Curve/surface intersection problem by means of matrix representations

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Suppose given a parametrization

$$\begin{array}{ccc} \mathbb{P}^2_{\mathbb{K}} & \xrightarrow{\phi} & \mathbb{P}^3_{\mathbb{K}} \\ (s:t:u) & \mapsto & (f_1:f_2:f_3:f_4)(s,t,u) \end{array}$$

of a surface **S** such that

i) f_i are the homogeneous polynomial with the same degree d.

ii) $\gcd(f_1,\ldots,f_4) \in \mathbb{K} \setminus \{0\}$.

We have $\mathbf{S} := \overline{\operatorname{Im}\phi} := \{(x : y : z : w) \in \mathbb{P}^3_{\mathbb{K}} : S(x, y, z, w) = 0\}$ where $S(x, y, z, w) \in \mathbb{K}[x, y, z, w]$ is irreducible homogeneous polynomial.

The equationle $S(x, y, z, w) = 0$ is called the implicit equation of $\mathbf{S}$.

Definition

A matrix $\mathtt{M}(\mathbf{f})$ with entries in $\mathbb{K}[x, y, z, w]$ is said to be a representation of a given homogeneous polynomial $S \in \mathbb{K}[x, y, z, w]$ if

i) $\mathtt{M}(\mathbf{f})$ is generically full rank,

ii) the rank of $\mathtt{M}(\mathbf{f})$ drops exactly on the surface of equation $S = 0$,

iii) the GCD of the maximal minors of $\mathtt{M}(\mathbf{f})$ is equal to S, up to multiplication by a nonzero constant in $\mathbb{K}$.

'Moving Plane ' :

For all $\nu \in \mathbb{N}$, consider the set $\mathcal{L}_\nu$ of polynomials of the form

$$a_1(s,t,u)x + a_2(s,t,u)y + a_3(s,t,u)z + a_4(s,t,u)w$$

such that

- $a_i(s,t,u) \in \mathbb{K}[s,t,u]$ is homogeneous of degree ν for all $i = 1,\ldots,4$,
- $\sum_{i=1}^{4} a_i(s,t,u)f_i(s,t,u) \equiv 0$ in $\mathbb{K}[s,t,u]$.

Denote by $L^{(1)},\ldots,L^{(n_\nu)}$ a basis of $\mathbb{K}$-vector space $\mathcal{L}_\nu$. Then, define the matrix $\mathtt{M}(\mathbf{f})_\nu$ by the equality

$$\begin{bmatrix} s^\nu & s^{\nu-1}t & \cdots & u^\nu \end{bmatrix} \mathtt{M}(\mathbf{f})_\nu = \begin{bmatrix} L^{(1)} & L^{(2)} & \cdots & L^{(n_\nu)} \end{bmatrix}$$

'Aproximate Complexes' :

Denote $A := \mathbb{K}[s,t,u]$ with naturally graded by $\deg(s) = \deg(t) = \deg(u) = 1$.
We consider the Koszul complex
$(K_\bullet(f_1, f_2, f_3, f_4), d_\bullet))$:

$$0 \to A[-4d] \xrightarrow{d_4} A[-3d]^4 \xrightarrow{d_3} A[-2d]^6 \xrightarrow{d_2} A[-d]^4 \xrightarrow{d_1} A$$

$(K_\bullet(f_1, f_2, f_3, f_4), u_\bullet))$:

$$0 \to A[\underline{x}][-4d] \xrightarrow{u_4} A[\underline{x}][-3d]^4 \xrightarrow{u_3} A[\underline{x}][-2d]^6 \xrightarrow{u_2} A[\underline{x}][-d]^4 \xrightarrow{u_1} A[\underline{x}]$$

$(K_\bullet(x, y, z, w), v_\bullet))$:

$$0 \to A[\underline{x}][-4] \xrightarrow{v_4} A[\underline{x}][-3]^4 \xrightarrow{v_3} A[\underline{x}][-2]^6 \xrightarrow{v_2} A[\underline{x}][-1]^4 \xrightarrow{v_1} A[\underline{x}]$$

Define $Z_i := \ker(d_i)$ and $\mathcal{Z}_i := Z_i \bigotimes_A A[\underline{x}]$. We obtain the bi-graded complex : $(\mathcal{Z}_\bullet, v_\bullet)$:

$$0 \to \mathcal{Z}_4[-4] \xrightarrow{v_4} \mathcal{Z}_3[-3]^4 \xrightarrow{v_3} \mathcal{Z}_2[-2]^6 \xrightarrow{v_2} \mathcal{Z}_1[-1]^4 \xrightarrow{v_1} \mathcal{Z}_0 = A[\underline{x}]$$

Theorem

Suppose that $I = (f_1, f_2, f_3, f_4)A$ is of codimension at least 2 and $\mathbf{P} = Proj(A/I)$ is locally defined by 3 equations. Then for all $v \geqslant v_0 := 2(d-1) - indeg(I_{\mathbf{P}})$, the matrix of surjective map

$$\begin{array}{ccc} \mathcal{Z}_{1[v]}[-1]^4 & \xrightarrow{v_1} & \mathcal{Z}_{0[v]} = A[\underline{x}] \\ (g_1, g_2, g_3, g_4) & \longmapsto & xg_1 + yg_2 + zg_3 + wg_4. \end{array}$$

is matrix representation of $\mathcal{S}$.

Suppose given an algebraic surface **S** with represented by a parameterization and a rational space curve **C** represented by a parameterization

$$\Psi : \mathbb{P}^1_{\mathbb{K}} \to \mathbb{P}^3_{\mathbb{K}} : (s:t) \mapsto (x(s,t) : y(s,t) : z(s,t) : w(s,t))$$

where $x(s,t), y(s,t), z(s,t), w(s,t)$ are homogeneous polynomials of the same degree and without common factor in $\mathbb{K}[s,t]$.
Determine the set $\mathbf{C} \cap \mathbf{S} \subset \mathbb{P}^3_{\mathbb{K}}$

Assume that $M(x, y, z, w)$ is a matrix representation of the surface **S**, meaning a representation of implicit equation $S(x, y, z, w)$. By replacing the variables x, y, z, w by the homogeneous polynomials $x(s, t), y(s, t), z(s, t), w(s, t)$ respectively, we get the matrix

$$M(s, t) = M(x(s, t), y(s, t), z(s, t), w(s, t)).$$

Lemma

For all point $(s_0 : t_0) \in \mathbb{P}^1_{\mathbb{K}}$ the rank of the matrix $M(s_0, t_0)$ drops if and only if the point $(x(s_0, t_0) : y(s_0, t_0) : z(s_0, t_0) : w(s_0, t_0))$ belongs to the intersection locus $\mathbf{C} \cap \mathbf{S}$.

It follows that points in $\mathbf{C} \cap \mathbf{S}$ associated to points $(s : t)$ such that $s \neq 0$, are in correspondence with the set of values $t \in \mathbb{K}$ such that $M(1, t)$ drops of rank strictly less than its row and column dimensions.

Given an $m \times n$-matrix $M(t) = (a_{i,j}(t))$ with $a_{i,j}(t) \in \mathbb{K}[t]$.

$$M(t) = M_d t^d + M_{d-1} t^{d-1} + \ldots + M_0$$

where $M_i \in \mathbb{K}^{m \times n}$ and $d = \max_{i,j}\{\deg(a_{i,j}(t))\}$.

Definition

The generalized companion matrices A, B of the matrix $M(t)$ are the matrices with coefficients in $\mathbb{K}$ of size $((d-1)m+n) \times dm$ that are given by

$$A = \begin{pmatrix} 0 & I & \dots & \dots & 0 \\ 0 & 0 & I & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & \dots & I \\ M_0^t & M_1^t & \dots & \dots & M_{d-1}^t \end{pmatrix}$$

$$B = \begin{pmatrix} I & 0 & \dots & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \\ 0 & 0 & \dots & \dots & -M_d^t \end{pmatrix}$$

Theorem

$$\operatorname{rank} M(t_0) < m \Leftrightarrow \operatorname{rank}(A - t_0 B) < dm.$$

We recall some known properties of the Kronecker form of pencils of matrices.

$$L_k(t) = \begin{pmatrix} 1 & t & 0 & \dots & 0 \\ 0 & 1 & t & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & 1 & t & 0 \\ 0 & 0 & \dots & 1 & t \end{pmatrix},$$

$$\Omega_k(t) = \begin{pmatrix} 1 & t & 0 & \dots & 0 \\ 0 & 1 & t & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \dots & \dots & 1 & t \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Theorem

$P(A - tB)Q = diag\{L_{i_1}, ..., L_{i_s}, L^t_{j_1}, ..., L^t_{j_u}, \Omega_{k_1}, ..., \Omega_{k_v}, A' - tB'\}$
where A', B' are square matrices and B' is invertible.

Remark : The dimension $i_1, ..., i_s, j_1, ..., j_u, k_1, .., k_v$ and the determinant of $A' - tB'$ (up to a scalar) are independent of the representation and $A' - tB'$ is a *square regular pencil*.

Theorem

We have

$$\operatorname{rank}(A - tB)\ \mathit{drops}\ \Leftrightarrow \operatorname{rank}(A' - tB')\ \mathit{drops}.$$

We start with a pencil $A - tB$ where A, B are constant matrices of size $p \times q$. Set $\rho = \operatorname{rank} B$. In the following algorithm, all computational steps are easily realized via the classical LU-decomposition.

Step 1.

$$B_1 = P_0 B Q_0 = [\underbrace{B_{1,1}}_{\rho} \,|\, \underbrace{0}_{q-\rho}]$$

where $B_{1,1}$ is an echelon matrix. Then, compute

$$A_1 = P_0 A Q_0 = [\underbrace{A_{1,1}}_{\rho} \,|\, \underbrace{A_{1,2}}_{q-\rho}]$$

Step 2.

Matrices A_1 and B_1 are represented under the form

$$P_1 A_1 Q_1 = \left(\begin{array}{c|c} A'_{1,1} & A'_{1,2} \\ \hline A_2 & 0 \end{array}\right) \quad P_1 B_1 Q_1 = \left(\begin{array}{c|c} B'_{1,1} & 0 \\ \hline B_2 & 0 \end{array}\right)$$

where

- $A'_{1,2}$ has full row rank,
- $\left(\begin{array}{c} B'_{1,1} \\ \hline B_2 \end{array}\right)$ has full column rank,
- $\left(\begin{array}{c} B'_{1,1} \\ \hline B_2 \end{array}\right)$ and B_2 are in echelon form.

After steps 1 and 2, we obtain a new pencil of matrices, namely $A_2 - tB_2$.

Starting from $j = 2$, repeat the above steps 1 and 2 for the pencil $A_j - tB_j$ until the $p_j \times q_j$ matrix B_j has full column rank, that is to say until rank $B_j = q_j$.
If B_j is not a square matrix, then we repeat the above procedure with the transposed pencil $A_j^t - tB_j^t$.
At last, we obtain the regular pencil $A' - tB'$ where A', B' are two square matrices and B' is invertible.

Matrix intersection algorithm

Input : A matrix representation of a surface **S** and a parametrization of a rational space curve **C**.

Output : The intersection points of **S** and **C**.

1. Compute the matrix representation $M(t)$.

2. Compute the generalized companion matrices A and B of $M(t)$.

3. Compute the companion regular matrices A' and B'.

4. Compute the eigenvalues of (A', B').

5. For each eigenvalue t_0, the point $P(x(t_0) : y(t_0) : z(t_0) : w(t_0))$ is one of the intersection points.

Let **S** be the rational surface which is parametrized by

$$\phi : \mathbb{P}^2 \rightarrow \mathbb{P}^3 : (s : t : u) \mapsto (f_1 : f_2 : f_3 : f_4)$$

where

$$f_1 = s^3 + t^2 u, f_2 = s^2 t + t^2 u, f_3 = s^3 + t^3, f_4 = s^2 u + t^2 u.$$

the rational space curve **C** given by the parameterization

$$x(t) = 1, y(t) = t, z(t) = t^2, w(t) = t^3.$$

First, on computes a matrix representation of **S** :

$$\begin{pmatrix} 0 & 0 & 0 & w-y & 0 & 0 & z-x \\ w & 0 & 0 & x & w-y & 0 & 0 \\ x-y-z & 0 & 0 & -z & 0 & w-y & 0 \\ 0 & w & 0 & 0 & x & 0 & -y \\ 0 & x-y-z & w & 0 & -z & x & y+z-x \\ 0 & 0 & x-y-z & 0 & 0 & -z & 0 \end{pmatrix}$$

$$M(t) := \begin{pmatrix} 0 & 0 & 0 & t^3 - t & 0 & 0 & t^2 - 1 \\ t^3 & 0 & 0 & 1 & t^3 - t & 0 & 0 \\ 1 - t - t^2 & 0 & 0 & -t^2 & 0 & t^3 - t & 0 \\ 0 & t^3 & 0 & 0 & 1 & 0 & -t \\ 0 & 1 - t - t^2 & -t^3 & 0 & -t^2 & 1 & t^2 + t - 1 \\ 0 & 0 & 1 - t - t^2 & 0 & 0 & -t^2 & 0 \end{pmatrix}$$

We have $M(t) = M_3 t^3 + M_2 t^2 + M_1 t + M_0$

The generalized companion matrices of $M(t)$ are

$$A = \begin{pmatrix} 0 & I & 0 \\ 0 & 0 & I \\ M_0^t & M_1^t & M_2^t \end{pmatrix}, B = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & -M_3^t \end{pmatrix}$$

We find that the regular part of the pencil $A - tB$ is the pencil $A' - tB'$ where A' is given by

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 \\
0 & 2 & 0 & 0 & 2 & 0 & 2 & 0 & -1 & -1 & -2 & -2 & 1 \\
0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 2 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & -1 & 0 \\
0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & -1 & -1 & 0 \\
0 & 1 & -1 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & -1 & -1 & 0 \\
0 & -1 & 0 & 0 & -2 & 0 & -1 & 0 & 0 & 0 & 1 & 2 & -1
\end{pmatrix},$$

and B' is the identity matrix.

Then, we compute the following eigenvalues : $t_1 = 1$, $t_2 = -1$ and the roots of the equation $Z^7 + 3Z^6 - Z^5 - Z^3 + Z^2 - 2Z + 1 = 0$.

- Introduce new matrix based representation of rational surfaces that are allowed to be non square.
- Transfer the solving of the curve/surface intersection problem into the eigenvalues computing problems
- Develop a symbolic/numeric algorithm to manipulate these new representations.

Thank you for attention