CURVE COUNTING VIA STABLE PAIRS IN THE DERIVED CATEGORY

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Abstract. For a nonsingular projective 3-fold X, we define integer invariants virtually enumerating pairs (C, D) where $C \subset X$ is an embedded curve and $D \subset C$ is a divisor. A virtual class is constructed on the associated moduli space by viewing a pair as an object in the derived category of X. The resulting invariants are conjecturally equivalent, after universal transformations, to both the Gromov-Witten and DT theories of X. For Calabi-Yau 3-folds, the latter equivalence should be viewed as a wall-crossing formula in the derived category.

Several calculations of the new invariants are carried out. In the Fano case, the local contributions of nonsingular embedded curves are found. In the local toric Calabi-Yau case, a completely new form of the topological vertex is described.

The virtual enumeration of pairs is closely related to the geometry underlying the BPS state counts of Gopakumar and Vafa. We prove that our integrality predictions for Gromov-Witten invariants agree with the BPS integrality. Conversely, the BPS geometry imposes strong conditions on the enumeration of pairs.

Contents

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0. Introduction

There are several ways to compactify the space of embedded curves in a nonsingular projective variety X. The moduli of stable maps $\overline{M}(X)$ provides one such compactification.[1] The Gromov-Witten invariants of X are defined via integration against the virtual class of $\overline{M}(X)$. Because of nontrivial automorphisms, $\overline{M}(X)$ is a Deligne-Mumford stack, and the Gromov-Witten invariants are rational numbers. Underlying these rational numbers should be integers which, in a regularized sense, count embedded curves.

Curve counting problems take special form for 3-folds since the expected dimension of $\overline{M}(X)$ is independent of the genus of the map domain. For Calabi-Yau 3-folds, the expected dimension of $\overline{M}(X)$ is always 0. The study of the underlying integer invariants in the Calabi-Yau case appears already in the quintic 3-fold calculations of Candelas, de la Ossa, Green, and Parks [12] via the Aspinwall-Morrison formula [3] in genus 0.

A second compactification of the space of embedded curves is provided by the Hilbert scheme $I(X)$. For 3-folds, $I(X)$ carries a virtual class [50] and yields invariants via integration. The resulting theory is *integer* valued. However, since 1-dimensional subschemes of X contain 0-dimensional subschemes which roam over all of X, the invariants do

[1]The locus of embedded curves need not be dense in any of the compactifications we consider here.

not directly count curves. In [43, 44], a formal *reduced* theory is defined by dividing by the generating series of 0-dimensional invariants. For many reasons, a direct geometrical approach to the reduced theory would be preferable.

A more recent compactification of the space of embedded curves is provided by the work of Honsen [21]. Honsen defines a proper algebraic space $H(X)$ parameterizing Cohen-Macaulay curves[2] with finite maps to X which are generically embeddings. There are no automorphisms of the map and no roaming 0-dimensional subschemes. However, a finite number of points of the curve may be identified in the image in X. Unfortunately, even for 3-folds, Honsen's space does not appear to carry a natural virtual class.

Honsen's space provides a connection to a fourth compactification. The push-forward $f_*\mathcal{O}_C$ of the structure sheaf associated to an element

$$[f : C \to X] \in H(X)$$

is a sheaf on X. What distinguishes $f_*\mathcal{O}_C$ from arbitrary sheaves is the canonical section $s \in H^0(X, f_*\mathcal{O}_C)$ obtained from $1 \in H^0(C, \mathcal{O}_C)$. The section s may have cokernel at a finite number of points where the curve is not embedded.

We are led to consider the moduli space $P(X)$ of *stable pairs* (F, s) where F is a sheaf of fixed Hilbert polynomial supported in dimension 1 and $s \in H^0(X, F)$ is a section. The two stability conditions are:

(i) the sheaf F is pure,
(ii) the section $\mathcal{O}_X \xrightarrow{s} F$ has 0-dimensional cokernel.

By definition, *purity* (i) means every nonzero subsheaf of F has support of dimension 1 [23]. In particular, purity implies the (scheme theoretic) support C_F of F is a Cohen-Macaulay curve. A projective moduli space of stable pairs can be constructed by a standard GIT analysis of Quot scheme quotients [32]. The relationship between our stability (i)-(ii) and GIT stability is discussed in Section 1.

The nicest stable pairs are obtained from the data of an embedded Cohen-Macaulay curve

$$\iota : C \hookrightarrow X$$

together with a Cartier divisor $D \subset C$. The associated stable pair is

$$(\iota_*\mathcal{O}_C(D), s_D) \tag{0.1}$$

where $s_D \in H^0(X, \iota_*\mathcal{O}_C(D))$ is the canonical (up to isomorphism) section determined by D. We will often use the abbreviated notation

[2] A Cohen-Macaulay curve is of pure dimension 1 and possibly nonreduced, but with no embedded points.

$(\mathcal{O}_C(D), s_D)$ to denote the pair (0.1). A characterization of all stable pairs is provided by Proposition 1.8.

In case D is empty, we obtain the pair $(\mathcal{O}_C, 1)$ associated to a Cohen-Macaulay subcurve C alone. Here, the pair data is equivalent to the kernel of the section

$$\mathcal{O}_X \xrightarrow{1} \mathcal{O}_C,$$

which is the ideal sheaf $\mathscr{I}_C$ of C. In fact in $D^b(X)$, the bounded derived category of coherent sheaves on X, the *complex* $\{\mathcal{O}_X \xrightarrow{1} \mathcal{O}_C\}$ formed from the pair is quasi-isomorphic to $\mathscr{I}_C$. More general stable pairs give rise to more complicated 2-term complexes

$$I^\bullet = \{\mathcal{O}_X \xrightarrow{s} F\} \quad \in \ D^b(X)$$

which may be viewed as Cohen-Macaulay curves C_F obtained from the kernel of s decorated with a finite number of points obtained from the cokernel. The points replace the 0-dimensional subschemes which appear in [43, 44, 50]. However, they are now constrained to lie on the curve C rather than being free to wander over all of X.

For a 3-fold X, the natural obstruction theory of stable pairs (F, s) fails to be 2-term and does *not* admit a virtual class. However, in Section 2, we show that the fixed-determinant obstruction theory of the complex $I^\bullet$ in the derived category provides an *alternative* obstruction theory for $P(X)$. The moduli space $P(X)$ may be naturally viewed as a well-behaved component of the generally ill-behaved moduli space of complexes in $D^b(X)$ [41]. Indeed, $P(X)$ provides a rare example where a component of the moduli of complexes is explicitly constructed as a projective variety.[3] Arguments parallel to [50] then show the alternative obstruction theory of $P(X)$ *does* admits a virtual class of the correct dimension.

Integration against the virtual class of $P(X)$ provides a theory of 3-folds which is deformation invariant and integer valued. We conjecture our pair theory for a 3-fold X to be equivalent to the Gromov-Witten theory and equal to the reduced DT theory. Integrality constraints on the Gromov-Witten theory of 3-folds have been predicted by the conjectural BPS invariants [19, 20, 45]. The integrality predicted by the pair theory and the BPS invariants are entirely equivalent.

No direct cohomological definition of BPS invariants satisfying all expected properties has yet been proposed.[4] Amongst other technical

[3]Earlier examples of well-behaved components of the moduli space of complexes in the derived category can be found in [10, 25].

[4]For example, the remarkable cohomological and motivic proposals of [22, 51] are unlikely to be deformation invariant and almost certainly do not satisfy the Gopakumar-Vafa relationship with Gromov-Witten theory.

issues, problems with semistability of sheaves on reducible curves are difficult to overcome. Our moduli problem stabilizes such sheaves by picking a section. The stable pairs invariants are closely related to the heuristic interpretation of BPS counts given in [29]. Indeed, we provide a rigorous definition of BPS counts via virtual Euler characteristics modulo a conjectured vanishing in Section 3.

In Section 4, we calculate the contribution of a nonsingular embedded curve to the pair theory in complete agreement with the Gromov-Witten calculations of [45]. In Section 5, we present the topological vertex for our invariants. The 3-legged topological vertex takes a completely new form from the pairs point of view. The agreement with Gromov-Witten theory is still conjectural.

Past work and future directions. A special case of the moduli space of pairs $P(X)$ arises naturally in Diaconescu's work on local curves [14]. He compactifies a rank 2 vector bundle over a curve C to a $\mathbb{P}^2$-bundle and then uses a relative Beilinson transform on the fibers to map ideal sheaves of curves (flat over C) to certain quiver sheaves. An appropriate stability condition for quiver sheaves then translates into ours for stable pairs on the $\mathbb{P}^2$-bundle.

The branch maps of Thaddeus and Alexeev-Knutsen [1] are *reduced* curves with finite maps to X. The moduli space of branch maps $B(X)$ is a proper algebraic space, providing yet another compactification of the space of embedded curves in X. As in the case of Honsen's space, even in dimension 3, the moduli space $B(X)$ seems to lack a natural virtual class.

Following the connection of Honsen's space to $P(X)$, we may push-forward the structure sheaf of a branch map

$$[f : C \to X] \in B(X).$$

The result $f_*\mathcal{O}_C$ together with the canonical section $s \in H^0(X, f_*\mathcal{O}_C)$ suggests considering pairs on X consisting of sheaves of possibly higher rank supported on curves with sections whose cokernel may be supported in dimension 1. While the construction of such moduli spaces is easily obtained by varying the stability condition on the space of pairs [33], we have been unable to prove the existence of an appropriate virtual class. In particular, the approach to the virtual class of $P(X)$ via derived category deformations discussed in Section 2 does not immediately succeed for higher rank pairs. Perhaps some variant can be pursued.

Finally, we point out the Gromov-Witten theory of Calabi-Yau 4 and 5-folds is also known conjecturally to be governed by integers [30, 48].

Finding equivalent integer valued sheaf theories in higher dimensions is an interesting problem.

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1. Definitions

1.1. **Stability.** Let X be a nonsingular projective 3-fold over $\mathbb{C}$ with a fixed polarization L. As usual, for sheaves F on X,

$$F(k) = F \otimes L^k.$$

Let $q \in \mathbb{Q}[k]$ with positive leading coefficient be a stability parameter. For $n \in \mathbb{Z}$ and nonzero $\beta \in H_2(X, \mathbb{Z})$, let $P_n^q(X, \beta)$ denote the moduli space of semistable pairs

$$\mathcal{O}_X \xrightarrow{s} F,$$

where F is a pure sheaf with Hilbert polynomial

$$\chi(F(k)) = k \int_\beta c_1(L) + n$$

and s is a nonzero section. The moduli space $P_n^q(X, \beta)$ can be constructed by GIT [33].

The stability condition for the GIT problem of pairs is defined as follows. For a sheaf G with support of dimension at most 1, we let $r(G)$ denote the coefficient of k in

$$\chi(G(k)) = r(G)k + c.$$

A *proper* subsheaf $G \subset F$ is nonzero and not equal to F. Since F is pure, G has 1-dimensional support and therefore $r(G) > 0$. The pair

(F,s) is q-stable if, for every proper subsheaf $G \subset F$,

$$\frac{\chi(G(k))}{r(G)} < \frac{\chi(F(k))+q(k)}{r(F)}, \qquad k \gg 0 \tag{1.1}$$

holds, and for every proper subsheaf G through which s factors,

$$\frac{\chi(G(k))+q(k)}{r(G)} < \frac{\chi(F(k))+q(k)}{r(F)}, \qquad k \gg 0 \tag{1.2}$$

holds. The q-semistability conditions are obtained from (1.1)-(1.2) after replacing $<$ with $\leq$.

1.2. **Degree 0.** Le Potier's treatment of the moduli of pairs

$$\mathcal{O}_X \xrightarrow{s} F$$

is undertaken for arbitrary Hilbert polynomial[5] of F. If the class β is taken to be 0, $\chi(F(k))$ is a constant n. Setting the top order coefficient $r(F)$ to be n, we obtain (semi)-stability conditions identical to (1.1)-(1.2) on proper subsheaves $G \subset F$.

The q-stable pairs with $\beta = 0$ are precisely those obtained from the structure sheaf of a length n subscheme $S \subset X$ with canonical section,

$$\mathcal{O}_X \xrightarrow{1} \mathcal{O}_S.$$

Indeed, condition (i) is always satisfied and condition (ii) is excluded since the section 1 generates $\mathcal{O}_S$. The converse is left to the reader. Hence, $P_n^q(X,0)$ is simply the Hilbert scheme $\mathrm{Hilb}(X,n)$ of n points.

1.3. **Limits.** Let $\beta \neq 0$. We are interested in the moduli space of pairs $P_n^q(X,\beta)$ in the large q limit. In the degree 1 case,

$$q(k) = Ak + B,$$

the limit is achieved for A and B sufficiently large (for fixed Hilbert polynomial of F). The limit is always achieved if q to has degree at least 2.

Lemma 1.3. *For q sufficiently large, as described above, stability and semistability coincide. A pair (F,s) is limit stable if and only if*

(i) *the sheaf F is pure,*
(ii) *the section $\mathcal{O}_X \xrightarrow{s} F$ has 0-dimensional cokernel.*

Proof. For q sufficiently large the inequality (1.1) is always strictly satisfied. After rearranging inequality (1.2) for semistability, we obtain

$$(r(F)-r(G))\big(\chi(F(k))+q(k)\big) \leq r(F)\ \chi((F/G)(k)), \tag{1.4}$$

[5]Le Potier also treats arbitrary numbers of sections, not just 1.

which, for q large, shows both that

$$r(F) - r(G) = 0 \tag{1.5}$$

and that equality can never occur. Setting $G = \mathrm{Im}(s)$ in (1.5) implies that s has 0-dimensional cokernel. □

We define a *stable pair* (F, s) to be limit stable. Then, $P_n(X, \beta)$ is the moduli space of stable pairs. Let

$$C_F = \mathrm{Supp}(F) \subset X$$

be the scheme theoretic support of F. By condition (ii) of Lemma 1.3, F is isomorphic to the structure sheaf of C_F away from finitely many points, and so has rank 1 on C_F.

Lemma 1.6. *For a stable pair (F, s), the support of $\mathrm{Im}(s)$ is C_F.*

Proof. The issue is local on X, so we may consider the geometry on an affine open on which F is a module. The supports of F and $\mathrm{Im}(s)$ are defined by the annihilators of F and s respectively. The annihilator of F certainly annihilates s. Conversely, let $a \in \mathrm{Ann}(s)$ be a function. If $a \notin \mathrm{Ann}(F)$, let $f \in F$ be a section for which $af \in F$ does not vanish. Then, the submodule of F generated by af has dimension 0 support (away from the nonempty open set on which s generates F guaranteed by condition (ii) of pair stability). Hence, the purity of F is violated. □

Since $\mathrm{Im}(s)$ is a quotient of $\mathcal{O}_X$, $\mathrm{Im}(s)$ is a structure sheaf. By Lemma 1.6, $\mathrm{Im}(s) \cong \mathcal{O}_{C_F}$. As a subsheaf of a pure sheaf, $\mathrm{Im}(s)$ is also pure. Therefore, C_F is Cohen-Macaulay.

The following kernel/cokernel exact sequence is associated to the stable pair (F, s),

$$0 \to \mathscr{I}_{C_F} \to \mathcal{O}_X \xrightarrow{s} F \to Q \to 0. \tag{1.7}$$

The cokernel Q has dimension 0 support by stability. The *reduced* support scheme, $\mathrm{Supp}^{red}(Q)$, is called the *zero locus* of the pair. The zero locus lies on C_F.

Let $C \subset X$ be a fixed Cohen-Macaulay curve, and $\mathfrak{m} \subset \mathcal{O}_C$ the ideal of a finite union of closed points. We now characterize stable pairs with support C and zero locus supported at these points.

Since

$$\mathscr{H}om(\mathfrak{m}^r/\mathfrak{m}^{r+1}, \mathcal{O}_C) = 0$$

by the purity of $\mathcal{O}_C$, we obtain an inclusion

$$\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C) \subset \mathscr{H}om(\mathfrak{m}^{r+1}, \mathcal{O}_C).$$

The inclusion $\mathfrak{m}^r \hookrightarrow \mathcal{O}_C$ induces a canonical section

$$\mathcal{O}_C \hookrightarrow \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C).$$

Proposition 1.8. *A stable pair (F, s) with support C satisfying*

$$\mathrm{Supp}^{red}(Q) \subset \mathrm{Supp}(\mathcal{O}_C/\mathfrak{m})$$

is equivalent to a subsheaf of $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$, $r \gg 0$.

Alternatively, we may work with coherent subsheaves of the quasi-coherent sheaf $\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$.

Proof. Let Q denote the 0-dimensional cokernel of the stable pair. Its dual $\mathscr{H}om(Q, \mathcal{O}_C)$ vanishes, since $\mathcal{O}_C$ is pure. Therefore, applying $\mathscr{H}om(\,\cdot\,, \mathcal{O}_C)$ to

$$0 \to \mathcal{O}_C \to F \to Q \to 0 \tag{1.9}$$

yields the inclusion

$$0 \to \mathscr{H}om(F, \mathcal{O}_C) \to \mathcal{O}_C. \tag{1.10}$$

Hence, $\mathscr{H}om(F, \mathcal{O}_C)$ is the pushfoward to X of an ideal sheaf $\mathscr{I}_Z$ on C. Since (1.10) is a generic isomorphism, Z is 0-dimensional and

$$Z^{red} \subset \mathrm{Supp}^{red}(Q).$$

Dualizing on C again gives

$$0 \to \mathcal{O}_C \to \mathscr{H}om(\mathscr{I}_Z, \mathcal{O}_C). \tag{1.11}$$

The obvious double dual map

$$F \to \mathscr{H}om(\mathscr{H}om(F, \mathcal{O}_C), \mathcal{O}_C) = \mathscr{H}om(\mathscr{I}_Z, \mathcal{O}_C)$$

is generically an isomorphism, so is an injection by the purity of F. The map (1.11) factors through the original section $\mathcal{O}_C \to F$. Thus, we have the data

$$\mathcal{O}_C \to F \subseteq \mathscr{H}om(\mathscr{I}_Z, \mathcal{O}_C),$$

with the composition being the canonical section of $\mathscr{H}om(\mathscr{I}_Z, \mathcal{O}_C)$.

For $r \gg 0$, there is an inclusion $\mathfrak{m}^r \subset \mathscr{I}_Z$ with 0-dimensional cokernel. Therefore by purity we get inclusions

$$\mathscr{H}om(\mathscr{I}_Z, \mathcal{O}_C) \subset \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C) \subset \mathscr{H}om(\mathfrak{m}^{r+1}, \mathcal{O}_C).$$

We obtain the subsheaf

$$F \subset \varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C), \tag{1.12}$$

containing the canonical section of $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)$. Dividing by the canonical section gives a coherent subsheaf

$$Q \subset \varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C.$$

Conversely, given such a Q in $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)/\mathcal{O}_C$, the sheaf F is recovered as its inverse image (1.12) in $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)$. Moreover, F has a section s fitting into an exact sequence (1.9). As a subsheaf of $\mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_C)$, which is pure since C is Cohen-Macaulay, F is also pure. The pair (F, s) supported on C is stable by Lemma 1.3. □

1.4. **Derived category.** Let $D^b(X)$ be the bounded derived category of coherent sheaves on X. Let $I^\bullet \in D^b(X)$ be determined by the complex

$$\{\mathcal{O}_X \stackrel{s}{\longrightarrow} F\}$$

associated to the stable pair (F, s) with $\mathcal{O}_X$ in degree 0. In what follows, all Hom and Ext groups are considered in $D^b(X)$.

We have the following exact triangles in $D^b(X)$ associated to $I^\bullet$:

$$F[-1] \to I^\bullet \to \mathcal{O}_X \stackrel{s}{\to} F \to \ldots, \tag{1.13}$$

$$\mathscr{I}_C \to I^\bullet \to Q[-1] \to \mathscr{I}_C[1] \to \ldots, \tag{1.14}$$

the second coming from (1.7).

Lemma 1.15. *$\mathscr{E}xt^{\leq -1}(I^\bullet, I^\bullet) = 0$ and $\mathscr{H}om(I^\bullet, I^\bullet) = \mathcal{O}_X$.*

Proof. Applying $\mathscr{H}om(\,\cdot\,, \mathcal{O}_X)$ to (1.13) yields

$$\mathscr{H}om(F, \mathcal{O}_X) \to \mathscr{H}om(\mathcal{O}_X, \mathcal{O}_X) \to \mathscr{H}om(I^\bullet, \mathcal{O}_X) \to \mathscr{E}xt^1(F, \mathcal{O}_X). \tag{1.16}$$

The first and last terms vanish since F has support of codimension 2. The identity generates the second term, and maps in the third term to the canonical map $I^\bullet \to \mathcal{O}_X$ of (1.13). This canonical map therefore generates

$$\mathscr{H}om(I^\bullet, \mathcal{O}_X) \cong \mathcal{O}_X. \tag{1.17}$$

It is the image of the identity in the exact sequence

$$\mathscr{E}xt^{-1}(I^\bullet, F) \to \mathscr{H}om(I^\bullet, I^\bullet) \to \mathscr{H}om(I^\bullet, \mathcal{O}_X) \tag{1.18}$$

obtained from (1.13) by applying $\mathscr{H}om(I^\bullet, \,\cdot\,)$.

Therefore to show that $\mathscr{H}om(I^\bullet, I^\bullet) = \mathcal{O}_X$ we need only prove the vanishing of $\mathscr{E}xt^{-1}(I^\bullet, F)$. But $\mathscr{H}om(\,\cdot\,, F)$ applied to (1.14) gives

$$\mathscr{E}xt^{-1}(I^\bullet, F) \cong \mathscr{H}om(Q, F),$$

which vanishes by the purity of F.

The same sequences in lower degrees prove the vanishing of the sheaves $\mathscr{E}xt^{\leq -1}(I^\bullet, I^\bullet)$. □

After tensoring the result of Lemma 1.15 by K_X, we obtain

$$\mathscr{E}xt^{\leq -1}(I^\bullet, I^\bullet \otimes K_X) = 0 \text{ and } \mathscr{H}om(I^\bullet, I^\bullet \otimes K_X) = K_X. \tag{1.19}$$

Lemma 1.20. *$\mathscr{E}xt^{\leq -1}(I^\bullet, \mathcal{O}_X) = 0$ and $\mathscr{H}om(I^\bullet, \mathcal{O}_X) = \mathcal{O}_X$.*

Proof. The second claim is (1.17). The vanishing of $\mathscr{E}xt^{\leq -1}(I^\bullet, \mathcal{O}_X)$ is obtained from (1.16) in lower degrees. □

By the local-to-global spectral sequence and Lemma 1.20, we obtain

$$\mathrm{Hom}(I^\bullet, \mathcal{O}_X) = \mathbb{C}.$$

The exact triangle (1.13) is obtained canonically from the unique (up to scalars) nonzero element $\mathrm{Hom}(I^\bullet, \mathcal{O}_X)$ with F quasi-isomorphic to the mapping cone M. The pair (F, s) can be recovered from the complex $I^\bullet \in D^b(X)$ from the 0^{th} cohomology of the induced map

$$\mathcal{O}_X \to M.$$

Hence, by considering $I^\bullet \in D^b(X)$, no information about the original pair is lost. We have proven the following result.

Proposition 1.21. *The stable pairs (F, s) and (F', s') are isomorphic if and only if the complexes $\{\mathcal{O}_X \xrightarrow{s} F\}$ and $\{\mathcal{O}_X \xrightarrow{s'} F'\}$ are quasi-isomorphic.*

2. Deformation theory and the virtual class

2.1. **Pairs and complexes.** Let X be a 3-fold. As before, to each stable pair

$$[\mathcal{O}_X \xrightarrow{s} F] \in P_n(X, \beta)$$

we associate a complex

$$I^\bullet = \{\mathcal{O}_X \to F\} \in D^b(X).$$

The first order deformation theory of the moduli space $P_n(X, \beta)$ of stable pairs is governed by the tangent space $\mathrm{Ext}^0(I^\bullet, F)$ and the obstruction space $\mathrm{Ext}^1(I^\bullet, F)$ [32]. The deformation theory of the complex $I^\bullet$ *with fixed determinant* $\mathcal{O}_X$ in $D^b(X)$ is governed by $\mathrm{Ext}^1(I^\bullet, I^\bullet)_0$ and $\mathrm{Ext}^2(I^\bullet, I^\bullet)_0$ [25, 41]. The subscript 0 in the latter two groups denotes trace-free Ext.

Since a deformation of the pair (F, s) induces a deformation of the complex $I^\bullet$ with trivial determinant, there is a map from the first pair of groups to the second,

$$\mathrm{Ext}^i(I^\bullet, F) \to \mathrm{Ext}^{i+1}(I^\bullet, I^\bullet)_0\,,$$

obtained by applying $\mathrm{Hom}(I^\bullet, \ \cdot\ )$ to the canonical map $F[-1] \to I^\bullet$ of (1.13) and showing the image is in the trace-free part of $\mathrm{Ext}^*(I^\bullet, I^\bullet)$.

2.2. **Tangent spaces.** We first show that to all orders the deformations of pairs (F, s) equal the deformations of complexes $I^\bullet$ of fixed determinant. Hence, $P_n(X, \beta)$ is a locally complete moduli space of *complexes* $I^\bullet$ of fixed determinant and the map

$$\mathrm{Ext}^0(I^\bullet, F) \to \mathrm{Ext}^1(I^\bullet, I^\bullet)_0$$

is an isomorphism.

A *family of stable pairs* over a quasi-projective base scheme B is a pair

$$\mathcal{O}_{X\times B} \xrightarrow{s} F,$$

on $X \times B$, for which

(i) F is flat over B,
(ii) for all closed points $b \in B$, the restriction (F_b, s_b) to the fiber $X \times \{b\}$ is a stable pair.

Let Q be the cokernel of s. The sheaf Q is supported in relative dimension 0 over B.

A *family of complexes* over a base B is a *perfect complex* $I^\bullet$ on $X \times B$. By definition, a perfect complex has a finite resolution by locally free sheaves.[6] No additional flatness condition over B is required.

For the deformation question, we study how families extend over nilpotent thickenings of the base. Let

$$B \supset B_0$$

be a thickening of base schemes where B_0 is defined by a nilpotent ideal J (satisfying $J^N = 0$ for some $N > 0$). A family of stable pairs (F, s) over B is a deformation of a family (F_0, s_0) over B_0 if the restriction of (F, s) over B_0 is isomorphic to (F_0, s_0). A family of complexes $I^\bullet$ over B is a deformation of a family $I_0^\bullet$ over B_0 if the derived restriction of $I^\bullet$ to $X \times B_0$ is quasi-isomorphic to $I_0^\bullet$.

Let (F_0, s_0) be a family of stable pairs over B_0. Let

$$I_0^\bullet = \{\mathcal{O}_X \to F_0\}$$

be the associated family of complexes over B_0. The flatness of F_0 implies that $I_0^\bullet$ is perfect.

Lemma 2.1. *Every deformation $I^\bullet$ over B of $I_0^\bullet$ is quasi-isomorphic to a 2-term complex of sheaves $\{A \to E^1\}$ satisfying*

(i) *E^1 is locally free,*

[6] Since $X \times B$ is quasi-projective, the local and global existence of a resolution are equivalent.

(ii) *A is a pure sheaf of local projective dimension at most 1,*
(iii) *$h^1(I^\bullet)$ has support of relative dimension 0 over B,*
(iv) *$h^0(I^\bullet)$, away from the support of $h^1(I^\bullet)$, is flat over B.*

Proof. Since F_0 is pure of relative dimension 1 on

$$\pi_0 \colon X \times B_0 \to B_0,$$

F_0 has depth 1 on the fibers of π_0. By the Auslander-Buchsbaum formula, F_0 has local projective dimension at most 2 on the fibers of π_0. Let

$$K^\bullet \to F_0 \to 0$$

be a locally free resolution of F_0. Consider the cut-off

$$0 \to \mathrm{Ker} \to K^{-1} \to K^0 \to F_0 \to 0.$$

By flatness of F_0 and the projective dimension computation, the restriction of Ker to each fiber $\pi_0^{-1}(b_0)$ is locally free. Since Ker is also flat over B_0, we conclude Ker is locally free. Hence, F_0 has a locally free resolution of length 3 on $X \times B_0$. Since $\mathcal{O}_{X\times B_0}$ is already locally free, $I_0^\bullet$ is quasi-isomorphic to a complex of locally free sheaves of length 3 on X.

Let $E^\bullet$ be a finite complex of locally free sheaves on $X \times B$ quasi-isomorphic to $I^\bullet$. We will use standard base change, semicontinuity, and Nakayama Lemma arguments to trim the complex $E^\bullet$ down to length 3.

Let E^n be the last nonzero term of $E^\bullet$. If $n > 1$, then

$$E^{n-1}|_{X\times B_0} \to E^n|_{X\times B_0}$$

is surjective. So $E^{n-1} \to E^n$ is surjective in a neighbourhood of $X \times B_0$ and thus on all of $X \times B$. The kernel of $E^{n-1} \to E^n$ is then locally free, and $E^\bullet$ can be trimmed. We can thus assume that E^1 is the last term.

Similarly, let E^m be the first nonzero term. If $m < -1$, then

$$E^m|_{X\times B_0} \to E^{m+1}|_{X\times B_0}$$

is injective on *fibers* by base change and the 3 term result for $I_0^\bullet$. So

$$E^m \to E^{m+1}$$

is injective on fibers with locally free cokernel, and $E^\bullet$ may be again trimmed. We therefore assume that E^{-1} is the first term.

We conclude that $I^\bullet$ is quasi-isomorphic to a length 3 complex of locally free sheaves

$$E^{-1} \to E^0 \to E^1$$

on $X \times B$. The first map

$$E^{-1}|_{X\times B_0} \to E^0|_{X\times B_0}$$

is injective as a map of sheaves since $h^{-1}(I_0^\bullet) = 0$. Hence, $E^{-1} \to E^0$ is injective in a neighbourhood of $X \times B_0$ and thus on all of $X \times B$. So $I^\bullet$ is quasi-isomorphic to

$$A \to E^1$$

for some sheaf $A = E^0/E^{-1}$ of projective dimension at most 1. Moreover, $E^{-1} \to E^0$ is actually injective on fibers away from the relative curve

$$C_0 \times B_0 \subset X \times B$$

on which F_0 is supported. Hence, A is locally free away from C_0.

To establish the purity of A, we must show that any subsheaf $A' \subset A$ with support of codimension at least 1 is in fact zero. Since A is locally free away from C_0, A' must be supported on C_0 and so has codimension at least 2. Let π denote the projection

$$\pi\colon X \times B \to B$$

and let K_X be the relative dualizing sheaf. Then,

$$\mathrm{Hom}(A', A) = H^0(\pi_*\mathscr{H}om(A', A)),$$

and we need only prove $\pi_*\mathscr{H}om(A', A) = 0$.

By relative Serre duality for the smooth map π, the derived dual of $R\pi_*R\mathscr{H}om(A', A)[3]$ is quasi-isomorphic to

$$R\pi_*R\mathscr{H}om(R\mathscr{H}om(A', A), K_X) = R\pi_*(R\mathscr{H}om(A, A' \otimes K_X)).$$

The latter's kth cohomology sheaf can be calculated by the local-to-global spectral sequence with E_2 term

$$R^i\pi_*\mathscr{E}xt^j(A, A' \otimes K_X), \qquad i + j = k. \tag{2.2}$$

As A has projective dimension at most 1, we have vanishing for $j \geq 2$. Since A' is supported in relative dimension at most 1, we have vanishing for $i \geq 2$. Therefore, the sheaves (2.2) vanish for $i + j \geq 3$.

It follows that $R\pi_*R\mathscr{H}om(A, A' \otimes K_X)$ is quasi-isomorphic to a complex supported in degrees $0, 1$ and 2. Taking the derived dual gives a complex in degrees ≥ -2. Shifting by $[-3]$ then shows that $R\pi_*R\mathscr{H}om(A', A)$ is supported in degrees ≥ 1. Therefore its 0th degree cohomology sheaf $\pi_*\mathscr{H}om(A', A)$ vanishes, and A is indeed pure.

The complement of the closed subscheme $Z_0 = \mathrm{Supp}(Q_0)$ determines open sets

$$U_0 = (X \times B_0)\backslash Z_0, \quad U = (X \times B)\backslash Z_0.$$

Though U_0 and U have the same closed points, U_0 is a closed subscheme of U. Certainly,

$$E^0|_{U_0} \to E^1|_{U_0}$$

is surjective. By Nakayama's Lemma,

$$E^0|_U \to E^1|_U \tag{2.3}$$

is also surjective. Thus, $h^1(I^\bullet)$ has relative dimension 0 support, which is property (iii).

Let $K|_U$ be the locally free kernel of (2.3). The sheaf $h^0(I^\bullet)|_U$ is quasi-isomorphic to

$$E^{-1}|_U \to K|_U$$

by the established injectivity. The complex

$$E^{-1}|_{U_0} \to K|_{U_0} \tag{2.4}$$

is the derived restriction of $h^0(I^\bullet)|_U$ to U_0. By repeating the argument for B_0 instead of B, we find (2.4) is quasi-isomorphic to its cokernel

$$h^0(I_0^\bullet)|_{U_0} \cong h^0(I^\bullet)|_{U_0}.$$

Since $h^0(I_0^\bullet)|_{U_0}$ is the kernel of the surjection

$$\mathcal{O}_{U_0} \to F_0|_{U_0}$$

and F_0 is flat over B_0, we see that $h^0(I_0^\bullet)|_{U_0}$ is flat over B_0. By Lemma 2.5 below, $h^0(I^\bullet)|_U$ is flat over B, which is (iv). □

Lemma 2.5. *Let $\iota\colon B_0 \hookrightarrow B$ be a nilpotent thickening, and let F be a coherent sheaf on an open set*

$$U \subset X \times B.$$

Let $F_0 = \iota^* F$ be the restriction to $U_0 = \iota^*(U)$. Then, F is flat over B if and only if

(i) *$L\iota^* F \cong F_0$ and*
(ii) *F_0 is flat over B_0.*

Proof. Flatness clearly implies (i) and (ii). For the converse we must show that

$$F \overset{L}{\otimes} M \cong F \otimes M \tag{2.6}$$

for any $\mathcal{O}_B$-module M. For $M = \iota_* M_0$, where M_0 is an $\mathcal{O}_{B_0}$-module, (2.6) is clear:

$$F \overset{L}{\otimes} \iota_* M_0 \cong \iota_*(L\iota^* F \overset{L}{\otimes} M_0) \cong \iota_*(F_0 \overset{L}{\otimes} M_0) \cong \iota_*(F_0 \otimes M_0) \cong F \otimes \iota_* M_0.$$

Here, the second isomorphism comes from (i) and the third from (ii).

Since B is a nilpotent thickening of B_0, M can be written as a finite series of extensions of such $\mathcal{O}_{B_0}$-modules, which gives (2.6). □

Theorem 2.7. *Every deformation $I^\bullet$ over B of $I_0^\bullet$ with trivial determinant is quasi-isomorphic to a complex*

$$\{\mathcal{O}_{X\times B} \xrightarrow{s} F\},$$

where F is a flat deformation of F_0 with section s.

Proof. The rank of $I^\bullet$ is 1. Since

$$Q = h^1(I^\bullet)$$

has rank 0 on $X \times B$, the rank of $h^0(I^\bullet)$ must be 1. As a subsheaf of A, $h^0(I^\bullet)$ is also pure so injects into its double dual,

$$0 \to h^0(I^\bullet) \to h^0(I^\bullet)^{\vee\vee}.$$

By (iii) and (iv) of Lemma 2.1, $h^0(I^\bullet)$ is flat over B away from a set of codimension 3. Therefore by [31, Lemma 6.13] the double dual is locally free away from the codimension 3 set. Moreover, in the proof of [31, Lemma 6.13], Kollár shows that a reflexive rank 1 sheaf which is locally free away from a set of codimension 3 is in fact locally free globally. Finally $h^0(I^\bullet)^{\vee\vee}$ has trivial determinant since $I^\bullet$ does, so we find that

$$h^0(I^\bullet)^{\vee\vee} \cong \mathcal{O}_{X\times B}$$

and $h^0(I^\bullet)$ is an ideal sheaf $\mathscr{I}_C \subset \mathcal{O}_{X\times B}$.

The exact triangle $h^0(I^\bullet) \to I^\bullet \to h^1(I^\bullet)[-1]$, written as

$$\mathscr{I}_C \to I^\bullet \to Q[-1],$$

describes $I^\bullet$ as the cone of a map $Q[-2] \to \mathscr{I}_C$. The latter is an element $\alpha \in \mathrm{Ext}^2(Q, \mathscr{I}_C)$.

Consider the exact sequence obtained from the ideal sequence

$$0 \to \mathscr{I}_C \to \mathcal{O}_{X\times B} \to \mathcal{O}_C \to 0 \tag{2.8}$$

by applying $\mathrm{Hom}(Q, \,\cdot\,)$:

$$\mathrm{Ext}^1(Q, \mathcal{O}_{X\times B}) \to \mathrm{Ext}^1(Q, \mathcal{O}_C) \to \mathrm{Ext}^2(Q, \mathscr{I}_C) \to \mathrm{Ext}^2(Q, \mathcal{O}_{X\times B}).$$

The first and last terms vanish since Q has codimension 3 support. Thus,

$$\mathrm{Ext}^1(Q, \mathcal{O}_C) \cong \mathrm{Ext}^2(Q, \mathscr{I}_C),$$

and α is the cup product of an element $\epsilon \in \mathrm{Ext}^1(Q, \mathcal{O}_C)$ with the extension class in $\mathrm{Ext}^1(\mathcal{O}_C, \mathscr{I}_C)$ of the ideal sequence (2.8). Thus it is represented by the splicing together of the exact sequences

$$\begin{array}{ccccccccc} & & & & 0 \to \mathcal{O}_C \to F \to Q \to 0 \\ & & & & \| \\ 0 \to \mathscr{I}_C \to \mathcal{O}_{X\times B} \to & & & & \mathcal{O}_C \to 0\,, \end{array}$$

where F is the extension defined by ϵ. The result is the exact sequence

$$0 \to \mathscr{I}_C \to \mathcal{O}_{X\times B} \to F \to Q \to 0.$$

Hence, $I^\bullet$ is quasi-isomorphic to the complex

$$\{\mathcal{O}_{X\times B} \to F\}.$$

Finally, we show F is flat over B. By Lemma 2.5, flatness follows if

$$L\iota^* F \cong F_0,$$

where $\iota\colon B_0 \hookrightarrow B$ is the inclusion.

F can be described as the cone of the canonical map (1.13),

$$I^\bullet \to \mathcal{O}_{X\times B},$$

so the derived restriction of F to $X \times B_0$ is the cone on the induced map

$$I_0^\bullet \to \mathcal{O}_{X\times B_0}.$$

Similarly F_0 is the cone on such a map (1.13), so we need only check the two maps coincide.

By sequence (1.16), $\mathscr{H}om(I^\bullet, \mathcal{O}_{X\times B}) \cong \mathcal{O}_{X\times B}$. In lower degrees, (1.16) yields the vanishing of $\mathscr{E}xt(I^\bullet, \mathcal{O}_{X\times B})^{\leq -1}$. Therefore,

$$\operatorname{Hom}(I^\bullet, \mathcal{O}_{X\times B}) \cong \Gamma(\mathcal{O}_B)$$

is generated over $\mathcal{O}_B$ by the canonical map (1.13). Similarly,

$$\operatorname{Hom}(I_0^\bullet, \mathcal{O}_{X\times B_0}) \cong \Gamma(\mathcal{O}_{B_0})$$

is generated by the canonical map. The restriction to $X \times B_0$ of the canonical map on $X \times B$ is ϕ times the canonical map on $X \times B_0$ for some $\phi \in \Gamma(\mathcal{O}_{B_0})$. However, away from the codimension 2 support C of F, both maps are just the standard isomorphism $I^\bullet \cong \mathcal{O}$, so ϕ is invertible. $\square$

2.3. **Obstruction theory.** In Section 1.4, the association of the complex

$$I^\bullet = \left\{\mathcal{O}_X \xrightarrow{s} F\right\} \in D^b(X)$$

to the pair (F, s) was seen to be injective on objects. By Theorem 2.7, the fixed determinant deformation theory in $D^b(X)$ matches the deformation theory of pairs to all orders. Hence, $P(X)$ is a component[7] of the moduli space of complexes of trivial determinant in $D^b(X)$. We will use the obstruction theory $\operatorname{Ext}^*(I^\bullet, I^\bullet)_0$ to define a virtual fundamental class on $P(X)$.

[7]We drop the subscripts n and $\beta \neq 0$ for notational convenience. Formally, a component is used here to signify a union of connected components.

Since stable pairs have no nontrivial automorphisms, the moduli space $P(X)$ is fine. There is a *universal* stable pair[8]

$$\mathcal{O}_{X\times P(X)} \to \mathbb{F} \tag{2.9}$$

on $X \times P(X)$. The moduli space is the GIT quotient of a subset of the product of a Quot scheme and a Grassmannian [32]. There is a universal sheaf pulled back from the Quot scheme, with a universal section over the product. Over the stable locus there are no stabilizers, so by Kempf's lemma (see for instance [23, Theorem 4.2.15]) the universal pair descends to $P(X)$.

The universal pair (2.9) determines a universal complex in the derived category,

$$\mathbb{I}^\bullet = \{\mathcal{O}_{X\times P(X)} \to \mathbb{F}\} \in D^b(X \times P(X)),$$

with $\mathbb{F}$ flat over $P(X)$. Let π denote the projection

$$\pi\colon X \times P(X) \to P(X).$$

We will prove that the complex $R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ determines an obstruction theory on $P(X)$.

Since $X \times P(X)$ is projective and $\mathbb{I}^\bullet$ is perfect we may resolve it by a finite complex of locally free sheaves $A^\bullet$, and form

$$(A^\bullet)^\vee \otimes A^\bullet \;\cong\; \mathcal{O}_{X\times P} \,\oplus\, ((A^\bullet)^\vee \otimes A^\bullet)_0.$$

The first summand is the image of the identity map and the second is the kernel of the trace map. These split each other since

$$\operatorname{tr}\circ \operatorname{id} = \operatorname{rank}(A^\bullet) = \operatorname{rank}(\mathbb{I}^\bullet) = 1 = \operatorname{id}\circ\operatorname{tr}.$$

We define the quasi-isomorphism class of the trace-free Homs by

$$R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \simeq ((A^\bullet)^\vee \otimes A^\bullet)_0.$$

The following result is a standard consequence of Lemma 1.15 and the Nakayama Lemma, but we give the argument in full.

Lemma 2.10. *The complex $R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$ on $P(X)$ is quasi-isomorphic to a 2-term complex of locally free sheaves $\{E_1 \to E_2\}$.*

Proof. Let $B^\bullet$ be a sufficiently negative locally free resolution of the complex $((A^\bullet)^\vee \otimes A^\bullet)_0$ trimmed to start at least 4 places earlier than $((A^\bullet)^\vee \otimes A^\bullet)_0$. Then, by standard arguments, for all j,

(i) $R^{\le 2}\pi_* B^j = 0$,
(ii) $R^3\pi_* B^j$ is locally free.

[8]Pairs are better than stable sheaves which have scalar automorphisms. In general, there is only a universal *twisted* sheaf on the product of X and the moduli space of stable sheaves.

The complex $E^\bullet$ with

$$E^k \cong R^3\pi_* B^{k+3}$$

is finite, locally free, and quasi-isomorphic to $R\pi_* R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0$.

By base change, the restriction of $E^\bullet$ over a point $[I^\bullet] \in P(X)$ is a complex of vector spaces computing $\mathrm{Ext}^*(I^\bullet, I^\bullet)_0$. By Lemmas 1.15 and 1.20, the local-to-global spectral sequence, and Serre duality, $\mathrm{Ext}^i(I^\bullet, I^\bullet)$ is nonzero only for i between 0 and 3.

By (1.19) and the local-to-global spectral sequence, we have the isomorphisms

$$\begin{aligned} K_X &\xrightarrow{\mathrm{id}} \mathscr{H}om(I^\bullet, I^\bullet \otimes K_X), \\ H^0(K_X) &\xrightarrow{\mathrm{id}} \mathrm{Hom}(I^\bullet, I^\bullet \otimes K_X). \end{aligned}$$

Then, by Serre duality,

$$\mathrm{Ext}^3(I^\bullet, I^\bullet) \xrightarrow{\mathrm{tr}} H^3(\mathcal{O}_X)$$

is also an isomorphism and $\mathrm{Ext}^3(I^\bullet, I^\bullet)_0$ vanishes. Similarly, the composition

$$\mathbb{C} \xrightarrow{\mathrm{id}} \mathrm{Hom}(I^\bullet, I^\bullet) \xrightarrow{\mathrm{tr}} H^0(\mathcal{O}_X)$$

is multiplication by $\mathrm{rank}(I^\bullet) = 1$. Hence $\mathrm{Hom}(I^\bullet, I^\bullet)_0$ also vanishes and the trace-free extensions $\mathrm{Ext}^i(I^\bullet, I^\bullet)_0$ are concentrated entirely in degrees $i = 1$ and 2. After base change to any point $[I^\bullet] \in P(X)$, the complex $E^\bullet$ has cohomology only in degrees 1 and 2.

If $E^{n>2}$ is the last nonzero term of $E^\bullet$, then on each fiber

$$E^{n-1} \to E^n \tag{2.11}$$

is surjective. Hence, the map (2.11) is surjective globally with locally free kernel. Replacing E^{n-1} by the kernel and E^n by zero, we can inductively assume $n = 2$. Similarly if the first nonzero term is $E^{m<1}$, then $E^m \to E^{m+1}$ is injective on fibers with locally free cokernel. We conclude $E^\bullet$ is quasi-isomorphic to a 2-term complex $\{E_1 \to E_2\}$. □

The Atiyah class of the universal complex $\mathbb{I}^\bullet$ gives an element of

$$\mathrm{Ext}^1(\mathbb{I}^\bullet, \mathbb{I}^\bullet \otimes L^\bullet_{X\times P(X)}), \tag{2.12}$$

where $L^\bullet_{X\times P(X)}$ is (a locally free resolution of) the cotangent complex of $X \times P(X)$. Since the cotangent complex of a product is the sum of the (pullbacks of the) cotangent complexes of the factors, the Ext group (2.12) maps to

$$\mathrm{Ext}^1(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0, \pi^* L^\bullet_{P(X)}),$$

which by Serre duality along π is isomorphic to

$$\mathrm{Ext}^{-2}(R\pi_*(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_\pi), L^\bullet_{P(X)}).$$

The relative dualizing sheaf ω_π is the pull-back of the canonical bundle of X. We obtain a map

$$R\pi_*(R\mathscr{H}om(\mathbb{I}^\bullet, \mathbb{I}^\bullet)_0 \otimes \omega_\pi)[2] \to L^\bullet_{P(X)}. \tag{2.13}$$

We claim that (2.13) provides a perfect obstruction theory for $P(X)$ in the sense of [7] via the results of [24, 41, 42]. More precisely, the obstruction classes of [41, 42] are expressed in [24] as a product of Atiyah and Kodaira-Spencer classes. As a result, compatibility with (2.13) is obtained.[9] By Lemma 2.10 and Serre duality, the obstruction theory (2.13) is *perfect*: representable by a 2-term complex of locally free sheaves

$$E_2^\vee \to E_1^\vee$$

in degrees -1 and 0 respectively. Therefore, the results of [7, 39] yield a virtual class.

Theorem 2.14. *$P_n(X, \beta)$ carries an algebraic virtual class*

$$[P_n(X,\beta)]^{vir} \in A_{c_\beta}(P_n(X,\beta), \mathbb{Z})$$

where

$$c_\beta = \int_\beta c_1(X) = -\chi(R\operatorname{Hom}(I^\bullet, I^\bullet)_0)$$

is the virtual dimension of $P_n(X,\beta)$ with the obstruction theory inherited from the moduli space of fixed determinant complexes $I^\bullet \in D^b(X)$.

Consider a smooth family of projective 3-folds,

$$\pi : \mathcal{X} \to B,$$

over a base B with central fiber $X \cong \mathcal{X} \times_B \{0\}$. Let

$$\mathcal{P}_n(\mathcal{X}, \beta) \to B$$

denote the relative moduli space of stable pairs on the fibers of π, and let

$$i_0 \colon P_n(X,\beta) \to \mathcal{P}_n(\mathcal{X},\beta)$$

be the inclusion of the space of stable pairs on the central fiber X. The deformation invariance of the virtual class in the following form is obtained from [24, Corollary 4.3].

Theorem 2.15. *There is π-relative virtual class*

$$[\mathcal{P}_n(\mathcal{X},\beta)]^{vir} \in A_{c_\beta + \dim B}(\mathcal{P}_n(\mathcal{X},\beta), \mathbb{Z})$$

for which

$$i_0^![\mathcal{P}_n(\mathcal{X},\beta)]^{vir} = [P_n(X,\beta)]^{vir}.$$

[9]See Theorem 4.1 [24] for a careful discussion of the obstruction class and the results needed to prove that (2.13) is an obstruction theory for $P(X)$.

2.4. **Invariants.** If X is a Calabi-Yau 3-fold, $c_\beta = 0$ for every curve class $\beta \in H_2(X, \mathbb{Z})$.

Definition 2.16. *If $c_\beta = 0$, the stable pairs invariants $P_{n,\beta} \in \mathbb{Z}$ are defined to be the degree of the virtual cycle*

$$P_{n,\beta} = \int_{[P_n(X,\beta)]^{vir}} 1.$$

We define similar invariants with primary field insertions in case $c_\beta > 0$ in Section 3.6.

For future reference, we list the letters we use for the various curve counting invariants associated to the class $\beta \in H^2(X, \mathbb{Z})$. The definitions and references will be given later.

- $N_{g,\beta}$: genus g connected Gromov-Witten invariant.
- $N^\bullet_{g,\beta}$: genus g disconnected Gromov-Witten invariant with no contracted contributions.
- $n_{g,\beta}$: genus g Gopakumar-Vafa BPS invariant.
- $I_{n,\beta}$: DT invariant with Euler characteristic n.
- $P_{n,\beta}$: stable pairs invariant with Euler characteristic n.

3. Conjectures

3.1. **GW/DT for Calabi-Yau 3-folds.** We recall the conjectural GW/DT correspondence from [43] for a Calabi-Yau 3-fold X.

The disconnected Gromov-Witten invariants of X for nonzero curve classes $\beta \in H_2(X, \mathbb{Z})$ are

$$N^\bullet_{g,\beta} = \int_{[\overline{M}^\bullet_g(X,\beta)]^{vir}} 1,$$

where $\overline{M}^\bullet_g(X, \beta)$ is the moduli space of stable maps with possibly disconnected domains and *no* contracted connected components.[10] Let

$$Z_{GW,\beta}(u) = \sum_g N^\bullet_{g,\beta}\ u^{2g-2}.$$

be the partition function. Alternatively, the partition function may be defined via the exponential of the connected potential,

$$Z_{GW}(u, v) = 1 + \sum_{\beta \neq 0} Z_{GW,\beta}(u) v^\beta = \exp F_{GW}(u, v),$$

where

$$F_{GW}(u, v) = \sum_{\beta \neq 0} \sum_g N_{g,\beta}\ u^{2g-2} v^\beta,$$

[10] We follow here the notation of [9].

and

$$N_{g,\beta} = \int_{[\overline{M}_g(X,\beta)]^{vir}} 1$$

is the standard connected Gromov-Witten invariant.

Let $I_n(X,\beta)$ be the Hilbert scheme of subschemes

$$Z \subset X$$

with holomorphic Euler characteristic n and fundamental class β. To obtain a virtual class [50], the Hilbert scheme is viewed as a moduli space of ideal sheaves.[11] The DT invariants are defined by

$$I_{n,\beta} = \int_{[I_n(X,\beta)]^{vir}} 1.$$

The partition function of DT theory is

$$Z_{DT,\beta}(q) = \sum_n I_{n,\beta}\ q^n.$$

By the boundedness of the Hilbert scheme $I_n(X,\beta)$, the number of free and embedded points of any element Z — the length of the maximal 0-dimensional subsheaf of $\mathcal{O}_Z$ — is bounded above. Therefore $I_n(X,\beta)$ is empty for n sufficiently negative, and $Z_{DT,\beta}(q)$ is a Laurent series in q.

The irreducible components of the subscheme Z consist of curves and 0-dimensional subschemes which wander all over X. A *reduced* partition function is defined [43] by dividing out by the degree 0 series,

$$Z'_{DT,\beta}(q) = \frac{Z_{DT,\beta}(q)}{Z_{DT,0}(q)}.$$

The degree 0 series is evaluated by the formula

$$Z_{DT,0}(q) = M(-q)^{\chi(X)},$$

conjectured in [43] and proved in [8, 34, 38]. Here,

$$M(q) = \prod_{n\geq 1}(1-q^n)^{-n}$$

is the MacMahon function.

The reduced series $Z'_{DT,\beta}(q)$ is conjectured in [43] to be the Laurent expansion of a rational function in q invariant under the transformation $q \leftrightarrow q^{-1}$. The GW/DT correspondence of [43] is the conjectural equality

$$Z'_{DT,\beta}(q) \ = \ Z_{GW,\beta}(u)$$

[11]As in the case of stable pairs, the straightforward obstruction theory of the Hilbert scheme is not appropriate and an alternative is required.

after the variable change $-q = e^{iu}$.

3.2. **Stable pairs conjectures for Calabi-Yau 3-folds.** Our new invariants $P_{n,\beta}$ counting stable pairs on X do not have the drawback of freely roaming points, so a reduced partition function is not necessary. For nonzero $\beta \in H_2(X, \mathbb{Z})$, let

$$Z_{P,\beta}(q) = \sum_n P_{n,\beta}\, q^n \tag{3.1}$$

be the partition function of the stable pairs theory. The moduli spaces $P_n(X, \beta)$ are empty for the same sufficiently negative n as for the $I_n(X, \beta)$, so $Z_{P,\beta}(q)$ is a Laurent series in q.

Conjecture 3.2. *The partition function $Z_{P,\beta}(q)$ is the Laurent expansion of a rational function in q invariant under $q \leftrightarrow q^{-1}$.*

In fact, the above rationality Conjecture will be significantly refined after our discussion of the BPS state counts of Gopakumar and Vafa.

Conjecture 3.3. *All the partition functions coincide,*

$$Z_{P,\beta}(q) \;=\; Z'_{DT,\beta}(q) \;=\; Z_{GW,\beta}(u),$$

after the variable change $-q = e^{iu}$.

It appears all reasonable enumerative theories of curves on Calabi-Yau 3-folds are actually equivalent.

3.3. **Wall crossing formula.** The first equality of Conjecture 3.3,

$$Z_{P,\beta}(q) \cdot Z_{DT,0}(q) \;=\; Z_{DT,\beta}(q) \tag{3.4}$$

can be expanded to yield

$$\sum_m P_{n-m,\beta} \cdot I_{m,0} = I_{n,\beta}. \tag{3.5}$$

Relation (3.5) should be interpreted as a wall-crossing formula for counting invariants in the derived category of coherent sheaves $D^b(X)$ under a change of stability condition [11]. So far, however, counting invariants have yet to be defined for general complexes, and Bridgeland stability conditions have not been shown to exist for compact 3-folds.

Let us assume that there is a Bridgeland stability condition for which ideal sheaves $\mathscr{I}_Z$ of subschemes $Z \subset X$ satisfying

$$\chi(\mathcal{O}_Z) = n, \quad [Z] = \beta$$

are stable and constitute the moduli space of semistable objects of the same phase and Chern character

$$(\mathrm{rk}, ch_1, ch_2, ch_3) = (1, 0, -\beta, -n)$$

and trivial determinant. The counting invariant here exists [50] and is $I_{n,\beta} \in \mathbb{Z}$. We also assume the structure sheaves of single points are stable. Since structure sheaves of subschemes of length $m > 1$ are only semistable (and have automorphisms), it is not clear what their counting invariant should be, but the virtual number of *ideal* sheaves of such subschemes is $I_{m,0}$.

Now move the central charge of [11] across a codimension 1 wall along which the phase of $\mathscr{I}_Z$ equals the phase of the structure sheaf of a point $\mathcal{O}_p$ minus 1:

$$\phi(\mathscr{I}_Z) = \phi(\mathcal{O}_p) - 1 = \phi(\mathcal{O}_p[-1]).$$

Any free or embedded points p in Z give rise to exact sequences of the form

$$0 \to \mathscr{I}_Z \to \mathscr{I}_{Z'} \to \mathcal{O}_p \to 0$$

where Z' is Z with the point p removed. In $D^b(X)$, we obtain an exact triangle

$$\mathcal{O}_p[-1] \to \mathscr{I}_Z \to \mathscr{I}_{Z'},$$

destabilizing $\mathscr{I}_Z$ as we cross the wall.

More generally, let $Q \subset \mathcal{O}_Z$ denote the maximal subsheaf with 0-dimensional support (roughly the structure sheaf of the union of all free and embedded points). Then

$$Q[-1] \to \mathscr{I}_Z \to \mathscr{I}_{Z'} \tag{3.6}$$

is the maximal destabilizing extension making up $\mathscr{I}_Z$. Conversely, previously unstable objects become stable as we cross the wall. These are objects $I^\bullet \in D^b(X)$ which are extensions of the form (3.6) but in the opposite direction,

$$\mathscr{I}_{Z'} \to I^\bullet \to Q[-1], \tag{3.7}$$

classified by elements of $\mathrm{Ext}^2(Q, \mathscr{I}_{Z'})$. We recognize (3.7) as the form of (1.14). We expect that the moduli space of pairs $P_n(X,\beta)$ gives precisely the space of stable objects for the new stability condition.[12] The virtual numbers $P_{n,\beta}$ should then be the right counting invariants.

Relation (3.5) has the form of a wall crossing formula envisaged by Joyce [27] for invariants counting stable objects in $D^b(X)$. The formula expresses all of the possible ways that a stable object on one side of the wall – an ideal sheaf $\mathscr{I}_Z$ – can be written as extensions of objects in $D^b(X)$ which are stable on the other side of the wall. No worse configurations occur since the K-theory classes of $\mathscr{I}_Z$, $\mathscr{I}_{Z'}$ and $\mathcal{O}_p$ are all primitive and distinct. So the m^{th} term in (3.5) is the contribution

[12] Recently, Bayer [4] and Toda [52] have defined variants of Bridgeland's axioms for a stability condition within which the above wall crossing occurs.

from subschemes Z whose maximal 0-dimensional subscheme (or total number of free and embedded points) is of length m – the length of Q. We expect that in Joyce's theory the space of extensions between the semistable pieces $\mathscr{I}_{Z'}$ and $Q[-1]$ contributes

$$-\chi(\mathscr{I}_{Z'}, Q[-1]) = m. \tag{3.8}$$

As Z' and Q vary this gets multiplied by the product of the invariants associated to $\mathscr{I}_{Z'}$ and $Q[-1]$. The former should be $P_{n-m,\beta}$. Formula (3.5) requires the latter to contribute $I_{m,0}/m$. That is, the automorphisms of the structure sheaves of length-m points should affect the virtual count for ideal sheaves of points by the factor $1/m$.

Until Joyce's theory can be fully extended to the derived category and made to include the virtual class, our entire discussion here remains conjectural.

Similar phenomena have been observed recently in a noncommutative example [49]. The paper [13] also studies counting invariants (and more generally BPS states) and wall crossing in $D^b(X)$ from a physical point of view. In fact, Denef and Moore predict a wall crossing of precisely the form (3.4) as the Kähler form and, crucially, the B-field, cross a certain wall in the space of stability conditions (formulae (6.21)–(6.24) of [13]). Their predictions are based on a supergravity analysis rather than algebraic geometry. For a certain stability condition, the virtual number of stable objects in $D^b(X)$ is predicted without identifying the form of these objects. Other wall crossings studied in [13, 15] have not yet been considered mathematically.

3.4. **BPS refinement.** Motivated by the M-theory prediction of Gopakumar and Vafa [19, 20] for Calabi-Yau 3-folds X, we conjecture a much stronger rationality statement for $Z_{P,\beta}(q)$.

Let $F_P(q, v)$ denote the *connected* series for the stable pairs invariants,

$$F_P(q, v) = \log Z_P(q, v) = \log \Big(1 + \sum_{\beta \neq 0} Z_{P,\beta}(q) v^\beta\Big).$$

The functions $F_{P,\beta}$ are defined by

$$F_P(q, v) = \sum_{\beta \neq 0} F_{P,\beta}(q) v^\beta,$$

While $F_{P,\beta}(q)$ should be thought of as counting pairs whose support is connected, we do not have a geometric method to define the invariants. Though $F_{P,\beta}(q)$ is still a Laurent series, the coefficients of $F_{P,\beta}(q)$ need not be integral due to the logarithm.

Let $V_0 \subset \mathbb{Q}(q)$ be the linear space of Laurent polynomials invariant under $q \leftrightarrow q^{-1}$. Certainly,

$$\Phi(q) = \frac{(1-q)^2}{q} = q - 2 + \frac{1}{q} \in V_0.$$

Lemma 3.9. *Every Laurent series $L(q) \in \mathbb{Q}((q))$ can be* uniquely *written as a sum*

$$L(q) = \sum_{g > -\infty} l_g \; \Phi(-q)^{g-1}, \quad l_g \in \mathbb{Q} \tag{3.10}$$

with only finitely *many positive g terms.*

Proof. Any $g > 1$ term

$$\Phi(-q)^{g-1} = \big((-q) - 2 + (-q)^{-1}\big)^{g-1} \in V_0$$

has a pole of order $g-1$ at $q = 0$. Together the $g > 1$ terms in (3.10) specify the finite polar part of $L(q)$. The $g \leq 1$ term

$$\Phi((-q))^{g-1} = \left(\frac{(-q)}{(1-(-q))^2}\right)^{1-g} = (-q)^{1-g} + 2(1-g)(-q)^{2-g} + \ldots$$

is regular at $q = 0$ with a zero of order $1 - g$. The right side of (3.10) is thus uniquely determined. □

For nonzero $\beta \in H_2(X, \mathbb{Z})$, let $\mathrm{div}(\beta)$ denote the divisibility of β. For simplicity, we either assume $H_2(X, \mathbb{Z})$ is torsion-free or allow β take values in the torsion-free quotient

$$H_2(X, \mathbb{Z})/\tau(H_2(X, \mathbb{Z}))$$

so each stable pairs invariant is a sum of all classes differing from β by torsion.[13]

Lemma 3.11. *The set of equations*

$$\left\{ F_{P,\beta}(q) \;=\; \sum_{g > -\infty} \; \sum_{r \mid \mathrm{div}(\beta)} n_{g, \frac{\beta}{r}} \frac{(-1)^{g-1}}{r} \big((-q)^r - 2 + (-q)^{-r}\big)^{g-1} \right\}_{\beta \neq 0},$$

has a unique solution $\{n_{g,\beta}\}_{g > -\infty, \beta \neq 0}$.

[13] In fact, no such assumption is necessary. The full torsion information can be kept. Then, the interior sum in Lemma 3.11 is over all elements of $H_2(X, \mathbb{Z})$ whose r^{th} multiple is β.

Proof. We proceed inductively on the divisibility of β. If $\mathrm{div}(\beta) = 1$, then the statement is a direct consequence of Lemma 3.9 applied to $F_{P,\beta}(q)$. For the inductive step, we apply Lemma 3.9 to

$$F_{P,\beta}(q) - \sum_{g>-\infty} \sum_{r|\,\mathrm{div}(\beta),\ r>1} n_{g,\frac{\beta}{r}} \frac{(-1)^{g-1}}{r} \left((-q)^r - 2 + (-q)^{-r}\right)^{g-1}$$

to conclude the existence and uniqueness of the solution. □

From vanishing of Lemma 3.9 applied inductively, we obtain the following result.

Lemma 3.12. *For fixed β, $n_{g,\beta} = 0$ for all sufficiently large g.*

We *define* the Gopakumar-Vafa BPS state counts in genus g and class $\beta \neq 0$ for X to be the solutions $n_{g,\beta}$ of Lemma 3.11. Our motivation for the definition is the agreement of the formula

$$F_P(q,v) = \sum_{g>-\infty} \sum_{\gamma\neq 0} \sum_{r\geq 1} n_{g,\gamma} \frac{(-1)^{g-1}}{r}\ \Phi((-q)^r)^{g-1} v^{r\gamma},$$

after truncation by Conjecture 3.14 below and the variable change

$$-q = e^{iu},$$

with the string theoretic Gopakumar-Vafa formula

$$F_{GW}(u,v) = \sum_{g\geq 0} \sum_{\gamma\neq 0} n_{g,\gamma}\ u^{2g-2} \sum_{r\geq 1} \frac{1}{r} \left(\frac{\sin(ru/2)}{u/2}\right)^{2g-2} v^{r\gamma} \tag{3.13}$$

for Gromov-Witten theory via BPS counts.

Philosophically, we view the integrals $P_{n,\beta}$ as giving a rigorous treatment of the heuristic approach[14] to BPS state counting proposed in [29] via virtual Euler characteristics of Hilbert schemes of points on curves in X. Certainly, $P_n(X,\beta)$ may be thought of as a compactification of the Hilbert scheme of points on curves in X. And, $P_{n,\beta}$ is precisely[15] a signed virtual Euler characteristic.

We replace the rationality statement of Conjecture 3.2 with a much stronger and more geometric vanishing.

Conjecture 3.14. *The invariants $n_{g,\beta}$ vanish for $g < 0$.*

---

[14] S. Katz [28] has proposed a mathematical definition of the $g = 0$ BPS state counts via virtual Euler characteristics of moduli of semistable sheaves on curves in X. Katz's construction, in which neither pairs nor derived categories appear, is nevertheless not unrelated to our perspective.

[15] In the Calabi-Yau case, the obstruction theory on $P_n(X,\beta)$ is self-dual [5, 16] and the virtual count $P_{n,\beta}$ is the virtual Euler characteristic, up to sign.

Define the linear subspace $V_d \subset \mathbb{Q}(q)$ for $d > 0$ by the following spanning set:

$$V_d = \mathrm{Span}_{\mathbb{Q}}\big\{\Phi((-q)^r)^{g-1}\big\}_{g\geq 0,\ 1\leq r\leq d}\,. \tag{3.15}$$

Since V_0 is the linear space of Laurent polynomials invariant under $q \leftrightarrow q^{-1}$, V_0 is spanned by $\{\Phi(-q)^{g-1}\}_{g\geq 1}$ and contained in all V_d. A basis of V_d/V_0 is given by the $g = 0$ terms of (3.15),

$$\frac{(-q)}{(1-(-q))^2}\,,\ \frac{(-q)^2}{(1-(-q)^2)^2}\,,\ \ldots,\ \frac{(-q)^d}{(1-(-q)^d)^2}\,.$$

Conjecture 3.14 implies the nontrivial inclusion

$$F_{P,\beta}(q) \in V_{\mathrm{div}(\beta)}. \tag{3.16}$$

As a consequence, we conclude that both $F_{P,\beta}(q)$ and $Z_{P,\beta}(q)$ have possible poles *only at r^{th} roots of unity* where $r \leq \mathrm{div}(\beta)$.

In fact, inclusion (3.16) is much stronger than the statement about poles. For example,

$$\Psi(q) = \frac{q}{1+q+q^2}$$

is invariant under $q \leftrightarrow q^{-1}$, has poles at only 3^{rd} roots of unity, but is easily seen to satisfy

$$\forall d, \quad \Psi(q) \notin V_d.$$

Conjecture 3.14 implies the following two effectivity statements. The asterisk denotes the dependence on the conjecture.

Lemma* 3.17. *$F_{P,\beta}(q)$ is uniquely and effectively determined by the coefficients of order q^n for $-\infty < n \leq \mathrm{div}(\beta)$.*

Proof. We can uniquely write $F_{P,\beta}$ as an element of $\Theta(q) \in V_0$ plus a linear combination of

$$\frac{(-q)}{(1-(-q))^2}\,,\ \frac{(-q)^2}{(1-(-q)^2)^2}\,,\ \ldots,\ \frac{(-q)^{\mathrm{div}(\beta)}}{(1-(-q)^{\mathrm{div}(\beta)})^2}\,. \tag{3.18}$$

The functions (3.18) are regular at $q = 0$. Therefore, the coefficients of q^n for $n \leq 0$ uniquely determine $\Theta(q)$, and the coefficients of q^n for $1 \leq n \leq \mathrm{div}(\beta)$ determine the linear combination (3.18). □

Lemma* 3.19. *$Z_P(q,v)$ is uniquely and effectively determined by the invariants $\big\{P_{n,\beta}\colon\ -\infty < n \leq 1,\ \forall\beta\big\}$.*

Proof. The argument follows the proof of Lemma 3.17. Since data is given for all classes β simultaneously, we may work by induction on the degree of β. The coefficients of v^β when β is primitive are dealt with by Lemma 3.17. For general β we need only handle the $r=1$ terms in

$$F_{P,\beta}(q) \;=\; \sum_{g\geq 0}\;\sum_{r|\,\mathrm{div}(\beta)} n_{g,\frac{\beta}{r}}\frac{(-1)^{g-1}}{r}\left((-q)^r-2+(-q)^{-r}\right)^{g-1},$$

since the others are dealt with by the induction assumption. □

The proof of Lemma* 3.19 shows the statement is local in the following sense. To determine $Z_{P,\gamma}(q)$, the data $\left\{P_{n,\beta}\right\}_{n\leq 1}$ is required only for classes β which are effective summands of γ.

3.5. **BPS integrality.** We show the integrality constraints on the stable pairs counts $P_{n,\beta}$ exactly matches the BPS integrality of $n_{g,\beta}$. We do not require Conjecture 3.14 for the result.

Theorem 3.20. *The invariants $P_{n,\beta}$ are integers for all $n,\beta\neq 0$ if and only if the coefficients $n_{g,\beta}$ are integers for all $g,\beta\neq 0$.*

Proof. By definition, $Z_{P,\beta}$ is the v^β coefficient of

$$\exp\left(\sum_{g>-\infty}\sum_{\gamma\neq 0}\sum_{k\geq 1} n_{g,\gamma}\frac{(-1)^{g-1}}{k}\left((-q)^k-2+(-q)^{-k}\right)^{g-1}v^{k\gamma}\right).$$

To simplify notation, let

$$\tilde{n}_{g,\gamma}=(-1)^{g-1}n_{g,\gamma},\quad Q=-q.$$

Then, $Z_{P,\beta}$ is the v^β coefficient of

$$\exp\left(\sum_{g>-\infty}\sum_{\gamma\neq 0}\sum_{k\geq 1}\frac{\tilde{n}_{g,\gamma}}{k}\Phi(Q^k)^{g-1}v^{k\gamma}\right). \tag{3.21}$$

We have seen already in the proof of Lemma 3.9 that for all $k\in\mathbb{Z}$,

$$\Phi(Q)^k=\sum_{l\geq -k}\phi_l^k Q^l,\quad \phi_l^k\in\mathbb{Z}, \tag{3.22}$$

with leading coefficient $\phi_{-k}^k=1$.

Let L be a very ample line bundle on X. We prove the Theorem inductively on the degree

$$L_\beta=\int_\beta c_1(L)$$

of β.

Classes $\beta \neq 0$ of minimal degree are necessarily primitive. Hence, only terms with $\gamma = \beta$ and $k = 1$ contribute to the v^β coefficient of (3.21),

$$Z_{P,\beta} = \sum_{g > -\infty} \tilde{n}_{g,\beta} \Phi(Q)^{g-1}.$$

By the expansion (3.22), the coefficients of $Z_{P,\beta}$ and the invariants $\tilde{n}_{g,\beta}$ are related by a triangular transformation with 1s along the diagonal. Hence,

$$\big\{P_{n,\beta}\big\}_{n\in\mathbb{Z}} \text{ are integral } \iff \big\{\tilde{n}_{g,\beta}\big\}_{g\in\mathbb{Z}} \text{ are integral.}$$

For the induction step, we write $Z_{P,\beta}(q)$ with the $\gamma = \beta$ and $k = 1$ term in front:

$$(3.23) \quad \sum_{g>-\infty} \tilde{n}_{g,\beta}\Phi(Q)^{g-1} + \left(\exp \sum_{g>-\infty} \sum_{\gamma\neq\beta} \sum_{k\geq 1} \frac{\tilde{n}_{g,\gamma}}{k} \Phi(Q^k)^{g-1} v^{k\gamma}\right)_{v^\beta},$$

where the suffix denotes taking the v^β coefficient. The term inside the brackets in (3.23) is

$$\exp \sum_{g>-\infty} \sum_{\gamma\neq\beta} \sum_{k\geq 1} \sum_{l\geq 1-g} \frac{\tilde{n}_{g,\gamma}}{k} \phi_l^{g-1} Q^{kl} v^{k\gamma}.$$

Doing the sum over k first, we obtain

$$\exp \sum_{g>-\infty} \sum_{\gamma\neq\beta} \sum_{l\geq 1-g} \tilde{n}_{g,\gamma}\ \phi_l^{g-1} \left(-\log(1 - Q^l v^\gamma)\right).$$

Substitution in (3.23) yields the following expression for $Z_{P,\beta}(q)$:

$$(3.24) \quad \sum_{g>-\infty} \tilde{n}_{g,\beta}\Phi(Q)^{g-1} + \left(\prod_{g>-\infty} \prod_{\gamma\neq\beta} \prod_{l\geq 1-g} \left(\frac{1}{1-Q^l v^\gamma}\right)^{\tilde{n}_{g,\gamma}\cdot\phi_l^{g-1}}\right)_{v^\beta}.$$

By the argument in the base case, if all the $\tilde{n}_{g,\gamma}$ contributing to the second term are integral, then

$$\big\{P_{n,\beta}\big\}_{n\in\mathbb{Z}} \text{ are integral } \iff \big\{\tilde{n}_{g,\beta}\big\}_{g\in\mathbb{Z}} \text{ are integral.}$$

But all the $\tilde{n}_{g,\gamma}$ contributing to the second term correspond to classes γ of strictly lower degree than β, so the induction hypothesis applies. □

Since the stable pair invariants $P_{n,\beta}$ are certainly integral, we conclude the BPS counts defined by $n_{g,\beta}$ are also integral. Conversely, the integrality of $n_{g,\beta}$ imposes no further conditions on $P_{n,\beta}$.

If the vanishing of Conjecture 3.14 is assumed, we conclude the integrality placed on Gromov-Witten theory by Conjecture 3.3 exactly coincides with the integrality predicted by the Gopakumar-Vafa formula (3.13).

3.6. **The Fano case.** For an arbitrary 3-fold X, the virtual dimension of the moduli space of pairs is

$$\dim_{\mathbb{C}}[P_n(X,\beta)]^{vir} = c_\beta,$$

where

$$c_\beta = \int_\beta c_1(X).$$

For a nontrivial theory, we must have $c_\beta \geq 0$. In the *local Calabi-Yau* case satisfying $c_\beta = 0$, the discussion is identical to the global Calabi-Yau case treated in Sections 3.2-3.5. In the *local Fano* case satisfying $c_\beta > 0$, insertions in the theory are necessary. We explain here the conjectural structure of the local Fano case with primary field insertions.[16]

Let $T_1, \ldots, T_m \in H^*(X,\mathbb{Z})$ be a basis of the cohomology[17] of X mod torsion. Let

$$N^\bullet_{g,\beta}(T_1^{e_1}\ldots T_m^{e_m}) = \int_{[\overline{M}^\bullet_{g,\sum_i e_i}(X,\beta)]^{vir}} \prod_{i=1}^m \prod_{j=1}^{e_i} \mathrm{ev}^*_{i,j}(T_i)$$

denote the disconnected Gromov-Witten invariant with primary field insertions and no degree 0 connected components. The primary fields should be viewed as simple incidence conditions. The curves are required to intersect the Poincaré duals of the insertions T_i.

For nonzero $\beta \in H_2(X,\mathbb{Z})$, let

$$Z_{GW,\beta}(u,t_i) = \sum_g \sum_{e_\bullet} N^\bullet_{g,\beta}(T_1^{e_1}\ldots T_m^{e_m}) \frac{t_1^{e_1}\ldots t_m^{e_m}}{e_1!\ldots e_m!} u^{2g-2}.$$

The sum is over finitely many negative and infinitely many positive g. The second sum is over all vectors $(e_1,\ldots,e_m)$ of non-negative integers. Let

$$Z_{GW}(u,v,t_i) = 1 + \sum_{\beta\neq 0} Z_{GW,\beta}(u,t_i)v^\beta$$

be the full partition function.

Alternatively, $Z_{GW} = \exp F_{GW}$ is the exponential of the connected Gromov-Witten series,

$$F_{GW}(u,v,t_i) = \sum_{\beta\neq 0} F_{GW,\beta}(u,t_i)v^\beta,$$

[16]There exists a full descendent theory which we will treat elsewhere.

[17]For simplicity, we assume there is no odd cohomology. There is no difficulty including the odd cohomology with supercommuting variables.

where

$$F_{GW,\beta}(u,t_i)=\sum_{g\geq 0}\sum_{e_\bullet} N_{g,\beta}(T_1^{e_1}\dots T_m^{e_m})\frac{t_1^{e_1}\dots t_m^{e_m}}{e_1!\dots e_m!}u^{2g-2}.$$

To define the corresponding series via stable pairs, let

$$\mathbb{F}\to X\times P_n(X,\beta)$$

denote the universal sheaf (2.9). For a pair

$$[\mathcal{O}_X\to F]\in P_n(X,\beta),$$

the restriction of $\mathbb{F}$ to the fiber

$$X\times[\mathcal{O}_X\to F]\subset X\times P_n(X,\beta)$$

is canonically isomorphic to F. Let

$$\pi_X\colon X\times P_n(X,\beta)\to X,$$
$$\pi_P\colon X\times P_n(X,\beta)\to P_n(X,\beta)$$

be the projections on the first and second factors. By definition, the operation

$$\pi_{P*}\big(\pi_X^*(T_i)\cdot \mathrm{ch}_2(\mathbb{F})\cap(\pi_P^*(\ \cdot\ ))\big)\colon H_*(P_n(X,\beta))\to H_*(P_n(X,\beta))$$

is the action of the primary field $\tau_0(T_i)$.

For nonzero $\beta\in H_2(X,\mathbb{Z})$, define the stable pairs invariant with primary field insertions by

$$\begin{aligned}P_{n,\beta}(T_1^{e_1}\dots T_m^{e_m}) &= \int_{[P_n(X,\beta)]^{vir}}\prod_{i=1}^m \tau_0(T_i)^{e_i}\\ &= \int_{P_n(X,\beta)}\prod_{i=1}^m \tau_0(T_i)^{e_i}\big([P_n(X,\beta)]^{vir}\big).\end{aligned}$$

The partition function is

$$Z_P(q,v,t_i)=1+\sum_{\beta\neq 0} Z_{P,\beta}(q,t_i)v^\beta,$$

where

$$Z_{P,\beta}(q,t_i)=\sum_{n}\sum_{e_\bullet} P_{n,\beta}(T_1^{e_1}\dots T_m^{e_m})\frac{t_1^{e_1}\dots t_m^{e_m}}{e_1!\dots e_m!}q^n.$$

Since $P_n(X,\beta)$ is empty for sufficiently negative n, $Z_{P,\beta}(q,t_i)$ is a Laurent series in q. The connected stable pair invariants are obtained formally via the logarithm,

$$F_P(q,v,t_i)=\sum_{\beta\neq 0}F_{P,\beta}(q,t_i)v^\beta=\log Z_P(q,v,t_i),$$

as in the Calabi-Yau case.

We state the rationality conjecture for $Z_{P,\beta}$ in BPS form following the Gromov-Witten structure explained in [45, 46]. The equation

$$F_{P,\beta}(q,t_i) = \sum_{g>-\infty} n_{g,\beta}(T_1^{e_1}\dots T_m^{e_m})\frac{t_1^{e_1}\dots t_m^{e_m}}{e_1!\dots e_m!} \cdot (-1)^{g-1}\left((-q)-2+(-q)^{-1}\right)^{g-1}\left(1+q\right)^{c_\beta}. \tag{3.25}$$

uniquely determines the invariants $n_{g,\beta}(T_1^{e_1}\dots T_m^{e_m}) \in \mathbb{Z}$. The latter are *defined* to be the BPS state counts. Moreover, for fixed β,

$$n_{g,\beta}(T_1^{e_1}\dots T_m^{e_m}) = 0$$

for all sufficiently large g. The proofs are identical to those in the Calabi-Yau case.

Conjecture 3.26. *The invariants $n_{g,\beta}(T_1^{e_1}\dots T_m^{e_m})$ vanish for $g<0$.*

Conjecture 3.26 is the stronger form of rationality obtained from the BPS perspective. As before, we obtain an effectivity statement as a consequence.

Lemma* 3.27. *$Z_P(q,v,t_i)$ is uniquely and effectively determined by the invariants $\left\{P_{n,\beta}(T_1^{e_1}\dots T_m^{e_m})\right\}_{n\leq 1}$.*

Conjecture 3.28. *All the partition functions coincide*

$$\begin{aligned}(-q)^{-\frac{c_\beta}{2}}Z_{P,\beta}(q,v,t_i) &= (-q)^{-\frac{c_\beta}{2}}Z'_{DT,\beta}(q,v,t_i)\\ &= (-iu)^{c_\beta}Z_{GW,\beta}(u,v,t_i)\end{aligned}$$

after the change of variables $-q=e^{iu}$.

The second equality in Conjecture 3.28 is the GW/DT correspondence for primary fields in the local Fano case [44]. We refer the reader to [44] for the definitions of the reduced partition function $Z'_{DT,\beta}(q,v,t_i)$.

3.7. **Variants.** Let G be a linearized algebraic group action on X. The construction of the virtual class for the moduli space of stable pairs in Section 2 is valid in the equivariant setting,

$$[P_n(X,\beta)]^{vir} \in A^G_{c_\beta}(P_n(X,\beta),\mathbb{Z}).$$

We may then define an equivariant theory of stable pairs and an equivariant correspondence following [9]. We leave the details to the reader.

Another standard direction is the relative theory. While relative Gromov-Witten theory has well-developed foundations [26, 35, 36, 37], relative DT theory awaits a definitive treatment. A sketch of the relative theory for ideal sheaves (following suggestions of J. Li) is given in [44]. A very similar discussion holds for the theory of stable pairs.

Let X be a nonsingular projective 3-fold, and let $S \subset X$ be a nonsingular divisor. The moduli space $P_n(X/S, \beta)$ parameterizes stable relative pairs

$$\mathcal{O}_{X[k]} \xrightarrow{s} F \tag{3.29}$$

on $X[k]$, the k-step degeneration [37] along S. F is a sheaf on $X[k]$ with

$$\chi(F) = n$$

and whose support pushes down to the class

$$\beta \in H_2(X, \mathbb{Z}).$$

The stability conditions for the data are more complicated in the relative geometry:

(i) F is pure with finite locally free resolution,
(ii) the higher derived functors of the restriction of F – to the singular loci of $X[k]$ and to the relative divisor $S_\infty \subset X[k]$ – vanish,
(iii) the section s has 0-dimensional cokernel supported away from the singular loci of $X[k]$.
(iv) the pair (3.29) has only finitely many automorphisms covering the automorphisms of $X[k]/X$.

The moduli space $P_n(X/S, \beta)$ is a complete Deligne-Mumford stack equipped with a map to the Hilbert scheme of points of S via the relative geometry. We expect a perfect obstruction theory of virtual dimension $\int_\beta c_1(X)$ to be induced from the deformation theory of complexes. All of these topics would benefit from foundational work.

The relative theory of stable pairs should admit a degeneration formula [40] and a correspondence to relative Gromov-Witten and relative DT theory following [44].

Several other variants can be considered: families invariants for 3-folds, equivariant relative invariants, residue invariants in the presence of a torus action, and so on. We expect the stable pairs theory to be equivalent in all reasonable cases to the corresponding Gromov-Witten and DT theories.

4. First examples

4.1. **Local $\mathbb{P}^1$.** The simplest possible example is to consider the local Calabi-Yau 3-fold X given by the total space of the bundle

$$\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2} \to \mathbb{P}^1$$

in the class $[\mathbb{P}^1]$ of the zero section. Then, $P_n(X, [\mathbb{P}^1])$ parameterizes nonzero sections, up to scale, of $\mathcal{O}_{\mathbb{P}^1}(n-1)$ supported on $\mathbb{P}^1$. Thus,

$$P_n(X, [\mathbb{P}^1]) \cong \mathrm{Sym}^{n-1}(\mathbb{P}^1) \cong \mathbb{P}^{n-1}$$

is nonsingular. As noted earlier, in the Calabi-Yau case, the obstruction theory of the moduli space of stable pairs is self-dual. Hence, if the moduli space is nonsingular, the obstruction bundle is the cotangent bundle. So

$$P_{n,[\mathbb{P}^1]} = (-1)^{n-1}\chi_{top}(\mathbb{P}^{n-1}) = (-1)^{n-1}n,$$

for $n \geq 1$, and is 0 otherwise. Therefore

$$Z_{P,[\mathbb{P}^1]}(q) = q - 2q^2 + 3q^3 - \ldots = \frac{q}{(1+q)^2}\,,$$

in complete agreement with $Z_{GW,[\mathbb{P}^1]}$ and $Z'_{DT,[\mathbb{P}^1]}$ [17, 43].

For curve class $\beta = 2[\mathbb{P}^1]$, the lowest possible value of the holomorphic Euler characteristic is 3. The moduli space $P_3(X, 2[\mathbb{P}^1])$ is a copy of $\mathbb{P}^1$ corresponding to the choice of a sub-bundle

$$\mathcal{O}_{\mathbb{P}^1}(-1) \subset \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}.$$

The associated stable pair is the structure sheaf of the curve obtained by doubling along the sub-bundle with the canonical section. So the potential starts as

$$Z_{P,2[\mathbb{P}^1]}(q) = -2q^3 + \ldots\,.$$

The next moduli space $P_4(X, 2[\mathbb{P}^1])$ is more interesting. Most stable pairs correspond to a choice of sub-bundle

$$\mathcal{O}_{\mathbb{P}^1}(-2) \subset \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}.$$

We then take the structure sheaf of the doubling of the zero section along the sub-bundle with the canonical section. We obtain an open set in $\mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(1))^{\oplus 2}) \cong \mathbb{P}^3$ consisting of pairs of sections of $\mathcal{O}_{\mathbb{P}^1}(1)$ which are not proportional to each other. Along the quadric where the sections become proportional, we find instead an $\mathcal{O}_{\mathbb{P}^1}(-1)$ sub-bundle along which we again double the zero section. We then take the ideal sheaf of a reduced point $p \in \mathbb{P}^1$ and twist by $\mathcal{O}_{\mathbb{P}^1}(1)$. The resulting sheaf F satisfies $\chi(F) = 4$ and has a unique section (up to automorphisms of the sheaf). We find a $\mathbb{P}^1 \times \mathbb{P}^1$ of pairs which glues into $\mathbb{P}^3$ as the quadric.

However, if the moduli space were really $\mathbb{P}^3$, then the invariant would be -4, and we would have

$$Z_{P,2[\mathbb{P}^1]}(q) = -2q^3 - 4q^4 + \ldots$$

in disagreement with the predictions

$$Z_{GW,2[\mathbb{P}^1]}(q) = Z'_{DT,2[\mathbb{P}^1]}(q) = -2q^3 + 4q^4 + \dots$$

of [17, 43]. In fact the moduli space has a thickening along the quadric

$$\mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3$$

corresponding to the first order movement of the point p in the direction of the $\mathcal{O}_{\mathbb{P}^1}(-1)$ sub-bundle. A computation shows there are no more deformations: the moduli space $P_4(X, 2[\mathbb{P}^1])$ has Zariski tangent spaces of dimension 4 along the quadric (and 3 over the rest of $\mathbb{P}^3$).

The natural $(\mathbb{C}^*)^3$ acting on $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$ fixing the Calabi-Yau form acts on $P_4(X, 2[\mathbb{P}^1])$ with 4 isolated fixed points, all lying on the quadric. In the virtual localisation formula of [18], the fixed points each count as $(-1)^4$, where 4 is the dimension of the corresponding Zariski tangent space. Thus, the extra thickened dimension along the quadric indeed makes the invariant 4 instead of -4.

4.2. **Contribution of an isolated curve.** Let X be a 3-fold, and let

$$C \subset X$$

be a nonsingular embedded curve of genus g which represents an infinitesimally isolated solution of the incidence[18] conditions $\prod_i T_i^{e_i}$. The curve C has a well-defined contribution to the Gromov-Witten potential $Z_{GW,[C]}$ by [45],

$$Z^C_{GW,[C]}(u) = \left(\frac{\sin(u/2)}{u/2}\right)^{2g-2+\int_C c_1(X)} u^{2g-2}. \tag{4.1}$$

We will calculate the contribution of C to the stable pairs theory to be

$$Z^C_{P,[C]}(q) = q^{1-g}(1+q)^{2g-2+\int_C c_1(X)}, \tag{4.2}$$

in agreement with our conjectures in the Calabi-Yau and Fano cases.

Since all pure rank 1 sheaves on C are locally free, the stable pairs with support on C are of the form

$$(F, s) = (\mathcal{O}_C(D), s_D),$$

where $D \subset C$ is a divisor and s_D is the canonical section of $\mathcal{O}_C(D)$. The cokernel of s_D is $\mathcal{O}_D(D)$. Thus, the moduli space

$$P^C_{1-g+d}(X, [C]) \subset P_{1-g+d}(X, [C]) \tag{4.3}$$

of stable pairs cut by the incidence conditions is simply the space $\mathrm{Sym}^d(C)$ of degree d divisors on C. The infinitesimal isolation of C implies the inclusion (4.3) is both open and closed.

[18] In the Calabi-Yau case, no incidence conditions are required.

To calculate the contribution of C, we must identify the obstruction bundle

$$\text{Obs} \to \text{Sym}^d(C)$$

determined by the deformation theory of the complex

$$I^\bullet = \{\mathcal{O}_X \to \mathcal{O}_C(D)\}.$$

In the Calabi-Yau case,

$$\text{Obs} = T^\vee_{\text{Sym}^d(C)}$$

by the self-duality of the obstruction theory. Then,

$$Z^C_{P,[C]}(q) \ = \ \sum_{d\geq 0} q^{1-g+d}(-1)^d \chi_{top}(\text{Sym}^d(C)). \tag{4.4}$$

Lemma 4.5. $\sum_{d\geq 0} \chi_{top}(\text{Sym}^d(C))q^d \ = \ (1-q)^{-\chi_{top}(C)}$.

Proof. There are many elementary derivations of the result. By expressing $\text{Sym}^d(X)$ as the quotient of X^d by the symmetric group, we see $\chi_{top}(\text{Sym}^d(X))$ depends only on $\chi_{top}(X)$ for any topological space. From the identity

$$\text{Sym}^d(X \sqcup \{p\}) \ \cong \ \text{Sym}^d(X) \ \sqcup \ \text{Sym}^{d-1}(X \sqcup \{p\}) \ ,$$

we deduce

$$\sum_{d\geq 0} \chi_{top}(\text{Sym}^d(X))q^d \ = (1-q) \sum_{d\geq 0} \chi_{top}(\text{Sym}^d(X \sqcup \{p\}))q^d \ .$$

Hence, we need only prove the identity for X with positive Euler characteristic, for which we may replace X by a finite number of points. □

Therefore (4.4) becomes $Z^C_{P,[C]}(q) = q^{1-g}(1+q)^{2g-2}$ and the verification of (4.2) in the Calabi-Yau case is complete. The stable pairs calculation provides a simple geometric interpretation of the Hodge integrals entering in (4.1).

The obstruction bundle in the Fano case is determined in the next result.

Proposition 4.6. *The obstruction space* $\text{Ext}^2(I^\bullet, I^\bullet)_0$ *sits inside a canonical exact sequence*

$$0 \to H^1(\nu_C) \to \text{Ext}^2(I^\bullet, I^\bullet)_0 \to H^0(\mathcal{O}_D(D) \otimes K_X)^\vee \to 0$$

where ν_C *is the normal bundle to* $C \subset X$.

Proof. Applying $\mathrm{Hom}(\,\cdot\,, I^\bullet)$ to the exact triangle $I^\bullet \to \mathcal{O}_X \to F$ yields the exact sequence

$$
\begin{aligned}
(4.7)\quad \mathrm{Ext}^2(F, I^\bullet) \to \mathrm{Ext}^2(\mathcal{O}_X, I^\bullet) \to \mathrm{Ext}^2(I^\bullet, I^\bullet) \to& \\
\mathrm{Ext}^3(F, I^\bullet) \to \mathrm{Ext}^3(\mathcal{O}_X, I^\bullet) \to \mathrm{Ext}^3(I^\bullet, I^\bullet).&
\end{aligned}
$$

The exact sequence obtained by applying $\mathrm{Hom}(\mathcal{O}_X, \,\cdot\,)$ to the same triangle and the vanishing $H^2(F) = H^3(F) = 0$ together yield

$$\mathrm{Ext}^3(\mathcal{O}_X, I^\bullet) = \mathrm{Ext}^3(\mathcal{O}_X, \mathcal{O}_X) \cong H^3(\mathcal{O}_X).$$

The last map in (4.7) is the identity $H^3(\mathcal{O}_X) \to \mathrm{Ext}^3(I^\bullet, I^\bullet)$ and an injection since the composition

$$H^3(\mathcal{O}_X) \xrightarrow{\mathrm{id}} \mathrm{Ext}^3(I^\bullet, I^\bullet) \xrightarrow{\mathrm{tr}} H^3(\mathcal{O}_X)$$

is multiplication by $\mathrm{rank}\,(I^\bullet) = 1$. Thus, we may replace (4.7) by the exact sequence

$$(4.8)\quad \mathrm{Ext}^2(F, I^\bullet) \to \mathrm{Ext}^2(\mathcal{O}_X, I^\bullet) \to \mathrm{Ext}^2(I^\bullet, I^\bullet) \to \mathrm{Ext}^3(F, I^\bullet) \to 0,$$

fitting into the diagram

$$
\begin{array}{cccccccc}
 & & & & 0 & & & \\
 & & & & \downarrow & & & \\
\mathrm{Ext}^1(F,F) & \longrightarrow & H^1(F) & \longrightarrow & \mathrm{Ext}^2(I^\bullet, I^\bullet)_0 & & & \\
\downarrow & & \downarrow & & \downarrow & & & \\
\mathrm{Ext}^2(F, I^\bullet) & \longrightarrow & \mathrm{Ext}^2(\mathcal{O}_X, I^\bullet) & \longrightarrow & \mathrm{Ext}^2(I^\bullet, I^\bullet) & \longrightarrow & \mathrm{Ext}^3(F, I^\bullet) & \longrightarrow 0 \\
 & & \downarrow & & \downarrow{\scriptstyle \mathrm{tr}} & & & \\
 & & H^2(\mathcal{O}_X) & = & H^2(\mathcal{O}_X) & & & \\
 & & \downarrow & & \downarrow & & & \\
 & & 0 & & 0 & & &
\end{array}
$$

The first two vertical sequences are obtained from the exact triangle $I^\bullet \to \mathcal{O}_X \to F$ and the third is obtained from the trace map. The diagram commutes because the lower square does. The commutation for the lower square holds since $\mathrm{Ext}^2(\mathcal{O}_X, I^\bullet) = H^2(\mathscr{I}_C)$ and the map to $\mathrm{Hom}(I^\bullet, I^\bullet)$ induced by the canonical maps $\mathscr{I}_C \to I^\bullet \to \mathcal{O}_X$ takes $f \in \mathscr{I}_C$ to $f \cdot \mathrm{id}$.

The first horizontal map in the above diagram, restricted to

$$H^1(\mathcal{O}_C) = H^1(\mathscr{H}om(F, F)) \subset \mathrm{Ext}^1(F, F),$$

is multiplication by the section $s\colon H^1(\mathcal{O}_C) \to H^1(F)$ which is onto because $H^1(Q) = 0$ since Q is supported in dimension 0. Therefore the diagram gives

$$\mathrm{Ext}^2(I^\bullet, I^\bullet)_0 \;\cong\; \mathrm{Ext}^3(F, I^\bullet).$$

The latter is Serre dual to $\mathrm{Hom}(I^\bullet, F \otimes K_X)$. The exact sequence

$$(4.9) \quad 0 \to \mathscr{E}xt^{-1}(I^\bullet, F \otimes K_X) \to K_X|_C \xrightarrow{s} F \otimes K_X \\ \to \mathscr{H}om(I^\bullet, F \otimes K_X) \to \mathscr{E}xt^1(F, F \otimes K_X) \to 0.$$

is obtained by applying $\mathscr{H}om(\,\cdot\,, F \otimes K_X)$ to $I^\bullet \to \mathcal{O}_X \to F$. Since s is injective, $\mathscr{E}xt^i(I^\bullet, F \otimes K_X) = 0$ for $i < 0$ and

$$(4.10) \qquad \mathrm{Hom}(I^\bullet, F \otimes K_X) = H^0(\mathscr{H}om(I^\bullet, F \otimes K_X)).$$

Then, by (4.9),

$$0 \to \mathcal{O}_D(D) \otimes K_X \to \mathscr{H}om(I^\bullet, F \otimes K_X) \to \nu_C \otimes K_X \to 0.$$

Taking H^0 we have, by (4.10),

$$0 \to H^0(\mathcal{O}_D(D) \otimes K_X) \to \mathrm{Hom}(I^\bullet, F \otimes K_X) \to H^0(\nu_C \otimes K_X) \to 0.$$

Dualizing gives the required result. □

Since C is assumed to be infinitesimally isolated, the obstruction space $H^1(\nu_C)$ vanishes. Therefore,

$$\mathrm{Obs}_{[D]} = H^0(\mathcal{O}_D(D) \otimes K_X)^\vee$$

for $[D] \in \mathrm{Sym}^d(C)$.

Let $Z \subset \mathrm{Sym}^d(C) \times C$ be the universal divisor, and let $\mathcal{O}_Z(Z)$ be its normal bundle. Let

$$\pi_d \colon Z \to \mathrm{Sym}^d(C), \quad \pi_C \colon Z \to C$$

be the projections. The first projection π_d is a d-fold cover.

Let L be a line bundle on C of degree l. A rank d vector bundle L_d on $\mathrm{Sym}^d(C)$ is obtained tautologically by

$$L_d = \pi_{d*}(\mathcal{O}_Z(Z) \otimes \pi_C^* L^\vee)^\vee,$$

The identity

$$\int_{\mathrm{Sym}^d(C)} c_d(L_d) = \binom{2g-2+l}{d} = \frac{(2g-2+l)\ldots(2g-2+l-d+1)}{d!},$$

where we use the formula on the right if $2g-2+l < 0$, is easily obtained via intersection theory on the $d!$-fold cover

$$\epsilon : C^d \to \mathrm{Sym}^d(C).$$

The product of the d Chern roots of $\epsilon^*(L_d)$ is

$$\prod_{i=1}^d \left(\pi_i^*(c_1(K_C) + c_1(L)) - \sum_{j=i+1}^d \triangle_{ij} \right),$$

where

$$\pi_i : C^d \to C$$

is the i^{th} projection and $\triangle_{ij}$ is the codimension 1 diagonal. The leading term

$$\frac{(2g-2+l)^d}{d!} = \frac{1}{d!}\int_{C^d}\prod_{i=1}^{d}\pi_i^*\big(c_1(K_C)+c_1(L)\big)$$

is obtained from the leading terms of the Chern roots, and the lower order terms are obtained from the diagonals.

In our case, Obs $= L_d$ where $L = K_X^\vee|_C$ has degree $l = \int_C c_1(X)$. The stable pairs invariant is

$$P^C_{1-g+d}(X,[C]) = \frac{(2g-2+l)\dots(2g-2+l-d+1)}{d!}\,.$$

Therefore

$$\begin{aligned} Z_{P,C}(q) &= \sum_{d\geq 0}\frac{(2g-2+l)\dots(2g-2+l-d+1)}{d!}q^{1-g+d} \\ &= q^{1-g}(1+q)^{2g-2+l} \\ &= q^{1-g}(1+q)^{2g-2+\int_C c_1(X)}, \end{aligned}$$

in perfect agreement with (4.2).

Finally, we see here that the stable pairs invariant is truly local with respect to C. As in the Gromov-Witten case, the local calculation (4.2) is valid in any ambient geometry in which C is infinitesimally isolated. The DT invariant always probes the whole manifold because of the wandering points. To extract simple local curve results valid in ambient geometries for DT theory is technically very difficult [6].

5. The vertex

5.1. **Overview.** The study of 3-fold theories of nonsingular toric varieties naturally leads to the notion of a vertex. The vertex takes its simplest form when restricted to the toric Calabi-Yau case.[19] In Gromov-Witten theory, the Calabi-Yau vertex is an evaluation of special Hodge integrals [2]. The DT vertex, calculated in [43, 44], is related to normalized box counting. A full development of the stable pairs vertex will be presented in [47]. We give a short summary of the results and conjectures in the Calabi-Yau case here.

5.2. **DT vertex.** Let $\mathbb{C}^3$ have coordinates x_1, x_2, x_3. Let the torus

$$\mathbf{T} = (\mathbb{C}^*)^3$$

[19] Only non-compact geometries can be both toric and Calabi-Yau.

acting diagonally on $\mathbb{C}^3$. Let $C \subset \mathbb{C}^3$ be a $\mathbf{T}$-fixed subscheme of dimension at most 1. The subscheme C is defined by a monomial ideal

$$\mathscr{I}_C \subset \mathbb{C}[x_1, x_2, x_3].$$

The latter may be visualized as a 3-dimensional partition π. The localisations

$$(\mathscr{I}_C)_{x_1} \subset \mathbb{C}[x_1, x_2, x_3]_{x_1},$$
$$(\mathscr{I}_C)_{x_2} \subset \mathbb{C}[x_1, x_2, x_3]_{x_2},$$
$$(\mathscr{I}_C)_{x_3} \subset \mathbb{C}[x_1, x_2, x_3]_{x_3},$$

are all $\mathbf{T}$-fixed, and each corresponds to a 2-dimensional partition μ^i. Alternatively, the 2-dimensional partitions μ^i can be defined as the infinite limits of the x_i-constant cross-sections of π. If all the μ^i are empty, then C must be 0-dimensional.

Given a triple $\vec{\mu} = (\mu^1, \mu^2, \mu^3)$ of outgoing partitions, there exists a unique minimal $\mathbf{T}$-fixed subscheme

$$C_{\vec{\mu}} \subset \mathbb{C}^3$$

with outgoing partitions μ^i. If the μ^i are not all empty, then $C_{\vec{\mu}}$ is easily seen to be a Cohen-Macaulay curve.

Let $S_{\vec{\mu}}$ be the set of $\mathbf{T}$-fixed subschemes $C \subset \mathbb{C}^3$ with outgoing partitions μ^i. Since the $\mathbf{T}$-fixed subschemes are isolated, $S_{\vec{\mu}}$ is a discrete set. Each $[C] \in S_{\vec{\mu}}$ contains the minimal subscheme as a quotient

$$\mathcal{O}_C \to \mathcal{O}_{C_{\vec{\mu}}} \to 0.$$

The minimal subscheme is the unique Cohen-Macaulay element of $S_{\vec{\mu}}$. Let $|\vec{\mu}|$ denote the renormalized volume[20] of the partition π corresponding to $\mathscr{I}_{C_{\vec{\mu}}}$. For $[C] \in S_{\vec{\mu}}$, define the length by

$$\ell(C) = \dim_{\mathbb{C}}(\mathscr{I}_{C_{\vec{\mu}}}/\mathscr{I}_C) < \infty.$$

Consider the series

$$Z^{DT}_{\vec{\mu}}(q) = (-q)^{|\vec{\mu}|} \sum_{[C] \in S_{\vec{\mu}}} (-q)^{\ell(C)}.$$

[20]The renormalized volume $|\pi|$ is defined by

$$|\pi| = \#\left\{\pi \cap [0, \ldots, N]^3\right\} - (N+1) \sum_1^3 |\mu^i|, \quad N \gg 0.$$

The renormalized volume is independent of the cut-off N as long as N is sufficiently large. The number $|\pi|$ so defined may be negative.

Since there are only finitely many elements of given length, $Z^{DT}_{\vec{\mu}}(q)$ is well-defined.

The series $Z^{DT}_{\emptyset,\emptyset,\emptyset}(q)$ is the well-known MacMahon function enumerating finite 3-dimensional partitions,

$$Z^{DT}_{\emptyset,\emptyset,\emptyset}(q) = \prod_{n\geq 1} \frac{1}{(1-(-q)^n)^n}.$$

The normalized vertex [43, 44] governing the DT theory in the toric Calabi-Yau case is

$$\mathsf{W}^{DT}_{\vec{\mu}}(q) = \frac{Z^{DT}_{\vec{\mu}}}{Z^{DT}_{\emptyset,\emptyset,\emptyset}}(q).$$

The DT vertex can be viewed a normalized count of 3-dimensional partitions.

5.3. **Stable pairs vertex.** Given a $\mathbf{T}$-fixed stable pair on $\mathbb{C}^3$,

$$\mathcal{O}_{\mathbb{C}^3} \stackrel{s}{\to} F,$$

the scheme theoretic support must be of the form

$$C_{\vec{\mu}} \subset \mathbb{C}^3.$$

Moreover, the quotient Q must be supported at the origin.

We will use the characterization of Proposition 1.8 to study $\mathbf{T}$-fixed stable pairs with support $C_{\vec{\mu}}$. Let

$$\mathfrak{m} \subset \mathcal{O}_{C_{\vec{\mu}}}$$

be the ideal sheaf of the origin in $C_{\vec{\mu}}$. Let

$$M_1 = \mathbb{C}[x_1,x_2,x_3]_{x_1}/\mathscr{I}_{\mu^1},$$
$$M_2 = \mathbb{C}[x_1,x_2,x_3]_{x_2}/\mathscr{I}_{\mu^2},$$
$$M_3 = \mathbb{C}[x_1,x_2,x_3]_{x_3}/\mathscr{I}_{\mu^3},$$

be the quotients by the $\mathbf{T}$-fixed ideals $\mathscr{I}_{\mu^i}$ determined by the respective outgoing partitions, and let

$$M_{\vec{\mu}} = \oplus_{i=1}^3 M_i$$

be viewed as a $\mathbb{C}[x_1,x_2,x_3]$-module. There is a canonical homomorphism

$$\mathcal{O}_{C_{\vec{\mu}}} \to M_{\vec{\mu}}$$

given by

$$1 \mapsto (1,1,1).$$

The limit in Proposition 1.8 has a simple description,

$$\varinjlim \mathscr{H}om(\mathfrak{m}^r, \mathcal{O}_{C_{\vec{\mu}}})/\mathcal{O}_{C_{\vec{\mu}}} = M_{\vec{\mu}}/\mathcal{O}_{C_{\vec{\mu}}}$$

as a $\mathbb{C}[x_1, x_2, x_3]$-module.

By Proposition 1.8, **T**-fixed stable pairs on $\mathbb{C}^3$ correspond to **T**-fixed $\mathbb{C}[x_1, x_2, x_3]$-submodules $Q \subset M_{\vec{\mu}}/\mathcal{O}_{C_{\vec{\mu}}}$ of finite length

$$\ell(Q) = \dim_{\mathbb{C}}(Q).$$

Unlike the DT case, such **T**-fixed submodules *need not be isolated.* However, the geometry of the moduli of **T**-fixed submodules modules of $M_{\vec{\mu}}/\mathcal{O}_{C_{\vec{\mu}}}$ is elementary: the components are simply products[21] of $\mathbb{P}^1$.

Let the set $S^M_{\vec{\mu}}$ index the components $\mathcal{Q}$ of the moduli space of **T**-fixed submodules of $M_{\vec{\mu}}/\mathcal{O}_{C_{\vec{\mu}}}$. The components are easily indexed by a box counting strategy explained in [47]. The stable pairs vertex is conjectured[22] in [47] to be the series

$$\mathsf{W}^P_{\vec{\mu}}(q) = (-q)^{|\vec{\mu}|} \sum_{[\mathcal{Q}] \in S^M_{\vec{\mu}}} \chi_{top}(\mathcal{Q})(-q)^{\ell(\mathcal{Q})}.$$

The topological Euler characteristic above is always a power of 2. Since the length is constant in components, $\ell(\mathcal{Q})$ is well-defined. Again, there are only finitely many components of given length.

We conjecture a correspondence between the DT and stable pairs vertices.

Conjecture 5.1. $\mathsf{W}^{DT}_{\mu^1,\mu^2,\mu^3}(q) = \mathsf{W}^P_{\mu^1,\mu^2,\mu^3}(q)$.

Independent of the geometric and string theoretic framework, Conjecture 5.1 is a striking statement about the algebro-combinatorics of $\mathbb{C}[x_1, x_2, x_3]$.

5.4. **Example.** The simplest 3-leg example occurs when

$$\mu^1 = \mu^2 = \mu^3 = (1).$$

The Cohen-Macaulay curve $C_{(1),(1),(1)}$ is the union of the 3 coordinate axes. From the definitions, we find

$$\frac{M_{(1),(1),(1)}}{\mathcal{O}_{C_{(1),(1),(1)}}} = \frac{\mathbb{C}[x_1, x_1^{-1}] \oplus \mathbb{C}[x_2, x_2^{-1}] \oplus \mathbb{C}[x_3, x_3^{-1}]}{(1,1,1) \cdot \mathbb{C}[x_1, x_2, x_3]}. \tag{5.2}$$

[21] The 0^{th} product is a point. In fact, the **T**-fixed modules are isolated if at least one of the μ^i is empty. Indeed, the proof Conjecture 5.1 in the 1 and 2-leg case is not difficult [47]. Non-trivial moduli of **T**-fixed submodules appear only in the full 3-leg case.

[22] Proofs are given in the 1 and 2-leg cases.

To compute the stable pairs vertex, we must study the **T**-fixed submodules of (5.2). Of course, 0 is the unique submodule of length 0. The first interesting case is length 1. Then,

$$(\gamma_1, \gamma_2, \gamma_3) \in \mathbb{C}^3 \setminus \langle\ (1,1,1)\ \rangle$$

determines a **T**-fixed submodule of length 1 of (5.2). So the moduli space for length 1 is isomorphic to $\mathbb{P}^1$. For length 2, the **T**-fixed submodules are

$$\langle\ (x_1^{-1}, 0, 0)\ \rangle, \quad \langle\ (0, x_2^{-1}, 0)\ \rangle, \quad \langle\ (0, 0, x_3^{-1})\ \rangle,$$
$$\langle\ (1,0,0),\ (0,1,0),\ (0,0,1)\ \rangle$$

all of which are isolated points. In general, the **T**-fixed submodules are elementary to enumerate since the generators must be vectors of monomials. We find

$$\begin{aligned}(-q)^2 \cdot \mathsf{W}^P_{(1),(1),(1)}(q) &= 1 + 2(-q) + \sum_{n\geq 2} \left(\binom{n}{2} + 3 \right) (-q)^n \\ &= \frac{1-q^5}{(1-q)(1+q)^3} \\ &= 1 - 2q + 4q^4 - 6q^3 + \ldots \end{aligned}$$

Indeed, the vertex $\mathsf{W}^P_{\vec{\mu}}(q)$ is always a rational function.

The DT vertex $\mathsf{W}^{DT}_{(1),(1),(1)}$ counts boxes added to the Cohen-Macaulay curve $C_{(1),(1),(1)}$. We find

$$(-q)^2 \cdot \mathsf{W}^{DT}_{(1),(1),(1)}(q) = 1 - 3q + 9q^2 - 22q^3 + \ldots.$$

Since the MacMahon function is

$$\mathsf{W}^{DT}_{\emptyset,\emptyset,\emptyset}(q) = 1 - q + 3q^2 - 6q^3 + \ldots,$$

Conjecture 5.1 is verified to order 3 by

$$1 - 2q + 4q^4 - 6q^3 + \ldots = \frac{1 - 3q + 9q^2 - 22q^3 + \ldots}{1 - q + 3q^2 - 6q^3 + \ldots}.$$

In fact, the exact equality of Conjecture 5.1 is not hard to obtain by box counting for the example ((1), (1), (1)).

References

[1] V. Alexeev and A. Knutson. *Complete moduli spaces of branchvarieties*, math.AG/0602626.

[2] M. Aganagic, A. Klemm, M. Mariño, and C. Vafa. *The topological vertex*, Comm. Math. Phys., **254**, 425–478, 2005. hep-th/0305132.

[3] P. S. Aspinwall and D. R. Morrison. *Topological field theory and rational curves*, Comm. Math. Phys., **151**, 245–262, 1993.

[4] A. Bayer, *Polynomial Bridgeland stability conditions and the large volume limit*, arXiv:0712.1083.

[5] K. Behrend. *Donaldson-Thomas invariants via microlocal geometry*, to appear in Ann. of Math., 2009. math.AG/0507523.

[6] K. Behrend and J. Bryan. *Super-rigid Donaldson-Thomas invariants*, Math. Res. Lett. **14**, 559–571, 2007. math.AG/0601203.

[7] K. Behrend and B. Fantechi. *The intrinsic normal cone*, Invent. Math., **128**, 45–88, 1997. math.AG/9601010.

[8] K. Behrend and B. Fantechi. *Symmetric obstruction theories and Hilbert schemes of points on threefolds*, Algebra Number Theory **2**, 313–345, 2008. math.AG/0512556.

[9] J. Bryan and R. Pandharipande. *The local Gromov-Witten theory of curves*, J. Amer. Math. Soc. **21**, 101–136, 2008. math.AG/0411037.

[10] T. Bridgeland. *Flops and derived categories*, Invent. Math., **147**, 613–632, 2002. math.AG/0009053.

[11] T. Bridgeland. *Stability conditions on triangulated categories*, Ann. of Math. **166**, 317–345, 2007. math.AG/0212237.

[12] P. Candelas, X. C. de la Ossa, P. S. Green, and L. Parkes. *A pair of Calabi-Yau manifolds as an exactly soluble superconformal theory*, Nuclear Phys. B, **359**, 21–74, 1991.

[13] F. Denef and G. W. Moore. *Split states, entropy enigmas, holes and halos*, hep-th/0702146.

[14] D. E. Diaconescu. *Moduli of ADHM sheaves and local Donaldson-Thomas theory*, arXiv:0801.0820.

[15] D. E. Diaconescu and G. W. Moore. *Crossing the wall: Branes vs. bundles*, hep-th/07063193.

[16] S. K. Donaldson and R. P. Thomas. *Gauge theory in higher dimensions.* In *The geometric universe (Oxford, 1996)*, 31–47. Oxford Univ. Press, Oxford, 1998.

[17] C. Faber and R. Pandharipande. *Hodge integrals and Gromov-Witten theory*, Invent. Math., **139**, 173–199, 2000. math.AG/9810173.

[18] T. Graber and R. Pandharipande. *Localization of virtual classes*, Invent. Math., **135**, 487–518, 1999. math.AG/9708001.

[19] R. Gopakumar and C. Vafa. *M-theory and topological strings–I*, hep-th/9809187.

[20] R. Gopakumar and C. Vafa. *M-theory and topological strings–II*, hep-th/9812127.

[21] M. Honsen. *A compact moduli space parameterizing Cohen-Macaulay curves in projective space*. PhD thesis, MIT, 2004.

[22] S. Hosono, M.-H. Saito, and A. Takahashi. *Relative Lefschetz actions and BPS state counting*, IMRN, **15**, 783–816, 2001.

[23] D. Huybrechts and M. Lehn. *The geometry of moduli spaces of shaves.* Aspects of Mathematics, E31. Friedr. Vieweg & Sohn, Braunschweig, 1997.

[24] D. Huybrechts and R. P. Thomas. *Deformation-obstruction theory for complexes via Atiyah and Kodaira–Spencer classes*, arXiv:0805.3527.

[25] M. Inaba. *Moduli of stable objects in a triangulated category*, math.AG/0612078.

[26] E. Ionel and T. H. Parker. *Relative Gromov-Witten invariants*, Ann. of Math. (2), **157**, 45–96, 2003. math.SG/9907155.
[27] D. Joyce. *Configurations in abelian categories. IV. Invariants and changing stability conditions*, Adv. Math. **217**, 125–204, 2008. math.AG/0410268.
[28] S. Katz. *Genus zero Gopakumar-Vafa invariants of contractible curves*, J. Differential Geom. **79**, 185–195, 2008. math.AG/0601193.
[29] S. Katz, A. Klemm, and C. Vafa. *M-theory, topological strings and spinning black holes*, Adv. Theor. Math. Phys., **3**, 1445–1537, 1999. hep-th/9910181.
[30] A. Klemm and R. Pandharipande. *Enumerative geometry of Calabi-Yau 4-folds*, Comm. Math. Phys. **281**, 621–653, 2008. math.AG/0702189.
[31] J. Kollár. *Projectivity of complete moduli*, J. Differential Geom., **32**, 235–268, 1990.
[32] J. Le Potier. *Systèmes cohérents et structures de niveau*, Astérisque, **214**, 143, 1993.
[33] J. Le Potier. *Faisceaux semi-stables et systèmes cohérents.* In *Vector bundles in algebraic geometry (Durham, 1993)*, vol 208 of London Math. Soc. Lecture Note Ser., 179–239. Cambridge Univ. Press, Cambridge, 1995.
[34] M. Levine and R. Pandharipande. *Algebraic cobordism revisited*, math.AG/0605196.
[35] A.-M. Li and Y. Ruan. *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds I*, Invent. Math. **145**, 151–218, 2001.
[36] J. Li. *Stable morphisms to singular schemes and relative stable morphisms*, J. Differential Geom., **57**, 509–578, 2001. math.AG/0009097.
[37] J. Li. *A degeneration formula of GW-invariants*, J. Differential Geom., **60**, 199–293, 2002. math.AG/0110113.
[38] J. Li. *Zero dimensional Donaldson-Thomas invariants of threefolds*, Geom. Topol. **10**, 2117–2171, 2006. math.AG/0604490.
[39] J. Li and G. Tian. *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties*, J. Amer. Math. Soc., **11**, 119–174, 1998. math.AG/9602007.
[40] J. Li and B. Wu. *Degeneration of Donaldson-Thomas invariants*, in preparation.
[41] M. Lieblich. *Moduli of complexes on a proper morphism*, J. Algebraic Geom., **15**, 175–206, 2006. math.AG/0502198.
[42] W. Lowen. *Obstruction theory for objects in abelian and derived categories*, Comm. Algebra **33** (2005), 3195–3223. math.KT/0407019.
[43] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. *Gromov-Witten theory and Donaldson-Thomas theory. I*, Compos. Math., **142**, 1263–1285, 2006. math.AG/0312059.
[44] D. Maulik, N. Nekrasov, A. Okounkov, and R. Pandharipande. *Gromov-Witten theory and Donaldson-Thomas theory. II*, Compos. Math., **142**, 1286–1304, 2006. math.AG/0406092.
[45] R. Pandharipande. *Hodge integrals and degenerate contributions*, Comm. Math. Phys., **208**, 489–506, 1999. math.AG/9811140.
[46] R. Pandharipande. Three questions in Gromov-Witten theory. In *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)* 503–512, Beijing, 2002. Higher Ed. Press. math.AG/0302077.

[47] R. Pandharipande and R. P. Thomas. *The 3-fold vertex via stable pairs*, to appear in Geom. Topol. arXiv:0709.3823.
[48] R. Pandharipande and A. Zinger. *Enumerative geometry of Calabi-Yau 5-folds*, arXiv:0802.1640.
[49] B. Szendrői. *Non-commutative Donaldson-Thomas theory and the conifold*, Geom. Topol. **12**, 1171–1202, 2008. math.AG/07053419.
[50] R. P. Thomas. *A holomorphic Casson invariant for Calabi-Yau 3-folds, and bundles on $K3$ fibrations*, J. Differential Geom., **54**, 367–438, 2000. math.AG/9806111.
[51] Y. Toda. *Birational Calabi-Yau 3-folds and BPS state counting*, Commun. Number Theory Phys. **2**, 63–112, 2008. math.AG/07071643.
[52] Y. Toda, *Limit stable objects on Calabi-Yau 3-folds,* arXiv:0803.2356.

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