

CURVE COUNTING VIA STABLE OBJECTS IN DERIVED CATEGORIES OF CALABI-YAU 4-FOLDS

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ABSTRACT. In our previous paper with Maulik, we proposed a conjectural Gopakumar-Vafa (GV) type formula for the generating series of stable pair invariants on Calabi-Yau (CY) 4-folds. The purpose of this paper is to give an interpretation of the above GV type formula in terms of wall-crossing phenomena in the derived category. We introduce invariants counting LePotier's stable pairs on CY 4-folds, and show that they count certain stable objects in D0-D2-D8 bound states in the derived category. We propose a conjectural wall-crossing formula for the generating series of our invariants, which recovers the conjectural GV type formula. Examples are computed for both compact and toric cases to support our conjecture.

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0. INTRODUCTION

0.1. Background on GV/PT formula on CY 3-folds. The notion of stable pairs was introduced by Pandharipande-Thomas (PT) [PT09] in order to give a better formulation of Maulik-Nekrasov-Okounkov-Pandharipande (MNOP) conjecture [MNOP06] relating Gromov-Witten (GW) invariants and Donaldson-Thomas (DT) curve counting invariants on Calabi-Yau (CY) 3-folds. By definition, a stable pair on a variety X consists of a pair

$$(0.1) \quad (F, s), \quad s: \mathcal{O}_X \rightarrow F$$

satisfying PT stability condition: F is a pure one dimensional sheaf and s is surjective in dimension one. When X is a CY 3-fold, we have integer valued invariants $P_{n,\beta} \in \mathbb{Z}$ (called PT invariants) which virtually count stable pairs (0.1) satisfying $([F], \chi(F)) = (\beta, n)$. The generating series

$$(0.2) \quad \text{PT}(X) = \sum_{n,\beta} P_{n,\beta} y^n q^\beta$$

is conjectured to be equal to the generating series of GW invariants under some variable change, which was proved by Pandharipande-Pixton [PP17] in many cases including quintic 3-folds.

On the other hand, the generating series (0.2) on a CY 3-fold is expected to be written as an infinite product (see e.g. [Tod12, Conj. 6.2]) with powers given by Gopakumar-Vafa (GV) invariants $n_{g,\beta} \in \mathbb{Z}$ [GV, MT18]:

$$(0.3) \quad \text{PT}(X) = \prod_{\beta} \left(\prod_{j=1}^{\infty} (1 - (-y)^j q^\beta)^{jn_{0,\beta}} \cdot \prod_{g=1}^{\infty} \prod_{k=0}^{2g-2} (1 - (-y)^{g-1-k} q^\beta)^{(-1)^{k+g} n_{g,\beta} \binom{2g-2}{k}} \right).$$

In fact, such an infinite product can be explained from wall-crossing phenomena. In the second author's previous works [Tod09, Tod10a, Tod10b, Tod12], he investigated wall-crossing phenomena of stable D0-D2-D6 bound states in the derived category of coherent sheaves, by introducing one parameter family of weak stability conditions on them. The wall-crossing formula of associated DT counting invariants can be studied using the works of Joyce-Song [JS12] and Kontsevich-Soibelman [KS]. As a result, it turned out that the factor $\prod_{j=1}^{\infty} (1 - (-y)^j q^\beta)^{jn_{0,\beta}}$ is the wall-crossing term (up to showing multiple cover conjecture of Joyce-Song's generalized DT invariants [JS12] for one dimensional semistable sheaves), so giving an intrinsic understanding of the GV formula (0.3) via wall-crossing.

0.2. Motivation on GV/PT formula on CY 4-folds. The purpose of this paper is to give a similar interpretation of GV formula for stable pair invariants on CY 4-folds, using DT_4 virtual classes defined in general cases by Borisov-Joyce [BJ] and in special cases by Cao-Leung [CL14]. In our previous paper with Maulik [CMT19], we studied stable pair invariants on CY 4-folds X and conjectured that the generating series of these invariants with exponential insertions $\exp(\gamma)$ for $\gamma \in H^4(X, \mathbb{Z})$ is written as a similar infinite product

$$(0.4) \quad \text{PT}(X)(\exp(\gamma)) = \prod_{\beta} \left(\exp(yq^\beta)^{n_{0,\beta}(\gamma)} \cdot M(q^\beta)^{n_{1,\beta}} \right).$$

Here $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function, and the invariants

$$n_{0,\beta}(\gamma) \in \mathbb{Q}, \quad n_{1,\beta} \in \mathbb{Q}$$

are GV type invariants on CY 4-folds defined by Klemm-Pandharipande [KP] from GW invariants on CY 4-folds, which are conjectured to be integers.

We will consider a family of LePotier stability conditions on pairs (0.1), parametrized by a stability parameter $t \in \mathbb{R}$, and construct DT_4 type invariants counting such pairs. Indeed our invariants count certain stable objects in the category of D0-D2-D8 bound states, which is an abelian subcategory in the derived category of coherent sheaves. Here we observe a new phenomenon for CY 4-folds: stable objects on D0-D2-D8 bound states on CY 4-folds are always written as a pair (0.1) while this is not the case for stable D0-D2-D6 bound states on CY 3-folds. We then propose a conjectural wall-crossing formula of generating series of our invariants, and explain that it recovers the GV formula (0.4).

0.3. DT_4 type invariants counting LePotier stable pairs. Let (X, ω) be a polarized smooth projective CY 4-fold over \mathbb{C} . For $t \in \mathbb{R}$, a pair (0.1) for a pure one dimensional sheaf F is called Z_t -stable if the following conditions holds (here we denote $\mu(F) = \chi(F)/(\omega \cdot [F])$):

- (i) for any subsheaf $0 \neq F' \subseteq F$, we have $\mu(F') < t$,
- (ii) for any subsheaf $F' \subsetneq F$ such that s factors through F' , we have $\mu(F/F') > t$.

Indeed we will see that the above stability condition is a special case of LePotier's stability conditions [Pot93] (ref. Proposition 1.8). For a given $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, we denote by

$$P_n^t(X, \beta)$$

the moduli space of Z_t -stable pairs (F, s) with $([F], \chi) = (\beta, n)$. It has a wall-chamber structure and for a generic $t \in \mathbb{R}$ (generic means outside a finite subset of \mathbb{R}), the above moduli space is a projective scheme, and coincides with the moduli space of PT stable pairs for the $t \rightarrow \infty$ limit.

An important question is whether we can define DT_4 type counting invariants of $P_n^t(X, \beta)$. When $t \rightarrow \infty$, this was done in [CMT19] by using the well-known fact that moduli spaces of PT stable pairs can be regarded as moduli spaces of objects in derived categories of coherent sheaves [PT09] (note that the natural pair deformation-obstruction theory of PT moduli spaces $P_n(X, \beta)$ does not seem to give rise to a virtual class even when X is 3-dimensional). Therefore we are allowed to apply Pantev-Töen-Vaquière-Vezzosi's construction of (-2) -shifted symplectic structures [PTVV13] and Borisov-Joyce's virtual classes [BJ].

Our first result is that the moduli space $P_n^t(X, \beta)$ is indeed a moduli space of two term complexes in the derived category for any choice of $t \in \mathbb{R}$:

Theorem 0.1. (Proposition 1.3, Theorem 1.4) *Let \mathcal{M}_0 be the moduli stack of $E \in D^b(\text{Coh}(X))$ satisfying $\text{Ext}^{<0}(E, E) = 0$ and $\det E \cong \mathcal{O}_X$. Then the natural morphism*

$$P_n^t(X, \beta) \rightarrow \mathcal{M}_0, \quad (F, s) \mapsto (\mathcal{O}_X \xrightarrow{s} F)$$

is an open immersion.

As we mentioned in the previous subsection, this is a new phenomenon for CY 4-folds, as the similar statement is not true for CY 3-folds except the $t \rightarrow \infty$ limit, i.e. PT stable pairs. Indeed the moduli space $P_n^t(X, \beta)$ is regarded as a moduli space of certain stable objects in the extension closure

$$\mathcal{A}_X := \langle \mathcal{O}_X, \text{Coh}_{\leq 1}(X)[-1] \rangle_{\text{ext}} \subset D^b(\text{Coh}(X)),$$

called the category of *D0-D2-D8 bound states*. We will show that the above category is equivalent to the category of pairs $(\mathcal{V} \rightarrow F)$, where \mathcal{V} is an iterated extensions of \mathcal{O}_X (ref. Proposition 1.1). The above mentioned equivalence is not true for CY 3-folds, as we need the vanishing $\text{Ext}^2(F, \mathcal{O}_X) = 0$ for any one dimensional sheaf F .

Thanks to Theorem 0.1, we are able to define a virtual class (ref. Theorem 2.2)

$$[P_n^t(X, \beta)]^{\text{vir}} \in H_{2n}(P_n(X, \beta), \mathbb{Z}),$$

depending on the choice of orientation on certain real line bundle on it [CGJ]. In order to define invariants from the above virtual class, we need to involve insertions: define the map

$$\tau: H^4(X, \mathbb{Z}) \rightarrow H^2(P_n^t(X, \beta), \mathbb{Z}), \quad \tau(\gamma) := (\pi_P)_*(\pi_X^* \gamma \cup \text{ch}_3(\mathbb{F})),$$

where π_X, π_P are projections from $X \times P_n^t(X, \beta)$ onto corresponding factors, $\mathbb{F} = (\pi_X^* \mathcal{O}_X \rightarrow \mathbb{F})$ is the universal pair and $\text{ch}_3(\mathbb{F})$ is the Poincaré dual to the fundamental cycle of \mathbb{F} .

For a generic $t \in \mathbb{R}$, we define the Z_t -stable pair invariant by

$$P_{n,\beta}^t(\gamma) := \int_{[P_n^t(X,\beta)]^{\text{vir}}} \tau(\gamma)^n \in \mathbb{Z}.$$

Here we also write $P_{0,\beta}^t := P_{n,\beta}^t(\gamma)$ when $n = 0$.

When $t \rightarrow \infty$, Z_t -stable pairs are PT stable pairs. So we denote

$$(0.5) \quad P_{n,\beta}(\gamma) := P_{n,\beta}^t(\gamma)|_{t \rightarrow \infty},$$

which is nothing but the stable pair invariant on CY 4-fold X studied in [CMT19].

0.4. Conjectures. The main conjecture of this paper is the following:

Conjecture 0.2. (Conjecture 3.4) *Let (X, ω) be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}_{\geq 0}$. Choose a generic $t \in \mathbb{R}_{>0}$. Then for certain choice of orientation, we have*

$$(0.6) \quad P_{n,\beta}^t(\gamma) = \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_n = \beta \\ \omega \cdot \beta_i > \frac{1}{2}, i=1, \dots, n}} P_{0,\beta_0} \cdot \prod_{i=1}^n n_{0,\beta_i}(\gamma).$$

In particular, $P_{0,\beta}^t = P_{0,\beta}$ is independent of the choice of $t > 0$.

As in the previous paper [CMT19], the conjecture is based on a heuristic argument given in Section 4.1, where we verify it assuming the CY 4-fold X to be ‘ideal’, i.e. curves in X deform in some family of expected dimensions and have expected generic properties.

The conjectural formula (0.6) can be expressed in terms of generating series as follows. Set

$$\mathrm{PT}^t(X)(\exp(\gamma)) := \sum_{n,\beta} \frac{P_{n,\beta}^t(\gamma)}{n!} y^n q^\beta.$$

Then for each $t_0 \in \mathbb{R}_{>0}$, the formula (0.6) implies the wall-crossing formula

$$(0.7) \quad \lim_{t \rightarrow t_0^+} \mathrm{PT}^t(X)(\exp(\gamma)) = \prod_{\omega \cdot \beta = \frac{1}{t_0}} \exp(yq^\beta)^{n_{0,\beta}(\gamma)} \cdot \lim_{t \rightarrow t_0^-} \mathrm{PT}^t(X)(\exp(\gamma)).$$

In the $t \rightarrow \infty$ limit,

$$\mathrm{PT}(X)(\exp(\gamma)) = \lim_{t \rightarrow \infty} \mathrm{PT}^t(X)(\exp(\gamma))$$

is the generating series of PT stable pair invariants on X by (0.5).

In the $t \rightarrow +0$ limit, by (0.6), we have

$$\lim_{t \rightarrow +0} \mathrm{PT}^t(X)(\exp(\gamma)) = \sum_{\beta} P_{0,\beta} q^\beta = \prod_{\beta} M(q^\beta)^{n_{1,\beta}},$$

where the second identity is conjectured in [CMT19, Conj. 1.2]. Therefore the wall-crossing formula from $t \rightarrow +0$ to $t \rightarrow \infty$ recovers the conjectural GV formula (0.4) of stable pair invariants on CY 4-folds, giving an interpretation of (0.4) in terms of wall-crossing in the derived category.

A particularly interesting choice of $t \in \mathbb{R}$ is $t = n/(\omega \cdot \beta) + 0$, which sits in the first non-trivial chamber for a fixed β and n . In this case, the moduli space

$$P_n^t(X, \beta), \quad t = \frac{n}{\omega \cdot \beta} + 0$$

is the moduli space of ‘Joyce-Song type’ stable pairs, i.e. one dimensional semistable sheaves with sections satisfying certain property (ref. Definition 1.10, compared with [JS12, Def. 12.2] which deals with pairs from a very negative line bundle instead of \mathcal{O}_X). We define JS stable pair invariant by

$$P_{n,\beta}^{\mathrm{JS}}(\gamma) := \lim_{t \rightarrow n/(\omega \cdot \beta) + 0} P_{n,\beta}^t(\gamma).$$

Then Conjecture 0.2 in particular implies the following:

Conjecture 0.3. (Conjecture 3.4) *In the same situation of Conjecture 0.2, we have the identity*

$$(0.8) \quad (1) P_{n,\beta}^{\mathrm{JS}}(\gamma) = \sum_{\substack{\beta_1 + \dots + \beta_n = \beta \\ \omega \cdot \beta_i = \frac{\omega \cdot \beta}{n}, i=1, \dots, n}} \prod_{i=1}^n n_{0,\beta_i}(\gamma), \text{ if } n \geq 1, \quad (2) P_{0,\beta}^{\mathrm{JS}} = P_{0,\beta}.$$

In particular, $P_{1,\beta}^{\mathrm{JS}}(\gamma) = n_{0,\beta}(\gamma)$.

Remarkably, JS stable pair invariants (based on above conjecture) also encode information of all genus GV type invariants.

0.5. Verifications of conjectures. In Section 4.2, we give a ‘heuristic’ master space argument for Conjecture 0.2 in the case of simple wall-crossing. There we discuss the construction of master spaces and virtual classes locally and ‘heuristic’ means we assume they extend to the global moduli spaces. We show that the wall-crossing formula given by the master space heuristics coincides with the formula in our main conjecture (Proposition 4.1).

Besides that, in Section 5, we check Conjecture 0.2 or Conjecture 0.3 in several examples: for some compact CY 4-folds (sextic 4-fold, elliptic fibered CY 4-fold), local Fano 3-folds, local surfaces ($\mathrm{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$, $\mathrm{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1)^{\oplus 2})$). The most important check among them is the comparison between JS stable pair invariants $P_{1,\beta}^{\mathrm{JS}}(\gamma)$ and DT_4 invariants counting one dimensional stable sheaves.

Theorem 0.4. (Theorem 5.7, 5.9, 5.13) *Suppose that X is either $Y \times E$ where Y is a projective CY 3-fold and E is an elliptic curve, or $\mathrm{Tot}_Y(K_Y)$ for a Fano 3-fold Y . Then for any curve class $\beta \in H_2(Y) \subseteq H_2(X)$, we have the identity*

$$(0.9) \quad P_{1,\beta}^{\mathrm{JS}}(\gamma) = \int_{[M_1(X,\beta)]^{\mathrm{vir}}} \tau(\gamma),$$

for certain choice of orientation. Here $M_1(X, \beta)$ is the moduli space of one dimensional stable sheaves F on X with $([F], \chi(F)) = (\beta, 1)$.

The right hand side of (0.9) is conjectured to be equal to $n_{0,\beta}(\gamma)$ in [CMT18] (see also Conjecture 3.1). If this is the case, Theorem 0.4 implies Conjecture 0.3 in the case of $X = Y \times E$, $X = \text{Tot}_Y(K_Y)$ with $\beta \in H_2(Y) \subseteq H_2(X)$ (see Corollary 5.8, 5.10).

Apart from them, we also study Conjecture 0.3 for local \mathbb{P}^1 , i.e.

$$X = \text{Tot}_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

In this case, the four dimensional complex torus $(\mathbb{C}^*)^4$ acts on X , and we denote by $T \subset (\mathbb{C}^*)^4$ the subtorus preserving the CY 4-form. We will define the T -equivariant JS stable pair invariant by (see Definition 6.7):

$$P_{n,d}^{\text{JS}} := \sum_{I \in P_n^{\text{JS}}(X,d[\mathbb{P}^1])^T} (-1)^{d+1} e_T(\chi_X(I, I)_0^{\frac{1}{2}}) \in \frac{\mathbb{Q}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)}{(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)}.$$

Here we make a particular choice of square root $\chi_X(I, I)_0^{\frac{1}{2}}$ as in Lemma 6.6 and the sign $(-1)^{d+1}$ denotes a choice of orientation to normalize the expression. We will give an explicit computation of the above invariant.

Theorem 0.5. (Theorem 6.8) *We have:*

$$P_{n,d}^{\text{JS}} = \frac{(-1)^{k(d+1)}}{1! 2! \dots k!} \cdot \frac{1}{\lambda_0^{k(k+1)/2} \lambda_3^d} \cdot \sum_{\substack{d_0 + \dots + d_k = d \\ d_0, \dots, d_k \geq 0}} \frac{1}{d_0! \dots d_k!} \cdot \prod_{\substack{i < j \\ 0 \leq i, j \leq k}} \left((j-i)\lambda_0 + (d_i - d_j)\lambda_3 \right) \\ \times \prod_{i=0}^k \left(\prod_{\substack{1 \leq a \leq d_i \\ 1 \leq b \leq k-i}} \frac{1}{a\lambda_3 + b\lambda_0} \cdot \prod_{\substack{1 \leq a \leq d_i \\ 1 \leq b \leq i}} \frac{1}{a\lambda_3 - b\lambda_0} \right), \quad \text{if } n = d(k+1), \quad k \geq 0,$$

and $P_{n,d}^{\text{JS}} = 0$ otherwise.

The formula in Theorem 0.5 is complicated, but we expect significant cancellations of rational functions. Indeed as an analogy of Conjecture 0.3, we should have the identities:

$$(0.10) \quad P_{n,d}^{\text{JS}} = \begin{cases} \frac{1}{d!(\lambda_3)^d}, & n = d, \\ 0, & n \neq d. \end{cases}$$

By an residue argument and a ‘Mathematica’ program, we show the following:

Theorem 0.6. (Theorem 6.10) *The identity (0.10) holds in the following cases*

- $d \nmid n$,
- $d = 1, 2$ with any n ,
- $n = d, 2d$ with any d .

The identity (0.10) is also checked in many other cases by Mathematica (see Proposition 6.11).

Remark 0.7. *Recently (0.10) has been proved in full generality in [CT20b].*

Finally we remark that one issue of the current proposal (this also happened in previous related works, e.g. [CMT19]) is that we do not have a general way to fix the choice of orientation in the virtual classes and invariants. Our choice of orientation in verifications is based on case by case studies. Nevertheless, we expect our wall-crossing interpretation in this paper will shed new light on this issue, i.e. we expect the choice of orientation on different moduli spaces should be compatible with wall-crossing. In fact, motivated by this, explicit choice of orientation for moduli spaces of PT stable pairs on K_Y (where Y is Fano 3-fold) is given in [CKM20, (1.5)] (at least when stable pairs are scheme theoretically supported on Y) and used to verify (0.4) in examples. We hope to explore this more in the future.

0.6. Notation and convention. In this paper, all varieties and schemes are defined over \mathbb{C} . For a morphism $\pi: X \rightarrow Y$ of schemes, and for $\mathcal{F}, \mathcal{G} \in \text{D}^b(\text{Coh}(X))$, we denote by $\mathbf{R}\mathcal{H}om_\pi(\mathcal{F}, \mathcal{G})$ the functor $\mathbf{R}\pi_* \mathbf{R}\mathcal{H}om_X(\mathcal{F}, \mathcal{G})$. We also denote by $\text{ext}^i(\mathcal{F}, \mathcal{G})$ the dimension of $\text{Ext}_X^i(\mathcal{F}, \mathcal{G})$.

A class $\beta \in H_2(X, \mathbb{Z})$ is called *irreducible* (resp. *primitive*) if it is not the sum of two non-zero effective classes (resp. if it is not a positive integer multiple of an effective class).

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1. DEFINITIONS

Throughout this paper, unless stated otherwise, (X, ω) is always denoted to be a smooth projective Calabi-Yau 4-fold (i.e. $K_X \cong \mathcal{O}_X$) with an ample divisor ω on it.

1.1. Category of D0-D2-D8 bound states. We define the category of D0-D2-D8 bound states on X to be the extension closure in the derived category

$$\mathcal{A}_X := \langle \mathcal{O}_X, \text{Coh}_{\leq 1}(X)[-1] \rangle_{\text{ext}} \subset \text{D}^b(\text{Coh}(X)).$$

Here $\text{Coh}_{\leq 1}(X)$ is the category of coherent sheaves F on X whose support have dimension less than or equal to one. The argument in [Tod10a, Lem. 3.5] shows that \mathcal{A}_X is the heart of a bounded t-structure on the triangulated subcategory of $\text{D}^b(\text{Coh}(X))$ generated by \mathcal{O}_X and $\text{Coh}_{\leq 1}(X)$. In particular, \mathcal{A}_X is an abelian category.

We also define the category \mathcal{B}_X , whose objects consist of triples

$$(\mathcal{V}, F, s), \quad \mathcal{V} \in \langle \mathcal{O}_X \rangle_{\text{ext}}, \quad F \in \text{Coh}_{\leq 1}(X), \quad s: \mathcal{V} \rightarrow F.$$

Note that if $H^1(\mathcal{O}_X) = 0$, the vector bundle \mathcal{V} is of the form $V \otimes \mathcal{O}_X$ for a finite dimensional vector space V . The set of morphisms in \mathcal{B}_X is given by commutative diagrams of coherent sheaves

$$(1.1) \quad \begin{array}{ccc} \mathcal{V} & \xrightarrow{s} & F \\ \alpha \downarrow & & \downarrow \beta \\ \mathcal{V}' & \xrightarrow{s'} & F'. \end{array}$$

We compare the categories \mathcal{A}_X and \mathcal{B}_X in the following proposition:

Proposition 1.1. *There exists a natural equivalence of categories*

$$(1.2) \quad \Phi: \mathcal{B}_X \xrightarrow{\sim} \mathcal{A}_X.$$

Proof. For an object $E = (\mathcal{V}, F, s)$ in \mathcal{B}_X , we have the associated two term complex $\Phi(E) = (\mathcal{V} \xrightarrow{s} F) \in \text{D}^b(\text{Coh}(X))$, where \mathcal{V} is located in degree zero. By the distinguished triangle

$$F[-1] \rightarrow \Phi(E) \rightarrow \mathcal{V},$$

the object $\Phi(E)$ lies in \mathcal{A}_X , hence we obtain the functor (1.2). Indeed the above sequence is a short exact sequence in the abelian category \mathcal{A}_X . Below we show that Φ is an equivalence along with the argument of [Tod10c, Prop. 2.2].

We first show that Φ is fully-faithful. Let us take another triple $E' = (\mathcal{V}', F', s')$, and take a morphism $\gamma: \Phi(E) \rightarrow \Phi(E')$ in \mathcal{A}_X . By the Serre duality, we have the vanishing $\text{Hom}(F[-1], \mathcal{V}') = \text{Ext}^3(\mathcal{V}', F)^\vee = 0$, hence we have the unique morphisms (α, β) which make the following diagram commutative

$$(1.3) \quad \begin{array}{ccccc} F[-1] & \longrightarrow & \Phi(E) & \longrightarrow & \mathcal{V} \\ \beta \downarrow \cdots & & \downarrow \gamma & & \downarrow \alpha \cdots \\ F'[-1] & \longrightarrow & \Phi(E') & \longrightarrow & \mathcal{V}'. \end{array}$$

By taking cones, we obtain the diagram (1.1). Conversely given a diagram (1.1), there is a morphism γ which makes the diagram (1.3) commutative. Because of $\text{Hom}(\mathcal{V}, F'[-1]) = 0$, such γ is uniquely determined. Therefore the functor Φ is fully-faithful.

It remains to show that the functor Φ is essentially surjective. For an object $M \in \mathcal{A}_X$, by the definition of \mathcal{A}_X , there is a filtration

$$M_0 \subset M_1 \subset \cdots \subset M_k = M,$$

such that each $N_i = M_i/M_{i-1}$ is isomorphic to \mathcal{O}_X or an object in $\text{Coh}_{\leq 1}(X)[-1]$. We show that, by the induction on j , each M_j is isomorphic to an object of the form $\Phi(E_j)$ for an object

$E_j = (\mathcal{V}_j \rightarrow F_j)$ in \mathcal{B}_X . The case of $j = 0$ is obvious. Suppose that M_{j-1} is isomorphic to $\Phi(E_{j-1})$. If $N_j = \mathcal{O}_X$, then by taking the cones of the commutative diagram

$$\begin{array}{ccccc} & & \mathcal{O}_X[-1] & & \\ & & \downarrow & \searrow & \\ F_{j-1}[-1] & \longrightarrow & M_{j-1} & \longrightarrow & \mathcal{V}_{j-1}, \end{array}$$

we obtain the exact sequences in \mathcal{A}_X

$$0 \rightarrow F_{j-1}[-1] \rightarrow M_j \rightarrow \mathcal{V}_j \rightarrow 0, \quad 0 \rightarrow \mathcal{V}_{j-1} \rightarrow \mathcal{V}_j \rightarrow \mathcal{O}_X \rightarrow 0.$$

Therefore M_j is isomorphic to $\Phi(\mathcal{V}_j \rightarrow F_{j-1})$. If $N_j = F[-1]$ for $F \in \text{Coh}_{\leq 1}(X)[-1]$, we have a commutative diagram

$$\begin{array}{ccccc} & & F[-2] & & \\ & \swarrow & \downarrow & & \\ F_{j-1}[-1] & \longrightarrow & M_{j-1} & \longrightarrow & \mathcal{V}_{j-1}, \end{array}$$

since $\text{Hom}(F[-2], \mathcal{V}_{j-1}) = H^2(\mathcal{V}_{j-1}^\vee \otimes F)^\vee = 0$. By taking cones, we obtain exact sequences in \mathcal{A}_X :

$$0 \rightarrow F_j[-1] \rightarrow M_j \rightarrow \mathcal{V}_{j-1} \rightarrow 0, \quad 0 \rightarrow F_{j-1}[-1] \rightarrow F_j[-1] \rightarrow F[-1] \rightarrow 0.$$

Therefore M_j is isomorphic to $\Phi(\mathcal{V}_{j-1} \rightarrow F_j)$. \square

Below we will be interested in objects in \mathcal{A}_X with Chern character of the following form

$$(1.4) \quad v = (1, 0, 0, -\beta, -n) \in H^0(X) \oplus H^2(X) \oplus H^4(X) \oplus H^6(X) \oplus H^8(X).$$

We also identify β with an element in $H_2(X)$ by Poincaré duality. Note that for an object $(\mathcal{V} \rightarrow F)$ in \mathcal{B}_X , we have

$$\text{ch} \Phi(\mathcal{V} \rightarrow F) = v \Leftrightarrow \mathcal{V} = \mathcal{O}_X, \quad ([F], \chi(F)) = (\beta, n).$$

Here $[F] \in H_2(X, \mathbb{Z})$ is the fundamental one cycle of F .

We relate objects in \mathcal{A}_X with Chern character of the form (1.4) to objects in a tilting of $\text{Coh}(X)$ with respect to the slope stability. For $E \in \text{Coh}(X)$, with respect to the ample divisor ω on X , we set

$$\widehat{\mu}_\omega(E) = \frac{c_1(E) \cdot \omega^3}{\text{rank}(E)} \in \mathbb{Q} \cup \{\infty\}.$$

As usual, an object $E \in \text{Coh}(X)$ is called $\widehat{\mu}_\omega$ -semistable if for any non-zero subsheaf $E' \subset E$, we have $\widehat{\mu}_\omega(E') \leq \widehat{\mu}_\omega(E)$. We define subcategories of $\text{Coh}(X)$:

$$\begin{aligned} \mathcal{T}_\omega &= \langle \widehat{\mu}_\omega\text{-semistable } E \text{ with } \widehat{\mu}_\omega(E) > 0 \rangle_{\text{ext}}, \\ \mathcal{F}_\omega &= \langle \widehat{\mu}_\omega\text{-semistable } E \text{ with } \widehat{\mu}_\omega(E) \leq 0 \rangle_{\text{ext}}. \end{aligned}$$

By the existence of Harder-Narasimhan filtrations, the pair of subcategories $(\mathcal{T}_\omega, \mathcal{F}_\omega)$ forms a torsion pair of $\text{Coh}(X)$. By taking the tilting [HRS96], we obtain the heart of a bounded t-structure

$$\widehat{\mathcal{A}}_X = \langle \mathcal{F}_\omega, \mathcal{T}_\omega[-1] \rangle_{\text{ext}} \subset \text{D}^b(\text{Coh}(X)).$$

Note that we have $\mathcal{A}_X \subset \widehat{\mathcal{A}}_X$ by their definitions.

Lemma 1.2. *Let $v \in H^*(X)$ be of the form (1.4). For an object $E \in \text{D}^b(\text{Coh}(X))$ with $\text{ch}(E) = v$ and $\det(E) \cong \mathcal{O}_X$, it is an object in \mathcal{A}_X if and only if it is an object in $\widehat{\mathcal{A}}_X$.*

Proof. Since $\mathcal{A}_X \subset \widehat{\mathcal{A}}_X$, it is enough to show that an object $E \in \widehat{\mathcal{A}}_X$ with $\text{ch}(E) = v$ and $\det(E) \cong \mathcal{O}_X$ is an object in \mathcal{A}_X . We have an exact sequence in $\widehat{\mathcal{A}}_X$

$$(1.5) \quad 0 \rightarrow \mathcal{H}^0(E) \rightarrow E \rightarrow \mathcal{H}^1(E)[-1] \rightarrow 0,$$

such that $\mathcal{H}^0(E) \in \mathcal{F}_\omega$ and $\mathcal{H}^1(E) \in \mathcal{T}_\omega$. Since

$$\text{ch}_1(\mathcal{H}^0(E)) \cdot \omega^3 \leq 0, \quad \text{ch}_1(\mathcal{H}^1(E)[-1]) \cdot \omega^3 \leq 0,$$

and their sum is zero, we have $\text{ch}_1(\mathcal{H}^i(E)) \cdot \omega^3 = 0$ for $i = 0, 1$. It follows that $\mathcal{H}^0(E)$ is a μ_ω -semistable sheaf and $\mathcal{H}^1(E) \in \text{Coh}_{\leq 2}(X)$. Moreover we have

$$\text{ch}_2(\mathcal{H}^0(E)) \cdot \omega^2 \leq 0, \quad \text{ch}_2(\mathcal{H}^1(E)[-1]) \cdot \omega^2 \leq 0,$$

where the first inequality follows from the Bogomolov-Gieseker inequality. As their sum is also zero, we have $\text{ch}_2(\mathcal{H}^i(E)) \cdot \omega^2 = 0$ for $i = 0, 1$. Therefore $\mathcal{H}^1(E)[-1] \in \text{Coh}_{\leq 1}(X)[-1]$. As for $\mathcal{H}^0(E)$, since it is of rank one and has trivial determinant, we have the exact sequence of coherent sheaves

$$0 \rightarrow \mathcal{H}^0(E) \rightarrow \mathcal{H}^0(E)^{\vee\vee} \cong \mathcal{O}_X \rightarrow T \rightarrow 0,$$

for some $T \in \text{Coh}_{\leq 2}(X)$. The vanishing of $\text{ch}_2(\mathcal{H}^0(E)) \cdot \omega^2$ implies that $T \in \text{Coh}_{\leq 1}(X)$, so we have $\mathcal{H}^0(E) \in \mathcal{A}_X$. From the exact sequence (1.5), we conclude $E \in \mathcal{A}_X$. \square

1.2. Moduli stacks of objects on \mathcal{A}_X . Let \mathcal{M} be the 2-functor

$$\mathcal{M}: \text{Sch}/\mathbb{C} \rightarrow \text{Groupoid},$$

which sends a \mathbb{C} -scheme T to the groupoid of perfect complexes \mathcal{E} on $X \times T$ such that the derived restriction $\mathcal{E}_t = \mathcal{E}|_{X \times \{t\}}$ for each closed point $t \in T$ is an object in $\text{D}^b(\text{Coh}(X))$ satisfying $\text{Ext}^{<0}(\mathcal{E}_t, \mathcal{E}_t) = 0$. By a result of Lieblich [Lie06], the 2-functor \mathcal{M} is an Artin stack locally of finite type.

By taking the determinant of \mathcal{E} , we have a morphism of stacks

$$\det: \mathcal{M} \rightarrow [\text{Pic}(X)/\mathbb{C}^*].$$

We define the Artin stack \mathcal{M}_0 by the following Cartesian square

$$\begin{array}{ccc} \mathcal{M}_0 & \longrightarrow & \mathcal{M} \\ \downarrow & & \downarrow \det \\ \text{Spec } \mathbb{C} & \longrightarrow & [\text{Pic}(X)/\mathbb{C}^*]. \end{array}$$

Here the bottom arrow corresponds to the trivial line bundle \mathcal{O}_X . We have the decomposition into open and closed substacks

$$\mathcal{M}_0 = \coprod_{v \in H^*(X)} \mathcal{M}_0(v),$$

where $\mathcal{M}_0(v)$ parametrizes objects in $\text{D}^b(\text{Coh}(X))$ with Chern character v and trivial determinant. We have substacks

$$\underline{\mathcal{A}}_X(v) \subset \widehat{\mathcal{A}}_X(v) \subset \mathcal{M}_0(v),$$

where $\underline{\mathcal{A}}_X(v)$ (resp. $\widehat{\mathcal{A}}_X(v)$) parametrizes objects in \mathcal{A}_X (resp. $\widehat{\mathcal{A}}_X$), with Chern characters v and trivial determinant. We have the following proposition.

Proposition 1.3. *Suppose that v is of the form (1.4). Then we have $\underline{\mathcal{A}}_X(v) = \widehat{\mathcal{A}}_X(v)$, and they are open substacks of $\mathcal{M}_0(v)$.*

Proof. The identity $\underline{\mathcal{A}}_X(v) = \widehat{\mathcal{A}}_X(v)$ follows from Lemma 1.2. It remains to show that $\widehat{\mathcal{A}}_X(v)$ is an open substack of $\mathcal{M}_0(v)$. This can be proved literally following the proof of [Tod08, Lem. 4.7], where the similar statement is proved for K3 surfaces. Alternatively, the stacks of objects in $(\mathcal{T}_\omega, \mathcal{F}_\omega)$ determine the open stack of torsion theories in the sense of [AB13, App. A, Definition] on the moduli stack of objects in $\text{Coh}(X)$. Therefore the openness of $\widehat{\mathcal{A}}_X(v)$ follows from [AB13, Thm. A.3]. \square

Let us take v of the form (1.4). We define the moduli stack of objects in \mathcal{B}_X with Chern character v to be the 2-functor

$$\underline{\mathcal{B}}_X(v): \text{Sch}/\mathbb{C} \rightarrow \text{Groupoid},$$

which sends a \mathbb{C} -scheme T to the groupoid of pairs $(\mathcal{O}_{X \times T} \xrightarrow{s} \mathcal{F})$, where \mathcal{F} is a flat family of objects in $\text{Coh}_{\leq 1}(X)$ such that $(\mathcal{O}_X \rightarrow \mathcal{F}_t)$ has Chern character v for any closed point $t \in T$. The isomorphisms in $\underline{\mathcal{B}}_X(v)$ are given by commutative diagrams

$$\begin{array}{ccc} \mathcal{O}_{X \times T} & \xrightarrow{s} & \mathcal{F} \\ \parallel & & \downarrow \cong \\ \mathcal{O}_{X \times T} & \xrightarrow{s'} & \mathcal{F}'. \end{array}$$

We have a morphism of stacks

$$(1.6) \quad \Phi: \underline{\mathcal{B}}_X(v) \rightarrow \underline{\mathcal{A}}_X(v),$$

by sending pairs $(\mathcal{O}_{X \times T} \xrightarrow{s} \mathcal{F})$ to the associated two term complexes.

Theorem 1.4. *The morphism of stacks (1.6) is an isomorphism of stacks.*

Proof. By Proposition 1.1, the morphism (1.6) induces an equivalence of groupoid of \mathbb{C} -valued points of $\underline{\mathcal{B}}_X(v)$ and $\underline{\mathcal{A}}_X(v)$. It is enough to show that the infinitesimal deformation theories of \mathbb{C} -valued points in $\underline{\mathcal{B}}_X(v)$ and $\underline{\mathcal{A}}_X(v)$ are equivalent. Namely, let R_0 be an Artinian local \mathbb{C} -algebra and $0 \rightarrow I \rightarrow R \rightarrow R_0 \rightarrow 0$ a square zero extension. Take a R_0 -valued point

$$(1.7) \quad (\mathcal{O}_{X \times \text{Spec } R_0} \rightarrow \mathcal{F}_0) \in \underline{\mathcal{B}}_X(v)(\text{Spec } R_0).$$

Suppose that the associated two term complex

$$\mathcal{E}_0 = \Phi(\mathcal{O}_{X \times \text{Spec } R_0} \rightarrow \mathcal{F}_0) \in \underline{\mathcal{A}}_X(v)(\text{Spec } R_0)$$

extends to a R -valued point $\mathcal{E} \in \underline{\mathcal{A}}_X(v)(\text{Spec } R)$. Then we show that there is an extension of (1.7) to a R -valued point of $\underline{\mathcal{B}}_X(v)$, unique up to isomorphisms, and corresponds to \mathcal{E} under Φ .

Let $\mathfrak{m} \subset R_0$ be the maximal ideal, $F = \mathcal{F}_0 \otimes_{R_0} R_0/\mathfrak{m}$ and take $E = \mathcal{E}_0 \otimes_{R_0} R_0/\mathfrak{m} = \Phi(\mathcal{O}_X \rightarrow F)$. From the distinguished triangle $F[-1] \rightarrow E \rightarrow \mathcal{O}_X$, we have the following commutative diagram

$$\begin{array}{ccccc} & & \mathbf{R}\text{Hom}(\mathcal{O}_X, \mathcal{O}_X)[1] & \xlongequal{\quad} & \mathbf{R}\text{Hom}(\mathcal{O}_X, \mathcal{O}_X)[1] \\ & & \downarrow & & \downarrow \\ \mathbf{R}\text{Hom}(E, F) & \longrightarrow & \mathbf{R}\text{Hom}(E, E)[1] & \longrightarrow & \mathbf{R}\text{Hom}(E, \mathcal{O}_X)[1] \\ & & \downarrow & & \downarrow \\ & & \mathbf{R}\text{Hom}(E, E)_0[1] & \longrightarrow & \mathbf{R}\text{Hom}(F, \mathcal{O}_X)[2]. \end{array}$$

Here $(-)_0$ means taking the traceless part. From the above diagram, we obtain a distinguished triangle

$$\mathbf{R}\text{Hom}(E, F) \rightarrow \mathbf{R}\text{Hom}(E, E)_0[1] \rightarrow \mathbf{R}\text{Hom}(F, \mathcal{O}_X)[2].$$

Since X is a Calabi-Yau 4-fold, we have the following vanishing by Serre duality:

$$\text{Hom}(F, \mathcal{O}_X[1]) = H^3(X, F)^\vee = 0, \quad \text{Hom}(F, \mathcal{O}_X[2]) = H^2(X, F)^\vee = 0.$$

Therefore we have an isomorphism and an injection:

$$(1.8) \quad \text{Hom}(E, F) \xrightarrow{\cong} \text{Ext}^1(E, E)_0, \quad \text{Ext}^1(E, F) \hookrightarrow \text{Ext}^2(E, E)_0.$$

From the deformation-obstruction theory of pairs, the obstruction class of extending (1.7) to a R -valued point of $\underline{\mathcal{B}}_X(v)$ lies in $\text{Ext}^1(E, F) \otimes I$. Its image under the map $\text{Ext}^1(E, F) \otimes I \rightarrow \text{Ext}^2(E, E)_0 \otimes I$ is the obstruction class extending \mathcal{E}_0 to a R -valued point of $\underline{\mathcal{A}}_X(v)$, which vanishes as we assumed \mathcal{E}_0 extends to \mathcal{E} . Thus by the injectivity of the right map in (1.8), the obstruction class of extending (1.7) vanishes, hence it extends to a R -valued point of $\underline{\mathcal{B}}_X(v)$. Moreover all possible extensions of (1.7) form $\text{Hom}(E, F) \otimes I$ -torsor, and those of \mathcal{E}_0 form $\text{Ext}^1(E, E)_0 \otimes I$ -torsor. Therefore the uniqueness also holds by the left isomorphism in (1.8). \square

1.3. Moduli spaces of Z_t -stable pairs. We recall Le Potier's stability (such pairs were studied in low dimensions by S. Bradlow, M. Thaddeus and A. Bertram, etc.) for pairs on X (ref. [Pot93], [PT09, Sect. 1.1], [JS12, pp. 164]). Let us fix an ample line bundle $\mathcal{O}_X(1)$ on X with corresponding divisor ω . For a coherent sheaf F on X , its Hilbert polynomial is defined by

$$\chi(F(m)) = \sum_{i \in \mathbb{Z}} (-1)^i \dim H^i(F(m)) \in \mathbb{Q}[m].$$

We denote by $r(F)$ the leading coefficient of the above polynomial.

Definition 1.5. *A pair $(s: \mathcal{O}_X \rightarrow F)$ in \mathcal{B}_X is called q -(semi)stable for a polynomial $q \in \mathbb{Q}[m]$ if the following conditions hold:*

(i) *For any subsheaf $F' \subset F$, we have the inequality*

$$(1.9) \quad \frac{\chi(F'(m))}{r(F')} < (\leq) \frac{\chi(F(m)) + q(m)}{r(F)}, \quad m \gg 0.$$

(ii) *For any subsheaf $F' \subsetneq F$ such that s factors through F' , we have the inequality*

$$(1.10) \quad \frac{\chi(F'(m)) + q(m)}{r(F')} < (\leq) \frac{\chi(F(m)) + q(m)}{r(F)}, \quad m \gg 0.$$

On the other hand for each $t \in \mathbb{R}$, we define the map

$$Z_t: K(\mathcal{A}_X) \rightarrow \mathbb{C},$$

by sending $E \in K(\mathcal{A}_X)$ with $\text{ch}(E) = (r, 0, 0, -\beta, -n)$ to

$$Z_t(E) := \begin{cases} r(-t + \sqrt{-1}), & r \neq 0, \\ -n + (\beta \cdot \omega)\sqrt{-1}, & r = 0. \end{cases}$$

By the definition of \mathcal{A}_X , we have $Z_t(E) \in \mathcal{H} \cup \mathbb{R}_{<0}$ for any non-zero $E \in \mathcal{A}_X$, where $\mathcal{H} \subset \mathbb{C}$ is the upper half plane. The pair (\mathcal{A}_X, Z_t) is a weak stability condition introduced in [Tod10b] (see also [Tod12]), which generalizes Bridgeland's notion of stability conditions [Bri].

Definition 1.6. *An object $E \in \mathcal{A}_X$ is called Z_t -(semi)stable if for any exact sequence $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ in \mathcal{A}_X with $E' \neq 0$, $E'' \neq 0$, we have*

$$\arg Z_t(E') < (\leq) \arg Z_t(E'') \in (0, \pi].$$

For an object $F \in \text{Coh}_{\leq 1}(X)$ with $([F], \chi(F)) = (\beta, n)$, we set

$$\mu(F) = \frac{n}{\beta \cdot \omega} \in \mathbb{Q} \cup \{\infty\},$$

where $\mu(F) = \infty$ if $\beta = 0$. We have the following characterization of Z_t -(semi)stable objects.

Lemma 1.7. *For an object $(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{B}_X$, the object $\Phi(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{A}_X$ is Z_t -(semi)stable if and only if the following conditions hold:*

- (i) *for any subsheaf $0 \neq F' \subseteq F$, we have $\mu(F') < (\leq)t$.*
- (ii) *for any subsheaf $F' \subsetneq F$ such that s factors through F' , we have $\mu(F/F') > (\geq)t$.*

Proof. Since the object $E = \Phi(\mathcal{O}_X \rightarrow F)$ is of rank one and Φ is an equivalence, any exact sequence $0 \rightarrow E' \rightarrow E \rightarrow E'' \rightarrow 0$ in \mathcal{A}_X is given by the image of either one of the following exact sequences in \mathcal{B}_X

$$\begin{aligned} 0 \rightarrow (0 \rightarrow F') \rightarrow (\mathcal{O}_X \xrightarrow{s} F) \rightarrow (\mathcal{O}_X \rightarrow F'') \rightarrow 0, \\ 0 \rightarrow (\mathcal{O}_X \rightarrow F') \rightarrow (\mathcal{O}_X \xrightarrow{s} F) \rightarrow (0 \rightarrow F'') \rightarrow 0. \end{aligned}$$

Therefore the lemma follows from the definition of Z_t -(semi)stability. \square

Proposition 1.8. *Let us fix $v \in H^*(X)$ of the form (1.4) and set $q_t(m) \in \mathbb{Q}[m]$ to be the constant polynomial*

$$(1.11) \quad q_t(m) \equiv (\beta \cdot \omega)t - n.$$

Then a pair $(\mathcal{O}_X \rightarrow F) \in \mathcal{B}_X$ with $([F], \chi(F)) = (\beta, n)$ is q_t -(semi)stable if and only if $\Phi(\mathcal{O}_X \rightarrow F) \in \mathcal{A}_X$ is Z_t -(semi)stable.

Proof. For $F \in \text{Coh}_{\leq 1}(X)$ with $([F], \chi(F)) = (\beta, n)$, its Hilbert polynomial is written as

$$\chi(F(m)) = (\beta \cdot \omega)m + n.$$

Therefore we have

$$\frac{\chi(F(m))}{r(F)} = m + \mu(F), \quad \frac{\chi(F(m)) + q_t(m)}{r(F)} = m + t.$$

Thus (1.9), (1.10) are equivalent to the conditions (i), (ii) in Lemma 1.7 respectively. \square

Let us take an element $v \in H^*(X)$ of the form (1.4). For each $t \in \mathbb{R}$, we denote by

$$(1.12) \quad \mathcal{P}_n^t(X, \beta) \subset \mathcal{P}_n^t(X, \beta) \subset \underline{\mathcal{A}}_X(v)$$

the open substacks of Z_t -stable (semistable) objects in \mathcal{A}_X with Chern character v . Because of Theorem 1.4 and Proposition 1.8, they are also identified with open substacks of $\underline{\mathcal{B}}_X(v)$ parametrizing pairs satisfying (i), (ii) in Lemma 1.7. Below we write a \mathbb{C} -valued point of the stacks in (1.12) as a pair $(\mathcal{O}_X \rightarrow F)$ by the above identification.

Theorem 1.9. *For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, the moduli space $\mathcal{P}_n^t(X, \beta)$ is a quasi-projective scheme, and $\mathcal{P}_n^t(X, \beta)$ admits a good moduli space*

$$\mathcal{P}_n^t(X, \beta) \rightarrow \overline{\mathcal{P}}_n^t(X, \beta),$$

where $\overline{\mathcal{P}}_n^t(X, \beta)$ is a projective scheme which parametrizes Z_t -polystable objects.

Proof. By Theorem 1.4, Proposition 1.8, $P_n^t(X, \beta)$, $\mathcal{P}_n(X, \beta)$ are isomorphic to the moduli spaces of q_t -stable (semistable) pairs in \mathcal{B}_X (with $q_t \equiv (\beta \cdot \omega) t - n$). The latter has a GIT construction due to Le Potier's work on semistable coherent systems ([Pot93, Thm. 4.11], [JS12, pp. 164]). \square

Here by Proposition 1.8, a rank one Z_t -polystable object in \mathcal{A}_X is of the following form

$$(1.13) \quad (\mathcal{O}_X \rightarrow F_0) \oplus \bigoplus_{i=1}^k V_i \otimes F_i[-1],$$

where $(\mathcal{O}_X \rightarrow F_0)$ is Z_t -stable, each F_i for $1 \leq i \leq k$ are mutually non-isomorphic μ -stable one dimensional sheaves with $\mu(F_i) = t$, and V_i are finite dimensional vector spaces.

As usual, there is a wall-chamber structure for Z_t -stability, where moduli spaces of stable objects stay unchanged inside chambers. Namely there is a finite set of points $W \subset \mathbb{R}$ such that we have

$$P_n(X, \beta) = \mathcal{P}_n^t(X, \beta) = \overline{P}_n^t(X, \beta), \quad t \notin W.$$

In particular, $P_n(X, \beta)$ is a projective scheme for $t \notin W$. Let $t_0 \in W$ and set $t_{\pm} = t_0 \pm \varepsilon$ for $0 < \varepsilon \ll 1$. We have open immersions

$$P_n^{t_0+}(X, \beta) \subset \mathcal{P}_n^{t_0}(X, \beta) \supset P_n^{t_0-}(X, \beta),$$

which induce the following flip type diagram of good moduli spaces

$$(1.14) \quad \begin{array}{ccc} P_n^{t_0+}(X, \beta) & & P_n^{t_0-}(X, \beta) \\ & \searrow \pi^+ & \swarrow \pi^- \\ & \overline{P}_n^{t_0}(X, \beta) & \end{array}$$

1.4. PT stable pairs and JS stable pairs. We discuss two interesting chambers for Z_t -stability. Recall the following two notions of stable pairs.

Definition 1.10. ([PT09, JS12])

(i) A pair $(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{A}_X$ is called a *PT stable pair* if F is a pure one dimensional sheaf and s is surjective in dimension one.

(ii) A pair $(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{A}_X$ is called a *JS stable pair* if s is a non-zero morphism, F is μ -semistable and for any subsheaf $0 \neq F' \subsetneq F$ such that s factors through F' we have $\mu(F') < \mu(F)$.

Remark 1.11. Strictly speaking, JS stable pairs in [JS12, Def. 12.2] are of type $(\mathcal{O}_X(-N) \rightarrow F)$ for a sufficiently negative line bundle $\mathcal{O}_X(-N)$, different from the definition given here.

The above stable pairs appear as Z_t -stable objects in some chambers.

Proposition 1.12. For a fixed $v \in H^*(X, \mathbb{Q})$ of the form (1.4), we have the following:

(i) There exists $t(v) > 0$ such that an object $(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{A}_X$ with Chern character v is Z_t -stable for $t > t(v)$ if and only if it is a PT stable pair.

(ii) There exists $\varepsilon(v) > 0$ such that an object $(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{A}_X$ with Chern character v is Z_t -stable for $\frac{n}{\omega \cdot \beta} < t < \frac{n}{\omega \cdot \beta} + \varepsilon(v)$ if and only if it is JS stable pair.

(iii) For $t < \frac{n}{\omega \cdot \beta}$, there is no Z_t -semistable object $(\mathcal{O}_X \xrightarrow{s} F) \in \mathcal{A}_X$ with Chern character v .

Proof. (i) follows from the same argument for CY 3-folds, see e.g. [Tod12, Prop. 5.4 (i)]. (ii) easily follows from Lemma 1.7, by setting $\epsilon(v)$ satisfying the following condition

$$\frac{n'}{\omega \cdot \beta'} + \epsilon(v) < \frac{n}{\omega \cdot \beta},$$

for any effective class $\beta' \in H_2(X, \mathbb{Z})$ and $n' \in \mathbb{Z}$ satisfying $n'/(\omega \cdot \beta') < n/(\omega \cdot \beta)$. (iii) also follows by setting $F' = F$ in Lemma 1.7. \square

Following Proposition 1.12, we discuss three distinguished chambers as follows.

(i) **Pandharipande-Thomas chamber.** For $t \rightarrow \infty$, we have the moduli space of PT stable pairs (see Proposition 1.12 (i))

$$P_n(X, \beta) := P_n^t(X, \beta)|_{t \rightarrow \infty}.$$

(ii) **Joyce-Song chamber.** For $t = \frac{n}{\omega \cdot \beta} + 0$, we have the moduli space of JS stable pairs (see Proposition 1.12 (ii))

$$P_n^{\text{JS}}(X, \beta) := P_n^t(X, \beta)|_{t = \frac{n}{\omega \cdot \beta} + 0}.$$

(iii) **Empty chamber.** For $t < \frac{n}{\omega \cdot \beta}$, we have (see Proposition 1.12 (iii))

$$P_n^t(X, \beta) = \emptyset.$$

For (β, n) , we denote by

$$\iota_M: \mathcal{M}_n(X, \beta) \rightarrow M_n(X, \beta)$$

the moduli stack of μ -semistable one dimensional sheaves F on X with $([F], \chi(F)) = (\beta, n)$, and its good moduli space parametrizing μ -polystable objects. Since the target of JS stable pair $(\mathcal{O}_X \rightarrow F)$ is μ -semistable, we have a natural morphism

$$(1.15) \quad P_n^{\text{JS}}(X, \beta) \rightarrow M_n(X, \beta), \quad (\mathcal{O}_X \rightarrow F) \mapsto \iota_M(F).$$

In this way, the wall-crossing diagrams (1.14) relate $P_n(X, \beta)$, $P_n^{\text{JS}}(X, \beta)$ and $M_n(X, \beta)$ in terms of flip type diagrams as in (1.14) (see also the diagram in Example 1.14).

Nevertheless, in some cases there is no wall in $t > \frac{n}{\omega \cdot \beta}$ so that PT stable pairs and JS stable pairs coincide. For an effective curve class $\beta \in H_2(X, \mathbb{Z})$, we define

$$n(\beta) := \inf\{\chi(\mathcal{O}_C) : C \subset X \text{ is a one dimensional closed subscheme with } [C] = \beta\} > -\infty.$$

Then we state the following proposition:

Proposition 1.13. *Let β be an effective curve class and $(\beta, n) \in H_2(X, \mathbb{Z}) \oplus \mathbb{Z}$. Suppose the following inequality*

$$(1.16) \quad \frac{n}{\omega \cdot \beta} \leq \frac{n(\beta')}{\omega \cdot \beta'}$$

holds for any effective class $0 < \beta' < \beta$. Then $P_n^t(X, \beta)$ is independent of t if $t > \frac{n}{\omega \cdot \beta}$. In particular in this case, we have $P_n(X, \beta) = P_n^{\text{JS}}(X, \beta)$.

Proof. We first show that any Z_t -stable pair $(\mathcal{O}_X \xrightarrow{s} F) \in P_n^t(X, \beta)$ is a PT stable pair if $t > \frac{n}{\omega \cdot \beta}$. Note that the Z_t -stability always implies that F is a pure one dimensional sheaf. The image of s is written as $\mathcal{O}_{C'}$ for a one dimensional subscheme C' such that $\beta' = [C'] \leq \beta$. Suppose by contradiction that $\beta' < \beta$ so that $\beta - \beta' > 0$, and set $Q = F/\mathcal{O}_{C'}$. We have two exact sequences in \mathcal{A}_X

$$\begin{aligned} 0 \rightarrow (0 \rightarrow \mathcal{O}_{C'}) \rightarrow (\mathcal{O}_X \rightarrow F) \rightarrow (\mathcal{O}_X \rightarrow Q) \rightarrow 0, \\ 0 \rightarrow (\mathcal{O}_X \rightarrow \mathcal{O}_{C'}) \rightarrow (\mathcal{O}_X \rightarrow F) \rightarrow (0 \rightarrow Q) \rightarrow 0. \end{aligned}$$

Then the Z_t -stability yields

$$\frac{n(\beta')}{\omega \cdot \beta'} \leq \frac{\chi(\mathcal{O}_{C'})}{\omega \cdot \beta'} < t < \frac{n - \chi(\mathcal{O}_{C'})}{\omega \cdot (\beta - \beta')} \leq \frac{n - n(\beta')}{\omega \cdot (\beta - \beta')}.$$

The above inequalities contradict with the inequality (1.16).

Conversely, we show that any PT stable pair $(\mathcal{O}_X \rightarrow F) \in P_n(X, \beta)$ is Z_t -stable for $t > \frac{n}{\omega \cdot \beta}$. It is enough to show that for any subsheaf $F' \subset F$ with $([F'], \chi(F')) = (\beta', n')$, we have

$$\mu(F') = \frac{n'}{\omega \cdot \beta'} \leq \frac{n}{\omega \cdot \beta} < t.$$

The above inequality is obvious if $\beta' = \beta$, so we may assume that $\beta' < \beta$ and set $\beta'' = \beta - \beta'$. The composition $\mathcal{O}_X \rightarrow F \rightarrow F/F'$ is surjective in dimension one, so we have $\chi(F/F') = n - n' \geq n(\beta'')$. Therefore we have

$$\frac{n'}{\omega \cdot \beta'} \leq \frac{n - n(\beta'')}{\omega \cdot (\beta - \beta'')} \leq \frac{n}{\omega \cdot \beta},$$

where the last inequality follows from (1.16) for β'' . Therefore we obtain the proposition. \square

Example 1.14. *Let X be the non-compact CY 4-fold given by*

$$X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2)).$$

Let $[l] \in H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^2, \mathbb{Z})$ be the class of a line. In this case, the numerical class $(4[l], 1)$ does not satisfy the condition (1.16) as $n(3[l]) = 0$. Indeed a PT stable pair in $P_1(X, 4[l])$ is destabilized at $t = 1$ if and only if it is of the form $(I_C \xrightarrow{s} \mathcal{O}_l)$ for a cubic curve $C \subset \mathbb{P}^2$, a line $l \subset \mathbb{P}^2$, and a non-zero morphism s . The destabilizing sequence is given by

$$0 \rightarrow \mathcal{O}_l[-1] \rightarrow (I_C \xrightarrow{s} \mathcal{O}_l) \rightarrow I_C \rightarrow 0.$$

We can show that $t = 1$ is the only wall and we have the following wall-crossing phenomena of moduli spaces of Z_t -stable objects

$$P_1^t(X, 4[l]) = \begin{cases} P_1(X, 4[l]), & t > 1, \\ P_1^{\text{JS}}(X, 4[l]), & \frac{1}{4} < t < 1, \\ \emptyset, & t < \frac{1}{4}. \end{cases}$$

The corresponding flip type diagram is

$$\begin{array}{ccc} P_1(X, 4[l]) & & P_1^{\text{JS}}(X, 4[l]) \\ & \searrow & \swarrow \quad \searrow \\ & \overline{P}_1^{t=1}(X, 4[l]) & M_1(X, 4[l]). \end{array}$$

2. DT₄ TYPE INVARIANTS FOR Z_t -STABLE PAIRS

2.1. Review of DT₄ invariants. Before defining DT₄ type counting invariants associated with moduli spaces $P_n^t(X, \beta)$ of Z_t -stable objects, we first introduce the set-up of DT₄ invariants. We fix an ample divisor ω on X and take a cohomology class $v \in H^*(X, \mathbb{Q})$.

The coarse moduli space $M_\omega(v)$ of ω -Gieseker semistable sheaves E on X with $\text{ch}(E) = v$ exists as a projective scheme. We always assume that $M_\omega(v)$ is a fine moduli space, i.e. any point $[E] \in M_\omega(v)$ is stable and there is a universal family

$$(2.1) \quad \mathcal{E} \in \text{Coh}(X \times M_\omega(v)).$$

For instance, moduli spaces of one dimensional stable sheaves E with $\chi(E) = 1$ and Hilbert schemes of closed subschemes satisfy this assumption [Cao2, CK18, CK19, CMT18].

In [BJ, CL14], under certain hypotheses, the authors construct a DT₄ virtual class

$$(2.2) \quad [M_\omega(v)]^{\text{vir}} \in H_{2-\chi(v,v)}(M_\omega(v), \mathbb{Z}),$$

where $\chi(-, -)$ is the Euler pairing. Notice that this class will not necessarily be algebraic.

Roughly speaking, in order to construct such a class, one chooses at every point $[E] \in M_\omega(v)$, a half-dimensional real subspace

$$\text{Ext}_+^2(E, E) \subset \text{Ext}^2(E, E)$$

of the usual obstruction space $\text{Ext}^2(E, E)$, on which the quadratic form Q defined by Serre duality is real and positive definite. Then one glues local Kuranishi-type models of form

$$\kappa_+ = \pi_+ \circ \kappa : \text{Ext}^1(E, E) \rightarrow \text{Ext}_+^2(E, E),$$

where κ is a Kuranishi map of $M_\omega(v)$ at E and π_+ is the projection according to the decomposition $\text{Ext}^2(E, E) = \text{Ext}_+^2(E, E) \oplus \sqrt{-1} \cdot \text{Ext}_+^2(E, E)$.

In [CL14], local models are glued in three special cases:

- (1) when $M_\omega(v)$ consists of locally free sheaves only;
- (2) when $M_\omega(v)$ is smooth;
- (3) when $M_\omega(v)$ is a shifted cotangent bundle of a derived smooth scheme.

And the corresponding virtual classes are constructed using either gauge theory or algebro-geometric perfect obstruction theory.

The general gluing construction is due to Borisov-Joyce [BJ] (one needs to assume that $M_\omega(v)$ can be given a (-2) -shifted symplectic structure as in Claim 3.29 [BJ] to apply their constructions. For the Z_t -stable pairs case, we show this can be done in Lemma 2.1), based on Pantev-Töen-Vaquié-Vezzosi's theory of shifted symplectic geometry [PTVV13] and Joyce's theory of derived C^∞ -geometry. The corresponding virtual class is constructed using Joyce's D-manifold theory (a machinery similar to Fukaya-Oh-Ohta-Ono's theory of Kuranishi space structures used in defining Lagrangian Floer theory).

In this paper, all computations and examples will only involve the virtual class constructions in situations (2), (3), mentioned above. We briefly review them as follows:

- When $M_\omega(v)$ is smooth, the obstruction sheaf $Ob \rightarrow M_\omega(v)$ is a vector bundle endowed with a quadratic form Q via Serre duality. Then the DT₄ virtual class is given by

$$[M_\omega(v)]^{\text{vir}} = \text{PD}(e(Ob, Q)).$$

Here $e(\mathcal{O}b, Q)$ is the half-Euler class of $(\mathcal{O}b, Q)$ (i.e. the Euler class of its real form $\mathcal{O}b_+$), and $\text{PD}(-)$ is its Poincaré dual. Note that the half-Euler class satisfies

$$\begin{aligned} e(\mathcal{O}b, Q)^2 &= (-1)^{\frac{\text{rk}(\mathcal{O}b)}{2}} e(\mathcal{O}b), \text{ if } \text{rk}(\mathcal{O}b) \text{ is even,} \\ e(\mathcal{O}b, Q) &= 0, \text{ if } \text{rk}(\mathcal{O}b) \text{ is odd.} \end{aligned}$$

- When $M_\omega(v)$ is a (-2) -shifted cotangent bundle of a derived smooth scheme, roughly speaking, this means that at any closed point $[E] \in M_\omega(v)$, we have Kuranishi map of type

$$\kappa : \text{Ext}^1(E, E) \rightarrow \text{Ext}^2(E, E) = V_E \oplus V_E^*,$$

where κ factors through a maximal isotropic subspace V_E of $(\text{Ext}^2(E, E), Q)$. Then the DT_4 virtual class of $M_\omega(v)$ is, roughly speaking, the virtual class of the perfect obstruction theory formed by $\{V_E\}_{E \in M_\omega(v)}$. When $M_\omega(v)$ is furthermore smooth as a scheme, then it is simply the Euler class of the vector bundle $\{V_E\}_{E \in M_\omega(v)}$ over $M_\omega(v)$.

To construct the above virtual class (2.2) with coefficients in \mathbb{Z} (instead of \mathbb{Z}_2), we need an orientability result for $M_\omega(v)$, which is stated as follows. Let

$$(2.3) \quad \mathcal{L} := \det(\mathbf{R}\mathcal{H}om_{\pi_M}(\mathcal{E}, \mathcal{E})) \in \text{Pic}(M_\omega(v)), \quad \pi_M : X \times M_\omega(v) \rightarrow M_\omega(v),$$

be the determinant line bundle of $M_\omega(v)$, equipped with a symmetric pairing Q induced by Serre duality. An *orientation* of (\mathcal{L}, Q) is a reduction of its structure group (from $O(1, \mathbb{C})$) to $SO(1, \mathbb{C}) = \{1\}$; in other words, we require a choice of square root of the isomorphism

$$(2.4) \quad Q : \mathcal{L} \otimes \mathcal{L} \rightarrow \mathcal{O}_{M_\omega(v)}$$

to construct the virtual class (2.2). An orientability result was first obtained for $M_\omega(v)$ when the CY 4-fold X satisfies $\text{Hol}(X) = SU(4)$ and $H^{\text{odd}}(X, \mathbb{Z}) = 0$ [CL17, Thm. 2.2] and it has recently been generalized to arbitrary CY 4-folds [CGJ, Cor. 1.17]. Note that the set of orientations forms a torsor for $H^0(M_\omega(v), \mathbb{Z}_2)$.

2.2. Z_t -stable pair invariants. For $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$, let

$$(2.5) \quad P_n^t(X, \beta)$$

be the moduli space of Z_t -stable objects (F, s) on X such that $[F] = \beta$, $\chi(F) = n$. By Theorem 1.9, it is a quasi-projective scheme whose closed points correspond to two-term complexes

$$I = (\mathcal{O}_X \xrightarrow{s} F) \in \text{D}^b(\text{Coh}(X)),$$

in the derived category of coherent sheaves on X , satisfying the Z_t -stability condition.

Similar to moduli spaces of stable sheaves, the moduli space $P_n^t(X, \beta)$ admits a deformation-obstruction theory, whose tangent, obstruction and ‘higher’ obstruction spaces are given by

$$\text{Ext}^1(I, I)_0, \text{Ext}^2(I, I)_0, \text{Ext}^3(I, I)_0,$$

where $(-)_0$ denotes the trace-free part. Note that Serre duality gives an isomorphism $\text{Ext}_0^1 \cong (\text{Ext}_0^3)^\vee$ and a non-degenerate quadratic form on Ext_0^2 . Moreover, we have the following lemma:

Lemma 2.1. *The moduli space $P_n^t(X, \beta)$ can be given the structure of a (-2) -shifted symplectic derived scheme in the sense of Pantev-Töen-Vaquié-Vezzosi [PTVV13].*

Proof. By Proposition 1.3, $P_n^t(X, \beta)$ is an open substack of the moduli stack of perfect complexes of coherent sheaves with trivial determinant on X , whose (-2) -shifted symplectic structure is constructed by [PTVV13, Thm. 0.1] (see also [PTVV13, Sect. 3.2, pp. 48] for pull-back to the determinant fixed substack). \square

Let \mathbb{I} be the universal pair

$$(2.6) \quad \mathbb{I} = (\mathcal{O}_{X \times P_n^t(X, \beta)} \rightarrow \mathbb{F}).$$

Then the determinant line bundle

$$\mathcal{L} := \det(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{I})_0) \in \text{Pic}(P_n^t(X, \beta))$$

is endowed with a non-degenerate quadratic form Q defined by Serre duality, where $\pi_P : X \times P_n^t(X, \beta) \rightarrow P_n^t(X, \beta)$ is the projection. Similarly as before, the orientability issue for the moduli space $P_n^t(X, \beta)$ is whether the structure group of the quadratic line bundle (\mathcal{L}, Q) can be reduced from $O(1, \mathbb{C})$ to $SO(1, \mathbb{C}) = \{1\}$.

As $P_n^t(X, \beta)$ is an open substack of the moduli stack of perfect complexes of coherent sheaves with trivial determinant on X , it is always orientable in the above sense by [CGJ, Cor. 1.17]. Combining this with Theorem 1.9 and Lemma 2.1, we can construct their virtual classes.

Theorem 2.2. *Let (X, ω) be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}$. For a generic choice of $t \in \mathbb{R}$ such that $P_n^t(X, \beta)$ is projective, there exists a virtual class*

$$(2.7) \quad [P_n^t(X, \beta)]^{\text{vir}} \in H_{2n}(P_n^t(X, \beta), \mathbb{Z}),$$

in the sense of Borisov-Joyce [BJ], depending on the choice of orientation.

As in [CMT19], we consider primary insertions: for integral classes $\gamma \in H^4(X, \mathbb{Z})$, let

$$(2.8) \quad \tau: H^4(X, \mathbb{Z}) \rightarrow H^2(P_n^t(X, \beta), \mathbb{Z}), \quad \tau(\gamma) := (\pi_P)_*(\pi_X^* \gamma \cup \text{ch}_3(\mathbb{F})),$$

where π_X, π_P are projections from $X \times P_n^t(X, \beta)$ to corresponding factors, \mathbb{F} is the target of the universal pair (2.6), and $\text{ch}_3(\mathbb{F})$ is the Poincaré dual to the fundamental cycle of \mathbb{F} .

Definition 2.3. *For a generic $t \in \mathbb{R}$, the Z_t -stable pair invariant is defined to be*

$$(2.9) \quad P_{n,\beta}^t(\gamma) := \int_{[P_n^t(X,\beta)]^{\text{vir}}} \tau(\gamma)^n \in \mathbb{Z}.$$

Here we write $P_{0,\beta}^t = P_{n,\beta}^t(\gamma)$ when $n = 0$.

As in Section 1.4, we are particularly interested in our invariants in the following two distinguished chambers.

(i) When $t \rightarrow \infty$, Z_t -stable pairs are PT stable pairs. We denote

$$(2.10) \quad P_{n,\beta}(\gamma) := P_{n,\beta}^t(\gamma)|_{t \rightarrow \infty}, \quad P_{0,\beta} := P_{0,\beta}^t|_{t \rightarrow \infty},$$

which has been studied before in [CMT19].

(ii) When $t = \frac{n}{\omega \cdot \beta} + 0$, Z_t -stable pairs are Joyce-Song stable pairs. We denote

$$(2.11) \quad P_{n,\beta}^{\text{JS}}(\gamma) := P_{n,\beta}^t(\gamma)|_{t = \frac{n}{\omega \cdot \beta} + 0}, \quad P_{0,\beta}^{\text{JS}} := P_{0,\beta}^t|_{t = +0}.$$

3. CONJECTURES

3.1. GW/GV conjecture. Let X be a smooth projective CY 4-fold. The genus 0 Gromov-Witten invariants on X are defined using insertions: for $\gamma \in H^4(X, \mathbb{Z})$, one defines

$$\text{GW}_{0,\beta}(\gamma) = \int_{[\overline{M}_{0,1}(X,\beta)]^{\text{vir}}} \text{ev}^*(\gamma) \in \mathbb{Q},$$

where $\text{ev}: \overline{M}_{0,1}(X, \beta) \rightarrow X$ is the evaluation map.

The *genus 0 Gopakumar-Vafa type invariants*

$$(3.1) \quad n_{0,\beta}(\gamma) \in \mathbb{Q}$$

are defined by Klemm-Pandharipande [KP] from the identity

$$\sum_{\beta > 0} \text{GW}_{0,\beta}(\gamma) q^\beta = \sum_{\beta > 0} n_{0,\beta}(\gamma) \sum_{d=1}^{\infty} d^{-2} q^{d\beta}.$$

For genus 1 case, virtual dimensions of moduli spaces of stable maps are zero, so Gromov-Witten invariants

$$\text{GW}_{1,\beta} = \int_{[\overline{M}_{1,0}(X,\beta)]^{\text{vir}}} 1 \in \mathbb{Q}$$

can be defined without insertions. The *genus 1 Gopakumar-Vafa type invariants*

$$(3.2) \quad n_{1,\beta} \in \mathbb{Q}$$

are defined in [KP] by the identity

$$\begin{aligned} \sum_{\beta > 0} \text{GW}_{1,\beta} q^\beta &= \sum_{\beta > 0} n_{1,\beta} \sum_{d=1}^{\infty} \frac{\sigma(d)}{d} q^{d\beta} + \frac{1}{24} \sum_{\beta > 0} n_{0,\beta} (c_2(X)) \log(1 - q^\beta) \\ &\quad - \frac{1}{24} \sum_{\beta_1, \beta_2} m_{\beta_1, \beta_2} \log(1 - q^{\beta_1 + \beta_2}), \end{aligned}$$

where $\sigma(d) = \sum_{i|d} i$ and $m_{\beta_1, \beta_2} \in \mathbb{Z}$ are called meeting invariants which can be inductively determined by genus 0 Gromov-Witten invariants. In [KP], both of the invariants (3.1), (3.2) are conjectured to be integers, and Gromov-Witten invariants on X are computed to support the conjectures in many examples by localization technique or mirror symmetry.

3.2. Katz/GV conjecture. In [CMT18], Maulik and the authors define DT_4 counting invariants for one dimensional stable sheaves and use them to give a sheaf theoretical interpretation to the genus 0 GV type invariants (3.1).

To be precise, we consider the moduli scheme $M_1(X, \beta)$ of one dimensional stable sheaves F on X with $[F] = \beta \in H_2(X, \mathbb{Z})$ and $\chi(F) = 1$. The spherical twist (here we need to assume $h^{0,1}(X) = h^{0,2}(X) = 0$, see [ST01, Def. 0.1]):

$$\Phi_{\mathcal{O}_X}(\bullet) = \text{cone}(\mathbf{R}\text{Hom}(\mathcal{O}_X, \bullet) \otimes \mathcal{O}_X \rightarrow \bullet)$$

identifies $M_1(X, \beta)$ with some moduli stack of rank one objects in $\text{D}^b(\text{Coh}(X))$. As in Theorem 2.2, there exists a virtual class

$$[M_1(X, \beta)]^{\text{vir}} \in H_2(M_1(X, \beta), \mathbb{Z}).$$

Consider primary insertions: for integral classes $\gamma \in H^4(X, \mathbb{Z})$, let

$$(3.3) \quad \tau: H^4(X, \mathbb{Z}) \rightarrow H^2(M_1(X, \beta), \mathbb{Z}), \quad \tau(\gamma) := (\pi_M)_*(\pi_X^* \gamma \cup \text{ch}_3(\mathbb{F})),$$

where π_X, π_M are projections from $X \times M_1(X, \beta)$ to corresponding factors, \mathbb{F} is the universal sheaf and $\text{ch}_3(\mathbb{F})$ is the Poincaré dual to the fundamental cycle of \mathbb{F} .

The following may be thought as an analogue of Katz/GV conjecture [Kat08] on CY 4-folds.

Conjecture 3.1. ([CMT18, Conjecture 0.2]) *For certain choice of orientation, we have*

$$\int_{[M_1(X, \beta)]^{\text{vir}}} \tau(\gamma) = n_{0, \beta}(\gamma),$$

where $n_{0, \beta}(\gamma)$ is the $g=0$ Gopakumar-Vafa type invariant (3.1).

Remark 3.2. See also [CT20a] for a discussion on the higher genus case.

3.3. PT/GV conjecture. In [CMT19], Maulik and the authors define Pandharipande-Thomas type invariants (2.10) on Calab-Yau 4-folds and conjecture the following PT/GV correspondence.

Conjecture 3.3. ([CMT19, Conj. 1.1, 1.2, Sect. 1.7]) *Let (X, ω) be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}_{\geq 0}$. Fix $\gamma \in H^4(X, \mathbb{Z})$, then for certain choice of orientation, we have*

$$P_{n, \beta}(\gamma) = \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_n = \beta, \\ \omega \cdot \beta_i > 0, i=1, \dots, n}} P_{0, \beta_0} \cdot \prod_{i=1}^n n_{0, \beta_i}(\gamma), \quad \sum_{\beta \geq 0} P_{0, \beta} q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1, \beta}}.$$

Here $n_{0, \beta}(\gamma), n_{1, \beta}$ are the Gopakumar-Vafa type invariants (3.1), (3.2) of X . And $M(q) = \prod_{k \geq 1} (1 - q^k)^{-k}$ is the MacMahon function.

3.4. Main conjecture. Pandharipande-Thomas stable pairs are Z_t -stable pairs when $t \rightarrow \infty$ (Definition 1.6), and the corresponding PT type invariants (2.10) are $Z_{t \rightarrow \infty}$ -stable pair invariants (2.9). The following main conjecture of this paper generalizes Conjecture 3.3.

Conjecture 3.4. *Let (X, ω) be a smooth projective Calabi-Yau 4-fold, $\beta \in H_2(X, \mathbb{Z})$ and $n \in \mathbb{Z}_{\geq 0}$. Choose a generic $t \in \mathbb{R}_{> 0}$. Then for certain choice of orientation, we have*

$$(3.4) \quad P_{n, \beta}^t(\gamma) = \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_n = \beta \\ \omega \cdot \beta_i > \frac{1}{t}, i=1, \dots, n}} P_{0, \beta_0} \cdot \prod_{i=1}^n n_{0, \beta_i}(\gamma),$$

where $\gamma \in H^4(X, \mathbb{Z})$, $n_{0, \beta}(\gamma)$ is the genus 0 Gopakumar-Vafa type invariant (3.1).

In particular, $P_{0, \beta}^t = P_{0, \beta}$ is independent of the choice of $t > 0$.

Combining with the second formula in Conjecture 3.3, we may express Z_t -stable pair invariants in terms of all genus GV type invariants of X . Indeed let us consider the generating series for $\gamma \in H^4(X, \mathbb{Z})$:

$$(3.5) \quad \text{PT}^t(X)(\exp(\gamma)) := \sum_{n \in \mathbb{Z}, \beta \geq 0} \frac{P_{n, \beta}^t(\gamma)}{n!} y^n q^\beta.$$

Then for a very generic $t \in \mathbb{R}$ (very generic means outside a countable subset of rational numbers in \mathbb{R}), the identity in Conjecture 3.4 together with the second formula in Conjecture 3.3 implies that

$$\text{PT}^t(X)(\exp(\gamma)) = \prod_{\omega \cdot \beta > \frac{1}{t}} \exp(yq^\beta)^{n_{0, \beta}(\gamma)} \cdot \prod_{\beta > 0} M(q^\beta)^{n_{1, \beta}}.$$

By taking the $t \rightarrow \infty$ limit, we recover the conjectural formula (0.4).

3.5. JS/GV conjecture. In the Joyce-Song chamber, there are two particularly interesting special cases of Conjecture 3.4.

Conjecture 3.5. (Special case of Conjecture 3.4) *In the same setting as Conjecture 3.4, we have*

$$(1) P_{n,\beta}^{\text{JS}}(\gamma) = \sum_{\substack{\beta_1+\dots+\beta_n=\beta \\ \omega \cdot \beta_i = \frac{\omega \cdot \beta}{n}, i=1,\dots,n}} \prod_{i=1}^n n_{0,\beta_i}(\gamma), \text{ if } n \geq 1, \quad (2) P_{0,\beta}^{\text{JS}} = P_{0,\beta}.$$

In particular, $P_{1,\beta}^{\text{JS}}(\gamma) = n_{0,\beta}(\gamma)$.

When $n = 1$, we recover genus 0 GV type invariants $n_{0,\beta}(\gamma)$ (3.1). While in the $n = 0$ case, we recover genus 1 GV type invariants $n_{1,\beta}$ (3.2) by assuming the conjectural relation between $P_{0,\beta}$ and $n_{1,\beta}$ (as in Conjecture 3.3):

$$\sum_{\beta \geq 0} P_{0,\beta} q^\beta = \prod_{\beta > 0} M(q^\beta)^{n_{1,\beta}}.$$

Therefore (conjecturally), we may use DT_4 counting invariants for (semi)stable one dimensional sheaves together with sections (more precisely JS stable pairs) to recover all genus GV type invariants of Calabi-Yau 4-folds.

4. HEURISTIC EXPLANATIONS OF THE MAIN CONJECTURE

Our main conjecture 3.4 is difficult to prove, due to the difficulties of DT_4 -virtual classes and the absence of wall-crossing formulae available in Donaldson invariants [Moc09] and Donaldson-Thomas invariants [JS12, KS]. Here we give heuristic explanations of our main conjecture from the viewpoint of ideal geometry, master space argument and a virtual push-forward formula.

4.1. Heuristic argument on ideal CY 4-folds. In this subsection, we give a heuristic argument to explain why we expect Conjecture 3.4 to be true in an ideal CY4 geometry. In this heuristic discussion, we ignore the issue of orientations.

Let X be an ‘ideal’ CY 4-fold in the sense that all curves of X deform in families of expected dimensions, and have expected generic properties, i.e.

- (1) any rational curve in X is a chain of smooth \mathbb{P}^1 with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, and moves in a compact 1-dimensional smooth family of embedded rational curves, whose general member is smooth with normal bundle $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$.
- (2) any elliptic curve E in X is smooth, super-rigid, i.e. the normal bundle is $L_1 \oplus L_2 \oplus L_3$ for general degree zero line bundle L_i on E satisfying $L_1 \otimes L_2 \otimes L_3 = \mathcal{O}_E$. Furthermore any two elliptic curves are disjoint and also disjoint with rational curve families.
- (3) there is no curve in X with genus $g \geq 2$.

For the moduli space $P_n^t(X, \beta)$ of Z_t -stable pairs, we want to compute

$$\int_{[P_n^t(X, \beta)]^{\text{vir}}} \tau(\gamma)^n, \quad \gamma \in H^4(X, \mathbb{Z}),$$

when X is an ideal CY 4-fold. Let $\{Z_i\}_{i=1}^n$ be 4-cycles which represent the class γ . For dimension reasons, we may assume for any $i \neq j$ the rational curves which meet with Z_i are disjoint from those with Z_j . The insertions cut out the moduli space and pick up stable pairs whose support intersects with all $\{Z_i\}_{i=1}^n$. We denote the moduli space of such ‘incident’ stable pairs by

$$Q_n^t(X, \beta; \{Z_i\}_{i=1}^n) \subseteq P_n^t(X, \beta).$$

Then we claim that

$$(4.1) \quad Q_n^t(X, \beta; \{Z_i\}_{i=1}^n) = \coprod_{\substack{\beta_0+\beta_1+\dots+\beta_n=\beta \\ \omega \cdot \beta_i > \frac{1}{t}, i=1,\dots,n}} P_0(X, \beta_0) \times Q_1(X, \beta_1; Z_1) \times \dots \times Q_1(X, \beta_n; Z_n),$$

where $Q_1(X, \beta_i; Z_i)$ is the (finite) set of rational curves (in class β_i) which meet with Z_i .

Indeed let us take a Z_t -stable pair $(\mathcal{O}_X \xrightarrow{s} F)$ in $Q_n^t(X, \beta; \{Z_i\}_{i=1}^n)$. Then F decomposes into a direct sum

$$F = F_0 \oplus \bigoplus_{i=0}^n F_i \oplus F_{n+1},$$

where F_0 is supported on elliptic curves, each F_i for $1 \leq i \leq n$ is supported on smooth rational curves which meet with Z_i , and F_{n+1} is supported on rational curves without incident condition.

Here each F_i for $1 \leq i \leq n$ is non-zero due to the incidence condition, but F_0 and F_{n+1} are possibly zero.

We take the Harder-Narasimhan filtration of F_i for $0 < i \leq n$

$$0 \subset F_{i,1} \subset F_{i,2} \subset \cdots \subset F_{i,n_i} = F_i.$$

If $s' = 0$ in the following diagram

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{s} & F_{i,n_i} \\ & \searrow s' & \downarrow \\ & & F_{i,n_i}/F_{i,n_i-1}, \end{array}$$

then Z_t stability and the HN filtration property gives

$$0 < t < \mu(F_{i,n_i}/F_{i,n_i-1}) < \mu(F_{i,n_i-1}/F_{i,n_i-2}) < \cdots < \mu(F_{i,1}),$$

hence also

$$0 < t < \mu(F_{i,n_i}) = \mu(F_i).$$

If $s' \neq 0$, then the semistable sheaf $F_{i,n_i}/F_{i,n_i-1}$ has a section. By noting that $F_{i,n_i}/F_{i,n_i-1}$ is supported on rational curves, we see that its Jordan-Hölder factors also have sections, so we conclude

$$(4.2) \quad \chi(F_{i,n_i}/F_{i,n_i-1}) > 0.$$

So we have the inequalities

$$0 < \mu(F_{i,n_i}/F_{i,n_i-1}) < \mu(F_{i,n_i-1}/F_{i,n_i-2}) < \cdots < \mu(F_{i,1}).$$

Hence we also have $\mu(F_i) > 0$. Thus in either case, we have

$$\chi(F_i) > 0, \quad 1 \leq i \leq n.$$

Similar argument also gives $\chi(F_0) \geq 0$, and $\chi(F_{n+1}) > 0$ if F_{n+1} is non-zero. Here $\chi(F_0)$ can be zero even if F_0 is non-zero, since the inequality (4.2) is replaced by $\chi(F_{0,n_0}/F_{0,n_0-1}) \geq 0$ as it is supported on an elliptic curve. From the identity

$$n = \chi(F) = \chi(F_0) + \sum_{i=1}^n \chi(F_i) + \chi(F_{n+1}),$$

we conclude that $\chi(F_0) = 0$, $\chi(F_i) = 1$ for $1 \leq i \leq n$, $F_{n+1} = 0$, and all F_i ($i \geq 0$) are semistable. Further argument using Jordan-Hölder filtration shows that F_i ($i \geq 1$) are stable (otherwise $\chi(F_i) > 1$). Hence $F_i \cong \mathcal{O}_{C_i}$ for some rational curve $\mathbb{P}^1 \cong C_i \subset X$.

Next, we discuss the role of section s . We write

$$(4.3) \quad s = s_0 \oplus \bigoplus_{i=1}^n s_i: \mathcal{O}_X \rightarrow F_0 \oplus \bigoplus_{i=0}^n F_i.$$

Note that s_i for $i \geq 1$ is either zero or surjective, since $F_i \cong \mathcal{O}_{C_i}$ as we mentioned above. If $s_i = 0$, then the pair $(\mathcal{O}_X \rightarrow F)$ decomposes as

$$(\mathcal{O}_X \rightarrow F) = (\mathcal{O}_X \rightarrow *) \oplus (0 \rightarrow F_i),$$

which violates the Z_t -stability of $(\mathcal{O}_X \rightarrow F)$. Hence s_i is surjective. Similarly, s_0 is also surjective, so $F_0 \cong \mathcal{O}_Z$ is an iterated extension of \mathcal{O}_E for an elliptic curve E .

The argument implies that, by setting $\beta_i = [F_i]$ for the pair (4.3), we have the inclusion

$$Q_n^t(X, \beta; \{Z_i\}_{i=1}^n) \subset \prod_{\substack{\beta_0 + \beta_1 + \cdots + \beta_n = \beta \\ \omega \cdot \beta_i > 0, i=1, \dots, n}} P_0(X, \beta_0) \times Q_1(X, \beta_1; Z_1) \times \cdots \times Q_1(X, \beta_n; Z_n).$$

In order to conclude (4.1), it remains to show that a pair of the form (4.3), where each s_i is surjective, $F_0 \cong \mathcal{O}_Z$ and $F_i \cong \mathcal{O}_{C_i}$ as mentioned above, is a Z_t -stable pair if and only if we have

$$(4.4) \quad \omega \cdot \beta_i > \frac{1}{t}, \quad 1 \leq i \leq n.$$

Since s is surjective, we only need to know when any $F' \subseteq F$ satisfies $\mu(F') < t$. Any $F' \subseteq F$ is of form

$$F' = \bigoplus_{i \in I} F'_i,$$

for some subset $I \subseteq \{0, 1, \dots, n\}$ such that each F'_i is a non-zero subsheaf of F_i . For a fixed I , the maximal $\mu(F')$ is achieved when $F'_i = F_i$. By taking $I = \{i\}$ for $i \geq 1$, the Z_t -stability of

the pair (4.3) implies the inequalities (4.4). Conversely suppose that (4.4) holds. Then for any $I \subseteq \{0, 1, \dots, n\}$, by setting $I' = I \cap \{1, \dots, n\}$, we have

$$\mu(F') \leq \frac{|I'|}{\sum_{i \in I'} \omega \cdot \beta_i} \leq \frac{|I'|}{\sum_{i \in I'} \omega \cdot \beta_i} < t,$$

if $I' \neq \emptyset$. If $I' = \emptyset$, then $I = \{0\}$ so $\mu(F') \leq 0 < t$. Therefore the pair (4.3) is Z_t -stable, and the identity (4.1) is justified.

Finally, note that each $Q_1(X, \beta_i; Z_i)$ consists of finitely many rational curves that meet with Z_i , whose number is exactly $n_{0, \beta_i}(\gamma)$. By counting the number of points in $P_0(X, \beta_0)$ and $Q_1(X, \beta_i; Z_i)$'s, we obtain

$$P_{n, \beta}^t(\gamma) := \int_{[P_n^t(X, \beta)]^{\text{vir}}} \tau(\gamma)^n = \int_{[Q_n^t(X, \beta; \gamma)]^{\text{vir}}} 1 = \sum_{\substack{\beta_0 + \beta_1 + \dots + \beta_n = \beta \\ \omega \cdot \beta_i > \frac{1}{t}, i=1, \dots, n}} P_{0, \beta_0} \cdot \prod_{i=1}^n n_{0, \beta_i}(\gamma).$$

Therefore we obtain the formula in Conjecture 3.4 from the above heuristic argument.

4.2. A master space argument. For each $t_0 \in \mathbb{R}_{>0}$, the formula (3.4) implies the wall-crossing formula

$$(4.5) \quad \lim_{t \rightarrow t_0^+} \text{PT}^t(X)(\exp(\gamma)) = \prod_{\omega \cdot \beta = \frac{1}{t_0}} \exp(yq^\beta)^{n_{0, \beta}(\gamma)} \cdot \lim_{t \rightarrow t_0^-} \text{PT}^t(X)(\exp(\gamma)).$$

In this subsection, we give a heuristic explanation of the above formula for a simple wall-crossing via master spaces, which are used by Mochizuki [Moc09] in proving wall-crossing formulae for Donaldson type invariants on algebraic surfaces.

For a fixed (β, n) , suppose that $t_0 \in \mathbb{R}$ is a wall with respect to the Z_t -stability. We say that t_0 is a *simple wall* if any point $p \in \overline{P}_n^{t_0}(X, \beta)$ corresponds to a Z_{t_0} -stable object, or Z_{t_0} -polystable object of the form

$$(4.6) \quad I = A \oplus B, \quad A = (\mathcal{O}_X \rightarrow F'), \quad B = F''[-1],$$

where A is Z_{t_0} -stable and F' is μ -stable with $\mu(F'') = t_0$. In other words in the description of polystable objects (1.13), we have $k \leq 1$, and if $k = 1$ then $\dim V_1 = \mathbb{C}$. In this case, $\overline{P}_n^{t_0}(X, \beta)$ is stratified as

$$(4.7) \quad \overline{P}_n^{t_0}(X, \beta) = P_n^{t_0}(X, \beta) \coprod \coprod_{\substack{(\beta', n') + (\beta'', n'') = (\beta, n) \\ \frac{n''}{\omega \cdot \beta''} = t_0}} (P_n^{t_0}(X, \beta') \times M_{n''}(X, \beta'')),$$

and $M_{n''}(X, \beta'')$ consists of only μ -stable one dimensional sheaves.

Let us take a point $p \in \overline{P}_n^{t_0}(X, \beta)$ corresponding to the polystable object I given in (4.6). Below we give a description of the diagram (1.14) locally around p , following similar arguments of [Tod17, Tod18]. Let κ be a Kuranishi map for the object I :

$$\kappa: \text{Ext}^1(I, I) \rightarrow \text{Ext}^2(I, I).$$

The above map describes the stack $\mathcal{P}_n^{t_0}(X, \beta)$ locally around $p \in \overline{P}_n^{t_0}(X, \beta)$. Namely by [Tod17, Theorem 1.1], the quotient stack

$$[\kappa^{-1}(0) / \text{Aut}(I)_0] \subset [\text{Ext}^1(I, I) / \text{Aut}(I)_0]$$

is isomorphic to the stack $\mathcal{P}_n^{t_0}(X, \beta)$ for the preimage of an analytic open neighbourhood of $p \in \overline{P}_n^{t_0}(X, \beta)$ under the map $\mathcal{P}_n^{t_0}(X, \beta) \rightarrow \overline{P}_n^{t_0}(X, \beta)$. Here $\text{Aut}(I)_0 \subset \text{Aut}(I)$ is the traceless part, given by

$$\mathbb{C}^* = \text{Aut}(I)_0 \subset \text{Aut}(I) = \text{Aut}(A) \times \text{Aut}(B), \quad u \rightarrow (\text{id}, u).$$

It acts on $\text{Ext}^1(I, I)$ by the conjugation. Note that

$$W := \text{Ext}^1(I, I) = \text{Ext}^1(A, A) \oplus \text{Ext}^1(B, B) \oplus \text{Ext}^1(A, B) \oplus \text{Ext}^1(B, A),$$

and the above \mathbb{C}^* = $\text{Aut}(I)_0$ -action on W is of weight $(0, 0, 1, -1)$. Let $W^\pm \subset W$ be the open subsets defined by

$$\begin{aligned} W^+ &= W \setminus (\text{Ext}^1(A, A) \oplus \text{Ext}^1(B, B) \oplus \{0\} \oplus \text{Ext}^1(B, A)), \\ W^- &= W \setminus (\text{Ext}^1(A, A) \oplus \text{Ext}^1(B, B) \oplus \text{Ext}^1(A, B) \oplus \{0\}). \end{aligned}$$

They are GIT stable loci with respect to different linearizations. We have the toric flip type diagram

$$\begin{array}{ccc} W^+/\mathbb{C}^* & & W^-/\mathbb{C}^* \\ & \searrow & \swarrow \\ & W//\mathbb{C}^* & \end{array}$$

Then locally around p (i.e. the preimage of an analytic open neighbourhood of $p \in \overline{P}_n^{t_0}(X, \beta)$ under the maps π^\pm in (1.14)), the moduli spaces $P_n^{t_\pm}(X, \beta)$ are isomorphic to M^\pm defined by (see the arguments of [Tod17, Thm. 7.7], [Tod18, Thm. 9.11]):

$$M^\pm := (\kappa^{-1}(0) \cap W^\pm)/\mathbb{C}^* \subset W^\pm/\mathbb{C}^*.$$

Since we have

$$\text{Ext}^2(I, I) = \text{Ext}^2(A, A) \oplus \text{Ext}^2(B, B) \oplus \text{Ext}^2(A, B) \oplus \text{Ext}^2(B, A)$$

and $\text{Ext}^2(A, B), \text{Ext}^2(B, A)$ are dual to each other, we may take

$$\text{Ext}^2(I, I)^{\frac{1}{2}} = \text{Ext}_+^2(A, A) \oplus \text{Ext}_+^2(B, B) \oplus \text{Ext}^2(A, B)$$

as a ‘half obstruction space (this half obstruction space is a mixture of positive real subspaces and maximal isotropic subspaces. We use it as it is \mathbb{C}^* -equivariant and can be descended. Its Euler class is the same as the half Euler class of $\text{Ext}^2(I, I)$). As $\text{Ext}^2(I, I)^{\frac{1}{2}} \times W \rightarrow W$ is a \mathbb{C}^* -equivariant vector bundle, it descends to a vector bundle $\text{Obs} \rightarrow [W/\mathbb{C}^*]$, which restricts to vector bundles

$$\text{Obs}^\pm \rightarrow W^\pm/\mathbb{C}^* =: \overline{W}^\pm.$$

Thus locally around p , the DT_4 virtual classes on $P_n^{t_\pm}(X, \beta)$ pushed forward to W^\pm/\mathbb{C}^* are Euler classes of the above half obstruction bundles

$$[M^\pm]^{\text{vir}} = e(\text{Obs}^\pm).$$

We compare the above virtual classes using the master space. Let \widetilde{W} be defined by

$$\widetilde{W} = (W \times \mathbb{C}^*) \sqcup (W^+ \times \{0\}) \sqcup (W^- \times \{\infty\}) \subset W \times \mathbb{P}^1.$$

Let $T_i = \mathbb{C}^*$ for $i = 1, 2$. Both of T_1 and T_2 acts on \widetilde{W} : for $t_i \in T_i$,

$$(4.8) \quad t_1 \cdot (x, [s_0, s_1]) = (t_1 x, [t_1 s_0, s_1]), \quad t_2 \cdot (x, [s_0, s_1]) = (x, [s_0, t_2 s_1]).$$

The above T_1 -action on \widetilde{W} is free, and the quotient space $Z := \widetilde{W}/T_1$ is called the *master space*. Since the two actions (4.8) commute, the T_2 -action on \widetilde{W} descends to a T_2 -action on Z . Its fixed locus is

$$Z^{T_2} = \overline{W}^+ \sqcup \overline{W}^- \sqcup W^{\mathbb{C}^*}.$$

Similarly as above, the $(T_1 \times T_2)$ -equivariant vector bundle $\text{Ext}^2(I, I)^{\frac{1}{2}} \times \widetilde{W} \rightarrow \widetilde{W}$ descends to the T_2 -equivariant vector bundle $\widetilde{\text{Obs}} \rightarrow Z$, whose Euler class is denoted by $[Z]^{\text{vir}}$. The T_2 -localization formula gives the identity in the localized T_2 -equivariant homology of Z

$$(4.9) \quad [Z]^{\text{vir}} = \frac{[M^+]^{\text{vir}}}{e(N_{\overline{W}^+}/Z)} + \frac{[M^-]^{\text{vir}}}{e(N_{\overline{W}^-}/Z)} + \frac{[W^{\mathbb{C}^*}]^{\text{vir}}}{e(N_{W^{\mathbb{C}^*}}^{\text{mov}}/Z)}.$$

Note that we have

$$W^{\mathbb{C}^*} = \text{Ext}^1(A, A) \oplus \text{Ext}^1(B, B) \oplus \{0\} \oplus \{0\},$$

and $[W^{\mathbb{C}^*}]^{\text{vir}} = [P_n^{t_0}(X, \beta')]^{\text{vir}} \times [M_{n''}(X, \beta'')]^{\text{vir}}$, viewed locally around the point (A, B) .

The above arguments are local around p , so M^\pm, Z are non-compact. However suppose that we have some globalization of the above argument (e.g. the construction of master space and its virtual class, for the global compact moduli spaces $P_n^{t_\pm}(X, \beta)$), and pretend that M^\pm, Z are compact. Then the integration of the left hand side of (4.9), after some insertions, is independent of the equivariant parameter t_2 .

Note that the real virtual dimension of M^\pm is $2n$, while that of $W^{\mathbb{C}^*}$ is $2n' + 2$. Therefore in order to obtain non-trivial contribution to the wall-crossing, by taking the insertions $\tau(\gamma)^n$ in (4.9) and the residue at $t_2 = 0$, we must have $2n = 2n' + 2$, i.e. $(n', n'') = (n - 1, 1)$. Then

$$N_{W^{\mathbb{C}^*}}^{\text{mov}}/Z = \text{Ext}^1(A, B) + \text{Ext}^1(B, A) - \text{Ext}^2(A, B)$$

has rank $-\chi(A, B) = n'' = 1$, with T_2 -weight 1, -1 , 1 respectively. Therefore the contribution of the denominator of the last term of (4.9) to the residue at $t_2 = 0$ is -1 . By taking the insertion $\tau(\gamma)^n$ and the residue at $t_2 = 0$ of (4.9), we obtain

$$\begin{aligned} & \int_{[P_n^{t_+}(X, \beta)]^{\text{vir}}} \tau(\gamma)^n - \int_{[P_n^{t_-}(X, \beta)]^{\text{vir}}} \tau(\gamma)^n \\ &= \sum_{\substack{\beta' + \beta'' = \beta \\ \frac{1}{\omega \cdot \beta''} = t_0}} \int_{[P_{n-1}^{t_0}(X, \beta')]^{\text{vir}} \times [M_1(X, \beta'')]^{\text{vir}}} (\tau(\gamma) \boxtimes 1 + 1 \boxtimes \tau(\gamma))^n. \end{aligned}$$

By expanding the RHS and assuming Conjecture 3.1, we obtain the wall-crossing formula

$$(4.10) \quad P_{n, \beta}^{t_+}(\gamma) - P_{n, \beta}^{t_-}(\gamma) = \sum_{\substack{\beta' + \beta'' = \beta \\ \frac{1}{\omega \cdot \beta''} = t_0}} n \cdot P_{n-1, \beta'}^{t_0}(\gamma) \cdot n_{0, \beta''}(\gamma).$$

Indeed this wall-crossing formula is compatible with our main conjecture.

Proposition 4.1. *Suppose that $t_0 \in \mathbb{R}_{>0}$ is a simple wall with respect to (β, n) . Then under Conjecture 3.1, the coefficient of $y^n q^\beta$ in the formula (4.5) is equivalent to (4.10).*

Proof. By expanding the formula (4.5), the identity at the coefficient of $y^n q^\beta$ is

$$(4.11) \quad P_{n, \beta}^{t_+}(\gamma) - P_{n, \beta}^{t_-}(\gamma) = \sum_{k=1}^n \frac{n!}{k!(n-k)!} \sum_{\substack{\beta' + \beta'' = \beta \\ \beta_1'' + \dots + \beta_k'' = \beta'', \omega \cdot \beta_i'' = \frac{1}{t_0}}} P_{n-k, \beta'}^{t_-}(\gamma) \cdot \prod_{i=1}^k n_{0, \beta_i''}(\gamma).$$

If $t_0 \in \mathbb{R}_{>0}$ is a simple wall, then a term $P_{n-k, \beta'}^{t_-}(\gamma) \cdot \prod_{i=1}^k n_{0, \beta_i''}(\gamma)$ is non-zero only if $k = 1$. Otherwise, we have

$$P_{n-k}^{t_-}(X, \beta') \times \prod_{i=1}^k M_1(X, \beta_i'') \neq \emptyset.$$

For a point $(I', F_1'', \dots, F_k'')$ in the above product, I' is Z_{t_0} -semistable. Therefore by denoting $\text{gr}(I')$ the associated graded with respect to Jordan-Hölder filtration of the Z_{t_0} -stability, we have

$$\text{gr}(I') \oplus F_1''[-1] \oplus \dots \oplus F_k''[-1] \in \overline{\mathcal{P}}_n^{t_0}(X, \beta).$$

The above Z_{t_0} -polystable object is of the form (4.6) only if $k = 1$ and I' is Z_{t_0} -stable. By the same reason, we have $P_{n-1}^{t_-}(X, \beta') = P_{n-1}^{t_0}(X, \beta') = \overline{\mathcal{P}}_{n-1}^{t_0}(X, \beta')$, hence $P_{n-1, \beta'}^{t_-}(\gamma) = P_{n-1, \beta'}^{t_0}(\gamma)$. Therefore the identity (4.11) is nothing but the formula (4.10) if t_0 is a simple wall. \square

If (β, n) satisfies the condition (1.16), there is no wall-crossing for $t > \frac{n}{\omega \cdot \beta}$, and PT and JS pairs are the same. Indeed in this case, our main conjecture is compatible with our previous PT/GV conjecture.

Proposition 4.2. *Suppose that (β, n) satisfies (1.16). Then we have*

$$(4.12) \quad P_{n, \beta}(\gamma) = P_{n, \beta}^t(\gamma) = P_{n, \beta}^{\text{JS}}(\gamma), \quad t > \frac{n}{\omega \cdot \beta},$$

for certain choice of orientation. And the first identity of Conjecture 3.3, Conjecture 3.4 for $t > \frac{n}{\omega \cdot \beta}$, and Conjecture 3.5 are equivalent.

Proof. The identities (4.12) follows from Proposition 1.13. In order to show the compatibilities of conjectures, by the argument of Proposition 4.1, it is enough to show that for any $t_0 > \frac{n}{\omega \cdot \beta}$ the right hand side of (4.11) vanishes. Suppose that it is non-zero, and take $(\beta_0, \beta_1, \dots, \beta_k)$ as in the right hand side of (4.11). Then from the inequalities

$$t_0 = \frac{1}{\omega \cdot \beta_i} > \frac{n}{\omega \cdot \beta}, \quad 1 \leq i \leq k,$$

together with the condition (1.16), we obtain the inequality

$$(4.13) \quad \frac{n-k}{\omega \cdot \beta_0} < \frac{n}{\omega \cdot \beta} \leq \frac{n(\beta')}{\omega \cdot \beta'}, \quad 0 < \beta' \leq \beta_0 < \beta.$$

Therefore the condition (1.16) is satisfied for $(\beta_0, n-k)$, so $P_{n-k}^{t_-}(X, \beta_0) = P_{n-k}(X, \beta_0) \neq \emptyset$. Thus $n(\beta_0) \leq n-k$ by the definition of $n(\beta_0)$, which contradicts to (4.13) for $\beta' = \beta_0$. \square

If the condition (1.16) is not satisfied, we have wall-crossing phenomena as observed in Example 1.14. In this example, there is nontrivial wall-crossing of our invariants.

Example 4.3. *In the situation of Example 1.14, the $t = 1$ is a simple wall, and the stratification (4.7) is given by*

$$\overline{P}_1^{t=1}(X, 4[l]) = P_1^{t=1}(X, 4[l]) \coprod (P_0(X, 3[l]) \times M_1(X, [l])).$$

Let $[\text{pt}] \in H^4(X, \mathbb{Z}) = H^4(\mathbb{P}^2, \mathbb{Z})$ be the point class. From the formula (4.10), we should have the identity

$$P_{1,4[l]}([\text{pt}]) - P_{1,4[l]}^{\text{JS}}([\text{pt}]) = P_{0,3[l]} \cdot n_{0,[l]}([\text{pt}]).$$

Indeed in this case, we can compute the invariants and conclude (ref. Proposition 5.15):

$$P_{1,4[l]}^t([\text{pt}]) = \begin{cases} n_{0,4}([\text{pt}]) + P_{0,3} \cdot n_{0,1}([\text{pt}]) = 3, & \text{if } t > 1 \\ n_{0,4}([\text{pt}]) = 2, & \text{if } \frac{1}{4} < t < 1. \end{cases}$$

4.3. A virtual pushforward formula. The formula for $n = 1$ in Conjecture 3.5 is

$$(4.14) \quad P_{1,\beta}^{\text{JS}}(\gamma) = n_{0,\beta}(\gamma),$$

which could be understood from both wall-crossing and virtual pushforward formulae.

In terms of wall-crossing, let us take $t_0 = \frac{1}{\omega \cdot \beta}$. Then $t = t_0$ is a simple wall with respect to $(\beta, 1)$, so the formula (4.10) gives

$$P_{1,\beta}^{t_0^+}(\gamma) - P_{1,\beta}^{t_0^-}(\gamma) = \sum_{\substack{\beta' + \beta'' = \beta, \\ \omega \cdot \beta'' = \omega \cdot \beta}} 1 \cdot P_{0,\beta'} \cdot n_{0,\beta''}(\gamma).$$

Note that $\omega \cdot \beta' = 0$ implies that $\beta' = 0$ as β' is an effective class or zero. Since $P_{1,\beta}^{t_0^+}(\gamma) = P_{1,\beta}^{\text{JS}}(\gamma)$, $P_{1,\beta}^{t_0^-}(\gamma) = 0$, and $P_{0,0} = 1$, we obtain the identity (4.14).

In terms of virtual push-forward formula, we consider the morphism in (1.15):

$$P_1^{\text{JS}}(X, \beta) \rightarrow M_1(X, \beta), \quad (\mathcal{O}_X \rightarrow F) \mapsto F.$$

We expect that the following virtual pushforward formula

$$(4.15) \quad f_*([P_1^{\text{JS}}(X, \beta)]^{\text{vir}}) = [M_1(X, \beta)]^{\text{vir}} \in H_2(M_1(X, \beta), \mathbb{Z})$$

holds for certain choice of orientation. Capping with insertions gives

$$P_{1,\beta}^{\text{JS}}(\gamma) = \int_{[M_1(X, \beta)]^{\text{vir}}} \tau(\gamma),$$

where τ is the primary insertion (3.3) for one dimensional stable sheaves. Assuming this, the equality $P_{1,\beta}^{\text{JS}}(\gamma) = n_{0,\beta}(\gamma)$ is reduced to the ‘Katz/GV’ conjecture mentioned in Conjecture 3.1.

Generally speaking, the virtual class in DT_4 theory is difficult to work with. In the special case when the moduli space is a (-2) -shifted cotangent bundle of some derived smooth scheme (as reviewed in Section 2.1), the virtual class can be described algebraically. The virtual pushforward formula (4.15) can be rigorously proved, due to the work of Manolache [Man12]. We review her formula in the following setting:

Theorem 4.4. (Manolache [Man12, Thm.]) *Given a proper morphism $f : P \rightarrow M$ between Deligne-Mumford stacks which possess perfect obstruction theories E_P^\bullet and E_M^\bullet . If f has a perfect relative obstruction theory compatible with E_P^\bullet and E_M^\bullet and M is connected. Assume the virtual dimension of the relative obstruction theory is zero, then*

$$f_*[P]^{\text{vir}} = c \cdot [M]^{\text{vir}},$$

where $c \in \mathbb{Q}$ is the degree of the relative perfect obstruction theory.

In Section 5, we will apply it to several examples and prove Conjecture 3.5 (with $n = 1$) in those cases.

5. EXAMPLES OF JS/GV FORMULA

Evidence of Conjecture 3.4 in the Pandharipande-Thomas chamber is given in [CMT19]. In this section, we give further verifications of Conjecture 3.4 mainly concentrated in the Joyce-Song chamber as stated in the form of Conjecture 3.5.

5.1. Irreducible curve class. When the curve class is irreducible, there is no difference between JS and PT chamber. We refer to our previous work [CMT19, Prop. 1.4, 1.8, Thm. 1.5, 1.7] for many checks of Conjecture 3.4 in such setting.

Proposition 5.1. *Let (X, ω) be a Calabi-Yau 4-fold and $\beta \in H_2(X, \mathbb{Z})$ be an irreducible curve class. Then we have*

$$P_{n, \beta}^t(\gamma) = P_{n, \beta}(\gamma), \quad t > \frac{n}{\omega \cdot \beta},$$

for certain choice of orientation.

Proof. Since β is irreducible, the condition (1.16) is automatically satisfied. Therefore we have $P_n^t(X, \beta) = P_n(X, \beta)$ for $t > \frac{n}{\omega \cdot \beta}$ by Proposition 1.13. \square

5.2. Degree two curve class. Let $X \subseteq \mathbb{P}^5$ be a smooth sextic 4-fold with hyperplane class ω . By Lefschetz hyperplane theorem, $H_2(X, \mathbb{Z}) \cong H_2(\mathbb{P}^5, \mathbb{Z}) = \mathbb{Z}[l]$, where l is the class of a line.

Proposition 5.2. *For $n = 0, 1, 2$ and degree two class $\beta = 2[l] \in H_2(X, \mathbb{Z})$, we have*

$$P_{n, \beta}^t(\gamma) = P_{n, \beta}(\gamma), \quad t > \frac{n}{2},$$

for certain choice of orientation. Furthermore, Conjecture 3.4 holds for $\beta = 2[l]$ and $n = 0, 1$.

Proof. The condition (1.16) is satisfied for $(2[l], n)$ with $n \leq 2$, since $n([l]) = 1$. Therefore we have $P_n^t(X, 2[l]) = P_n(X, 2[l])$ for $n \leq 2$. Under the isomorphism of moduli spaces, virtual classes are identified and invariants are the same by choosing same orientations and insertions. When $n = 0, 1$ and $t > \frac{1}{2}$, Conjecture 3.4 then reduces to [CMT19, Prop. 3.1, 3.2] (see also [Cao1]). The case $n = 0, 1$ and $t < \frac{1}{2}$ is obvious. \square

5.3. Elliptic fibration. For $Y = \mathbb{P}^3$, we take general elements

$$u \in H^0(Y, \mathcal{O}_Y(-4K_Y)), \quad v \in H^0(Y, \mathcal{O}_Y(-6K_Y)).$$

We define X to be the hypersurface

$$X = \{zy^2 = x^3 + uxz^2 + vz^3\} \subset \mathbb{P}(\mathcal{O}_Y(-2K_Y) \oplus \mathcal{O}_Y(-3K_Y) \oplus \mathcal{O}_Y).$$

Here $[x : y : z]$ is the homogeneous coordinate of the projective bundle over Y in the right hand side. Then X is a CY 4-fold, and the projection to Y gives an elliptic fibration

$$(5.1) \quad \pi: X \rightarrow Y.$$

A general fiber of π is a smooth elliptic curve, and any singular fiber is either a nodal or cuspidal plane curve. Moreover, π admits a section ι whose image corresponds to fiber point $[0 : 1 : 0]$.

Let h be a hyperplane in Y and $f := \pi^{-1}(p)$ for a general point $p \in \mathbb{P}^3$, set

$$B = \pi^*h, \quad E = \iota(Y) \in H_6(X, \mathbb{Z}).$$

We consider multiple fiber classes $r[f]$ ($r \geq 1$) below.

Proposition 5.3. *For any $t > 0$ and certain choice of orientation, we have*

$$P_{0, r[f]}^t = P_{0, r[f]}.$$

Furthermore, Conjecture 3.4 holds for $\beta = r[f]$ ($r \geq 1$) and $n = 0$.

Proof. The condition (1.16) is satisfied since $n(r'[f]) = 0$ for any $r' > 0$. Therefore $P_0^t(X, r[f]) = P_0(X, r[f])$ and their virtual classes are identified. Then Conjecture 3.4 follows from the statement for PT stable pairs [CMT19, Prop. 3.6]. \square

Proposition 5.4. *For certain choice of orientation, we have*

$$P_{1, r[f]}^{\text{JS}}(\gamma) = \int_{[M_1(X, r[f])]^{\text{vir}}} \tau(\gamma).$$

Moreover, Conjecture 3.5 holds for $\beta = r[f]$ ($r \geq 1$), $n = 1$ and $\gamma = B \cdot E$ or B^2 .

Proof. Let $M_1(X, r[f])$ be the moduli space of one dimensional stable sheaves F with $[F] = r[f]$ and $\chi(F) = 1$. An element $[F] \in M_1(X, r[f])$ is scheme theoretically supported on a fiber $\pi^{-1}(p)$ of π (ref. [CMT18, Lem. 2.2]). Write $F = i_*\mathcal{F}$ for the inclusion $i: \pi^{-1}(p) \rightarrow X$, then

$$H^1(X, F) \cong H^1(\pi^{-1}(p), \mathcal{F}) \cong \text{Hom}(\mathcal{F}, \mathcal{O}_{\pi^{-1}(p)})^\vee = 0,$$

since the slope of \mathcal{F} is bigger than the slope of $\mathcal{O}_{\pi^{-1}(p)}$.

The morphism

$$f : P_1^{\text{JS}}(X, r[f]) \rightarrow M_1(X, r[f]), \quad (\mathcal{O}_X \rightarrow F) \mapsto F$$

is then an isomorphism as the fiber is $\mathbb{P}(H^0(X, F))$ and $h^0(X, F) = h^1(X, F) + 1 = 1$. Conjecture 3.5 in this case reduces to Conjecture 3.1, which has been verified in [CMT18, Prop. 2.3]. \square

5.4. Product of CY 3-fold and elliptic curve. Let Y be a smooth projective Calabi-Yau 3-fold and E is an elliptic curve. Argument of Proposition 5.3 leads straightforward to:

Proposition 5.5. *Let $X = Y \times E$ be as above. Then for any $t > 0$, $r \geq 1$, we have*

$$P_{0,r[E]}^t = P_{0,r[E]},$$

for certain choice of orientation. Hence Conjecture 3.4 holds for $\beta = r[E]$ ($r \geq 1$) and $n = 0$.

Next, we discuss the $n = 1$ case. For any smooth projective variety Y , we may define the moduli space $P_n^{\text{JS}}(Y, \beta)$ of Joyce-Song stable pairs on Y by Definition 1.10 (ii). We have the following deformation-obstruction theory (called pair deformation-obstruction theory):

$$\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{F})^\vee \rightarrow \tau_{\geq -1} \mathbb{L}_{P_n^{\text{JS}}(X, \beta)}.$$

Here $\mathbb{I} = (\mathcal{O}_{X \times P_n^{\text{JS}}(X, \beta)} \rightarrow \mathbb{F})$ is the universal pair, and $\pi_P: X \times P_n^{\text{JS}}(X, \beta) \rightarrow P_n^{\text{JS}}(X, \beta)$ is the projection.

Lemma 5.6. *Let (Y, ω) be a smooth projective Calabi-Yau 3-fold and $\beta \in H_2(Y, \mathbb{Z})$. Then the truncated pair deformation-obstruction theory*

$$\tau_{\geq -1}(\mathbf{R}\mathcal{H}om_{\pi}(\mathbb{I}, \mathbb{F})^\vee) \rightarrow \tau_{\geq -1} \mathbb{L}_{P_1^{\text{JS}}(X, \beta)}$$

of $P_1^{\text{JS}}(Y, \beta)$ is perfect in the sense of [BF97, LT98]. Hence there exists an algebraic virtual class of virtual dimension zero

$$[P_1^{\text{JS}}(Y, \beta)]_{\text{pair}}^{\text{vir}} \in A_0(P_1^{\text{JS}}(Y, \beta), \mathbb{Z}).$$

Proof. For any JS stable pair $I_Y = (\mathcal{O}_Y \xrightarrow{s} F) \in P_1^{\text{JS}}(Y, \beta)$, F is stable as it is semistable with $\chi(F) = 1$, Therefore we have

$$\text{Ext}_Y^3(F, F) \cong \text{Hom}_Y(F, F)^\vee \cong \mathbb{C}.$$

Applying $\mathbf{R}\text{Hom}_Y(-, F)$ to $I_Y \rightarrow \mathcal{O}_Y \rightarrow F$, we obtain a distinguished triangle

$$(5.2) \quad \mathbf{R}\text{Hom}_Y(F, F) \rightarrow \mathbf{R}\text{Hom}_Y(\mathcal{O}_Y, F) \rightarrow \mathbf{R}\text{Hom}_Y(I_Y, F),$$

whose cohomology gives an exact sequence

$$0 = H^2(Y, F) \rightarrow \text{Ext}_Y^2(I_Y, F) \rightarrow \text{Ext}_Y^3(F, F) \rightarrow 0 \rightarrow \text{Ext}_Y^3(I_Y, F) \rightarrow 0.$$

Hence $\text{Ext}_Y^i(I_Y, F) = 0$ for $i \geq 3$ and $\text{Ext}_Y^2(I_Y, F) \cong \text{Ext}_Y^3(F, F) \cong \mathbb{C}$. By truncating $\text{Ext}_Y^2(I_Y, F) = \mathbb{C}$, the pair deformation-obstruction theory is perfect. \square

Theorem 5.7. *Let $X = Y \times E$ be as above. Assume Conjecture 3.1 holds for $\beta \in H_2(Y) \subseteq H_2(X)$. Then for any $\gamma \in H^4(X)$, we have*

$$P_{1,\beta}^{\text{JS}}(\gamma) = n_{0,\beta}(\gamma),$$

for certain choice of orientation, i.e. Conjecture 3.5 holds for $\beta \in H_2(Y) \subseteq H_2(X)$ and $n = 1$.

Proof. We take a JS stable pair $(\mathcal{O}_X \xrightarrow{s} F) \in P_1^{\text{JS}}(X, \beta)$. Then F is stable and scheme theoretically supported on $i_p : Y \times \{p\} \hookrightarrow X$ for some $p \in E$ (ref. [CMT18, Lem. 2.2]). Similar to [CMT19, Prop. 3.11], there exists an isomorphism

$$P_1^{\text{JS}}(X, \beta) \cong P_1^{\text{JS}}(Y, \beta) \times E, \\ (\mathcal{O}_X \xrightarrow{s} i_{p*} \mathcal{E}) \mapsto ((\mathcal{O}_Y = i_p^* \mathcal{O}_X \xrightarrow{s} \mathcal{E}), p),$$

under which virtual classes satisfy

$$[P_1^{\text{JS}}(X, \beta)]^{\text{vir}} = [P_1^{\text{JS}}(Y, \beta)]_{\text{pair}}^{\text{vir}} \times [E],$$

where $[P_1^{\text{JS}}(Y, \beta)]_{\text{pair}}^{\text{vir}}$ denotes the virtual class defined in Lemma 5.6.

There is a forgetful morphism

$$f : P_1^{\text{JS}}(Y, \beta) \rightarrow M_1(Y, \beta), \quad (s : \mathcal{O}_Y \rightarrow \mathcal{E}) \mapsto \mathcal{E},$$

to the moduli space $M_1(Y, \beta)$ of one dimensional stable sheaves F on Y with $[F] = \beta$ and $\chi(F) = 1$. The fiber of f over F is $\mathbb{P}(H^0(Y, F))$.

Let $\mathbb{F} \rightarrow M_1(Y, \beta) \times Y$ be the universal sheaf. Then the above map identifies $P_1^{\text{JS}}(Y, \beta)$ with $\mathbb{P}(\pi_{M*}\mathbb{F})$ where $\pi_M: M_1(Y, \beta) \times Y \rightarrow M_1(Y, \beta)$ is the projection. And the universal stable pair is given by

$$\mathbb{I} = (\mathcal{O}_{Y \times P_1^{\text{JS}}(Y, \beta)} \xrightarrow{s} \mathbb{F}^\dagger), \quad \mathbb{F}^\dagger := (\text{id}_Y \times f)^* \mathbb{F} \otimes \mathcal{O}(1),$$

where $\mathcal{O}(1)$ is the tautological line bundle on $\mathbb{P}(\pi_{M*}\mathbb{F})$ and s is the tautological map.

As in [CMT19, Prop. 3.10], we can apply Theorem 4.4 and obtain

$$f_*[P_1^{\text{JS}}(Y, \beta)]_{\text{pair}}^{\text{vir}} = c[M_1(Y, \beta)]^{\text{vir}},$$

where the coefficient c can be fixed by restricting the relative perfect obstruction theory to a fiber of f (ref. [Man12, pp. 2022 (18)]). The obstruction bundle over fiber $\mathbb{P}(H^0(Y, F))$ is $H^1(Y, F) \otimes \mathcal{O}(1)$ whose rank is $(h^0(Y, F) - 1)$ as $\chi(F) = 1$. Thus

$$c = \int_{\mathbb{P}(H^0(Y, F))} e(H^1(Y, F) \otimes \mathcal{O}(1)) = 1.$$

To sum up, we obtain

$$(f \times \text{id}_E)_*[P_1^{\text{JS}}(X, \beta)]^{\text{vir}} = [M_1(X, \beta)]^{\text{vir}},$$

where we use $M_1(X, \beta) \cong M_1(Y, \beta) \times E$ and $[M_1(X, \beta)]^{\text{vir}} = [M_1(Y, \beta)]^{\text{vir}} \times [E]$ (ref. [CMT18, Lem. 2.6]).

As the insertion τ (2.8) depends only on \mathbb{F} (not the section), we have

$$P_{1, \beta}^{\text{JS}}(\gamma) = \int_{[P_1^{\text{JS}}(X, \beta)]^{\text{vir}}} \tau(\gamma) = \int_{[M_1(X, \beta)]^{\text{vir}}} \tau(\gamma),$$

where we use same notation τ to denote the insertion for $M_1(X, \beta)$. By Conjecture 3.1, we have

$$\int_{[M_1(X, \beta)]^{\text{vir}}} \tau(\gamma) = n_{0, \beta}(\gamma),$$

for certain choice of orientation. Combining the two equalities, we are done. \square

Combining with previous verifications of Conjecture 3.1 (ref. [CMT18, Thm. 2.8]), we have:

Corollary 5.8. *Let Y be a complete intersection CY 3-fold in a product of projective spaces, and $X = Y \times E$ for an elliptic curve E . Then Conjecture 3.5 holds for primitive curve class $\beta \in H_2(Y) \subseteq H_2(X)$ and $n = 1$.*

5.5. Local Fano 3-folds. Let Y be a smooth Fano 3-fold and consider the total space $X = K_Y$ of canonical bundle of Y . Take ω to be the pullback ample line bundle from an ample line bundle on Y . The moduli space $P_1^{\text{JS}}(X, \beta)$ of Joyce-Song stable pairs is proper, since for any pair $(\mathcal{O}_X \rightarrow F)$, F is stable and scheme theoretically supported on Y . So we can still study Conjecture 3.5 on such non-compact Calabi-Yau 4-folds. Similar to Theorem 5.7, we have

Theorem 5.9. *Let $X = K_Y$ be as above. Assume Conjecture 3.1 holds for $\beta \in H_2(X)$. Then for any $\gamma \in H^4(X)$, we have*

$$P_{1, \beta}^{\text{JS}}(\gamma) = n_{0, \beta}(\gamma),$$

for certain choice of orientation, i.e. Conjecture 3.5 holds for $\beta \in H_2(X)$ and $n = 1$.

Combining with the previous verification of Conjecture 3.1 (ref. [Cao2, Prop. 0.3, 0.4, Thm. 0.6]), we have:

Corollary 5.10. *Conjecture 3.5 holds for $\beta \in H_2(K_Y)$ and $n = 1$ in the following cases:*

- $Y \subseteq \mathbb{P}^4$ is a smooth Fano hypersurface and β is irreducible.
- $Y = S \times \mathbb{P}^1$ and $\beta = n[\mathbb{P}^1]$ ($n \geq 1$), where S is a del Pezzo surface.
- $Y = S \times \mathbb{P}^1$, $\beta \in H_2(S) \subseteq H_2(Y)$ and $\gamma \in H^2(S) \otimes H^2(\mathbb{P}^1) \subset H^4(Y)$, where S is a toric del Pezzo surface.

5.6. Local surfaces. In this section, we consider two local surfaces:

$$(5.3) \quad X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2)), \quad \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1)).$$

For the first one, we choose ω to be the pullback of $\mathcal{O}_{\mathbb{P}^2}(1) \rightarrow \mathbb{P}^2$, and for the second one, we choose ω to be the pullback of $\mathcal{O}_{\mathbb{P}^1}(l_1) \boxtimes \mathcal{O}_{\mathbb{P}^1}(l_2) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$, where $l_1, l_2 > 0$.

Although X is non-compact, the moduli space of Z_t -semistable pairs on X is proper, so we can study Conjecture 3.4 on X .

Lemma 5.11. *For any $t > 0$, the moduli space $\overline{P}_n^t(X, \beta)$ of Z_t -semistable pairs on X is proper.*

Proof. Let \overline{X} be the compactification of X by adding section at infinity, i.e.

$$\overline{X} := \mathbb{P}(L_1 \oplus L_2 \oplus \mathcal{O}_S).$$

Then $\overline{P}_n^t(\overline{X}, \beta)$ is proper as \overline{X} is so. For a Z_t -semistable pair $(\mathcal{O}_{\overline{X}} \xrightarrow{s} F) \in \mathcal{P}_n^t(\overline{X}, \beta)$, F is set theoretically supported on the zero section by the the negativity of normal bundle of $S \subseteq X$ (ref. [CMT18, Prop. 3.1]). Therefore $\overline{P}_n^t(\overline{X}, \beta)$ is isomorphic to $\overline{P}_n^t(X, \beta)$ and $\overline{P}_n^t(X, \beta)$ is proper. \square

Proposition 5.12. *Assume $(n, \omega \cdot \beta) = 1$, then we have an isomorphism*

$$P_n^{\text{JS}}(X, \beta) \cong P_n^{\text{JS}}(S, \beta),$$

where $S = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$ is the zero section of X (5.3).

Moreover, the virtual class satisfies

$$[P_n^{\text{JS}}(X, \beta)]^{\text{vir}} = [P_n^{\text{JS}}(S, \beta)]^{\text{vir}} \cdot e\left(-\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}, \mathbb{F} \boxtimes L_1)\right),$$

for certain choice of orientation. Here $\mathbb{I}_S = (\mathcal{O}_{S \times P_n^{\text{JS}}(S, \beta)} \rightarrow \mathbb{F}) \in D^b(S \times P_n^{\text{JS}}(S, \beta))$ is the universal stable pair and $\pi_P: S \times P_n^{\text{JS}}(S, \beta) \rightarrow P_n^{\text{JS}}(S, \beta)$ is the projection. The virtual class $[P_n^{\text{JS}}(S, \beta)]^{\text{vir}}$ is constructed with respect to the perfect obstruction theory $\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}_S, \mathbb{F})$.

Proof. Given $(\mathcal{O}_X \xrightarrow{s} F) \in P_n^{\text{JS}}(X, \beta)$ such that $(n, \omega \cdot \beta) = 1$, F is stable and hence scheme theoretically supported on S (ref. [CMT18, Prop. 3.1]). Similar to [CMT19, Prop. 4.7], [CKM20, Prop. 4.2] we have an isomorphism

$$P_n^{\text{JS}}(X, \beta) \cong P_n^{\text{JS}}(S, \beta)$$

of moduli spaces, under which virtual classes have the desired property. \square

As $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$ and the primary insertion (2.8) is linear with respect to $\gamma \in H^4(X, \mathbb{Z})$, so we may simply take $\gamma = [\text{pt}]$ to be the generator $H^4(X, \mathbb{Z}) \cong \mathbb{Z}$. By the same argument as in Theorem 5.7, we have

Theorem 5.13. *Let X be one of the above two local surfaces (5.3). Assume Conjecture 3.1 holds for $\beta \in H_2(X)$. Then we have*

$$P_{1, \beta}^{\text{JS}}([\text{pt}]) = n_{0, \beta}([\text{pt}]),$$

for certain choice of orientation, i.e. Conjecture 3.5 holds for $\beta \in H_2(X)$ and $n = 1$.

Note also the following case where our conjecture holds by a trivial reason.

Proposition 5.14. *If $n > 1$ and $\omega \cdot \beta$ are coprime, then*

$$P_{n, \beta}^{\text{JS}}([\text{pt}]) = 0.$$

Moreover, Conjecture 3.5 holds in this setting.

Proof. As $(n, \omega \cdot \beta) = 1$, the coarse moduli space $M_n(X, \beta)$ of one dimensional semistable sheaves F on X with $[F] = \beta$ and $\chi(F) = n$ consists of stable sheaves only. By [CMT18, Prop. 3.1], F is scheme theoretically supported on the zero section S of X , and we have an isomorphism

$$M_n(X, \beta) \cong M_n(S, \beta),$$

to the (smooth) coarse moduli space $M_n(S, \beta)$ of one dimensional stable sheaves on S .

We have a surjective (as $n > 1$) forgetful map

$$f: P_n^{\text{JS}}(S, \beta) \rightarrow M_n(S, \beta), \quad (\mathcal{O}_X \rightarrow F) \mapsto F.$$

By Riemann-Roch formula, we know

$$\text{vir.dim}_{\mathbb{C}}(P_n(S, \beta)) = n + \beta^2, \quad \dim_{\mathbb{C}}(M_n(S, \beta)) = 1 + \beta^2.$$

By Proposition 5.12, we have

$$\begin{aligned} P_{n, \beta}^{\text{JS}}([\text{pt}]) &= \int_{[P_n^{\text{JS}}(S, \beta)]^{\text{vir}}} \tau([\text{pt}]) \cdot e\left(-\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{F}, \mathbb{F} \boxtimes L_1)\right) \\ &= \int_{[P_n^{\text{JS}}(S, \beta)]^{\text{vir}}} f^*\left(\tau([\text{pt}]) \cdot e\left(-\mathbf{R}\mathcal{H}om_{\pi_M}(\mathbb{F}, \mathbb{F} \boxtimes L_1)\right)\right) \\ &= \int_{f_*[P_n^{\text{JS}}(S, \beta)]^{\text{vir}}} \left(\tau([\text{pt}]) \cdot e\left(-\mathbf{R}\mathcal{H}om_{\pi_M}(\mathbb{F}, \mathbb{F} \boxtimes L_1)\right)\right) \\ &= 0. \end{aligned}$$

Here the second equality is because the insertion comes from the pull-back from $M_n(S, \beta)$ via f , and in the last equality we use $f_*[P_n^{\text{JS}}(S, \beta)]^{\text{vir}} = 0$ by a dimension counting.

In the coprime case $(n, \omega \cdot \beta) = 1$, the conjectural formula in Conjecture 3.5 obviously gives vanishing $P_{n, \beta}^{\text{JS}}([\text{pt}]) = 0$, which coincides with the above computations. \square

To sum up, we verify Conjecture 3.5 for low degree curve classes. In the following, we denote by $[l] \in H_2(X, \mathbb{Z}) = H_2(\mathbb{P}^2, \mathbb{Z})$ to be the line class.

Proposition 5.15. *Conjecture 3.5 holds for $X = \text{Tot}_{\mathbb{P}^2}(\mathcal{O}(-1) \oplus \mathcal{O}(-2))$ in the following cases*

- $\beta = 2[l]$ and $n = 0, 1, 2, 2k + 1$ ($k \geq 1$).
- $\beta = 3[l]$ and $n = 0, 1, 3k \pm 1$ ($k \geq 1$).
- $\beta = 4[l]$ and $n = 0, 2k + 1$ ($k \geq 1$).

Proof. When $(n, \omega \cdot \beta) = 1$, Conjecture 3.5 is reduced to Theorem 5.13, Proposition 5.14 and our previous verification of Conjecture 3.1 (ref. [CMT18, Sect. 3.2]). The $\beta = 2[l]$, $n = 2$ case follows from a similar argument as Proposition 5.2.

When $n = 0$, we discuss $\beta = 4[l]$ case (other cases follow from an easier argument). In this case, the condition (1.16) is satisfied since $n([l]) = n(2[l]) = 1$ and $n(3[l]) = 0$. Therefore $P_0(X, 4[l]) = P_0^{\text{JS}}(X, 4[l])$, and we have the identity $P_{0, 4[l]} = P_{0, 4[l]}^{\text{JS}}$ for certain choice of orientation and we then use [CKM20, Cor. 1.6]. \square

Similar to Proposition 5.15, we also have:

Proposition 5.16. *Conjecture 3.5 holds for $X = \text{Tot}_{\mathbb{P}^1 \times \mathbb{P}^1}(\mathcal{O}(-1, -1) \oplus \mathcal{O}(-1, -1))$ in the following cases*

- $\beta = (0, d)$ ($d \geq 1$) and $n = 0, 1, 2$.
- $\beta = (1, d)$ ($d \geq 1$) and $n = 0, 1$.
- $\beta = (2, 2)$ and $n = 0, 1$.

Proof. For $n = 1$ case, by Theorem 5.13, we are reduced to prove Conjecture 3.1 in those cases. When $\beta = (2, 2)$, this was done in [CMT18, Sect. 3.2]. When $\beta = (0, d)$, any one dimensional stable sheaf F in this class is scheme theoretically supported on one \mathbb{P}^1 factor (ref. [CMT18, Lem. 2.2]). This is possible only when $d = 1$, Conjecture 3.1 then follows easily. When $\beta = (1, d)$, this is not discussed in the previous literature. To be self-contained, we include the argument here. Any Cohen-Macaulay curve C in class $(1, d)$ has $\chi(\mathcal{O}_C) = 1$. Hence $M_1(X, \beta)$ is isomorphism to the moduli space

$$\mathcal{M} = \mathbb{P}\left(H^0(\mathbb{P}^1 \times \mathbb{P}^1, \mathcal{O}(1, d))\right) \cong \mathbb{P}^{2d+1}$$

of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ with curve class $\beta = (1, d)$. The universal curve

$$\mathcal{Z} \subset \mathcal{M} \times \mathbb{P}^1 \times \mathbb{P}^1$$

is the $(1, 1, d)$ divisor. By [CMT18, Prop. 3.1], any $F = \mathcal{O}_C \in M_1(X, \beta)$ is scheme theoretically supported on the zero section $\iota : S := \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow X$, then

$$\begin{aligned} \text{Ext}_X^2(F, F) &\cong \text{Ext}_S^2(\mathcal{O}_C, \mathcal{O}_C) \oplus \text{Ext}_S^1(\mathcal{O}_C, \mathcal{O}_C \otimes \mathcal{O}(-1, -1)^{\oplus 2}) \oplus \text{Ext}_S^0(\mathcal{O}_C, \mathcal{O}_C \otimes \mathcal{O}(-2, -2)) \\ &\cong \text{Ext}_S^1(\mathcal{O}_C, \mathcal{O}_C \otimes \mathcal{O}(-1, -1))^{\oplus 2}. \end{aligned}$$

By Section 2.1, for certain choice of orientation, we have

$$[M_1(X, \beta)]^{\text{vir}} = [\mathcal{M}] \cap e(\mathcal{E}xt_{\pi_M}^1(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{O}(-1, -1))),$$

where $\pi_M : \mathcal{M} \times \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathcal{M}$ is the projection. Therefore

$$\begin{aligned} &\int_{[M_1(X, \beta)]^{\text{vir}}} \tau([\text{pt}]) \\ &= \int_{\mathbb{P}^{2d+1}} e(\mathcal{E}xt_{\pi_M}^1(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{O}(-1, -1))) \cdot \tau([\text{pt}]) \\ &= \int_{\mathbb{P}^{2d+1}} e(-\mathbf{R}Hom_{\pi_M}(\mathcal{O}_{\mathcal{Z}}, \mathcal{O}_{\mathcal{Z}} \otimes \mathcal{O}(-1, -1))) \cdot c_1(\mathcal{O}_{\mathbb{P}^{2d+1}}(1)) \\ &= \int_{\mathbb{P}^{2d+1}} e(-\mathbf{R}Hom_{\pi_M}(\mathcal{O} - \mathcal{O}(-1, -1, -d), (\mathcal{O} - \mathcal{O}(-1, -1, -d)) \otimes \mathcal{O}(-1, -1))) \cdot c_1(\mathcal{O}_{\mathbb{P}^{2d+1}}(1)) \\ &= \int_{\mathbb{P}^{2d+1}} e(\mathbf{R}Hom_{\pi_M}(\mathcal{O}, \mathcal{O}(1, 0, d-1)) + \mathbf{R}Hom_{\pi_M}(\mathcal{O}, \mathcal{O}(-1, -2, -d-1))) \cdot c_1(\mathcal{O}_{\mathbb{P}^{2d+1}}(1)) \\ &= \int_{\mathbb{P}^{2d+1}} (1 + c_1(\mathcal{O}_{\mathbb{P}^{2d+1}}(1)))^{2d} \cdot c_1(\mathcal{O}_{\mathbb{P}^{2d+1}}(1)) = 1, \end{aligned}$$

where in the third equality, we use \mathcal{Z} is a $(1, 1, d)$ divisor in $\mathbb{P}^{2d+1} \times \mathbb{P}^1 \times \mathbb{P}^1$. This computation matches with $n_{0,(1,d)} = 1$ (ref. [KP, pp. 24]), i.e. Conjecture 3.1 also holds in this case.

In all other cases, we can identify $P_n^{\text{JS}}(X, \beta) \cong P_n(X, \beta)$ as in Proposition 5.15, so invariants are the same for certain choice of orientation. Conjecture 3.5 then follows. \square

6. EQUIVARIANT COMPUTATIONS ON LOCAL CURVES

Let C be a smooth projective curve and

$$(6.1) \quad p: X = \text{Tot}_C(L_1 \oplus L_2 \oplus L_3) \rightarrow C$$

be the total space of split rank three bundle on C . Denote the zero section by $\iota: C \rightarrow X$. Assuming that

$$(6.2) \quad L_1 \otimes L_2 \otimes L_3 \cong \omega_C,$$

then the variety (6.1) is a non-compact Calabi-Yau 4-fold and we set $l_i := \deg L_i$.

In this section, we consider the case that

$$C = \mathbb{P}^1, \quad l_1 + l_2 + l_3 = -2,$$

where the latter is equivalent to (6.2). Let $T \subset (\mathbb{C}^*)^4$ be the three dimensional subtorus (when the genus $g(C) > 0$, we can use fiberwise two dimensional CY torus action for (6.1) to define equivariant invariants) which preserves the Calabi-Yau 4-form of X . Let $\bullet = \text{Spec } \mathbb{C}$ with trivial T -action, $\mathbb{C} \otimes t_i$ be the one dimensional $(\mathbb{C}^*)^4$ -representation with weight t_i ($i = 0, 1, 2, 3$), and $\lambda_i \in H_{(\mathbb{C}^*)^4}^*(\bullet)$ be its first Chern class. They are generators of equivariant cohomology rings:

$$H_{(\mathbb{C}^*)^4}^*(\bullet) = \mathbb{Z}[\lambda_0, \lambda_1, \lambda_2, \lambda_3], \quad H_T^*(\bullet) = \frac{\mathbb{Z}[\lambda_0, \lambda_1, \lambda_2, \lambda_3]}{(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)}.$$

The Calabi-Yau torus T lifts to an action on the moduli space of Z_t -stable pairs on X which preserves Serre duality pairing. Since the moduli space is non-compact, we define (equivariant) stable pair invariants by a localization formula (as in [CL14, CMT19, CK19]):

$$(6.3) \quad P_{n,d}^t = [P_n^t(X, d[\mathbb{P}^1])^T]^{\text{vir}} \cdot e(\mathbf{R}\mathcal{H}om_{\pi_P}(\mathbb{I}, \mathbb{I})_0^{\text{mov}})^{1/2},$$

The PT and JS stable pair invariants are then the special limits

$$P_{n,d} := P_{n,d}^t|_{t \rightarrow \infty}, \quad P_{n,d}^{\text{JS}} := P_{n,d}^t|_{t = \frac{d}{2} + 0}.$$

Here $\mathbb{I} = (\mathcal{O}_{X \times P_n^t(X, d[\mathbb{P}^1])} \rightarrow \mathbb{F})$ is the universal stable pair and $\pi_P: X \times P_n^t(X, d[\mathbb{P}^1]) \rightarrow P_n^t(X, d[\mathbb{P}^1])$ is the projection. Of course, the equality (6.3) is not a definition as the virtual class of the fixed locus as well as the square root needs justification. We will make it precise in cases studied below. The PT moduli space $P_n(X, d[\mathbb{P}^1])$, i.e. $t \rightarrow \infty$ case is studied in [CMT19, Sect. 5.3], [CK19, Sect. 2.2, 2.3]. Here we concentrate on the moduli spaces of Joyce-Song stable pairs (ref. Definition 1.10):

$$P_n^{\text{JS}}(X, d[\mathbb{P}^1]) = \left\{ (\mathcal{O}_X \xrightarrow{s} F) \mid F \text{ is one dim compactly supported semistable, } \text{Im}(s) \neq 0, \right. \\ \left. \text{with } [p_* F] = d[\mathbb{P}^1], \chi(p_* F) = n \text{ and } \frac{\chi(p_* F')}{\deg(p_* F')} < \frac{n}{d} \text{ if } \text{Im}(s) \subseteq F' \subsetneq F \right\}.$$

6.1. When (l_1, l_2, l_3) general and $d = 1$. For the $d = 1$ case, as in Proposition 5.1, we have

$$P_n^{\text{JS}}(X, [\mathbb{P}^1]) = P_n(X, [\mathbb{P}^1]),$$

whose torus fixed loci are described by:

Lemma 6.1. *Let $\iota: \mathbb{P}^1 \rightarrow X$ be the zero section. Then*

$$P_n^{\text{JS}}(X, [\mathbb{P}^1])^T = \left\{ I = (\mathcal{O}_X \xrightarrow{s} \iota_* \mathcal{O}_{\mathbb{P}^1}(aZ_0 + bZ_\infty)) \mid a, b \geq 0 \text{ with } a + b = n - 1 \right\},$$

where s is given by the canonical section and $Z_0, Z_\infty \in \mathbb{P}^1$ are the torus fixed points.

Proof. As s is nonzero, it is surjective, then the result follows. \square

For an equivariant line bundle F on \mathbb{P}^1 and $I = (\mathcal{O}_X \xrightarrow{s} \iota_* F)$, we have

$$\chi_X(I, I)_0 = \chi_X(\iota_* F, \iota_* F) - \chi_X(\mathcal{O}_X, \iota_* F) - \chi_X(\iota_* F, \mathcal{O}_X) \in K_0^T(\bullet).$$

By the adjunction formula (ref. [CMT18, Lem. 4.1]), we have

$$(6.4)$$

$$\chi_X(\iota_* F, \iota_* F) = \chi_{\mathbb{P}^1}(F, F) - \chi_{\mathbb{P}^1}(F, F \otimes N_{\mathbb{P}^1/X}) + \chi_{\mathbb{P}^1}(F, F \otimes \wedge^2 N_{\mathbb{P}^1/X}) - \chi_{\mathbb{P}^1}(F, F \otimes \wedge^3 N_{\mathbb{P}^1/X}),$$

where

$$N_{\mathbb{P}^1/X} = \mathcal{O}_{\mathbb{P}^1}(l_1 Z_\infty) \otimes t_1 \oplus \mathcal{O}_{\mathbb{P}^1}(l_2 Z_\infty) \otimes t_2 \oplus \mathcal{O}_{\mathbb{P}^1}(l_3 Z_\infty) \otimes t_3.$$

We want to choose a *square root* of $\chi_X(I, I)_0$, i.e. finding $\chi_X(I, I)_0^{\frac{1}{2}} \in K_0^T(\bullet)$ such that

$$\chi_X(I, I)_0 = \chi_X(I, I)_0^{\frac{1}{2}} + \overline{\chi_X(I, I)_0^{\frac{1}{2}}} \in K_0^T(\bullet),$$

where $\overline{(\cdot)}$ denotes the involution on $K_0^T(\bullet)$ induced by \mathbb{Z} -linearly extending the map

$$t_0^{w_0} t_1^{w_1} t_2^{w_2} t_3^{w_3} \mapsto t_0^{-w_0} t_1^{-w_1} t_2^{-w_2} t_3^{-w_3}.$$

By Serre duality and (6.4), we can define

$$(6.5) \quad \begin{aligned} \chi_X(I, I)_0^{\frac{1}{2}} &:= \chi_X(\iota_* F, \iota_* F)^{\frac{1}{2}} - \chi_X(\mathcal{O}_X, \iota_* F) \\ &:= \chi_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}) - \chi_{\mathbb{P}^1}(N_{\mathbb{P}^1/X}) - \chi_{\mathbb{P}^1}(\mathcal{O}_{\mathbb{P}^1}, F). \end{aligned}$$

The ($d = 1$) T -equivariant JS stable pair invariant is defined in the following:

Definition 6.2. Let $\chi_X(I, I)_0^{\frac{1}{2}}$ be chosen as in (6.5). Then we define

$$P_{n,1}^{\text{JS}} := \sum_{I \in P_n^{\text{JS}}(X, [\mathbb{P}^1])^T} e_T(\chi_X(I, I)_0^{\frac{1}{2}}) \in \frac{\mathbb{Q}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)}{(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)}.$$

By Proposition 1.13, $P_{n,1}^{\text{JS}} = P_{n,1}$ which has been studied in [CK19, CKM19].

6.2. When (l_1, l_2, l_3) general and $d = 2$. In this case, we have

$$P_n^{\text{JS}}(X, 2[\mathbb{P}^1])^T = P_n(X, 2[\mathbb{P}^1])^T,$$

for $n \leq 2$ by Proposition 1.13, since $n([\mathbb{P}^1]) = 1$. Based on (6.3), one can then define T -equivariant JS stable pair invariants $P_{n,2}^{\text{JS}}$ such that

$$P_{n,2}^{\text{JS}} = P_{n,2} \in \frac{\mathbb{Q}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)}{(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)},$$

where $P_{n,2}$ has been rigorously defined in [CK19, Def. 2.9].

Remark 6.3.

(1) When $n < 0$, $P_{n,2}^{\text{JS}}$ and $P_{n,2}$ are not necessarily zero (e.g. the case $n = -1$, $l_1 = 3$, $l_2 = -2$, $l_3 = -3$).

(2) When $n \geq 3$, we can still define $P_{n,2}^{\text{JS}}$, $P_{n,2}$. But they are not necessarily the same (e.g. the case $n = 3$, $l_1 = 2$, $l_2 = l_3 = -2$). It is an interesting question to find a formula relating them.

6.3. When $(l_1, l_2, l_3) = (-1, -1, 0)$ and d is arbitrary. First of all, to have a nonempty moduli space $P_n^{\text{JS}}(X, d[\mathbb{P}^1])$, d must divide n by the Jordan-Hölder filtration. We first classify T -fixed JS stable pairs.

Lemma 6.4. Let $k \geq 0$, $n = d(k + 1)$ and $\{Z_0, Z_\infty\} = (\mathbb{P}^1)^T$ be the torus fixed points. Then a T -fixed JS stable pair $I = (\mathcal{O}_X \xrightarrow{s} F) \in P_n^{\text{JS}}(X, d[\mathbb{P}^1])^T$ is precisely of the form

$$(6.6) \quad F = \bigoplus_{i=0}^k \mathcal{O}_{\mathbb{P}^1}((k-i)Z_\infty + iZ_0) \left(\sum_{j=0}^{d_i-1} t_3^{-j} \right),$$

for some $d_0, \dots, d_k \geq 0$ with $\sum_{i=0}^k d_i = d$, and s is given by canonical sections.

Proof. Let ι be the inclusion

$$\iota = i \times \text{id}_{\mathbb{C}} : \mathbb{P}^1 \times \mathbb{C} \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1, -1) \times \mathbb{C} = X,$$

where i is the zero section of the projection $\mathcal{O}_{\mathbb{P}^1}(-1, -1) \rightarrow \mathbb{P}^1$. Let us take a T -fixed JS pair $I = (\mathcal{O}_X \rightarrow F)$. By the Jordan-Hölder filtration, F is written as

$$F = \bigoplus_{i=1}^l \iota_* F_i, \quad F_i = \mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{O}_{T_i},$$

where T_i is a zero dimensional subscheme of \mathbb{C} supported at $0 \in \mathbb{C}$. We write the section s as

$$s = (s_1, \dots, s_l), \quad 0 \neq s_i : \mathcal{O}_X \rightarrow \iota_*(\mathcal{O}_{\mathbb{P}^1}(k) \boxtimes \mathcal{O}_{T_i}).$$

Here each s_i is non-zero by the JS stability. By pushforward to \mathbb{P}^1 , we know s_i is described by commutative diagrams

$$\begin{array}{ccc} \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{s_i^j} & \mathcal{O}_{\mathbb{P}^1}(k) \otimes t_3^{-j} \\ \downarrow = & & \downarrow \cdot t_3^{-1} \\ \mathcal{O}_{\mathbb{P}^1} & \xrightarrow{s_i^{j+1}} & \mathcal{O}_{\mathbb{P}^1}(k) \otimes t_3^{-j-1}. \end{array}$$

Then s_i is determined by s_i^0 which gives $\mathcal{O}_{\mathbb{P}^1}(k)$ an equivariant structure of the form

$$\mathcal{O}_{\mathbb{P}^1}(a_i Z_0 + (k - a_i) Z_\infty), \quad 0 \leq a_i \leq k,$$

and s_i^0 is the canonical section. So each F_i is of the form

$$F_i = \mathcal{O}_{\mathbb{P}^1}(a_i Z_0 + (k - a_i) Z_\infty) \left(\sum_{i=0}^{d_i-1} t_3^{-i} \right),$$

for some $d_i \geq 0$. Furthermore we need $a_i \neq a_j$ for $i \neq j$ in order that (F, s) is JS stable. Indeed suppose that $a_i = a_j$, and set $\overline{F}_i = F_i/t_3^{-1}F_i$. Then there is an isomorphism $h: \overline{F}_j \xrightarrow{\cong} \overline{F}_i$ such that the composition

$$\mathcal{O}_X \xrightarrow{s} F \rightarrow \overline{F}_i \oplus \overline{F}_j \rightarrow \overline{F}_i$$

is zero, where the last arrow is $(x, y) \mapsto x - h(y)$. The above morphism destabilizes $(\mathcal{O}_X \xrightarrow{s} F)$ in JS stability, so a contradiction. Therefore F is of the form (6.6).

Conversely it is straightforward to check that any pair $(\mathcal{O}_X \xrightarrow{s} F)$ where (F, s) is as in (6.6) is a T -fixed JS stable pair. \square

To choose a square root for $\chi_X(I, I)_0$, we recall the following:

Lemma 6.5. *As elements in $K_0^T(\bullet)$, we have*

$$(6.7) \quad \chi(\mathcal{O}_{\mathbb{P}^1}(aZ_0 + bZ_\infty)) = \begin{cases} t_0^{-b} + \cdots + t_0^{-1} + 1 + t_0 + \cdots + t_0^a, & \text{if } a, b \geq 0, \\ t_0^a, & \text{if } a = -b > 0. \end{cases}$$

Proof. The T -equivariant Riemann-Roch formula gives

$$\text{ch} \left(\chi(\mathcal{O}_{\mathbb{P}^1}(aZ_0 + bZ_\infty)) \right) = \frac{e^{a\lambda_0}}{1 - e^{-\lambda_0}} + \frac{e^{-b\lambda_0}}{1 - e^{\lambda_0}} = \frac{e^{(a+1)\lambda_0} - e^{-b\lambda_0}}{e^{\lambda_0} - 1},$$

from which we can conclude the result. \square

Lemma 6.6. *Let $k \geq 0$, $n = d(k + 1)$ and $I = (\mathcal{O}_X \xrightarrow{s} F) \in P_n^{\text{JS}}(X, d[\mathbb{P}^1])^T$, where F is given by (6.6). Then we can choose a square root of $\chi_X(I, I)_0$ to be*

$$\chi_X(I, I)_0^{\frac{1}{2}} = - \sum_{i=0}^k \left(\sum_{j=-(k-i)}^i t_0^j \right) \left(\sum_{j=0}^{d_i-1} t_3^{-j} \right) + \sum_{i < j} t_0^{j-i} (1 - t_3^{d_i} - t_3^{-d_j} + t_3^{d_i-d_j}) + \sum_{i=0}^k (1 - t_3^{d_i}).$$

Proof. As in (6.5) we are left to choose a square root for $\chi_X(F, F)$ and then

$$(6.8) \quad \chi_X(I, I)_0^{\frac{1}{2}} := -\chi(F) + \chi_X(F, F)^{\frac{1}{2}},$$

where by (6.7) and (6.6), we have

$$(6.9) \quad \chi(F) = \sum_{i=0}^k \left(\sum_{j=-(k-i)}^i t_0^j \right) \left(\sum_{j=0}^{d_i-1} t_3^{-j} \right).$$

By Serre duality, we may define

$$\begin{aligned} \chi_X(F, F)^{\frac{1}{2}} &:= \sum_{\substack{i < j \\ 0 \leq i, j \leq k}} \chi_X(\mathcal{O}_{\mathbb{P}^1}(iZ_0 + (k-i)Z_\infty, \mathcal{O}_{\mathbb{P}^1}(jZ_0 + (k-j)Z_\infty))) \cdot \sum_{s=0}^{d_i-1} t_3^s \sum_{r=0}^{d_j-1} t_3^{-r} \\ &\quad + d \cdot \chi_X(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1})^{\frac{1}{2}} + \chi_X(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) \left(\sum_{i=0}^k \left(-d_i + \sum_{s=0}^{d_i-1} t_3^s \sum_{r=0}^{d_i-1} t_3^{-r} \right) \right)^{\frac{1}{2}}, \end{aligned}$$

where we use the fact that $\chi_X(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}) = 2 - t_3 - t_3^{-1}$ which is invariant under involution $(\overline{\cdot})$.

We choose

$$\chi_X(\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1})^{\frac{1}{2}} := 1 - t_3,$$

$$\begin{aligned}
 \left(\sum_{i=0}^k \left(-d_i + \sum_{s=0}^{d_i-1} t_3^s \sum_{r=0}^{d_i-1} t_3^{-r} \right) \right)^{\frac{1}{2}} &:= \sum_{i=0}^k \left(\frac{(1-t_3^{d_i})(1-t_3^{-d_i})}{2-t_3-t_3^{-1}} - d_i \right)^{\frac{1}{2}} \\
 &= \sum_{i=0}^k \left(\frac{2-t_3^{d_i}-t_3^{-d_i}-d_i(2-t_3-t_3^{-1})}{2-t_3-t_3^{-1}} \right)^{\frac{1}{2}} \\
 &:= \sum_{i=0}^k \left(\frac{1-t_3^{d_i}-d_i(1-t_3)}{2-t_3-t_3^{-1}} \right).
 \end{aligned}$$

By (6.7) and the adjunction formula, we have

$$\chi_X(\mathcal{O}_{\mathbb{P}^1}(iZ_0 + (k-i)Z_\infty), \mathcal{O}_{\mathbb{P}^1}(jZ_0 + (k-j)Z_\infty)) = 2t_0^{j-i} - t_0^{j-i}t_3 - t_0^{j-i}t_3^{-1}.$$

Then it is easy to see

$$(6.10) \quad \chi_X(F, F)^{\frac{1}{2}} = \sum_{i < j} t_0^{j-i} (1 - t_3^{d_i} - t_3^{-d_j} + t_3^{d_i-d_j}) + \sum_{i=0}^k (1 - t_3^{d_i}).$$

Combining with (6.8), (6.9), (6.10), we are done. \square

Definition 6.7. Let $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$ and T be the Calabi-Yau torus. The T -equivariant JS stable pair invariants are defined by

$$P_{n,d}^{\text{JS}} := \sum_{I \in P_n^{\text{JS}}(X, d[\mathbb{P}^1])^T} (-1)^{d+1} e_T(\chi_X(I, I)_0^{\frac{1}{2}}) \in \frac{\mathbb{Q}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)}{(\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3)},$$

where $\chi_X(I, I)_0^{\frac{1}{2}}$ is chosen as in Lemma 6.6 and the sign denotes a choice of orientation.

We can explicitly compute all T -equivariant JS stable pair invariants for $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$.

Theorem 6.8. In the setting of Definition 6.7, for $k \geq 0$ and $n = d(k+1)$, we have

$$\begin{aligned}
 P_{n,d}^{\text{JS}} &= \frac{(-1)^{k(d+1)}}{1!2! \dots k!} \cdot \frac{1}{\lambda_0^{k(k+1)/2} \lambda_3^d} \cdot \sum_{\substack{d_0 + \dots + d_k = d \\ d_0, \dots, d_k \geq 0}} \frac{1}{d_0! \dots d_k!} \cdot \prod_{\substack{i < j \\ 0 \leq i, j \leq k}} \left((j-i)\lambda_0 + (d_i - d_j)\lambda_3 \right) \\
 &\times \prod_{i=0}^k \left(\prod_{\substack{1 \leq a \leq d_i \\ 1 \leq b \leq k-i}} \frac{1}{a\lambda_3 + b\lambda_0} \cdot \prod_{\substack{1 \leq a \leq d_i \\ 1 \leq b \leq i}} \frac{1}{a\lambda_3 - b\lambda_0} \right).
 \end{aligned}$$

Proof. Let F be given by (6.6) and choose the square root $\chi_X(I, I)_0^{\frac{1}{2}}$ as in Lemma 6.6. A direct calculation gives the following:

$$\begin{aligned}
 e_T(\chi_X(I, I)_0^{\frac{1}{2}}) &= \frac{(-1)^{k(d+1)}}{1!2! \dots k!} \cdot \frac{1}{\lambda_0^{k(k+1)/2} \lambda_3^d} \cdot \frac{1}{d_0! \dots d_k!} \cdot \prod_{\substack{i < j \\ 0 \leq i, j \leq k}} \left((j-i)\lambda_0 + (d_i - d_j)\lambda_3 \right) \\
 &\times \prod_{i=0}^k \left(\prod_{\substack{1 \leq a \leq d_i \\ 1 \leq b \leq k-i}} \frac{1}{a\lambda_3 + b\lambda_0} \cdot \prod_{\substack{1 \leq a \leq d_i \\ 1 \leq b \leq i}} \frac{1}{a\lambda_3 - b\lambda_0} \right).
 \end{aligned}$$

Taking a sum over all torus fixed points (as described by Lemma 6.4) gives the result \square

For $\mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$, PT stable pair invariants satisfy $P_{0,d} = P_{1,d+1} = 0$ if $d > 0$ (ref. [CMT19, Prop. 5.2]). In view of Conjecture 3.3, the only non-zero ‘‘GV type invariant’’ exists in the $g = 0$, $d = 1$ case (and $n_{0,1} = \lambda_3^{-1}$). The following direct analogue of Conjecture 3.4 is expected to be true.

Conjecture 6.9. Let $X = \mathcal{O}_{\mathbb{P}^1}(-1, -1, 0)$ and T be the Calabi-Yau torus. Then the T -equivariant JS stable pair invariants (Definition 6.7) satisfy

$$P_{n,d}^{\text{JS}} = \begin{cases} \frac{1}{d!(\lambda_3)^d}, & \text{if } n = d \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

By Theorem 6.8, verification of Conjecture 6.9 reduces to an explicit combinatoric problem.

Theorem 6.10. Conjecture 6.9 is true in the following cases

- $d \nmid n$,
- $d = 1, 2$ with any n ,

- $n = d, 2d$ with any d .

Proof. By the Jordan-Hölder filtration, d must divide n , otherwise the moduli space is empty. The $d = 1$ case is discussed in Section 6.2. The $n = d$ case is easy. Here we show a proof when $n = 2d$ (the $d = 2$ case can be proved using a similar method).

In this case, the formula simplifies to

$$P_{2d,d}^{\text{JS}} = \frac{1}{\lambda_3^d} \cdot \sum_{\substack{d_0+d_1=d \\ d_1, d_1 \geq 0}} \frac{(-1)^{d_1}}{d_0! \cdot d_1!} \cdot \frac{\lambda_0 + (d_0 - d_1)\lambda_3}{(\lambda_0 + d_0\lambda_3) \cdots (\lambda_0 + 2\lambda_3) \cdot (\lambda_0 + \lambda_3) \cdot \lambda_0 \cdot (\lambda_0 - \lambda_3) \cdot (\lambda_0 - 2\lambda_3) \cdots (\lambda_0 - d_1\lambda_3)},$$

and we want to show it is zero. The rational function

$$\Phi = \sum_{\substack{d_0+d_1=d \\ d_1, d_1 \geq 0}} \frac{(-1)^{d_1}}{d_0! \cdot d_1!} \cdot \frac{\lambda_0 + (d_0 - d_1)\lambda_3}{(\lambda_0 + d_0\lambda_3) \cdots (\lambda_0 + 2\lambda_3) \cdot (\lambda_0 + \lambda_3) \cdot \lambda_0 \cdot (\lambda_0 - \lambda_3) \cdot (\lambda_0 - 2\lambda_3) \cdots (\lambda_0 - d_1\lambda_3)}$$

is homogenous in variable λ_0 and λ_3 and all possible poles are of order one. To prove cancellation of poles, we may set $\lambda_3 = 1$ and it is enough to prove the residue is zero at any pole.

Poles happen at $\lambda_0 = m \in \{0, \pm 1, \dots, \pm d\}$. Say $m \geq 0$ ($m \leq 0$ case is similar), terms involving $(\lambda_0 - m)$ exists only when $d_1 = m, m+1, \dots, d$. We consider the residue at $\lambda_0 = m$:

$$\begin{aligned} \text{Res}_{\lambda_0=m}(\Phi|_{\lambda_3=1}) &= \sum_{i=m}^d \frac{(-1)^i}{(d-i)! \cdot i!} \cdot \frac{m+d-2i}{(m+d-i) \cdots (m+1) \cdot m \cdot (m-1) \cdots (\widehat{m-m}) \cdots (m-i)} \\ &= \sum_{i=m}^d \frac{(-1)^m}{(d-i)! \cdot i!} \cdot \frac{m+d-2i}{(i-m)! \cdot (m+d-i)!} \\ &= (-1)^m \cdot \sum_{j=d-m}^{m-d} \frac{j}{\left(\frac{m+d-j}{2}\right)! \cdot \left(\frac{d-m+j}{2}\right)! \cdot \left(\frac{d-m-j}{2}\right)! \cdot \left(\frac{m+d+j}{2}\right)!} \\ &= 0, \end{aligned}$$

where we make a change of index $j = m + d - 2i$ in the third equality and the last equality is because the denominator is invariant under $j \rightarrow -j$ while the numerator gets a sign change. \square

In general, we may have higher order poles in the rational functions involved in $P_{n,d}^{\text{JS}}$ which we do not know how to deal with at the moment. Nevertheless, we verify our conjecture in a huge number of examples by direct calculations with the help of a ‘Mathematica’ program.

Proposition 6.11. *We have $P_{n,d}^{\text{JS}} = 0$ in the following cases:*

- $d = 3$ and $n/d \leq 30$,
- $d = 4$ and $n/d \leq 20$,
- $d = 5$ and $n/d \leq 14$,
- $d = 6$ and $n/d \leq 11$,
- $n = 3d$ and $d \leq 60$,
- $n = 4d$ and $d \leq 20$,
- $n = 5d, 6d, 7d$ and $d \leq 10$,
- $n = 8d$ and $d \leq 9$,
- $n = 9d$ and $d \leq 8$,
- $n = 10d$ and $d \leq 7$,

i.e. Conjecture 6.9 is true in all these cases.

Remark 6.12. *Recently Conjecture 6.9 has been proved in general by studying tautological invariants and their compact analogies [CT20b, CT20c].*

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