta}\Omega_{\varepsilon\gamma}{}^{\delta}$, ${}^{f}\widetilde{G}^{\alpha\varepsilon}$ are the contravariant components of the metric ${}^{f}\widetilde{G}$ with respect to the adapted frame.

Taking account of (3.3), (3.6) and (3.9), for various types of indices, we find the following relations

$$
\begin{aligned}
&{}^{f}\widetilde{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} + \tfrac{1}{2(f-1)} y^{p}(R_{pij}{}^{k} + R_{pji}{}^{k}) + \tfrac{1}{2(f-1)}\, {}^{f}A^{k}_{ij}\\
&{}^{f}\widetilde{\Gamma}^{k}_{\bar{i}j} = \tfrac{1}{2(f-1)} y^{p}R_{pij}{}^{k}\\
&{}^{f}\widetilde{\Gamma}^{k}_{i\bar{j}} = \tfrac{1}{2(f-1)} y^{p}R_{pji}{}^{k}\\
&{}^{f}\widetilde{\Gamma}^{\bar{k}}_{ij} = -\tfrac{1}{2(f-1)}\, {}^{f}A^{k}_{ij} - \tfrac{1}{2} y^{p}R_{ijp}{}^{k} - \tfrac{1}{2(f-1)} y^{p}(R_{pij}{}^{k} + R_{pji}{}^{k})\\
&{}^{f}\widetilde{\Gamma}^{\bar{k}}_{\bar{i}j} = -\tfrac{1}{2(f-1)} y^{p}R_{pij}{}^{k}\\
&{}^{f}\widetilde{\Gamma}^{\bar{k}}_{i\bar{j}} = \Gamma^{k}_{ij} - \tfrac{1}{2(f-1)} y^{p}R_{pji}{}^{k}\\
&{}^{f}\widetilde{\Gamma}^{k}_{\bar{i}\bar{j}} = 0\\
&{}^{f}\widetilde{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} = 0
\end{aligned} \qquad (3.10)
$$

with respect to the adapted frame, where ${}^{f}A^{k}_{ij}$ is a tensor field of type $(1,2)$ defined by ${}^{f}A^{k}_{ij} = (f_i\delta^{k}_{j} + f_j\delta^{k}_{i} - f_{.}^{\,k} g_{ji})$, $f_i = \partial_i f$.

4. The Riemannian curvature tensor

The Riemannian curvature tensor R of the connection ∇ is obtained from the well-known formula

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

for all $X, Y \in \Im^{1}_{0}(M)$. With respect to the adapted frame $\{E_\beta\}$, we write ${}^{f}\widetilde{\nabla}_{E_\alpha} E_\beta = {}^{f}\widetilde{\Gamma}^{\gamma}_{\alpha\beta} E_\gamma$, where ${}^{f}\widetilde{\Gamma}^{\gamma}_{\alpha\beta}$ denote the Levi-Civita connection constructed by ${}^{f}\widetilde{G}$. Then the Riemannian curvature tensor ${}^{f}\widetilde{R}$ has the components

$${}^{f}\widetilde{R}_{\alpha\beta\gamma}{}^{\sigma} = E_\alpha\, {}^{f}\widetilde{\Gamma}^{\sigma}_{\beta\gamma} - E_\beta\, {}^{f}\widetilde{\Gamma}^{\sigma}_{\alpha\gamma} + {}^{f}\widetilde{\Gamma}^{\sigma}_{\alpha\epsilon}\, {}^{f}\widetilde{\Gamma}^{\epsilon}_{\beta\gamma} - {}^{f}\widetilde{\Gamma}^{\sigma}_{\beta\epsilon}\, {}^{f}\widetilde{\Gamma}^{\epsilon}_{\alpha\gamma} - \Omega_{\alpha\beta}{}^{\epsilon}\, {}^{f}\widetilde{\Gamma}^{\sigma}_{\epsilon\gamma}.$$

From (3.6) and (3.10), we obtain the components of the Riemannian curvature tensor ${}^{f}\widetilde{R}$ of the metric ${}^{f}\widetilde{G}$ as follows:

$$
\begin{aligned}
&{}^{f}\widetilde{R}_{\overline{m}\bar{i}j}{}^{\bar{k}} = 0,\\
&{}^{f}\widetilde{R}_{\overline{m}i\bar{j}}{}^{k} = 0,\\
&{}^{f}\widetilde{R}_{\overline{m}\bar{i}j}{}^{k} = \tfrac{1}{f-1} R_{mij}{}^{k} + \tfrac{1}{4(f-1)^2} y^{p}y^{s}(R_{pmh}{}^{k}R_{sij}{}^{h} - R_{pih}{}^{k}R_{smj}{}^{h}),\\
&{}^{f}\widetilde{R}_{\overline{m}\bar{i}j}{}^{\bar{k}} = -\tfrac{1}{f-1} R_{mij}{}^{k} - \tfrac{1}{4(f-1)^2} y^{p}y^{s}(R_{pmh}{}^{k}R_{sij}{}^{h} - R_{pih}{}^{k}R_{smj}{}^{h}),\\
&{}^{f}\widetilde{R}_{\overline{m}i\bar{j}}{}^{k} = \tfrac{1}{2(f-1)} R_{mji}{}^{k} + \tfrac{1}{4(f-1)^2} y^{p}y^{s}R_{pmh}{}^{k}R_{sji}{}^{h},\\
&{}^{f}\widetilde{R}_{\overline{m}i\bar{j}}{}^{\bar{k}} = -\tfrac{1}{2(f-1)} R_{mji}{}^{k} - \tfrac{1}{4(f-1)^2} y^{p}y^{s}R_{pmh}{}^{k}R_{sji}{}^{h},
\end{aligned}
$$

(4.1)

$$
\begin{aligned}
{}^{f}\widetilde{R}_{m\bar{i}\bar{j}}{}^{k} &= -\tfrac{1}{2(f-1)}R_{ijm}{}^{k} - \tfrac{1}{4(f-1)^2}y^p y^s R_{pih}{}^{k} R_{sjm}{}^{h},\\
{}^{f}\widetilde{R}_{m\bar{i}\bar{j}}{}^{\bar{k}} &= \tfrac{1}{2(f-1)}R_{ijm}{}^{k} + \tfrac{1}{4(f-1)^2}y^p y^s R_{pih}{}^{k} R_{sjm}{}^{h},\\
{}^{f}\widetilde{R}_{\overline{mij}}{}^{\bar{k}} &= -\tfrac{1}{2}R_{ijm}{}^{k} - \tfrac{1}{2(f-1)}(R_{mij}{}^{k} + R_{mji}{}^{k}) + \tfrac{1}{2(f-1)}y^p \nabla_i R_{pmj}{}^{k}\\
&\quad - \tfrac{1}{4(f-1)^2}y^p y^s [R_{pmh}{}^{k}(R_{sij}{}^{h} + R_{sji}{}^{h}) - R_{smj}{}^{h}(R_{pih}{}^{k} + (f-1)R_{ihp}{}^{k})]\\
&\quad - \tfrac{1}{4(f-1)^2}y^p [2f_i R_{pmj}{}^{k} + R_{pmh}{}^{k}\, {}^{f}A^{h}_{ij} - R_{pmj}{}^{h}\, {}^{f}A^{k}_{ih}],\\
{}^{f}\widetilde{R}_{\overline{mij}}{}^{k} &= \tfrac{1}{2(f-1)}(R_{mij}{}^{k} + R_{mji}{}^{k}) - \tfrac{1}{2(f-1)}y^p \nabla_i R_{pmj}{}^{k}\\
&\quad + \tfrac{1}{4(f-1)^2}y^p y^s [R_{pmh}{}^{k}(R_{sij}{}^{h} + R_{sji}{}^{h}) - R_{smj}{}^{h} R_{pih}{}^{k}]\\
&\quad + \tfrac{1}{4(f-1)^2}y^p [2f_i R_{pmj}{}^{k} + R_{pmh}{}^{k}\, {}^{f}A^{h}_{ij} - R_{pmj}{}^{h}\, {}^{f}A^{k}_{ih}],\\
{}^{f}\widetilde{R}_{m\bar{i}j}{}^{\bar{k}} &= \tfrac{1}{2}R_{mji}{}^{k} + \tfrac{1}{2(f-1)}(R_{imj}{}^{k} + R_{ijm}{}^{k}) - \tfrac{1}{2(f-1)}y^p \nabla_m R_{pij}{}^{k}\\
&\quad - \tfrac{1}{4(f-1)^2}y^p y^s [R_{sij}{}^{h}((f-1)R_{mhp}{}^{k} + R_{pmh}{}^{k}) - R_{pih}{}^{k}(R_{smj}{}^{h} + R_{sjm}{}^{h})]\\
&\quad + \tfrac{1}{4(f-1)^2}y^p [2f_m R_{pij}{}^{k} - (R_{pij}{}^{h}\, {}^{f}A^{k}_{mh} - R_{pih}{}^{k}\, {}^{f}A^{h}_{mj})],\\
{}^{f}\widetilde{R}_{m\bar{i}j}{}^{k} &= -\tfrac{1}{2(f-1)}(R_{imj}{}^{k} + R_{ijm}{}^{k}) + \tfrac{1}{2(f-1)}y^p \nabla_m R_{pij}{}^{k}\\
&\quad + \tfrac{1}{4(f-1)^2}y^p [-2f_m R_{pij}{}^{k} + R_{pij}{}^{h}\, {}^{f}A^{k}_{mh} - R_{pih}{}^{k}\, {}^{f}A^{h}_{mj}]\\
&\quad - \tfrac{1}{4(f-1)^2}y^p y^s [R_{pih}{}^{k}(R_{smj}{}^{h} + R_{sjm}{}^{h}) - R_{sij}{}^{h} R_{pmh}{}^{k}]\\
{}^{f}\widetilde{R}_{mi\bar{j}}{}^{k} &= \tfrac{1}{2(f-1)}y^p(\nabla_m R_{pji}{}^{k} - \nabla_i R_{pjm}{}^{k}) + \tfrac{1}{4(f-1)^2}y^p y^s (R_{sji}{}^{h} R_{pmh}{}^{k} - R_{sjm}{}^{h} R_{pih}{}^{k})\\
&\quad + \tfrac{1}{4(f-1)^2}y^p [2f_i R_{pjm}{}^{k} - 2f_m R_{pji}{}^{k} + R_{pji}{}^{h}\, {}^{f}A^{k}_{mh} - R_{pjm}{}^{h}\, {}^{f}A^{k}_{ih}],\\
{}^{f}\widetilde{R}_{mij}{}^{\bar{k}} &= \tfrac{1}{2}y^p(\nabla_i R_{mjp}{}^{k} - \nabla_m R_{ijp}{}^{k}) + \tfrac{1}{2(f-1)}y^p[\nabla_i(R_{pmj}{}^{k} + R_{pjm}{}^{k})\\
&\quad - \nabla_m(R_{pij}{}^{k} + R_{pji}{}^{k})] + \tfrac{1}{4(f-1)^2}y^p[2f_m(R_{pij}{}^{k} + R_{pji}{}^{k})\\
&\quad - 2f_i(R_{pmj}{}^{k} + R_{pjm}{}^{k}) + (R_{pmj}{}^{h} + R_{pjm}{}^{h})\,{}^{f}A^{k}_{ih} - (R_{pij}{}^{h} + R_{pji}{}^{h})\,{}^{f}A^{k}_{mh}\\
&\quad + R_{pih}{}^{k}\,{}^{f}A^{h}_{mj} - R_{pmh}{}^{k}\,{}^{f}A^{h}_{ij} + (f-1)(R_{ihp}{}^{k}\,{}^{f}A^{h}_{mj} - R_{mhp}{}^{k}\,{}^{f}A^{h}_{ij})]\\
&\quad + \tfrac{1}{4(f-1)^2}y^p y^s[R_{pih}{}^{k}(R_{smj}{}^{h} + R_{sjm}{}^{h}) - R_{pmh}{}^{k}(R_{sij}{}^{h} + R_{sji}{}^{h})\\
&\quad + (f-1)(R_{ihp}{}^{k}(R_{smj}{}^{h} + R_{sjm}{}^{h}) - R_{mhp}{}^{k}(R_{sij}{}^{h} + R_{sji}{}^{h})\\
&\quad + R_{phm}{}^{k} R_{ijs}{}^{h} - R_{phi}{}^{k} R_{mjs}{}^{h} - 2R_{mis}{}^{h} R_{phj}{}^{k})]\\
&\quad + \tfrac{1}{4(f-1)^2}[2f_m\,{}^{f}A^{k}_{ij} - 2f_i\,{}^{f}A^{k}_{mj} + {}^{f}A^{k}_{ih}\,{}^{f}A^{h}_{mj} - {}^{f}A^{k}_{mh}\,{}^{f}A^{h}_{ij}\\
&\quad + 2(f-1)(\nabla_i\,{}^{f}A^{k}_{mj} - \nabla_m\,{}^{f}A^{k}_{ij})],\\
{}^{f}\widetilde{R}_{mi\bar{j}}{}^{\bar{k}} &= R_{mij}{}^{k} + \tfrac{1}{2(f-1)}y^p(\nabla_i R_{pjm}{}^{k} - \nabla_m R_{pji}{}^{k}) + \tfrac{1}{4(f-1)^2}y^p[2f_m R_{pji}{}^{k} - 2f_i R_{pjm}{}^{k}\\
&\quad - R_{pji}{}^{h}\,{}^{f}A^{k}_{mh} + R_{pjm}{}^{h}\,{}^{f}A^{k}_{ih}] + \tfrac{1}{4(f-1)^2}y^p y^s[(f-1)R_{ihp}{}^{k} R_{sjm}{}^{h} + R_{phi}{}^{k} R_{sjm}{}^{h}\\
&\quad - (f-1)R_{mhp}{}^{k} R_{sji}{}^{h} - R^{k}_{pmh} R_{sji}{}^{h}]\\
{}^{f}\widetilde{R}_{mij}{}^{k} &= R_{mij}{}^{k} + \tfrac{1}{2(f-1)}y^p[\nabla_m(R_{pij}{}^{k} + R_{pji}{}^{k}) - \nabla_i(R_{pmj}{}^{k} + R_{pjm}{}^{k})]\\
&\quad + \tfrac{1}{4(f-1)^2}y^p[-2f_m(R_{pij}{}^{k} + R_{pji}{}^{k}) + 2f_i(R_{pmj}{}^{k} + R_{pjm}{}^{k}) + (R_{pij}{}^{h} + R_{pji}{}^{h})\,{}^{f}A^{k}_{mh}\\
&\quad - (R_{pmj}{}^{h} + R_{pjm}{}^{h})\,{}^{f}A^{k}_{ih} + R_{pmh}{}^{k}\,{}^{f}A^{h}_{ij} - R_{pih}{}^{k}\,{}^{f}A^{h}_{mj}]\\
&\quad + \tfrac{1}{4(f-1)^2}y^p y^s[R_{pmh}{}^{k}(R_{sij}{}^{h} + R_{sji}{}^{h}) - (f-1)R_{phm}{}^{k} R_{ijs}{}^{h} - R_{pih}{}^{k}(R_{smj}{}^{h} + R_{sjm}{}^{h})\\
&\quad + (f-1)R_{phi}{}^{k} R_{mjs}{}^{h} + 2(f-1)R_{mis}{}^{h} R_{phj}{}^{k}] - \tfrac{1}{4(f-1)^2}[\,2f_m\,{}^{f}A^{k}_{ij} - 2f_i\,{}^{f}A^{k}_{mj}\\
&\quad + {}^{f}A^{k}_{ih}\,{}^{f}A^{h}_{mj} - {}^{f}A^{k}_{mh}\,{}^{f}A^{h}_{ij} + 2(f-1)(\nabla_i\,{}^{f}A^{k}_{mj} - \nabla_m\,{}^{f}A^{k}_{ij})]
\end{aligned}
$$

with respect to the adapted frame $\{E_\beta\}$.

We now compare the geometries of the Riemannian manifold (M, g) and its tangent bundle TM equipped with the Riemannian metric ${}^{f}\widetilde{G}$.

Theorem 4.1. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle with the Riemannian metric ${}^f\widetilde{G}$. Then TM is flat if M is flat and*

$$2f_m{}^fA_{ij}^k - 2f_i{}^fA_{mj}^k + {}^fA_{ih}^k{}^fA_{mj}^h - {}^fA_{mh}^k{}^fA_{ij}^h + 2(f-1)(\nabla_i{}^fA_{mj}^k - \nabla_m{}^fA_{ij}^k) = 0.$$

Proof. It follows from the equations (4.1) that if

$$2f_m{}^fA_{ij}^k - 2f_i{}^fA_{mj}^k + {}^fA_{ih}^k{}^fA_{mj}^h - {}^fA_{mh}^k{}^fA_{ij}^h + 2(f-1)(\nabla_i{}^fA_{mj}^k - \nabla_m{}^fA_{ij}^k) = 0,$$

then $R \equiv 0$ implies ${}^f\widetilde{R} \equiv 0$. □

Corollary 4.1. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle with the Riemannian metric ${}^f\widetilde{G}$. Assume that $f = C(const.)$. In the case, TM is flat if and only if M is flat.*

5. The scalar curvature

We now turn our attention to the Ricci tensor and scalar curvature of the Riemannian metric ${}^f\widetilde{G}$. Let ${}^f\widetilde{R}_{\alpha\beta} = {}^f\widetilde{R}_{\sigma\alpha\beta}{}^{\sigma}$ and ${}^f\widetilde{r} = {}^f\widetilde{G}^{\alpha\beta}\,{}^f\widetilde{R}_{\alpha\beta}$ denote the Ricci tensor and scalar curvature of the Riemannian metric ${}^f\widetilde{G}$, respectively. From (4.1), the components of the Ricci tensor ${}^f\widetilde{R}_{\alpha\beta}$ are characterized by

(5.1)

$$\begin{aligned}
&{}^f\widetilde{R}_{\bar{i}\bar{j}} = -\tfrac{1}{4(f-1)^2}y^py^sR_{pih}{}^mR_{sjm}{}^h,\\
&{}^f\widetilde{R}_{\bar{i}j} = -\tfrac{1}{2(f-1)}R_{ij} + \tfrac{1}{2(f-1)}y^p(\nabla_pR_{ij} - \nabla_iR_{pj}) - \tfrac{1}{4(f-1)^2}y^py^sR_{pih}{}^mR_{sjm}{}^h\\
&+\tfrac{1}{4(f-1)^2}y^p(n-4)f_mR_{pij}{}^m,\\
&{}^f\widetilde{R}_{i\bar{j}} = -\tfrac{1}{2(f-1)}R_{ji} + \tfrac{1}{2(f-1)}y^p(\nabla_pR_{ji} - \nabla_jR_{pi}) - \tfrac{1}{4(f-1)^2}y^py^sR_{sjm}{}^hR_{pih}{}^m\\
&+\tfrac{1}{4(\lambda-1)^2}y^p(n-4)f_mR_{pji}{}^m,\\
&{}^f\widetilde{R}_{ij} = \tfrac{f-2}{f-1}R_{ij} + \tfrac{1}{2(f-1)}y^p(2\nabla_pR_{ij} - \nabla_iR_{pj} - \nabla_jR_{pi})\\
&+\tfrac{1}{4(f-1)^2}y^p[(n-4)f_m(R_{pij}{}^m + R_{pji}{}^m)] + \tfrac{1}{4(f-1)^2}y^py^s[-R_{pih}{}^mR_{sjm}{}^h\\
&+(f-1)R_{phi}{}^mR_{mjs}{}^h + 2(f-1)R_{mis}{}^hR_{phj}{}^m + (f-1)R_{ihp}{}^mR_{smj}{}^h]\\
&-\tfrac{1}{4(f-1)^2}[2f_m{}^fA_{ij}^m - 2f_i{}^fA_{mj}^m - {}^fA_{mh}^m{}^fA_{ij}^h + {}^fA_{ih}^m{}^fA_{mj}^h\\
&+2(f-1)(\nabla_i{}^fA_{mj}^m - \nabla_m{}^fA_{ij}^m)]
\end{aligned}$$

with respect to the adapted frame $\{E_\beta\}$. From (3.3) and (5.1), the scalar curvature of the Riemannian metric ${}^f\widetilde{G}$ is given by

$$\begin{aligned}
{}^f\widetilde{r} = {} & \tfrac{1}{f-1}r - \tfrac{1}{2(f-1)^2}y^py^sR_{phik}R_s{}^{hik} - \tfrac{1}{4(f-1)^3}g^{ij}[2f_m{}^fA_{ij}^m - 2f_i{}^fA_{mj}^m\\
& -{}^fA_{mh}^m{}^fA_{ij}^h + {}^fA_{ih}^m{}^fA_{mj}^h + 2(f-1)(\nabla_i{}^fA_{mj}^m - \nabla_m{}^fA_{ij}^m)].
\end{aligned}$$

Thus we have the result as follows.

Theorem 5.1. *Let (M,g) be a Riemannian manifold and TM be its tangent bundle with the metric ${}^f\widetilde{G}$. Let r be the scalar curvature of g and ${}^f\widetilde{r}$ be the scalar curvature of ${}^f\widetilde{G}$. Then the following equation holds:*

$${}^f\widetilde{r} = \frac{1}{f-1}r - \frac{1}{2(f-1)^2}y^py^sR_{phik}R_s{}^{hik} - {}^fL,$$

where

$$\begin{aligned}
{}^fL = {} & \tfrac{1}{4(f-1)^3}g^{ij}[2f_m{}^fA_{ij}^m - 2f_i{}^fA_{mj}^m - {}^fA_{mh}^m{}^fA_{ij}^h\\
& +{}^fA_{ih}^m{}^fA_{mj}^h + 2(f-1)(\nabla_i{}^fA_{mj}^m - \nabla_m{}^fA_{ij}^m)].
\end{aligned}$$

From the Theorem 5.1, we have the following conclusion.

Corollary 5.1. *Let (M, g) be a Riemannian manifold and TM be its tangent bundle with the metric ${}^f\widetilde{G}$. If ${}^f\widetilde{r}= 0$, then ${}^f L = 0$ implies $r = 0$.*

Let (M, g), $n > 2$, be a Riemannian manifold of constant curvature κ, i.e.

$$R_{phi}{}^m = \kappa(\delta_p^m g_{hi} - \delta_h^m g_{pi})$$

and

$$r = n(n-1)\kappa$$

where δ is the Kronecker's. By virtue of Theorem 5.1, we have

$$\begin{aligned}
{}^f\widetilde{r} &= \frac{1}{f-1}r - \frac{1}{2(f-1)^2}y^p y^s R_{phik} R_s{}^{hik} - {}^f L \\
&= \frac{1}{f-1}r - \frac{1}{2(f-1)^2}y^p y^s\ g_{km} R_{phi}{}^m g^{hl} g^{it} R_{slt}{}^k - {}^f L \\
&= \frac{1}{f-1}n(n-1)\kappa - {}^f L \\
&\quad -\frac{1}{2(f-1)^2}y^p y^s\ g_{km}(\kappa(\delta_p^m g_{hi} - \delta_h^m g_{pi}))g^{hl} g^{it}(\kappa(\delta_s^k g_{lt} - \delta_l^k g_{st})) \\
&= \frac{1}{f-1}n(n-1)\kappa - {}^f L \\
&\quad -\frac{1}{2(f-1)^2}\kappa^2\ y^p y^s (g_{kp}\delta_i^l - g_{pi}\delta_k^l)(\delta_s^k \delta_l^i - \delta_l^k \delta_s^i) \\
&= \frac{1}{f-1}n(n-1)\kappa - \frac{1}{2(f-1)^2}2(n-1)\kappa^2\ g_{ps} y^p y^s - {}^f L \\
&= \frac{(n-1)\kappa}{f-1}(n - \frac{\kappa}{f-1}\ \|y\|^2) - \ {}^f L.
\end{aligned}$$

Hence we have the theorem below.

Theorem 5.2. *Let (M, g), $n > 2$, be a Riemannian manifold of constant curvature κ. Then the scalar curvature ${}^f\widetilde{r}$ of $(TM, {}^f\widetilde{G})$ is*

$${}^f\widetilde{r} = \frac{(n-1)\kappa}{f-1}(n - \frac{\kappa}{f-1}\ \|y\|^2) - \ {}^f L.$$

where $\|y\|^2 = g_{ps} y^p y^s$ and

$$\begin{aligned}
{}^f L = \tfrac{1}{4(f-1)^3} g^{ij} [&2 f_m {}^f A_{ij}^m - 2 f_i {}^f A_{mj}^m - {}^f A_{mh}^m {}^f A_{ij}^h \\
&+ {}^f A_{ih}^m {}^f A_{mj}^h + 2(f-1)(\nabla_i {}^f A_{mj}^m - \nabla_m {}^f A_{ij}^m)].
\end{aligned}$$

6. Locally conformally flat tangent bundles

In this section we investigate locally conformally flatness property of TM equipped with the Riemannian metric ${}^f\widetilde{G}$.

Theorem 6.1. *Let M be an n-dimensional Riemannian manifold with the Riemannian metric g and let TM be its tangent bundle with the Riemannian metric ${}^f\widetilde{G}$. The tangent bundle TM is locally conformally flat if and only if M is locally flat and $f = C(constant)$.*

Proof. The tangent bundle TM with the Riemannian metric ${}^f\widetilde{G}$ is locally conformally flat if and only if the components of the curvature tensor of TM satisfy the following equation:

$$\begin{aligned}{}^f\widetilde{R}_{\alpha\gamma\beta\sigma} &= -\tfrac{{}^f\widetilde{r}}{2(2n-1)(n-1)}\left\{{}^f\widetilde{G}_{\alpha\beta}\,{}^f\widetilde{G}_{\gamma\sigma} - {}^f\widetilde{G}_{\alpha\sigma}\,{}^f\widetilde{G}_{\gamma\beta}\right\} \\ &+ \tfrac{1}{2(n-1)}({}^f\widetilde{G}_{\gamma\sigma}\,{}^f\widetilde{R}_{\alpha\beta} - {}^f\widetilde{G}_{\alpha\sigma}\,{}^f\widetilde{R}_{\gamma\beta} + {}^f\widetilde{G}_{\alpha\beta}\,{}^f\widetilde{R}_{\gamma\sigma} - {}^f\widetilde{G}_{\gamma\beta}\,{}^f\widetilde{R}_{\alpha\sigma}),\end{aligned} \tag{6.1}$$

where ${}^f\widetilde{R}_{\alpha\gamma\beta\sigma} = {}^f\widetilde{G}_{\sigma\epsilon}\,{}^f\widetilde{R}_{\alpha\gamma\beta}{}^{\epsilon}$.

From (6.1), we have the following special cases:

$$\begin{aligned}{}^f\widetilde{R}_{\overline{m}\overline{i}\overline{j}k} &= -\frac{{}^f\widetilde{r}}{2(2n-1)(n-1)}(g_{mj}g_{ik} - g_{mk}g_{ij}) + \frac{1}{2(n-1)}(g_{ik}\,{}^f\widetilde{R}_{\overline{m}\overline{j}} \\ &\quad -g_{mk}\,{}^f\widetilde{R}_{\overline{i}\overline{j}} + g_{mj}\,{}^f\widetilde{R}_{\overline{i}k} - g_{ij}\,{}^f\widetilde{R}_{\overline{m}k})\end{aligned} \tag{6.2}$$

and

$$\begin{aligned}{}^f\widetilde{R}_{\overline{m}\overline{i}\overline{j}\overline{k}} &= -\frac{{}^f\widetilde{r}}{2(2n-1)(n-1)}(g_{mj}g_{ik} - g_{mk}g_{ij}) + \frac{1}{2(n-1)}(g_{ik}\,{}^f\widetilde{R}_{\overline{m}\overline{j}} \\ &\quad -g_{mk}\,{}^f\widetilde{R}_{\overline{i}\overline{j}} + g_{mj}\,{}^f\widetilde{R}_{\overline{i}\overline{k}} - g_{ij}\,{}^f\widetilde{R}_{\overline{m}\overline{k}}).\end{aligned} \tag{6.3}$$

By the first and second equation in (4.1) and (3.2), from ${}^f\widetilde{R}_{\alpha\gamma\beta\sigma} = {}^f\widetilde{G}_{\sigma\epsilon}\,{}^f\widetilde{R}_{\alpha\gamma\beta}{}^{\epsilon}$, we obtain ${}^f\widetilde{R}_{\overline{m}\overline{i}\overline{j}k} = 0$ and ${}^f\widetilde{R}_{\overline{m}\overline{i}\overline{j}\overline{k}} = 0$. Hence from (6.2) and (6.3), we obtain

$$\frac{{}^f\widetilde{r}}{(2n-1)}(g_{mj}g_{ik} - g_{mk}g_{ij}) = g_{ik}\,{}^f\widetilde{R}_{\overline{m}\overline{j}} - g_{mk}\,{}^f\widetilde{R}_{\overline{i}\overline{j}} + g_{mj}\,{}^f\widetilde{R}_{\overline{i}k} - g_{ij}\,{}^f\widetilde{R}_{\overline{m}k} \tag{6.4}$$

and

$$\frac{{}^f\widetilde{r}}{(2n-1)}(g_{mj}g_{ik} - g_{mk}g_{ij}) = g_{ik}\,{}^f\widetilde{R}_{\overline{m}\overline{j}} - g_{mk}\,{}^f\widetilde{R}_{\overline{i}\overline{j}} + g_{mj}\,{}^f\widetilde{R}_{\overline{i}\overline{k}} - g_{ij}\,{}^f\widetilde{R}_{\overline{m}\overline{k}}, \tag{6.5}$$

it follows that ${}^f\widetilde{R}_{\overline{i}k} = {}^f\widetilde{R}_{\overline{i}\overline{k}}$. By means of the first and second equations in (5.1), we get

$$R_{ij} = 0,\ f_m = 0, \text{ i.e. } f = C(constant)$$

and

$${}^f\widetilde{R}_{\overline{i}\overline{j}} = -\frac{1}{4(f-1)^2}y^p y^s R_{pih}{}^m R_{sjm}{}^h. \tag{6.6}$$

Transvecting (6.5) by g^{ik}, we obtain

$$\frac{(n-1){}^f\widetilde{r}}{(2n-1)}\,g_{mj} = (n-2){}^f\widetilde{R}_{\overline{m}\overline{j}} + g^{ik}g_{mj}\,{}^f\widetilde{R}_{\overline{i}\overline{k}}. \tag{6.7}$$

Transvecting (6.7) by g^{mj}, we get

$$\frac{n(n-1)}{(2n-1)}\,{}^f\widetilde{r} = 2(n-1)g^{ik}\,{}^f\widetilde{R}_{\overline{i}\overline{k}}. \tag{6.8}$$

On the other hand, from (6.6), we have

$$g^{ik}\ {}^f\widetilde{R}_{\overline{i}\overline{k}} = -\frac{1}{4(f-1)^2} y^p y^s g^{ik}\ R_{pih}{}^m R_{skm}{}^h = \frac{1}{4(f-1)^2} y^p y^s R_{pilh} R_s{}^{ilh} = -\frac{1}{2}\ {}^f\widetilde{r}. \tag{6.9}$$

Thus by (6.8) and (6.9), we obtain ${}^f\widetilde{r} = 0$, then it follows $R_{pilh}R_s{}^{ilh} = 0$ by using $f = C(constant)$. This shows $R_{pilh} = 0$. This completes the proof. □

7. Curvature properties of another metric connection of the Riemannian metric ${}^f\widetilde{G}$

Let ∇ be a linear connection on an $n-$dimensional differentiable manifold M. The connection ∇ is symmetric if its torsion tensor vanishes, otherwise it is non-symmetric. If there is a Riemannian metric g on M such that $\nabla g = 0$, then the connection ∇ is a metric connection, otherwise it is non-metric. It is well known that a linear connection is symmetric and metric if and only if it is the Levi-Civita connection. In section 4, we have considered the Levi-Civita connection ${}^f\widetilde{\nabla}$ of the Riemannian metric ${}^f\widetilde{G}$ on the tangent bundle TM over (M, g). The connection is the unique connection which satisfies ${}^f\widetilde{\nabla}_\alpha {}^f\widetilde{G}_{\beta\gamma} = 0$ and has a zero torsion. H. A.Hayden [8] introduced a metric connection with a non-zero torsion on a Riemannian manifold. Now we are interested in a metric connection ${}^{(M)}\widetilde{\nabla}$of the Riemannian metric ${}^f\widetilde{G}$ whose torsion tensor ${}^{(M)\nabla}T^{\varepsilon}_{\gamma\beta}$ is skew-symmetric in the indices γ and β. We denote components of the connection ${}^{(M)}\widetilde{\nabla}$ by ${}^{(M)}\widetilde{\Gamma}$. The metric connection ${}^{(M)}\widetilde{\nabla}$ satisfies

$${}^{(M)}\widetilde{\nabla}_\alpha {}^f\widetilde{G}_{\beta\gamma} = 0 \text{ and } {}^{(M)}\widetilde{\Gamma}^\gamma_{\alpha\beta} - {}^{(M)}\widetilde{\Gamma}^\gamma_{\beta\alpha} = {}^{(M)\nabla}T^\gamma_{\alpha\beta}. \tag{7.1}$$

On the equation (7.1) is solved with respect to ${}^{(M)}\widetilde{\Gamma}^\gamma_{\alpha\beta}$, one finds the following solution [8]

$${}^{(M)}\widetilde{\Gamma}^\gamma_{\alpha\beta} = {}^f\widetilde{\Gamma}^\gamma_{\alpha\beta} + \widetilde{U}^\gamma_{\alpha\beta}, \tag{7.2}$$

where ${}^f\widetilde{\Gamma}^\gamma_{\alpha\beta}$ is components of the Levi-Civita connection of the Riemannian metric ${}^f\widetilde{G}$,

$$\widetilde{U}_{\alpha\beta\gamma} = \frac{1}{2}({}^{(M)\nabla}T_{\alpha\beta\gamma} + {}^{(M)\nabla}T_{\gamma\alpha\beta} + {}^{(M)\nabla}T_{\gamma\beta\alpha}) \tag{7.3}$$

and

$$\widetilde{U}_{\alpha\beta\gamma} = U^\epsilon_{\alpha\beta}{}^f\widetilde{G}_{\epsilon\gamma},\ {}^{(M)\nabla}T_{\alpha\beta\gamma} = T^\epsilon_{\alpha\beta}{}^f\widetilde{G}_{\epsilon\gamma}.$$

If we put

$${}^{(M)\nabla}T^{\overline{r}}_{ij} = y^p R_{ijp}{}^r \tag{7.4}$$

all other ${}^{(M)\nabla}T^\gamma_{\alpha\beta}$ not related to ${}^{(M)\nabla}T^{\overline{r}}_{ij}$ being assumed to be zero. We choose this ${}^{(M)\nabla}T^\gamma_{\alpha\beta}$ in TM which is skew-symmetric in the indices γ and β as torsion tensor and determine a metric connection in TM with respect to the Riemannian metric

${}^{f}\widetilde{G}$ (see also, [16, p.151-155]. By using (7.3) and (7.4), we get non-zero components of $\widetilde{U}^{\gamma}_{\alpha\beta}$ as follows:

$$\begin{aligned}
\widetilde{U}^{k}_{ij} &= \frac{-1}{2(f-1)} y^{p}(R_{pij}{}^{k} + R_{pji}{}^{k}),\\
\widetilde{U}^{\bar{k}}_{ij} &= \frac{1}{2} y^{p} R_{ijp}{}^{k} + \frac{1}{2(f-1)} y^{p}(R_{pij}{}^{k} + R_{pji}{}^{k}),\\
\widetilde{U}^{k}_{\bar{i}j} &= \frac{-1}{2(f-1)} y^{p} R_{pij}{}^{k},\\
\widetilde{U}^{\bar{k}}_{\bar{i}j} &= \frac{1}{2(f-1)} y^{p} R_{pij}{}^{k},\\
\widetilde{U}^{k}_{i\bar{j}} &= \frac{-1}{2(f-1)} y^{p} R_{pji}{}^{k},\\
\widetilde{U}^{\bar{k}}_{i\bar{j}} &= \frac{1}{2(f-1)} y^{p} R_{pji}{}^{k}
\end{aligned}$$

with respect to the adapted frame. From (7.2) and (3.10), we have components of the metric connection ${}^{(M)}\widetilde{\nabla}$ with respect to ${}^{f}\widetilde{G}$ as follows:

$$\begin{aligned}
&{}^{(M)}\widetilde{\Gamma}^{k}_{ij} = \Gamma^{k}_{ij} + \tfrac{1}{2(f-1)} {}^{f}A^{k}_{ij},\\
&{}^{(M)}\widetilde{\Gamma}^{\bar{k}}_{ij} = -\tfrac{1}{2(f-1)} {}^{f}A^{k}_{ij},\\
&{}^{(M)}\widetilde{\Gamma}^{\bar{k}}_{i\bar{j}} = \Gamma^{k}_{ij},\\
&{}^{(M)}\widetilde{\Gamma}^{k}_{i\bar{j}} = 0,\ {}^{(M)}\widetilde{\Gamma}^{k}_{\bar{i}j} = 0\\
&{}^{(M)}\widetilde{\Gamma}^{\bar{k}}_{\bar{i}j} = 0, {}^{(M)}\widetilde{\Gamma}^{k}_{\bar{i}\bar{j}} = 0, {}^{(M)}\widetilde{\Gamma}^{\bar{k}}_{\bar{i}\bar{j}} = 0
\end{aligned}$$

with respect to the adapted frame, where $R_{hji}{}^{s}$ are the local coordinate components of the curvature tensor field R of g.

Remark 7.1. The metric connection ${}^{(M)}\widetilde{\nabla}$ and he Levi-Civita connection ${}^{f}\widetilde{\nabla}$ on TM of the Riemannian metric ${}^{f}\widetilde{G}$ coincide if and only if the base manifold M is flat.

The non-zero components of the curvature tensor ${}^{(M)}\widetilde{R}$ of the metric connection ${}^{(M)}\widetilde{\nabla}$ are given as follows:

$$\begin{aligned}
&{}^{(M)}\widetilde{R}_{mij}{}^{k} = R_{mij}{}^{k} - \tfrac{1}{4(f-1)^{2}}[\, 2f_{m}{}^{f}A^{k}_{ij} - 2f_{i}{}^{f}A^{k}_{mj}\\
&+{}^{f}A^{k}_{ih}{}^{f}A^{h}_{mj} - {}^{f}A^{k}_{mh}{}^{f}A^{h}_{ij} + 2(f-1)(\nabla_{i}{}^{f}A^{k}_{mj} - \nabla_{m}{}^{f}A^{k}_{ij})]\\
&{}^{(M)}\widetilde{R}_{mij}{}^{\bar{k}} = \tfrac{1}{4(f-1)^{2}}[\, 2f_{m}{}^{f}A^{k}_{ij} - 2f_{i}{}^{f}A^{k}_{mj}\\
&+{}^{f}A^{k}_{ih}{}^{f}A^{h}_{mj} - {}^{f}A^{k}_{mh}{}^{f}A^{h}_{ij} + 2(f-1)(\nabla_{i}{}^{f}A^{k}_{mj} - \nabla_{m}{}^{f}A^{k}_{ij})]\\
&{}^{(M)}\widetilde{R}_{mi\bar{j}}{}^{\bar{k}} = R_{mij}{}^{k}
\end{aligned}$$

with respect to the adapted frame.

The non-zero component of the contracted curvature tensor field (Ricci tensor field) ${}^{(M)}\widetilde{R}_{\gamma\beta} = {}^{(M)}\widetilde{R}_{\alpha\beta\gamma}{}^{\alpha}$ of the metric connection ${}^{(M)}\widetilde{\nabla}$ is as follows:

$$\begin{aligned}
&{}^{(M)}\widetilde{R}_{ij} = R_{ij} - \tfrac{1}{4(f-1)^{2}}[\, 2f_{m}{}^{f}A^{m}_{ij} - 2f_{i}{}^{f}A^{m}_{mj}\\
&+{}^{f}A^{m}_{ih}{}^{f}A^{h}_{mj} - {}^{f}A^{m}_{mh}{}^{f}A^{h}_{ij} + 2(f-1)(\nabla_{i}{}^{f}A^{m}_{mj} - \nabla_{m}{}^{f}A^{m}_{ij})]
\end{aligned}$$

For the scalar curvature ${}^{(M)}\widetilde{r}$ of the metric connection ${}^{(M)}\widetilde{\nabla}$ with respect to ${}^{f}\widetilde{G}$, we obtain

$$ {}^{(M)}\widetilde{r} = \frac{1}{f-1} r - {}^{f} L $$

where

$$ {}^{f}L = \frac{1}{4(f-1)^3} g^{ij} [2 f_m {}^{f}A^m_{ij} - 2 f_i {}^{f}A^m_{mj} - {}^{f}A^m_{mh} {}^{f}A^h_{ij} + {}^{f}A^m_{ih} {}^{f}A^h_{mj} $$
$$ +2(f-1)(\nabla_i {}^{f}A^m_{mj} - \nabla_m {}^{f}A^m_{ij})]. $$

Thus we have the following theorem.

Theorem 7.1. *Let M be an n-dimensional Riemannian manifold with the Riemannian metric g and let TM be its tangent bundle with the Riemannian metric ${}^{f}\widetilde{G}$. Then the tangent bundle TM with the metric connection ${}^{(M)}\widetilde{\nabla}$ has a vanishing scalar curvature with respect to the Riemannian metric ${}^{f}\widetilde{G}$ if the scalar curvature r of the Levi-Civita connection of g is zero and ${}^{f}L = 0$.*

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