Curvature Tensors and Covariant Derivatives.

SHŪKICHI TANNO (Sendai) (*)

Summary. – *The problems considered here are of two types.* (i) *What are implications of vanishing k-th covariant derivatives of curvature tensors?* (ii) *Under what conditions on curvature tensors, does the k-th covariant derivative* $\nabla^k T = 0$ *for a tensor* T *mean* $\nabla T = 0$*?*

1. – Introduction.

Let (M, g) be a Riemannian manifold with (positive definite) Riemannian metric tensor g or a pseudo-Riemannian manifold with (definite or indefinite) Riemannian metric tensor g. By $R = (R^i{}_{jkl})$ we denote the Riemannian curvature tensor:

$$R(X, Y)Z = \nabla_{[X,Y]}Z - [\nabla_X, \nabla_Y]Z ,$$

where X, Y, Z are vector fields on M and ∇ denotes the Riemannian connection defined by g. By $R_1 = (R_{jk} = R^r{}_{jkr})$ and $S = (g^{rs}R_{rs})$ we denote the Ricci curvature tensor and the scalar curvature, respectively.

NOMIZU and OZEKI [5] proved the following Proposition.

PROPOSITION (NOMIZU and OZEKI [5]). – *If a Riemannian manifold* (M, g) *is complete and irreducible, and if an arbitrary tensor* T *has the vanishing k-th covariant derivative, i.e.,* $\nabla^k T = 0$ *for some integer* $k \geq 1$*, then* $\nabla T = 0$.

We replace « completeness » by curvature tensor conditions.

THEOREM 1. – *Let* (M, g) *be a Riemannian manifold. Assume one of the following conditions:*

(i) *At some point* x *of* M, R_1 *is non-singular,*

(ii) *at some point* x *of* M *and for some tangent vectors* X, Y *at* x, $R(X, Y)$ *is non-singular,*

(ii′) *at some point* x *of* M*, the index of nullity is zero.*
Then, for an arbitrary tensor T, $\nabla^k T = 0$ *for some* $k \geq 1$ *implies* $\nabla T = 0$.

In the case $T = R$, NOMIZU and OZEKI [5] and later NOMIZU (without assuming completeness) proved the following Proposition.

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PROPOSITION (NOMIZU and OZEKI [5], NOMIZU). – *In a Riemannian manifold* (M, g), *if* $\nabla^k R = 0$ *for some* $k \geqslant 1$, *then* $\nabla R = 0$.

This is a generalization of a result of LICHNEROWICZ [2], [3] for the case $T = R$. Analogously we have

THEOREM 2. – *Let* (M, g) *be a Riemannian manifold. By* C *and* P *we denote the Weyl's conformal curvature tensor and projective curvature tensor.*

(1) *If* $\nabla^k R_1 = 0$ *for some* $k \geqslant 1$, *then* $\nabla R_1 = 0$.

(2) *If* $\nabla^k S = 0$ *for some* $k \geqslant 1$, *then* S = constant.

(3) *If* $\nabla^k C = 0$ *for some* $k \geqslant 1$, *then* $\nabla C = 0$.

(4) *If* $\nabla^k P = 0$ *for some* $k \geqslant 1$, *then* $\nabla P = 0$ *and* $\nabla R = 0$.

The author is grateful to Professor K. NOMIZU who gave him a letter containing a proof of the above Proposition. Proof of Theorem 2 is basically the same as one for the Proposition.

Generally, if T is a (homogeneous) tensor constructed by $[\nabla^r R, \nabla^s R_1, \nabla^t C, \nabla^u P;$ $r, s, t, u = 0, 1, \ldots$ finite$]$ and satisfies $\nabla^k T = 0$ for some $k \geqslant 1$, then $\nabla T = 0$, where $\nabla^0 R = R$, etc.

THEOREM 3. – *Let* (M, g) *be an irreducible Riemannian manifold. If*

(iii) *at some point* x *of* M, $(\nabla^j S)_x = 0$ *for some* $j \geqslant 1$ *and* $S_x \neq 0$,
then, for a tensor T, $\nabla^k T = 0$ *for some* $k \geqslant 1$ *implies* $\nabla T = 0$.

Next we consider pseudo-Riemannian manifolds.

THEOREM 4. – *Let* (M, g) *be a pseudo-Riemannian manifold of signature* (p, q). *Assume that*

(*a*) *the restricted homogeneous holonomy group is irreducible,*

(*b*) $[\dim M = m =$ odd *or* $m = 2]$ *or* $[m =$ even $\geqslant 4$ *and* $p \neq q]$,

(*c*) (M, g) *satisfies one of the conditions:* (i), (ii), (ii′) *in Theorem* 1, (iii) *for* $j = 1$ *in Theorem* 3.

Then, for a tensor T, $\nabla^2 T = 0$ *implies that* ∇T *is null and the inner product* (T, T) *is constant.*

THEOREM 5. – *Let* (M, g) *be a pseudo-Riemannian manifold of signature* (p, q). *Assume that* (*a*) *and* (*b*) *in Theorem* 4. *Then we have*

(1) $\nabla^2 R = 0$ *implies that* ∇R *is null and* (R, R) *is constant.*

(2) $\nabla^2 R_1 = 0$ *implies that* ∇R_1 *is null and* (R_1, R_1) *is constant.*

(3) $\nabla^2 S = 0$ *implies that* $S =$ *constant.*

(4) $\nabla^2 C = 0$ *implies that* ∇C *is null and* (C, C) *is constant.*

(5) $\nabla^2 P = 0$ *implies that* ∇P *is null and* (P, P) *is constant.*

Theorems 4 and 5 are generalised, if a pseudo-Riemannian manifold (M, g) is non-degenerately reducible in the sense of WU [9] and if respective part satisfies the required conditions.

Next, generalizing a result of GLODEK [1], we get

THEOREM 6. – *Let* (M, g) *be a pseudo-Riemannian manifold,* $m \geqslant 4$. *If the Weyl's conformal curvature tensor* C *is parallel, i.e.* $\nabla C = 0$, *then* $C = 0$ *or* $S =$ *constant.*

As an application of Theorem 1 we have

COROLLARY. – *Let* (M, g, J) *be an almost Hermitian manifold with almost complex structure tensor* J *and almost Hermitian metric tensor* g. *If the Ricci curvature tensor* R_1 *is non-singular at some point, and if* $\nabla^k J = 0$ *for some* $k \geqslant 1$, *then* (M, g, J) *is Kählerian.*

Finally, we have

THEOREM 7. – *Let* (M, g, J) *be a Kählerian manifold,* $m \geqslant 4$. *If* $\nabla^k C = 0$ *for some* $k \geqslant 1$, *then* $\nabla R = 0$, *i.e.,* (M, g, J) *is locally symmetric.*

In the proof of Theorem 7, we have also

THEOREM 8. – *Let* (M, g, J) *be a Kählerian manifold. Then* $\nabla_i R_{jk} - \nabla_j R_{ik} = 0$, *if and only if* $\nabla_i R_{jk} = 0$.

2. – Proof of Theorems 1, 2 and 3.

Let (M, g) be a Riemannian manifold. Let T be a tensor and let $T^{a\ldots b}{}_{c\ldots d}$ be its components in a local coordinate neighborhood U. Assume that $\nabla^k T = 0$ for some $k \geqslant 2$. We put $\nabla^0 T = T$. We define a scalar f by

$$f = (\nabla^{k-2} T, \nabla^{k-2} T) \tag{2.1}$$

$$= \nabla_r \ldots \nabla_s T^{a\ldots b}{}_{c\ldots d} \nabla^r \ldots \nabla^s T^{e\ldots f}{}_{u\ldots v} g_{ae} \cdots g^{dv}$$

where $\nabla^r = g^{rt} \nabla_t$ and $u, v, a, b, r, s, \ldots = 1, 2, \ldots, m = \dim M$. $\nabla^k T = 0$ implies

$$\nabla_w \nabla_v \nabla_u f = 0\,. \tag{2.2}$$

Assuming that U is sufficiently small, let $U = U_0 \times U_1 \times \ldots \times U_N$ be local decomposition of U corresponding to the restricted homogeneous holonomy group. Then the metric tensor g is decomposed into

$$g = \begin{pmatrix} g_0 & & & 0 \\ & g_1 & & \\ & & \ddots & \\ 0 & & & g_N \end{pmatrix}, \tag{2.3}$$

where g_0 is the flat part and $(U_1, g_1), \ldots, (U_N, g_N)$ are irreducible. The parallel symmetric tensor $\nabla^2 f = (\nabla_v \nabla_u f)$ is written as (cfr. EISENHART [11])

$$\nabla^2 f = \begin{pmatrix} \nabla^2 f | U_0 & & & 0 \\ & c_1 g_1 & & \\ & & \ddots & \\ 0 & & & c_N g_N \end{pmatrix}, \tag{2.4}$$

where $c_1, \ldots, c_N$ are constant.

Now, we define a subspace N_x of the tangent space M_x at x by

$$N_x = \{X \in M_x : R(X, Y) = 0 \text{ for all } Y \in M_x\}.$$

Then the dim N_x is called the index of nullity at x.

PROOF OF THEOREM 1. – If R_1 is non-singular at some point x, we consider U containing x. Then U has no flat part, i.e., $U = U_1 \times \ldots \times U_N$. This is the same for (ii) and (ii′). If we denote product coordinates (x^u) by

$$(x^u) = [x^\alpha \text{ in } U_1, x^\lambda \text{ in } U_2, \ldots, x^\omega \text{ in } U_N].$$

Then, (2.4) implies

$$\nabla_\alpha \nabla_\lambda f = 0. \tag{2.5}$$

Since the Clistoffel's symbol $\Gamma^u_{\alpha\lambda} = 0$, (2.5) implies

$$\partial^2 f / \partial x_\alpha \partial x_\lambda = 0.$$

Therefore, we can conclude that

$$f = f_1(x^\alpha) + f_2(x^\lambda) + \ldots + f_N(x^\omega). \tag{2.6}$$

Hence, we have $\nabla^2 f | U_\theta = c_\theta g_\theta = \nabla^2 f_\theta$, where ∇ denotes also the Riemannian connection on (U_θ, g_θ), $\theta = 1, \ldots, N$. That is, we get

$$\nabla_\mu \nabla_\nu f_\theta + \nabla_\nu \nabla_\mu f_\theta = 2c_\theta (g_\theta)_{\mu\nu}.$$

Indices μ, ν, ξ, η run from $\dim(U_1 \times \ldots \times U_{\theta-1}) + 1$ to $\dim(U_1 \times \ldots \times U_\theta)$. If we put $Z^\mu = (g_\theta)^{\mu\nu}\nabla_\nu f_\theta$, then Z is an infinitesimal homothety on (U_θ, g_θ). Consequently, denoting by L_Z the Lie derivation by Z, we get

$$L_Z(\Gamma_\theta)^\xi_{\mu\nu} = \nabla_\mu \nabla_\nu Z^\xi + (R_{(\theta)})^\xi{}_{\nu\mu\eta} Z^\eta = 0 . \tag{2.7}$$

Since $\nabla^3 f = 0$ implies $\nabla^3 f_\theta = 0$, we have $\nabla_\mu \nabla_\nu Z^\xi = 0$. Hence,

$$(R_{(\theta)})^\xi{}_{\nu\mu\eta} Z^\eta = 0 \quad \text{and} \quad (R_{1(\theta)})_{\nu\eta} Z^\eta = 0 .$$

Since $R = R_{(1)} + R_{(2)} + \ldots + R_{(N)}$ and $R_1 = R_{1(1)} + R_{1(2)} + \ldots + R_{1(N)}$, ((i), (ii) or (ii')) implies $Z = 0$. That is, f_θ is constant. Hence, f is constant on U. This means $\nabla_v \nabla_u f = 0$. Since $g^{vu}\nabla_v \nabla_u f = (\nabla^{k-1}T, \nabla^{k-1}T)$, we get $\nabla^{k-1}T = 0$ on U. Because $\nabla^{k-1}T$ is parallel, $\nabla^{k-1}T = 0$ holds on M. Continuing these steps, we get $\nabla T = 0$.

PROOF OF THEOREM 3. – In the proof of Theorem 1, we can put $N = 1$. Define f as before. Then we have an infinitesimal homothety Z: $L_Z g = 2cg$, and

$$L_Z S = L_Z(g^{rs} R_{rs}) = -2cS . \tag{2.8}$$

Since L_Z and the covariant differentiation are commutative, we get

$$L_Z(\nabla^{j-1} S) = -2c\nabla^{j-1} S . \tag{2.9}$$

On the other hand, we have

$$L_Z(\nabla^{j-1} S)_{rs\ldots t} = Z^u \nabla_u \nabla_r \nabla_s \ldots \nabla_t S + (j-1)\, c \nabla_r \nabla_s \ldots \nabla_t S . \tag{2.10}$$

By assumption $\nabla^j S = 0$ at a point x of M, (2.9) and (2.10) implies that $(j+1)\cdot c(\nabla^{j-1} S) = 0$ holds at x. Hence, we get $c = 0$ or $\nabla^{j-1} S = 0$ at x. Continuing these steps, finally we have $c = 0$ or $\nabla S = 0$ at x. If $\nabla S = 0$ at x, $L_Z S = Z^u \nabla_u S = 0$ holds at x. Since $S \neq 0$ at x, by (2.8) we have $c = 0$. Therefore in any case we have $c = 0$ and $\nabla_v \nabla_u f = 0$. Consequently $(\nabla^{k-1}T, \nabla^{k-1}T) = 0$ and $\nabla^{k-1}T = 0$. Finally $\nabla T = 0$.

PROOF OF THEOREM 2. – Since tensors we consider here are all curvature tensors, in a local decomposition corresponding to the restricted homogeneous holonomy group, if suffices to prove Theorem 2 in each part. So we assume (M, g) is irreducible. In stead of the Ricci curvature tensor R_1, the Weyl's conformal curvature tensor C, projective curvature tensor P, we write T. Put $f = (\nabla^{k-2}T, \nabla^{k-2}T)$. Then $Z^u = g^{vu}\nabla_v f$ is an infinitesimal homothety: $L_Z g = 2cg$. Hence $L_Z T = 0$. Since L_Z and ∇ are commutative,

$$L_Z(\nabla^{k-1}T) = \nabla^{k-1} L_Z T = 0 . \tag{2.11}$$

On the other hand, using $\nabla^k T = 0$ and $\nabla_v Z^u = c\delta^u_v$, we have

$$L_Z(\nabla^{k-1} T) = (k+1)\, c\nabla^{k-1} T\,. \tag{2.12}$$

By (2.11) and (2.12), we have $c = 0$ or $\nabla^{k-1}T = 0$. $c = 0$ implies $\nabla^{k-1}T = 0$. Continuing these steps we have $\nabla T = 0$. This proves (1), (3) and (4) for $[\nabla P = 0]$.

Next, we show (2). Put $f = (\nabla^{k-2}S, \nabla^{k-2}S)$. By (2.8), we have $L_Z\nabla^{k-1}S = -2c\nabla^{k-1}S$. On the other hand, we have

$$L_Z(\nabla^{k-1}S_{rs...t}) = (k-1)\, c(\nabla^{k-1}S_{rs...t})$$

by $\nabla^k S = 0$. Hence, $(k+1)c\nabla^{k-1}S = 0$ follows. $c = 0$ implies $\nabla^{k-1}S = 0$ on M. Therefore, we have $\nabla^{k-1}S = 0$ on M, and $\nabla S = 0$ on M. To complete our proof for (4), we need the following

PROPOSITION (MATSUMOTO [4]). – In a pseudo-Riemannian manifold (M, g), $\nabla P = 0$ implies $\nabla R = 0$.

REMARK. – R^*_{II}-spaces defined by ROTER [6] are locally symmetric, in the positive definite case, by the second Proposition in the Introduction.

3. – Proof of Theorems 4 and 5.

In a study of pseudo-Riemannian manifolds of signature (p, q), the following lemma is sometimes useful.

LEMMA (TANNO [7]). – *Assume that* [$\dim M = m =$ odd *or* $m = 2$] *or* [$m =$ even $\geqslant 4$ *and* $p \neq q$]. *If the restricted homogeneous holonomy group is irreducible and if a symmetric* $(0, 2)$*-tensor* g^* *is invariant by the group, then* $g^* = \sigma g$ *for some scalar* σ.

Further, if g^* *is parallel, then* σ *is constant.*

PROOF OF THEOREM 4. – Put $f = (T, T)$. Then $\nabla_v\nabla_u f$ is parallel. Hence, $\nabla_v\nabla_u f = cg_{vu}$ for some constant c. $g^{uv}\nabla_v f = Z^u$ is an infinitesimal homothety. Hence, (i), (ii), (ii′) of Theorem 1 imply that $c = 0$ (cf. (2.7), etc.). Consequently, $(\nabla T, \nabla T) = 0$. That is, ∇T is a null tensor. Next, $\nabla_v\nabla_u f = 0$ implies that $\nabla_u f$ is parallel. Since (M, g) is irreducible, we have $\nabla_u f = 0$. This means (T, T) is constant.

Next, assume that (iii) for $j = 1$ in Theorem 3. Then $L_Z S = -2cS$ gives $c = 0$. Thus, ∇T is null and (T, T) is constant.

PROOF OF THEOREM 5. – Let T be one of R, R_1, S, C, P. Put $f = (T, T)$. Then $Z^u = g^{uv}\nabla_v f$ satisfies $L_Z g = 2cg$. If T is one of R, R_1, C, P, we have $L_Z\nabla T = 0$. As in definite case, we have $c = 0$. Further, ∇T is null and (T, T) is constant.

As for $T = S$, we have $L_Z \nabla_r S = -2c\nabla_r S$ and

$$L_Z \nabla_r S = Z^u \nabla_u \nabla_r S + c\nabla_r S = c\nabla_r S .$$

Hence, $c = 0$ or $\nabla_r S = 0$ follows. $\nabla_r S = 0$ means that $S =$ constant. $c = 0$ means that $\nabla_u f$ is parallel, and $\nabla_u f = 0$. Consequently, (S, S), and hence, S is constant.

4. – Proof of Theorem 6.

GLODEK proved the following Proposition.

PROPOSITION (GLODEK [1]). – *Every conformally symmetric (i.e., $\nabla C = 0$) pseudo-Riemannian manifold (M, g) is conformally flat (i.e., $C = 0$) or $\nabla_r S$ is null.*

Put $C_{ijkl} = g_{ir} C^r{}_{jkl}$, $a = 1/(m-2)$ and $b = 1/(m-1)(m-2)$. Then

$$C_{ijkl} = R_{ijkl} - a[R_{jk}g_{il} - R_{jl}g_{ik} + g_{jk}R_{il} - g_{jl}R_{ik}] + bS[g_{jk}g_{il} - g_{jl}g_{ik}] . \tag{4.1}$$

To prove Theorem 6, we show that if $\nabla_r S$ is not vanishing, $C = 0$.

PROOF of THEOREM 6. – In [1] it is shown that

$$\nabla_i S C_{hjkl} - \nabla_j S C_{hikl} + \nabla_k S C_{hlij} - \nabla_l S C_{hkij} = 0 . \tag{4.2}$$

Assume that $\nabla_i S$ is not vanishing at some point x of M. Then we can take a suitable local coordinate system about x such that $(\nabla_i S)$ has components $(\nabla_1 S, 0, \ldots, 0)$, $\nabla_1 S \neq 0$, at x.

In (4.2), if we put $(i = 1)$ and $(j, k, l \neq 1)$, then we have $C_{hjkl} = 0$ for every h. That is

$$C_{1jkl} = 0 \qquad \text{for } j, k, l \neq 1 , \tag{4.3}$$

$$C_{hjkl} = 0 \qquad \text{for } h, j, k, l \neq 1 . \tag{4.4}$$

In (4.2), if we put $(h = i = k = 1)$ and $(j, l \neq 1)$, then we have

$$C_{1j1l} + C_{1l1j} = 0 . \tag{4.5}$$

Since $C_{ijkl} = C_{klij}$, (4.5) gives

$$C_{1j1l} = 0 \qquad \text{for } j, l \neq 1 . \tag{4.6}$$

Thus, (4.3), (4.4), and (4.6) show that $C = 0$ at x. Since $\nabla C = 0$, we have $C = 0$ on M. This completes the proof of Theorem 6.

5. - Proof of Corollary and Theorem 7.

Let (M, g, J) be an almost Hermitian manifold with almost complex structure tensor J and an almost Hermitian metric tensor g (which is positive definite). J and g satisfy

$$JJX = -X, \tag{5.1}$$

$$g(JX, JY) = g(X, Y). \tag{5.2}$$

(M, g, J) is Kählerian, if and only if $\nabla J = 0$. Then Corollary follows from Theorem 1.

PROOF OF THEOREM 7. - By (3) of Theorem 2, it suffices to show that $\nabla C = 0$ implies $\nabla R = 0$. So, assume that a Kählerian manifold (M, g, J), $m \geqslant 4$, satisfies $\nabla C = 0$. It is known that $\nabla_r C^r{}_{jkl} = 0$ implies

$$\nabla_l R_{jk} - \nabla_k R_{jl} = [1/2(m-1)](g_{jk}\nabla_l S - g_{jl}\nabla_k S). \tag{5.3}$$

By Glodek's theorem or Theorem 6, we have either $C = 0$ or $S = \text{constant}$. If $C = 0$ in a Kählerian manifold, we have (cf. YANO and MOGI [10])

(A) for $m \geqslant 6$, (M, g, J) is locally flat,

(B) for $m = 4$, $S = \text{constant}$.

Therefore, in any case, we see that $S = \text{constant}$. (5.3), then, gives

$$\nabla_l R_{jk} = \nabla_k R_{jl}. \tag{5.4}$$

It is known that (cf. YANO and MOGI [10])

$$R_{jk} J^j_r J^k_s = R_{rs}. \tag{5.5}$$

Since $\nabla S = 0$, operating ∇_i to (5.5) we get

$$\nabla_i R_{jk} J^j_r J^k_s = \nabla_i R_{rs}. \tag{5.6}$$

Now we show that $\nabla_i R_{jk} = 0$. In fact,

$$\begin{aligned} \nabla_i R_{jk} &= \nabla_i R_{rs} J^r_j J^s_k && \text{by (5.6)} \\ &= \nabla_r R_{is} J^r_j J^s_k && \text{by (5.4)} \\ &= (\nabla_r R_{pq} J^p_i J^q_s) J^r_j J^s_k && \text{by (5.6)} \end{aligned}$$

$$= \nabla_q R_{pr} J^p_i J^q_s J^r_j J^s_k \qquad \text{by (5.4)}$$

$$= \nabla_q R_{ab} J^a_p J^b_r J^p_i J^q_s J^r_j J^s_k \qquad \text{by (5.6)}$$

$$= -\nabla_k R_{ij} \qquad \text{by (5.1)}\,.$$

Hence, using (5.4), we have $\nabla_i R_{jk} = 0$. $\nabla_i S = 0$, $\nabla_i R_{jk} = 0$, (4.1), and $\nabla_h C_{ijkl} = 0$ give $\nabla_h R_{ijkl} = 0$. Therefore, we have $\nabla R = 0$.

PROOF OF THEOREM 8 is contained in the above Proof of Theorem 7.

REFERENCES

[1] E. GLODEK, *Some remarks on conformally symmetric Riemannian spaces*, Colloq. Math., **23** (1971), pp. 121-123.

[2] A. LICHNEROWICZ, *Courbure, nombres de Betti, et espaces symétriques*, Proc. Int. Cong. Math., **2** (1952), pp. 216-223.

[3] A. LICHNEROWICZ, *Géometrie des groupes de transformations*, Paris, Dunod, 1958.

[4] M. MATSUMOTO, *On Riemannian spaces with recurrent projective curvature*, Tensor, N.S., **19** (1968), pp. 11-18.

[5] K. NOMIZU - H. OZEKI, *A theorem on curvature tensor fields*, Proc. Nat. Acad. Sci., **48** (1962), pp. 206-207.

[6] W. ROTER, *Some remarks on second order recurrent spaces*, Bull. Polon. Sci. Ser. Sci. Math. Astr. Phys., **12** (1964), pp. 207-211.

[7] S. TANNO, *Strongly curvature-preserving transformations of pseudo-Riemannian manifolds*, Tôhoku Math. Journ., **19** (1967), pp. 245-250.

[8] S. TANNO, *Transformations of pseudo-Riemannian manifolds*, Journ. Math. Soc. Japan, **21** (1969), pp. 270-281.

[9] H. WU, *On the de Rham decomposition theorem*, Illinois Journ. Math., **8** (1964), pp. 291-311.

[10] K. YANO - I. MOGI, *On real representations of Kählerian manifolds*, Ann. Math., **61** (1955), pp. 170-189.

[11] L. P. EISENHART, *Symmetric tensors of the second order whose first covariant derives are zero*, Trans. Amer. Math. Soc., **25** (1923), pp. 297-306.